

TEICHMÜLLER CURVES, GALOIS ACTIONS AND \widehat{GT} -RELATIONS

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ABSTRACT. Teichmüller curves are geodesic discs in Teichmüller space that project to algebraic curves C in the moduli space M_g . Some Teichmüller curves can be considered as components of Hurwitz spaces. We show that the absolute Galois group $G_{\mathbb{Q}}$ acts faithfully on the set of these embedded curves.

We also compare the action of $G_{\mathbb{Q}}$ on $\pi_1(C)$ with the one on $\pi_1(M_g)$ and obtain a relation in the Grothendieck-Teichmüller group, seemingly independent of the known ones.

INTRODUCTION

Consider a complex geodesic for the Teichmüller metric $\tilde{j} : \mathbb{H} \rightarrow T_g$ from the upper half plane to Teichmüller space. These geodesics are generated by a pair (X, q) of a Riemann surface X of genus g and a quadratic differential q . The (rare) examples where the stabilizer in the mapping class group of \tilde{j} is a lattice $\Gamma \subset \text{Aut}(\mathbb{H})$ are called *Teichmüller curves*.

A particular case of these geodesics with lattice stabilizer can be described as follows: Take an assemblage of squares of paper and glue them along their edges to a surface without boundary, such that at each vertex about an even number of squares. If we provide the squares with a complex structure and glue the local quadratic differentials dz^2 , we obtain a pair (X, q) .

This description lead Lochak ([11]) to baptise them *origamis*. If the glueing is just by translations (and not by (-1) composed by a translation) the origami is said to be *oriented*. Oriented origamis are also known as square-tiled coverings or non-primitive Teichmüller curves. Lochak remarked that the corresponding *origami curves* $j : C = \mathbb{H}/\Gamma \rightarrow M_g$ in the moduli space of curves are defined over number fields and hence interesting not only in dynamical systems but also from a number theoretical viewpoint.

We study two examples (section 4) of origamis: the smallest example (called $L(2, 2)$) where X has genus greater than one and the smallest example (called \mathcal{S}_2) with a property relevant for the \widehat{GT} -comparisons, see below. In both cases we explicitly write down the equation of the origami curve in moduli space. This possibility seems rather unexpected from the 'geodesic' viewpoint.

The above definition suggests that – in the oriented case – origami curves are in fact (the analytic version of) some Hurwitz spaces for coverings of elliptic curves, ramified at most over ∞ . It is very natural to consider the geodesic discs then in $M_{g, [n]}$ with the n preimages of ∞ marked but unordered. The Hurwitz viewpoint reproves that origami curves and the map j are defined over number fields. It implies that there is an action of the absolute

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Galois group $G_{\mathbb{Q}}$ on the set of origami curves. Our first main result is that this action is faithful in the following sense (we abbreviate $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ by \mathbb{P}^* throughout):

Theorem 5.4 *For each $\sigma \in G_{\mathbb{Q}}$ there is an origami curve C isomorphic to \mathbb{P}^* , such that σ acts non-trivially on the map $j(C) \rightarrow M_{g,[n]}$, more precisely such that $j(C) \neq j^{\sigma}(C)$.*

There are examples of origami curves, that do not have genus 0. ([20]). It would be interesting to know, if the $G_{\mathbb{Q}}$ -action is faithful on the (abstract, not embedded) origami curves.

Let us give an overview over Grothendieck-Teichmüller theory. The technical details will be explained in section 6.

The absolute Galois group $G_{\mathbb{Q}}$ acts on the algebraic fundamental group $\pi_1(\widehat{\mathbb{P}^*_{\mathbb{Q}}})$, which is isomorphic to the profinite free group \widehat{F}_2 in two generators. This action can be described in terms of pairs $(\lambda_{\sigma}, f_{\sigma}) \in \widehat{\mathbb{Z}} \times (\widehat{F}_2)'$ with a suitable composition law. The subgroup of pairs satisfying three equations (due to Drinfel'd) is the Grothendieck-Teichmüller group \widehat{GT} . By construction it contains $G_{\mathbb{Q}}$ and the question is whether it is strictly larger than $G_{\mathbb{Q}}$ or not. One hence would like to find a set of equations that singles out precisely $G_{\mathbb{Q}}$. Where should these relations come from? Consider a category \mathcal{C} of varieties (or stacks) over \mathbb{Q} with some morphisms defined over \mathbb{Q} . For a large enough \mathcal{C} only $G_{\mathbb{Q}}$ acts equivariantly on all $\pi_1(\mathcal{C})$ by a result of F. Pop.

But only for a few types of 'geometric' morphisms one is able to express the equivariance in terms of (λ, f) . Among these are some coverings of curves ([17], [18]), and 'natural' morphisms between moduli spaces of curves ([5], [17], [22]). Maps from curves to moduli spaces should be considered next.

The map of some origami curves to moduli space are defined over \mathbb{Q} and 'geometric' in this sense. We need that the curve $C \cong \mathbb{P}^*$ and that it passes through a maximally degenerate point of the (compactified) moduli space. The smallest such origami is the \mathcal{S}_2 .

Let $\alpha_1, \dots, \alpha_5$ denote the standard generators (see section 7) of the profinite mapping class group $\widehat{\Gamma}_{2,0}$. For $\sigma \in G_{\mathbb{Q}}$ such that the Kummer cocycle $\rho_2(\sigma) = 0$ we obtain the following relation:

Theorem 7.1' *The element $(\lambda, f) \in \widehat{GT}$ respects the Galois actions on the morphism $j : C^{\text{orb}} \rightarrow \mathcal{M}_2$ induced from the two-steps origami \mathcal{S}_2 (see figure 4) if and only if*

$$f(\alpha_3, (\alpha_1^2 \alpha_2)^4) f(\alpha_1^2, \alpha_2^2) f(\alpha_5^2, \alpha_4^2) f(\alpha_2 \alpha_4, \alpha_1^2 \alpha_3 \alpha_5^2) = 1$$

holds in $\widehat{\Gamma}_{2,0}$. The elements (λ, f) satisfying this relation form a subgroup of \widehat{GT} containing $G_{\mathbb{Q}}$.

In section 7 we prove the refined relation for all $G_{\mathbb{Q}}$. As for all the relations in \widehat{GT} recently discovered it is not known, if the subgroup defined by this relation is properly between $G_{\mathbb{Q}}$ and \widehat{GT} .

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1. TEICHMÜLLER CURVES

A holomorphic quadratic differential $q \neq 0$ on a Riemann surface X of genus g determines on X minus the set of zeroes of q an atlas of open charts, whose transition functions are

of the form $z \mapsto \pm z + c$. Such an atlas is called a *flat structure* on X . Conversely a flat structure determines a quadratic differential by glueing the local dz^2 's. This correspondence is given in more details in [11] Ch. 2.

15:58 There is a natural $\mathrm{SL}_2(\mathbb{R})$ action on pairs (X, q) by postcomposing the charts of the flat structure with the linear map. The stabilizer of a pair (X, q) is precisely $\mathrm{SO}_2(\mathbb{R})$. We define the *Teichmüller space* \mathcal{T}_g as the space of Riemann surfaces plus an isotopy class of orientation-preserving diffeomorphism (a *Teichmüller marking*) to a reference surface Σ_g . If we choose a Teichmüller marking on X , the action $\mathrm{SL}_2(\mathbb{R}) \cdot (X, q)$ yields a geodesic curve (by Teichmüller's theorems) $\tilde{j} : \mathbb{H} \rightarrow \mathcal{T}_g$. We denote by \mathcal{M}_g the moduli stack of curves and the corresponding coarse moduli space by non-calligraphic letters, i.e. M_g .

