Übungen zu

Computational Finance II

Exercise 4 Tree for Two Assets

A two-asset extension of the binomial tree with (x, y)-coordinates representing the assets, and time-coordinate t, is assumed to develop as follows: Each node with position (x, y) may develop for $t \to t + \Delta t$ with equal probabilities 0.25 to the four positions

$$(xu, yA),\ (xu, yB),\ (xd, yC),\ (xd, yD) \qquad (*)$$

for constants u, d, A, B, C, D.

a) Show that the tree is recombining for $AD = BC$.

Hint: Sketch the possible values in a (x, y)-plane.

Following Rubinstein, a tree is defined for interest rate r, asset parameters σ_1, σ_2, correlation ρ, and dividend yield rates δ_1, δ_2, by

$$\begin{aligned}
\mu_i &:= r - \delta_i - \sigma_i^2/2 \quad \text{for } i = 1, 2 \\
u &:= \exp(\mu_1 \Delta t + \sigma_1 \sqrt{\Delta t}) \\
d &:= \exp(\mu_1 \Delta t - \sigma_1 \sqrt{\Delta t}) \\
A &:= \exp(\mu_2 \Delta t + \sigma_2 \sqrt{\Delta t}[\rho + \sqrt{1-\rho^2}]) \\
B &:= \exp(\mu_2 \Delta t + \sigma_2 \sqrt{\Delta t}[\rho - \sqrt{1-\rho^2}]) \\
C &:= \exp(\mu_2 \Delta t - \sigma_2 \sqrt{\Delta t}[\rho - \sqrt{1-\rho^2}]) \\
D &:= \exp(\mu_2 \Delta t - \sigma_2 \sqrt{\Delta t}[\rho + \sqrt{1-\rho^2}])
\end{aligned}$$

For initial prices $x^0 := S_1^0$, $y^0 := S_2^0$, and time level $t_\nu := \nu \Delta t$, the S_1-components of the grid according to $(*)$ distribute in the same way as for the one-dimensional tree,

$$x_i^\nu := S_1^0 u^i d^{\nu - i} \quad \text{for } i = 0, \ldots, \nu \, .$$

b) Verify that the above choice of A, B, C, D sets up a recombining tree.

c) Show that the second (S_2-)components belonging to x_i^ν are

$$y_{i,j}^\nu := S_2^0 \exp(\mu_2 \nu \Delta t) \, \exp\left(\sigma_2 \sqrt{\Delta t} \left[\rho(2i - \nu) + \sqrt{1-\rho^2}(2j - \nu)\right]\right)$$

for $j = 0, \ldots, \nu$.

Hint: For $\nu \to \nu + 1$, u corresponds to $i \to i + 1$, and d corresponds to $i \to i$.

d) Show that the first two moments of the continuous and the discrete model match: Verify that for the log-variables $\Delta Y_1 := \log u$ or $\log d$, and $\Delta Y_2 := \log A$, $\log B$, $\log C$, or $\log D$ the five equations

$$\mathsf{E}(\Delta Y_i) = \mu_i \Delta t \, , \quad \mathsf{Var}(\Delta Y_i) = \sigma_i^2 \Delta t \, , \quad \mathsf{Cov}(\Delta Y_1, \Delta Y_2) = \rho \sigma_1 \sigma_2 \Delta t$$

hold (for $i = 1, 2$ and the four probabilities $\frac{1}{4}$ associated to $(*)$).

Recommendation:

e) Set up a computer program that implements this binomial method. Analogously as for the binomial tree, work in a backward recursion for $\nu = M, \ldots, 0$. For each time level t_ν set up the (x, y)-grid with the above rules and $\Delta t = T/M$. For $t_M = T$ fix V by the payoff Ψ, and use for $\nu < M$

$$V_{i,j}^{\text{cont}} = \exp(-r\Delta t)\,\frac{1}{4}(V_{i,j}^{\nu+1} + V_{i+1,j}^{\nu+1} + V_{i,j+1}^{\nu+1} + V_{i+1,j+1}^{\nu+1})\,.$$

Test example: max call with $\Psi(S_1, S_2) = (\max(S_1, S_2) - K)^+$, $S_1^0 = S_2^0 = K = T = 1$, $r = 0.1$, $\sigma_1 = 0.2$, $\sigma_2 = 0.3$, $\rho = 0.25$, $\delta_1 = 0.05$, $\delta_2 = 0.3$. For $M = 2000$ an approximation of the American-style option is 0.130302, and for the European style 0.120036.

Illustrations from [R. Seydel: Tools for CF, 5th edition (2012)]:

Max call, with payoff $\Psi(S_1, S_2) = (\max(S_1, S_2) - K)^+$

top: (S_1, S_2)-plane with the grid of the tree for the payoff, $t = T$, with $M = 20$;

bottom: the payoff