Übungen zu
Computational Finance II

Background: Two versions of the same equation are

\[
\begin{align*}
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma_1^2 S_1 \frac{\partial^2 V}{\partial S_1^2} + (r - \delta_1) S_1 \frac{\partial V}{\partial S_1} - rV \\
+ \frac{1}{2} \sigma_2^2 S_2 \frac{\partial^2 V}{\partial S_2^2} + (r - \delta_2) S_2 \frac{\partial V}{\partial S_2} + \rho \sigma_1 \sigma_2 S_1 \frac{\partial^2 V}{\partial S_1 \partial S_2} = 0
\end{align*}
\]  

(1)

and

\[
-\nabla \cdot (D(x, y) \nabla u) + b(x, y)^* \nabla u + ru = u_t = -\frac{\partial}{\partial \tau} u.
\]  

(2)

Exercise 11

a) Prove the equivalence of (1) and (2). Specialize this to the one-dimensional case of the Black–Scholes equation.

b) Show

\[b^* \nabla u + ru = \nabla \cdot (bu) + \gamma u\]

and determine \(\gamma\).

c) With the transformation

\[x := \log\left(\frac{S_1}{K_1}\right), \quad y := \log\left(\frac{S_2}{K_2}\right)\]

and writing \(u(x, y, t)\) for \(V\) leads to the PDE

\[u_t + \frac{1}{2} \sigma_1^2 u_{xx} + (r - \delta_1 - \frac{1}{2} \sigma_1^2) u_x - ru \\
+ \frac{1}{2} \sigma_2^2 u_{yy} + (r - \delta_2 - \frac{1}{2} \sigma_2^2) u_y + \rho \sigma_1 \sigma_2 u_{xy} = 0.\]

What are the matrix \(D\) and the vector \(b\) such that we arrive at (2)?

Exercise 12

In the three-dimensional \((x, y, w)\)-space let the plane \(w(x, y) = c_1 + c_2 x + c_3 y\) interpolate the three points \((x_i, y_i, w_i), \ i = 1, 2, 3\). Show

\[
\begin{pmatrix}
1 & x_1 & y_1 \\
1 & x_2 & y_2 \\
1 & x_3 & y_3
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_2 \\
c_3
\end{pmatrix}
=
\begin{pmatrix}
w_1 \\
w_2 \\
w_3
\end{pmatrix}.
\]

By inversion, establish a formula for \(\nabla w = (c_2, c_3)^*\).
Exercise 13

Consider the domain $D := \{(x, y) \mid x \geq 0, y \geq 0, 1 \leq x + y \leq 2\}$ tilled by 12 triangles $D_k$, where triangles and nodes are numbered as in the figure below.

a) Set up the index set $I$ with entries $I_k = \{i_k, j_k, l_k\}$, which assigns node numbers to the $k$th triangle for $1 \leq k \leq 12$.

b) Formulate the assembling algorithm that builds up the global matrix out of the local matrices

$$
\begin{bmatrix}
  s_{11}^{(k)} & s_{12}^{(k)} & s_{13}^{(k)} \\
  s_{21}^{(k)} & s_{22}^{(k)} & s_{23}^{(k)} \\
  s_{31}^{(k)} & s_{32}^{(k)} & s_{33}^{(k)}
\end{bmatrix}
$$

for a general index set $I$ and $1 \leq k \leq m$.

c) The example of the figure leads to a banded stiffness matrix. What is the bandwidth?

Figure: Specific triangulation and numbering

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Besprechung dieses Blattes voraussichtlich am 4.2.2014.

Klausur: Mittwoch, 19.2.2014, 14-15 Uhr in H IV