Consider the projection of a geodesic $\mathbb{H} \rightarrow \mathcal{T}_g$ to M_g . We consider the moduli space in the first two sections in the analytic category. More precisely we should write $(M_g)_{\mathbb{C}}^{\mathrm{an}}$, etc. Let $\Gamma_{g,n}$ be the mapping class group of Riemann surfaces of genus g with n punctures. We define $\mathrm{Stab}(\tilde{j}) \subset \Gamma_{g,0}$ to be the (setwise) stabilizer of $\tilde{j}(\mathbb{H})$ and $\mathrm{Aut}(\tilde{j})$ to be the pointwise stabilizer. We summarize this by an exact sequence

$$1 \longrightarrow \mathrm{Aut}(\tilde{j}) \longrightarrow \mathrm{Stab}(\tilde{j}) \longrightarrow \overline{\mathrm{Stab}(\tilde{j})} \longrightarrow 1.$$

The map \tilde{j} descends to a map

$$j : C := \mathbb{H}/\overline{\mathrm{Stab}(\tilde{j})} \rightarrow M_g.$$

Definition 1.1. *This map j is called a Teichmüller curve if $\overline{\mathrm{Stab}(\tilde{j})}$ is a lattice in $\mathrm{Aut}(\mathbb{H})/\{\pm 1\}$.*

Later on it will sometimes be natural to fix say n marked points, consider $\tilde{j} : \mathbb{H} \rightarrow \mathcal{T}_{g,n}$ and consider the stabilizer in $\Gamma_{g,n}$ (or in $\Gamma_{g,[n]}$, if we allow permutation of the marked points). We then call the groups $\mathrm{Stab}(\tilde{j}, n)$ (or $\mathrm{Stab}(\tilde{j}, [n])$) etc.

We can also describe this group using the flat structure defined by (X, q) . Denote by $\mathrm{Aff}^+(X, q)$ the group of orientation preserving affine diffeomorphisms of X , i.e. diffeomorphisms which are affine with respect to the charts of the flat structure determined by q . Associating with $\varphi \in \mathrm{Aff}^+(X, q)$ the matrix part of the affine maps yields a well-defined map D to $\mathrm{PSL}_2(\mathbb{R})$. We denote the image of D by $\mathrm{PSL}(X, q)$. We obtain an exact sequence

$$1 \longrightarrow \mathrm{Aut}(X, q) \longrightarrow \mathrm{Aff}^+(X, q) \xrightarrow{D} \mathrm{PSL}(X, q) \longrightarrow 1,$$

where $\mathrm{Aut}(X, q)$ are the (conformal) automorphisms of X preserving q .

We will also consider the subgroup of $\mathrm{Aff}^+(X, q)$ that fixes n points (resp. up to permutation). We will denote these groups by $\mathrm{Aff}^+(X, q, n)$ (resp. $\mathrm{Aff}^+(X, q, [n])$) and their images under D by $\mathrm{PSL}(X, q, n)$ (resp. $\mathrm{PSL}(X, q, [n])$). Affine diffeomorphisms in $\mathrm{Aff}^+(X, q, [n])$ are called *balanced*.

Remark 1.2. i) The groups $\overline{\mathrm{Stab}(\tilde{j})}$ and $\mathrm{PSL}(X, q)$ are closely related, namely (see [14] Prop. 3.2) if we identify $\mathrm{Aut}(\mathbb{H})/\{\pm 1\}$ with $\mathrm{PSL}_2(\mathbb{R})$, we have

$$\overline{\mathrm{Stab}(\tilde{j})} = R \cdot \mathrm{PSL}(X, q) \cdot R, \quad \text{where } R = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The same holds for the corresponding groups with marked points.

ii) While $\overline{\mathrm{Stab}(\tilde{j})}$ is best suited for moduli problems, we will use the definition of $\mathrm{PSL}(X, q)$ for calculations of these groups, because affine diffeomorphisms are easily visualized.

2. ORIGAMIS

We consider now a special case of Teichmüller curves:

Definition 2.1. *An origami is a finite set of squares (say unit squares in \mathbb{R}^2) glued together along their edges to a surface without boundary, such that at each vertex about an even number of squares.*

Using also open charts covering the glueing edges and identifying \mathbb{R}^2 with \mathbb{C} , this construction defines a Riemann surface X with a flat structure. The local dz^2 's glue together to a quadratic differential q , which is holomorphic except for simple poles at vertices where only 2 squares abut.

We can distinguish two cases: If $q = \omega^2$ is a square of a one-form $\omega \in H^0(X, \Omega_X)$ the origami is said to be *oriented*, otherwise *non-oriented*.

If the origami is oriented, the number of squares abutting at each vertex is divisible by 4. This condition is not sufficient, as shown by the following example, where A is glued to A' etc. along the orientation of the arrow:

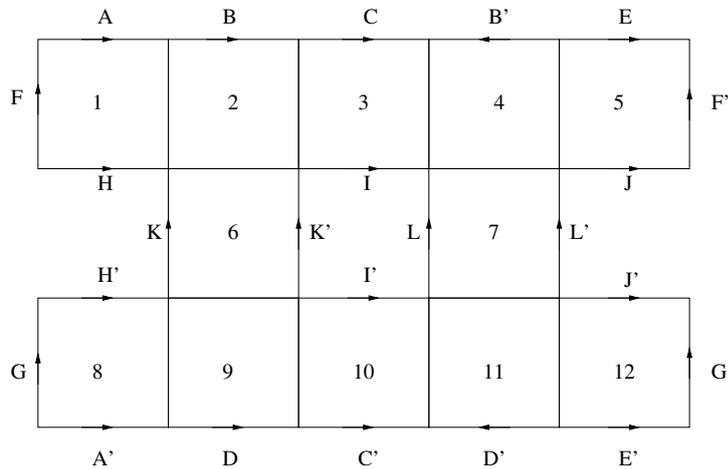


FIGURE 1. A non-oriented origami without poles

Note however, that if the origami is non-oriented, there is a canonical double covering (ramified precisely over the points, where 2 squares abutt), which is an oriented origami. *In the sequel we will treat only the oriented case.* A necessary and sufficient condition for an origami to be oriented is that the transition functions between the charts consist only of translations. In origami language this means that upper (resp. left) edges should be glued to lower (resp. right) edges preserving global orientation.

Definition 2.2. *An origami in the above sense defines an unramified covering $\pi : X^* \rightarrow E^*$ of a torus punctured at ∞ and a ramified covering $X \rightarrow E$, also denoted by π . We call this an origami covering.*

Note that by identifying \mathbb{R}^2 with \mathbb{C} we used the elliptic curve with $j(E) = 1728$ in the first definition. This choice plays no role in the sequel, because the geodesic curve generated by (X, q) is independent of this choice.

The fact that origamis indeed define Teichmüller curves, follows from the result of [4], that $\overline{\text{Stab}}(\tilde{j})$ is commensurable with $\text{PSL}_2(\mathbb{Z})$. This will also follow from the Hurwitz space description in section 3.

Remark 2.3. We will specify the (unramified) covering π of degree d by its monodromy: Fix two generators a and b of $\pi_1(E^*)$ and their images under the monodromy map $m : \pi_1(E^*) \rightarrow S_d$. These images determine π . Note that simultaneous conjugation in S_d (i.e. renumbering the preimages of a basepoint) gives the same covering. Note also that topologically different coverings may lead to the same origami curve.

The quadratic differential q on X is obtained using the origami covering as $q = \omega^2$, where $\omega = \pi^*\omega_E$ and ω_E is the unique (up to scalar multiple) holomorphic differential on E .

In the case of origamis it is natural to consider the groups $\text{Stab}(\tilde{j}, [n])$ (or $\text{Stab}(\tilde{j}, n)$), where the marked n points are the preimages $\pi^{-1}(\infty)$. This is for two reasons: First, this finite set of additional marked points replaces $\text{Stab}(\tilde{j})$ by a subgroup of finite index (see [4]). Note that not any set of additional marked points has this property, see [3]. Second, the following definitions are best suited with the Hurwitz space interpretation in section 3 and yield that $\overline{\text{Stab}}(\tilde{j})$ is contained in $\text{PSL}_2(\mathbb{Z})$.

Definition 2.4. The group $\Gamma(\pi) := \overline{\text{Stab}(\tilde{j}, [n])}$ is called the affine group of the origami. The quotient $C(\pi) = \mathbb{H}/\Gamma(\pi)$ (or simply C) is called the origami curve. If we consider it as orbifold quotient $C^{\text{orb}}(\pi) := \mathbb{H}/\text{Stab}(\tilde{j}, [n])$ it is called the orbifold origami curve.

The map $\mathbb{H}/\text{PSL}(X, q) \rightarrow M_g$ is generically injective and actually injective up to finitely many normal crossings, because it is the image of a geodesic locus under the quotient map of a discrete group. But the map $C \rightarrow M_g$ might be (see the following example) a composition of a covering and a generically injective map. If we want to reestablish injectivity, we will consider the origami curve in $M_{g,[n]}$, the moduli space of curves with n non-ordered points.

To give an example that $\Gamma(\pi)$ is a proper subgroup of $\overline{\text{Stab}}(\tilde{j})$ consider the following thick L (see figure 2: The sides L_i are glued with R_i and D_i with U_i , preserving the global orientation. One checks that for this origami the matrix $\begin{pmatrix} 0 & -1/2 \\ 2 & 0 \end{pmatrix}$ is in $\text{PSL}(X, q)$ but neither in $\Gamma(\pi)$ nor in $\text{PSL}_2(\mathbb{Z})$.

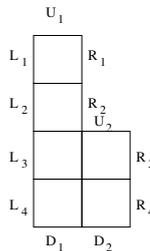


FIGURE 2. The thick L

Remark 2.5. The affine group also admits the following description (see also [20]): One can always lift a balanced affine diffeomorphism $\varphi \in \text{Aff}^+(X, q, [n])$ to its universal cover \mathbb{H} . Indeed one can lift an affine diffeomorphism locally to an unramified cover and

non-trivial paths provide the obstruction to do this globally. Denote by $q_{\mathbb{H}}$ the quadratic differential on \mathbb{H} obtained by pullback of the square of ω_E via the universal covering map $\pi_{\infty} : \mathbb{H} \rightarrow E$. One has a natural morphism

$$* : \begin{cases} \text{Aff}^+(\mathbb{H}, q_{\mathbb{H}}) & \rightarrow & \text{Aut}^+(\pi_1(E^*)) \\ \varphi & \mapsto & \varphi_* := (f \mapsto \varphi^{-1} \circ f \circ \varphi) \end{cases} ,$$

where we consider $\pi_1(E^*)$ as $\text{Aut}(\mathbb{H}/E)$ and composition is meant in $\text{Aut}(\mathbb{H})$. The 'plus' of $\text{Aut}^+(\pi_1(E^*))$ denotes the preimage of $\text{SL}_2(\mathbb{Z})$ under the quotient map by $D : \text{Aut}(\pi_1(E^*))/\text{Inn}(\pi_1(E^*)) \rightarrow \text{GL}_2(\mathbb{Z})$. Denote by $\text{Aut}_i(\mathbb{H}/E)$ the automorphisms of \mathbb{H} over the identity or the elliptic involution of i . The right and the left vertical morphism of the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \text{Aut}_i(\mathbb{H}/E) & \longrightarrow & \text{Aff}^+(\mathbb{H}, q_{\mathbb{H}}) & \longrightarrow & \text{PSL}_2(\mathbb{Z}) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow * & & \downarrow & & \\ 1 & \longrightarrow & \text{Inn}(\pi_1(E^*)) \rtimes \langle i_* \rangle & \longrightarrow & \text{Aut}^+(\pi_1(E^*)) & \longrightarrow & \text{PSL}_2(\mathbb{Z}) & \longrightarrow & 1 \end{array}$$

are isomorphisms, hence $*$ is an isomorphism, too.

In this language the affine group $\Gamma(\pi)$ is the image under D of the subgroup of $\text{Aut}^+(\pi_1(E^*))$ that fixes $\pi_1(X)$ (as set).

With this description the following Lemma is obvious:

Lemma 2.6. *Suppose an origami covering factors as $\pi = \psi \circ \tilde{\pi} : X \rightarrow Y \rightarrow E$. Denote the corresponding unramified coverings by $X^* \rightarrow Y^* \rightarrow E^*$. If the fundamental group $\pi_1(Y^*)$ is characteristic in $\pi_1(E^*)$ any balanced affine diffeomorphism $\varphi : X \rightarrow X$ descends to a balanced affine diffeomorphism $\varphi_Y : Y \rightarrow Y$.*

Remark 2.7. Recall that a holomorphic quadratic differential q is called *Strebel*, if its horizontal trajectories are compact or connect two zeroes of q . A direction $e^{2\pi i\theta}$ for $\theta \in \mathbb{R}$ is called Strebel for q , if $e^{2\pi i\theta}q$ is Strebel. A Strebel differential decomposes the Riemann surface into cylinders swept out by trajectories. With our convention for origamis ($\mathbb{R}^2 \cong \mathbb{C}$ and q is made by local dz^2), the direction $e^{2\pi i\theta}$ is Strebel if and only if $\theta \in \mathbb{Q}$. We will tacitly assume this in the sequel.

Remark 2.8. If the origami is given by its monodromy $m(a), m(b) \in S_d$ the horizontal trajectories decompose X into c_a disjoint unit height cylinders, where c_a is the number of cycles of the permutation $m(a)$. These cylinders lie in $c_a^{\max} \leq c_a$ maximal cylinders. Denote by $\alpha_i, i = 1, \dots, c_a^{\max}$ the core curves of the maximal cylinders. If both sides of a unit height cylinder contains a zero of ω , it is of course a maximal cylinder.

Consider the family of curves

$$X_t := \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \cdot X.$$

By Thm. 3 in [13] precisely the hyperbolic lengths of the homotopy classes of α_i tend to zero (as t tends to infinity). Hence the family of smooth curves X_t tends for $t \rightarrow \infty$ to the stable curve obtained by 'pinching' α_i to nodes.

3. ORIGAMI CURVES ARE COMPONENTS OF HURWITZ SPACES

Origami curves were defined in the previous sections as quotients $C = \mathbb{H}/\Gamma(\pi)$ of discs in Teichmüller space, hence in the analytic category. But the differential ω (or $q = \omega^2$) was defined by covering data of an elliptic curve (e.g. E_{1728}). One expects that the (orbifold) origami curve in the analytification of a component of an algebraic Hurwitz (stack resp.) space. We prove here that this is indeed the case.

We start with generalities on (algebraic) Hurwitz stacks. Roughly speaking they parametrize isomorphism classes of coverings. Isomorphism here means isomorphism over a *fixed* base curve. Notations in this section follow Wewers ([23]). We only need covers of smooth schemes and we assume all schemes to be schemes over \mathbb{Q} .

Let $f : \mathcal{E} \rightarrow \mathcal{M}$ denote the universal family of curves over the smooth stack \mathcal{M} . Fix a ("ramification") divisor \mathcal{D}/\mathcal{M} . Of course we have elliptic curves, maybe with additional structures in mind.

For any \mathcal{M} -scheme S let $\mathcal{H}_{\mathcal{E}}(S)$ be the category of finite covers $X \rightarrow E_S$ of fixed degree d and fixed genus g of X , ramified over $D_S := \mathcal{D} \times_{\mathcal{M}} S$, where $E_S := \mathcal{E} \times_{\mathcal{M}} S$. (Maybe the notation $\mathcal{H}_{\mathcal{E},g,d}$ would be more precise.) A morphism from $X \rightarrow E_S \rightarrow S$ to $X' \rightarrow E_{S'} \rightarrow S'$ is a morphism $S \rightarrow S'$ plus a morphism $X \rightarrow X'$ over the induced morphism $E_S \rightarrow E_{S'}$. We cite Th. 4.1.2 from [23]:

Proposition 3.1. *The Hurwitz stack $\mathcal{H}_{\mathcal{E}}$ is a smooth stack over \mathbb{Q} , étale over \mathcal{M} . The stack $\mathcal{H}_{\mathcal{E}}$ has a coarse moduli space $H_{\mathcal{E}}$, which is also defined over \mathbb{Q} .*

If we replace in the above definition \mathcal{M} by $\mathcal{M}_{\mathbb{C}}^{\text{an}}$ we obtain an analytic stack $\mathcal{H}_{\mathcal{E}}^{\text{an}}$, whose coarse moduli space is $(H_{\mathcal{E}})_{\mathbb{C}}^{\text{an}}$. In the sequel we do not distinguish between \mathcal{E} and $\mathcal{E}_{\mathbb{C}}$ for simplicity of notation.

Proposition 3.2. *Let \mathcal{E} be the universal family over the moduli stack $\mathcal{M}_{1,1}$ and \mathcal{D} the divisor corresponding to the section. Let \mathcal{H}_{π} be the connected component of $\mathcal{H}_{\mathcal{E}}$ which contains $\pi : X \rightarrow E$.*

Then the orbifold origami curve $C^{\text{orb}} = C^{\text{orb}}(\pi)$ coincides (as functor (analytic spaces) \mapsto (sets)) with $(\mathcal{H}_{\pi})_{\mathbb{C}}^{\text{an}}$.

Proof: Consider the analytic stack $\mathcal{TH}_{\mathcal{E}}$ whose objects (over S) are coverings $X \rightarrow E_S \rightarrow S$ in $(\mathcal{H}_{\mathcal{E}})_{\mathbb{C}}^{\text{an}}(S)$ with compatible Teichmüller markings on X and E_S . Because \mathcal{T}_g is étale over $(\mathcal{M}_g)_{\mathbb{C}}^{\text{an}}$ and because of Prop. 3.1 the functor 'forget X ' exhibits $(\mathcal{TH}_{\mathcal{E}})$ as an étale cover of $\mathcal{T}_{1,1} \cong \mathbb{H}$. Hence it consists of several disjoint copies of \mathbb{H} , one of which contains $\pi : X \rightarrow E$. We call this connected component \mathcal{TH}_{π} and we let $(\mathcal{H}_{\pi})_{\mathbb{C}}^{\text{an}}$ be the corresponding component of $(\mathcal{H}_{\mathcal{E}})_{\mathbb{C}}^{\text{an}}$ (after forgetting the Teichmüller marking). $(\mathcal{H}_{\pi})_{\mathbb{C}}^{\text{an}}$ stems from a uniquely determined connected component \mathcal{H}_{π} of $\mathcal{H}_{\mathcal{E}}$.

It remains to check that $(\mathcal{H}_{\pi})_{\mathbb{C}}^{\text{an}}$ is the quotient stack $\mathcal{TH}_{\pi}/\text{Stab}(\mathfrak{j}, [n])$. This easily follows from unwinding definitions, noting that Deck transformations respect the set $\pi^{-1}(\infty)$. \square

Corollary 3.3. *Origami curves are geometric components of a Hurwitz space defined over \mathbb{Q} . Hence they are defined over number fields and there is a natural $G_{\mathbb{Q}}$ -action on the set of origami curves.*

The morphism $j : C \rightarrow M_g$ (and its orbifold version) is defined over a number field and there is a natural $G_{\mathbb{Q}}$ -action on the set of embedded origami curves.

Proof: For the second claim note that the forgetful functor $\mathcal{H}_{\mathcal{E}} \rightarrow \mathcal{M}_g$ is defined over \mathbb{Q} . Hence the morphism between the geometric components of the (coarsely) representing schemes are defined over finite extensions of \mathbb{Q} . \square

Remark 3.4. There are obviously substacks of $\mathcal{H}_{\mathcal{E}}$ which are defined over \mathbb{Q} , for example the Hurwitz spaces with fixed monodromy (see e.g. [23]). Each Galois invariant additional structure describes a substack defined over \mathbb{Q} . Finding components of $\mathcal{H}_{\mathbb{P}^*}$ that are irreducible over \mathbb{Q} has been studied in the context of dessins d'enfants under the name of giving a *complete list of Galois invariants*. One can ask the same question for $\mathcal{H}_{\mathcal{E}}$.

Here is a list (certainly not complete) of Galois invariants known to the author. The given references consider the invariants from quite different viewpoints.

- Monodromy groups and ramification indices, or more generally the Nielsen classes are Galois invariant.
- If the ramification indices of π over ∞ are all odd, the parity of the spin structure (see [10]) is a Galois invariant.
- If $g(X) = 2$ there is a 2-division point μ_2 in E such that $\pi + \mu_2 : X \rightarrow E$ is equivariant with respect to the (hyper)elliptic involutions on X and E (see [9]). The property whether or not $\mu_2 = 0$ is a Galois invariant.

4. TWO EXAMPLES

In this section we examine two origamis, the smallest origami which is not an elliptic curve and the smallest one that can be used for \widehat{GT} -considerations. In both cases we explicitly describe the equation of the geodesic curve.

4.1. The $L(2, 2)$. We now study the simplest origami which is not an elliptic curve, i.e. such that π is not an isogeny. It is called $L(2, 2)$ in [11], see the left half of figure 3. The sides are glued 'naturally', i.e. L_i with R_i and U_i with D_i . The affine group of this origami $\Gamma(L(2, 2))$ contains the horizontal and vertical translation by 2, hence the modular group $\Gamma(2)$ and also

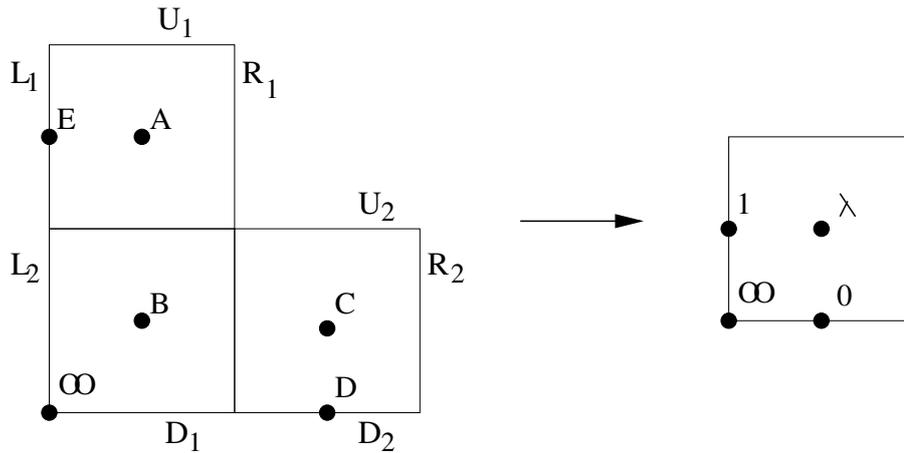
$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{but not } T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

In fact S corresponds to rotation by 90° . And if T were in $\Gamma(L(2, 2))$ one could map R_1 to one of the other vertical edges and extend this map to a diffeomorphism, that locally looks like T . But this leads to a contradiction in each of the cases. See [20] for an algorithm to determine the affine group of an origami. We conclude that $C \rightarrow M_{1,1}$ is a degree 3 cover.

We can illustrate Remark 2.3 here: Fix E and $a, b \in \pi_1(E)$. The coverings with monodromy $m(a) = (1)(23)$, $m(b) = (12)(3)$ (the $L(2, 2)$) and with monodromy $m'(a) = (123)$, $m'(b) = (12)(3)$ are topologically different (i.e. one cannot obtain one from the other by Deck transformations and renumbering of the preimages). Nevertheless the origami curves coincide.

We can see geometrically, that the corresponding origami covering $\pi := \pi_{L(2,2)}$ commutes with the hyperelliptic involutions h_X and h_E on X and E respectively (see figure 3).

When giving the equation of this family, we work for simplicity over $\mathbb{P}^* = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ (with coordinate t) instead of its quotient by S to have all Weierstraß points available.


 FIGURE 3. Weierstrass points of the $L(2,2)$

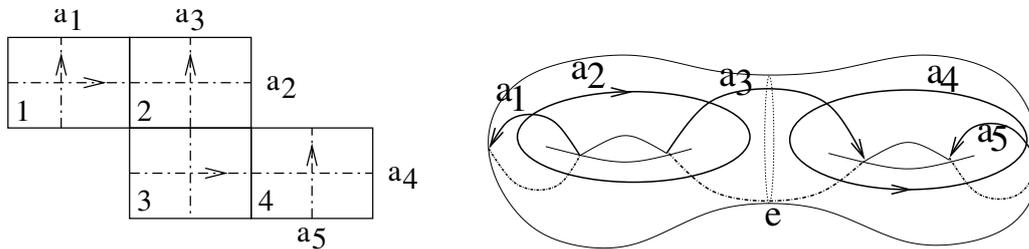
Proposition 4.1. *The equation of the family \mathcal{X}/\mathbb{P}^* is given by*

$$y^2 = x(x-4)(P(x) - t), \text{ where } P(x) = \frac{1}{4}x(x-3)^2.$$

together with the morphism $\pi(x, y) = (P(x), yQ(x))$, where $Q(x) = (x-3)(x-1)/4$.

Proof: Denote by $q : X \rightarrow X/h_X$ and $q_E : E \rightarrow E/h_E$ the quotient maps. To determine the equations it is sufficient to find a map $P : X/h_X \rightarrow E/h_E$ of degree 3 with the correct ramification behaviour: As the preimage of ∞ under $q_E \circ \pi$ is just one point (also denoted by ∞ in the above figure), we may suppose that P is a polynomial, and hence $q_X(\infty) = \infty$. Furthermore the preimage of 0 (resp. 1) under $q_E \circ \pi$ consists of a Weierstrass points of X (namely D resp. E) and two points exchanged by h_X . Hence among $P^{-1}(0)$ (resp. among $P^{-1}(1)$) there must be precisely one ramification point. The polynomial $P(X)$ does the job and the images in X/h_X of the Weierstrass points are ∞ , the ramification points of P over 0 and 1 and the three preimages of some $\lambda \notin \{0, 1, \infty\}$. \square

4.2. The two steps. We will study the following origami, let's call it \mathcal{S}_2 , given by the permutations $m(a) = (12)(34)$ and $m(b) = (1)(23)(4)$ or graphically by 'two steps'


 FIGURE 4. The origami \mathcal{S}_2

with the natural identifications, i.e. glueing left sides to right sides in the same row in the same orientation and glueing top sides to bottom sides in the same column. (The surface on the right does obviously not contain the origami grid. The loop e is added for later use.)

The origami \mathcal{S}_2 is of genus $g = 2$ and it is the origami of lowest degree, that has a Strebel direction with $3g - 3$ cylinders (namely e.g. the vertical ones). This means (see Rem. 2.8) that the origami curve passes through a maximally degenerate point of (the compactification of) M_2 and will be important in section 7. Here *maximally degenerate* means that the stable curve consists of a graph of \mathbb{P}^1 's with three marked points or normal crossings.

For each elliptic curve E the covering $\pi : X \rightarrow E$ is ramified at two points of order 2 (but not Galois). It factors as $\pi = \iota \circ \pi_1$ in a degree 2 covering $\pi_1 : X \rightarrow E_1$ and an isogeny $\iota : E_1 \rightarrow E$ of degree 2.

Proposition 4.2. $C = C(\mathcal{S}_2)$ is the modular curve $M_{1,1}^{[2]} = \mathbb{H}/\Gamma(2) =: \mathbb{P}^*$. C parametrizes the curves of genus 2 given by

$$y^2 = ([4x(x-1)]^2 - (1-t))(4x^2 - 4x - 1) = (P^2 - (1-t))(P-1),$$

where t is a coordinate on \mathbb{P}^* and where $P(x) = 4x(x-1)$. t is normalised such that $t = 0$ gives the maximally degenerate point.

The family of intermediate covers $\mathcal{E}_1/\mathbb{P}^*$ (containing E_1) is given by

$$y^2 = (x-1)(x+1)(x^2 - (1-t)).$$

The morphisms between these curves are

$$\pi_1 : (x, y) \mapsto (P(x), 2y(x - \frac{1}{2}))$$

and

$$\iota : (x, y) \mapsto (x^2, yx),$$

where the family of base curves is given by $y^2 = x(x-1)(x-(1-t))$. In particular C and $j : C \rightarrow M_2$ are defined over \mathbb{Q} .

Proof: One notices from figure 4 that $\Gamma(2)$ is contained in the affine group $\Gamma(\pi)$ of \mathcal{S}_2 and by inspection (using e.g. [20]) one finds, that it is not bigger.

To determine the equations note that the hyperelliptic involution h_X of X is compatible with π_1, ι and the elliptic involution h_{E_1} and h_E . The Weierstraß points of these curves are as follows:

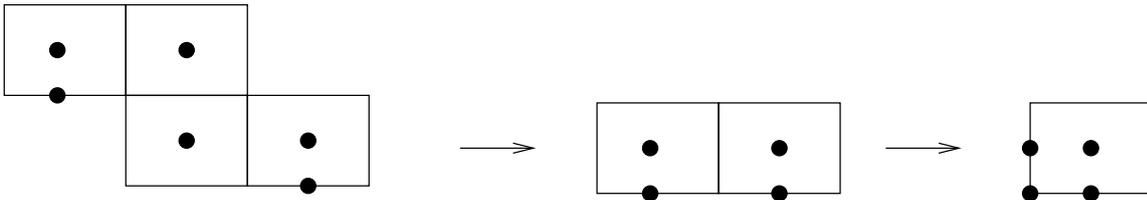


FIGURE 5. Weierstraß points

To determine π it is sufficient to find maps $X/h_X \rightarrow E_1/h_{E_1} \rightarrow E/h_E$ with the correct behaviour and in fact the polynomials $P = 4X(X-1)$ and X^2 do the job. \square

Remark 4.3. The orbifold structure of $C^{\text{orb}}(\mathcal{S}_2)$ consists precisely of a globally acting group $(\mathbb{Z}/2\mathbb{Z})^2$ given by the hyperelliptic involution and the automorphism $(x, y) \mapsto (1-x, y)$.

5. THE $G_{\mathbb{Q}}$ -ACTION ON ORIENTED ORIGAMIS IS FAITHFUL

Fix $\sigma \in G_{\mathbb{Q}}$ and let $K = \text{Fix}(\sigma)$ its fixed field. We first prove faithfulness in a weak sense:

Lemma 5.1. *For each $\sigma \in G_{\mathbb{Q}}$ and each elliptic curve E defined over K there is a covering $\pi : X \rightarrow E$, unramified except over one point, such that $\sigma\pi$ is not $\overline{\mathbb{Q}}$ -isomorphic to π .*

Proof: We will derive this from a corresponding result concerning dessins d'enfants: As shown in [21] Th. II.4 there exists a Belyi morphism $\beta : \mathbb{P}_{\overline{\mathbb{Q}}}^1 \rightarrow \mathbb{P}_{\overline{\mathbb{Q}}}^1$, i.e. unramified outside $\{0, 1, \infty\}$, such that $\sigma\beta \not\cong \beta$. We may also suppose that β is pure, i.e. that precisely the preimage of 1 consists only of points ramified of order 2.

Let $h : E \rightarrow \mathbb{P}^1$ be the double cover ramified over $\{0, 1, \lambda, \infty\}$. We take the pullback of β by the morphism h and call the desingularisation $\tilde{\pi} : X \rightarrow E$. The morphism $\tilde{\pi}$ is ramified at most over $\{0, 1, \lambda, \infty\}$ and hence after multiplication by 2 the map $\pi = [2] \circ \tilde{\pi}$ defines an origami.

Suppose there is a $\overline{\mathbb{Q}}$ -isomorphism $\varphi : X \rightarrow \sigma X$ with $\pi = \sigma\pi \circ \varphi$. We claim that this implies $\tilde{\pi} = \sigma\tilde{\pi} \circ \varphi$: As E and $[2]$ are defined over K the difference morphism $\tilde{\pi} - \sigma\tilde{\pi} \circ \varphi$ maps to the finite kernel of $[2]$. We can conclude because $\beta^{-1}(1)$ and hence $(\tilde{\pi})^{-1}(h^{-1}(1))$ is distinguished by the purity hypothesis.

Let τ be the involution on X induced by the elliptic involution of E . Once we have shown that $\varphi \circ \tau = \sigma\tau \circ \varphi$, the isomorphism φ descends to \mathbb{P}^1 and gives the desired contradiction. But both τ and $\varphi^{-1} \circ \sigma\tau \circ \varphi$ are involutions with the same number of fixed points. Since $X/\langle\tau\rangle \cong \mathbb{P}^1$ the involutions have to coincide by the uniqueness of the hyperelliptic involution. \square

Looking closer at the proof of Th. II.4 in [21], we can choose the morphism β in the above proof such that its ramification over 1 consists only of points of order 2, that β is totally ramified over ∞ and that the ramification behaviour over 0 is different from these two. Let $\mathcal{M}_{1,1}^{[2]}$ denote the moduli stack of elliptic curves with level 2 structure. We will use the non-calligraphic letters (e.g. $M_{1,1}^{[2]} \cong \mathbb{P}^*$) for the corresponding coarse moduli spaces.

Lemma 5.2. *The orbifold origami curve $C^{\text{orb}}(\pi)$ as constructed in the preceding lemma starting with a Belyi morphism β , whose ramification behaviour over 0, 1 and ∞ is pairwise distinct, is isomorphic to $\mathcal{M}_{1,1}^{[2]}$, hence $C(\pi) \cong \mathbb{P}^*$.*

Proof: We continue with the construction and the notations from the proof of the above lemma. First we prove that $\mathcal{M}_{1,1}^{[2]}$ surjects onto $C^{\text{orb}}(\pi)$:

For this purpose we need to construct a family over $\mathcal{M}_{1,1}^{[2]}$ of coverings of the universal family $\mathcal{E} \rightarrow \mathcal{M}_{1,1}^{[2]}$. Let $h : \mathcal{E} \rightarrow \mathbb{P}^1$ be the quotient by the elliptic involution h_E followed by the canonical projection $\mathbb{P}^1 \times \mathcal{M}_{1,1}^{[2]} \rightarrow \mathbb{P}^1$. (The quotient \mathcal{E}/h_E is indeed a trivial bundle as it contains 4 disjoint (Weierstraß) sections). Take the desingularisation of $\mathbb{P}^1 \times_{\mathbb{P}^1} \mathcal{E} \rightarrow \mathcal{M}_{1,1}^{[2]}$, where the morphisms in the fibred product are β and h respectively. The singular locus of the fibre product is étale over the base and thus the desingularisation is still a flat family of curves, which we denote by $\mathcal{X} \rightarrow \mathcal{M}_{1,1}^{[2]}$. Let $\pi : \mathcal{X} \rightarrow \mathcal{E}$ denote the composition of the second projection with the multiplication by 2. We thus constructed a topologically locally trivial family of coverings over $\mathcal{M}_{1,1}^{[2]}$. This family contains by construction the covering $X \rightarrow E$ of the preceding lemma (which we also denoted by π), hence does the job by Prop. 3.2.

It remains to exclude that $\mathcal{M}_{1,1}^{[2]} \rightarrow C^{\text{orb}}$ is a cover of degree greater than one or equivalently that $\mathbb{P}^* \rightarrow C(\pi)$ is a cover of degree greater than one. We have to exclude that the affine group of π is bigger than $\Gamma(2)$.

Fix a fibre $X \rightarrow E^{[2]} \rightarrow E$ of $\pi : \mathcal{X} \rightarrow \mathcal{E}$ and call this covering also $\pi = [2] \circ \tilde{\pi}$. The subgroup of $\pi_1(E^*) = \langle x, y \rangle$ corresponding to $[2]$ is generated by x^2, y^2, xy^2x and yx^2y . It is obviously characteristic and hence by Lemma 2.6 an affine diffeomorphism $\varphi : X \rightarrow X$ over $\bar{\varphi} : E \rightarrow E$ descends to an affine diffeomorphism $\varphi^{[2]} : E^{[2]} \rightarrow E^{[2]}$. If $D(\bar{\varphi})$ is not in $\Gamma(2)$, the morphism $\varphi^{[2]}$ has to permute the 2-division points of $E^{[2]}$.

But the fibres of $\tilde{\pi}$ over the 2-division points are different by our hypothesis on β : One fibre consists of unramified points that are all fixed by the hyperelliptic involution of X . One consists of unramified points that are pairwise interchanged by the hyperelliptic involution. One consists of 2 points and the last one has a ramification behaviour different from the above.

As an affine diffeomorphism has to preserve the ramification order and fixed points of the hyperelliptic involution, this leads to a contradiction. \square

As usual let $d = \deg(\pi)$. We need one more topological lemma. For the notation compare with the Remarks 2.7 and 2.8.

Lemma 5.3. *We may suppose that the differential $\omega_X = \pi^*\omega_E = \tilde{\pi}^*\omega_{E^{[2]}}$ has in the horizontal (resp. vertical, diagonal) direction 1 (resp. $d/8$, resp. $r \notin \{1, d/8\}$) maximal cylinders.*

Proof: Consider $h : E^{[2]} \rightarrow \mathbb{P}^1$ as unramified covering over the 4-punctured \mathbb{P}^1 . Denote the loops around $0, 1, \lambda, \infty \in \mathbb{P}^1$ by x_0, x_1, x_2, x_3 such that $x_3x_2x_1x_0 = 1$. Choose loops around the Weierstraß points on $E^{[2]}$ denoted by c_0, \dots, c_3 and $a, b \in \pi_1((E^{[2]})^*)$ such that $[a, b]c_3c_2c_1c_0 = 1$. This numbering is consistent with supposing that

$$h_*(a) = x_3x_2x_1x_3^{-1}, \quad h_*(b) = x_3^2x_2x_3^{-1}, \quad h_*(c_i) = x_i^2.$$

The isogeny $[2]$ doubles the number of unit height cylinders in each direction. As $\tilde{\pi}$ is totally ramified over $h^{-1}(\infty)$ all the maximal cylinders of π in each direction have height 2. What we need to ensure is hence that the monodromy images $m(a), m(ab), m(b) \in S_{d/4}$ corresponding to $\tilde{\pi}$ consist of 1 (resp. $d/8$, resp. $r \notin \{1, d/8\}$) cycles.

By construction of $\tilde{\pi}$ as fibre product the monodromy $m(c)$ for $\tilde{\pi}$, where $c \in \pi_1((E^{[2]})^*)$ coincides with the monodromy of $h_*(c)$ for β .

Since the monodromy of x_2 is trivial, the number of cycles of $m(a)$ equals the number of preimages $\beta^{-1}(1)$, which was $d/8$. The number of cycles of $m(b)$ equals the number of preimages $\beta^{-1}(\infty)$, which was one. Finally, the number of cycles of $m(ab)$ equals the number r of preimages of 0.

Going back to the construction of β , we show that we may choose $r \notin \{1, d/8\}$. Suppose β_0 is a Belyi morphism as constructed in [21], totally ramified over ∞ . If the number of preimages of 0 and 1 does not sum up to $\deg(\beta)/2$, we take $\beta = 4\beta_0(1 - \beta_0)$. Otherwise we take $\beta_1 = x^2 \circ \beta_0$ to ensure this condition and then take $\beta = 4\beta_1(1 - \beta_1)$. \square

Theorem 5.4. *For each $\sigma \in G_{\mathbb{Q}}$ there is an origami curve C isomorphic to \mathbb{P}^* , such that σ acts non-trivially on the map $j : C \rightarrow M_{g,[n]}$, more precisely such that $j(C) \neq j^\sigma(C)$.*

Proof: Take C as constructed in the preceding lemmas and assume that $j(C) = j^\sigma(C)$. We claim that this is equivalent to j being defined over K . To prove this, it is equivalent

to show that

$$j^\sigma \circ \sigma_C = \sigma_{M_{g,[n]}} \circ j$$

using descent for morphisms between varieties defined over K . The assumption implies that there exists an automorphism φ of $j(C)$, such that

$$\varphi \circ j^\sigma \circ \sigma_C = \sigma_{M_{g,[n]}} \circ j.$$

By Lemma 5.3 and Remark 2.8 the stable curves corresponding to the cusps of C have pairwise distinct number of nodes. This implies that $j(C) \cong \mathbb{P}^*$. Furthermore the number of nodes of a singular fibre is $G_{\mathbb{Q}}$ -invariant. Hence φ has to be the identity and this proves the claim.

Take a K -rational point x of C and denote the corresponding curve by X . The above claim implies the existence of an isomorphism $\sigma_X : X \rightarrow X^\sigma$. By construction we still have the morphisms $\pi : X \rightarrow E$ and $\pi^\sigma : X^\sigma \rightarrow E^\sigma$ plus the canonical morphism $\sigma_E : E \rightarrow E^\sigma$. If we knew that $\pi^\sigma \circ \sigma_X = \sigma_E \circ \pi$ then Lemma 5.1 would lead to a contradiction.

On X (resp. X^σ) we have the differentials $\omega_X = \pi^* \omega_E$ (resp. $\omega_{X^\sigma} = \pi^{*\omega_{E^\sigma}}$). Our second claim is that $\omega_X = \sigma_X^* \omega_{X^\sigma}$ (up to a multiplicative constant).

The map σ_{M_g} not only maps $[X] \in (\mathcal{M}_g)_{\overline{\mathbb{Q}}}$ to $[X^\sigma] \in (\mathcal{M}_g)_{\overline{\mathbb{Q}}}^\sigma$ but also C to C^σ . Hence the tangent vector t_X at $[X]$ to C is mapped to the tangent vector t_{X^σ} at $[X^\sigma]$ to C^σ . Now tensor the whole situation by \mathbb{C} for some fixed embedding $\overline{\mathbb{Q}} \rightarrow \mathbb{C}$. The Teichmüller metric induces a (non-linear, but functorial) duality between the projectivised tangent and cotangent space of \mathcal{M}_g at X . By construction $(\omega_X)^2$ is the unique (up to scalar multiple) quadratic differential corresponding to t_X via this duality. The same is true for X^σ , t_{X^σ} and $(\omega_{X^\sigma})^2$. This implies that $w_X^2 = (\sigma_X^* \omega_{X^\sigma})^2$ and establishes the second claim.

The periods of ω_X and ω_{X^σ} define a lattice in \mathbb{C} . Integration defines the morphisms

$$f_1 : X \rightarrow E_1 = \mathbb{C}/\text{Per}(\omega_X), \quad f_2 : X^\sigma \rightarrow E_2 = \mathbb{C}/\text{Per}(\omega_{X^\sigma}).$$

The second claim implies that the periods of ω_X and ω_{X^σ} are equal. Hence after suitable translation there exists an isomorphism $\sigma_{12} : E_1 \rightarrow E_2$ such that $f_2 \circ \sigma_X = \sigma_{12} \circ f_1$.

By construction there are isogenies $i_1 : E_1 \rightarrow E$ and $i_2 : E_2 \rightarrow E^\sigma$ such that $i_1 \circ f_1 = \pi$ and $i_2 \circ f_2 = \pi^\sigma$. All we need is to verify that $\sigma_E \circ i_1 = i_2 \circ \sigma_{12}$. But we can always lift σ_E to an isomorphism $E_1 \rightarrow E_2$, which has to coincide with σ_{12} if E_1 (or equivalently E) has no complex multiplication. As elliptic curves with CM have algebraic integers as j -invariants, we can exclude this by a suitable choice of $x \in C(K)$. \square

6. A SHORT REVIEW OF \widehat{GT}

We start with a summary on braid groups and mapping class groups:

Denote by B_n the (full) braid group on n strands and by P_n the pure subgroup, the kernel of the morphism $p : B_n \rightarrow S_n$. Let τ_i ($i = 1, \dots, n-1$) denote the standard (Artin) braid generators. We also need the following elements.

$$\begin{aligned} y_i &= \tau_{i-1} \cdots \tau_1 \tau_1 \cdots \tau_{i-1}, & i &= 2, \dots, n \\ w_i &= y_2 \cdots y_i, & i &= 2, \dots, n. \\ x_{ij} &= x_{ji} = (\tau_{j-1} \cdots \tau_{i+1}) \tau_i^2 (\tau_{j-1} \cdots \tau_{i+1})^{-1} & 1 \leq i < j \leq n \end{aligned}$$

We denote by $\Gamma_{g,[n]}$ (resp. $\Gamma_{g,n}$) the mapping class group of a Riemann surface of genus g with n unordered (resp. ordered) points. For a sphere $\Gamma_{0,[n]}$ equals $B_n / \langle w_n, y_n \rangle$ and this presentation is still valid, when we pass to the profinite completion $\widehat{\Gamma}_{0,n} = \pi_1(\mathcal{M}_{0,n})$.

We remark that only the *orbifold* fundamental groups of moduli stacks are mapping class groups and this is the reason to keep track of the orbifold structure of origamis.

In the sequel we use what is called 'σ-convention' in the appendix of [12], although we call the standard generators of the braid groups τ_i and use σ for elements of $G_{\mathbb{Q}}$: paths and braids are composed from the right to the left. Recall that we abbreviate $\mathbb{P}^* := \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and let

$$x, y, z \in \pi_1^{\text{top}}(\mathbb{P}_{\mathbb{Q}}^*, \vec{0}\vec{1}) \subset \pi_1(\mathbb{P}_{\mathbb{Q}}^*, \vec{0}\vec{1}) \cong \widehat{F}_2$$

denote the loops around 0, 1 and ∞ based at the tangential base point $\vec{0}\vec{1}$ (see below), such that $xyz = 1$. We denote their images in the algebraic fundamental group by the same letter. When using inner automorphisms, the exponent -1 is on the left.

The (\mathbb{Q} -rational) tangential base point $\vec{0}\vec{1}$ defines a splitting of the exact sequence

$$1 \rightarrow \pi_1(\mathbb{P}_{\mathbb{Q}}^*, \vec{0}\vec{1}) \rightarrow \pi_1(\mathbb{P}^*, \vec{0}\vec{1}) \rightarrow G_{\mathbb{Q}} \rightarrow 1$$

and with respect to this section the conjugate action of $G_{\mathbb{Q}}$ on $\pi_1(\mathbb{P}_{\mathbb{Q}}^*, \vec{0}\vec{1})$ is

$$\begin{aligned} x &\mapsto x^{\chi(\sigma)} \\ y &\mapsto f_{\sigma}(x, y)^{-1} y^{\chi(\sigma)} f_{\sigma}(x, y) \end{aligned} \quad (*)$$

where $\chi(\sigma) \in \widehat{\mathbb{Z}}$ is the cyclotomic character and $f_{\sigma} \in \widehat{F}_2$. Actually f_{σ} lies in the derived subgroup $(\widehat{F}_2)'$ and $(\chi(\sigma), f_{\sigma})$ are known to satisfy the following equations:

- (I) $f_{\sigma}(x, y) f_{\sigma}(y, x) = 1$
- (II) $f_{\sigma}(z, x) z^m f_{\sigma}(y, z) y^m f_{\sigma}(x, y) x^m = 1$, where $z = (xy)^{-1}$, $m = (\chi(\sigma) - 1)/2$.
- (III) $f_{\sigma}(x_{12}, x_{23}) f_{\sigma}(x_{34}, x_{45}) f_{\sigma}(x_{51}, x_{12}) f_{\sigma}(x_{23}, x_{34}) f_{\sigma}(x_{45}, x_{51}) = 1 \in \widehat{\Gamma}_{0,5}$

We thereby used the convention that $f(a, b)$ denotes the image of f under the morphism defined by $x \mapsto a, y \mapsto b$. One defines \widehat{GT} as the set of elements $F = (\lambda, f) \in \widehat{\mathbb{Z}} \times (\widehat{F}_2)'$ that satisfy (I), (II) and (III). F defines an endomorphism of \widehat{F}_2 via (*) and composition of endomorphisms makes \widehat{GT} into a monoid. We define \widehat{GT} as the group of invertible elements of \widehat{GT} .

Recently several other relations satisfied by the image of $G_{\mathbb{Q}}$ in \widehat{GT} have been found, in particular in [17] the following relation in \widehat{B}_3

$$\begin{aligned} (IV) \quad f_{\sigma}(\tau_1, \tau_2^2) &= \tau_2^{4\rho_2(\sigma)} f_{\sigma}(\tau_1^2, \tau_2^2) \tau_1^{2\rho_2(\sigma)} (\tau_1 \tau_2^2)^{-2\rho_2(\sigma)} \\ &= \tau_2^{-4\rho_2(\sigma)} f_{\sigma}(\tau_1, \tau_2^4) \tau_1^{-2\rho_2(\sigma)} (\tau_1 \tau_2^2)^{2\rho_2(\sigma)}. \end{aligned}$$

Here $\rho_p(\sigma)$ denotes the Kummer cocycle on the positive roots of $\sqrt[p]{p}$ for $n \in \mathbb{N}$. We will make use of the fact that this relation holds in the subgroup

$$\langle \tau_1, \tau_2^2 \mid [\tau_2^2, \tau_1 \tau_2^2 \tau_1] = 1 \rangle \subset \widehat{B}_3.$$

7. COMPARISON OF GALOIS ACTIONS

By its very definition we know to express Galois action on $\pi_1(\mathbb{P}_{\mathbb{Q}}^*, \vec{0}\vec{1})$ in terms of (λ, f) . On the other hand, from [5] and [17] we 'know' the conjugate action of $G_{\mathbb{Q}}$ on $\pi_1(\mathcal{M}_g)$ with respect to some tangential base points based at a maximally degenerate point.

We can thus use the two-steps origami to compare Galois actions: its origami curve is rational with 3 cusps, the inclusion $j : C^{\text{orb}} \rightarrow \mathcal{M}_2$ is defined over \mathbb{Q} and the extension

$j : \overline{C}^{\text{orb}} \rightarrow \overline{\mathcal{M}}_2$ goes through a maximally degenerate point. Here $\overline{\mathcal{M}}_2$ denotes the moduli stack of stable curves.

We may ignore the orbifold structure of C^{orb} (see Remark 4.3) for our considerations because the sequence

$$1 \rightarrow (\mathbb{Z}/2\mathbb{Z})^2 \rightarrow \pi_1(C^{\text{orb}}, \vec{0}\vec{1}) \rightarrow \pi_1(\mathbb{P}^*, \vec{0}\vec{1}) \rightarrow 1$$

is split. We simply define the \widehat{GT} -action on $\pi_1(C^{\text{orb}}, \vec{0}\vec{1})$ simply via a fixed splitting.

Consider the loops a_i and e drawn in figure 4. Denote by greek letters (i.e. $\alpha_i, \varepsilon \in \widehat{\Gamma}_{2,0}$) the corresponding Dehn twists.

Theorem 7.1. *The element $(\lambda, f) \in \widehat{GT}$ respects the Galois actions on the morphism $j : C^{\text{orb}} \rightarrow \mathcal{M}_2$ induced from the two-steps origami \mathcal{S}_2 (see figure 4) if and only if*

$$\begin{aligned} f(\alpha_3, (\alpha_1^2 \alpha_2)^4) f(\alpha_1^2, \alpha_2^2) f(\alpha_5^2, \alpha_4^2) \alpha_1^{-4\rho_2(\sigma)} \alpha_3^{-2\rho_2(\sigma)} \alpha_5^{-4\rho_2(\sigma)} f(\alpha_2 \alpha_4, \alpha_1^2 \alpha_3 \alpha_5^2) \\ = x_{16}^{2\rho_2(\sigma)} (x_{46} x_{56})^{\rho_2(\sigma)} (x_{12} x_{13})^{\rho_2(\sigma)} (\alpha_2 \alpha_4)^{2\rho_2(\sigma)} (\alpha_3 \alpha_2 \alpha_4)^{-2\rho_2(\sigma)} \end{aligned} \quad (\mathcal{S}_2)$$

holds in $\widehat{\Gamma}_{2,0}$. The elements (λ, f) satisfying this relation form a subgroup of \widehat{GT} containing $G_{\mathbb{Q}}$.

We will prove the theorem in the rest of this section.

We show now that equation (\mathcal{S}_2) is equivalent to the commutativity of the following diagram:

$$\begin{array}{ccc} \pi_1(C^{\text{orb}}, \vec{0}\vec{1}) & \xrightarrow{j_*} & \pi_1(\mathcal{M}_{2,0}, \vec{a}_*) \\ \downarrow F(C^{\text{orb}}) & & \downarrow F(\mathcal{M}_{2,0}) \\ \pi_1(C^{\text{orb}}, \vec{0}\vec{1}) & \xrightarrow{j_*} & \pi_1(\mathcal{M}_{2,0}, \vec{a}_*) \end{array} \quad (*)$$

Here $F = (\lambda, f) \in \widehat{GT}$ and $F(C^{\text{orb}})$ (resp. $F(\mathcal{M}_{2,0})$) are the induced automorphisms on the orbifold fundamental group of C^{orb} (resp. of $\mathcal{M}_{2,0}$) as explained in the previous section (resp. as will be explained in section 7.2). \vec{a}_* is a base point that will be conveniently chosen below.

Once we have shown this, the subgroup property is automatic: If $F, G \in \widehat{GT}$ satisfy \mathcal{S}_2 then

$$j_* \circ (G(C^{\text{orb}}) \circ F(C^{\text{orb}})) = G(\mathcal{M}_{2,0}) \circ j_* \circ F(C^{\text{orb}}) = G((\mathcal{M}_{2,0}) \circ F(\mathcal{M}_{2,0})) \circ j_*$$

and similarly one checks that the subset making the diagram commutative is closed under inversion.

7.1. Comparing tangential base points. We choose the coordinate t on the origami curve $C \cong \mathbb{P}^*$ such that 0 corresponds to a direction in which the trajectories of the Strebel differential $\pi^* \omega$ decompose the surface into 3 cylinders. The base point \vec{a}_* of the $G_{\mathbb{Q}}$ -action on \mathcal{M}_2 will hence correspond to the maximally degenerate point obtained by shrinking a_1, a_3 and a_5 in figure 4.

Somewhat more precisely: A $\mathbb{P}_{0,1,\infty}^1$ -diagram is a trivalent graph corresponding to a stable curve (maybe with marked points). Vertices correspond to \mathbb{P}^1 's, edges to normal crossings and 'loose ends' of the graph correspond to marked points. To each such diagram one can associate (see [8]) a versal deformation of the stable curve over the ring $\mathbb{Q}[[q_1, \dots, q_{3g-3+n}]]$. Taking $q_1 = \dots = q_{3g-3+n} =: q$ gives a $\mathbb{Q}[[q]]$ -valued point of $\mathcal{M}_{g,n}$, whose generic fibre

we call a *standard tangential base point* associated with the $\mathbb{P}_{0,1,\infty}^1$ -diagram. It is uniquely determined by the diagram up to the choice of signs of the q_i , or equivalently what is called a quilt over the corresponding pants decomposition in [17].

In our situation there is a unique quilt such that the resulting tangential base points \vec{a}_* and $j_*(\vec{0}\vec{1})$ are linked by a real path γ . Thus the sections $G_{\mathbb{Q}} \rightarrow \pi_1(\mathcal{M}_2, j(\vec{0}\vec{1}))$ induced by these tangential base points are related by

$$\gamma^{-1} s_{\vec{a}_*}(\sigma) \gamma = w_2^{d_l(\sigma)} x_{34}^{d_m(\sigma)} w_4^{d_r(\sigma)} s_{j(\vec{0}\vec{1})}(\sigma),$$

where d_l, d_r and d_m are products of Kummer cocycles. Indeed the morphism $\text{Spec } \mathbb{Q}[[t]] \rightarrow \text{Spec } \mathbb{Q}[[q_1, q_2, q_3]]$ to the base of the versal deformation induced by j is given by 3 power series $p_i(t)$ with leading coefficients say c_i ($i = 1, 2, 3$). If q_1 corresponds to w_2 and $c_1 = \prod_p p^{n_p}$, then $d_l = \sum_p n_p \rho_p(\sigma)$ and similarly for d_m and d_r .

The symmetry of the origami implies that $d_l = d_r$. We will obtain $d_l = d_r = -2\rho_2(\sigma)$ and $d_m = -\rho_2(\sigma)$ below automatically by group theory.

We remark that there is a geometric way to determine these exponents by comparing tangential base points in $\mathcal{M}_{0,6}$ and $\mathcal{M}_{2,0}$ using the methods of [16] and [7].

7.2. $G_{\mathbb{Q}}$ -action on $\Gamma_{2,0}$. We use the notion of A - and S -moves from [5]. There is an A -move between the pants decompositions of Σ_2 given by $\{a_1, a_3, a_5\}$ and $\{a_1, e, a_5\}$. Two S -moves change this into $\{a_2, e, a_5\}$ and into $\{a_2, e, a_4\}$. Hence by [7] and [17] the conjugate action of $s_{\vec{a}_*}$ on $\pi_1(\mathcal{M}_2, \vec{a}_*)$ is given by

$$\begin{aligned} \alpha_1 &\mapsto \alpha_1^{\chi(\sigma)}, & \alpha_3 &\mapsto \alpha_3^{\chi(\sigma)}, & \alpha_5 &\mapsto \alpha_5^{\chi(\sigma)} \\ \alpha_2 &\mapsto f(\alpha_3, \varepsilon)^{-1} f(\alpha_1^2, \alpha_2^2)^{-1} \alpha_2^{\chi(\sigma)} f(\alpha_1^2, \alpha_2^2) f(\alpha_3, \varepsilon) \\ \alpha_4 &\mapsto f(\alpha_3, \varepsilon)^{-1} f(\alpha_5^2, \alpha_4^2)^{-1} \alpha_4^{\chi(\sigma)} f(\alpha_5^2, \alpha_4^2) f(\alpha_3, \varepsilon) \end{aligned}$$

We should have written f_σ instead of f , but we will drop the subscript for simplicity. Using the topology of the origami, we see that j_* maps $x, y \in \pi_1(C^{\text{orb}}, \vec{0}\vec{1})$ to $\alpha_1^2 \alpha_3 \alpha_5^2$ and $\alpha_2 \alpha_4$ respectively. Hence the induced action on $\alpha_2 \alpha_4 \in \pi_1(\mathcal{M}_2, j(\vec{0}\vec{1}))$ is

$$\alpha_2 \alpha_4 \mapsto f(\alpha_1^2 \alpha_3 \alpha_5^2, \alpha_2 \alpha_4)^{-1} (\alpha_2 \alpha_4)^{\chi(\sigma)} f(\alpha_1^2 \alpha_3 \alpha_5^2, \alpha_2 \alpha_4).$$

Together with the comparison of the tangential base points we obtain that the diagram (*) commutes, if and only if

$$f(\alpha_3, \varepsilon) f(\alpha_1^2, \alpha_2^2) f(\alpha_5^2, \alpha_4^2) w_2^{d_l} x_{34}^{d_m} w_4^{d_r} f(\alpha_2 \alpha_4, \alpha_1^2 \alpha_3 \alpha_5^2) \in \text{Centr}_{\widehat{\Gamma}_{2,0}}(\alpha_2 \alpha_4).$$

It remains to write this as an equation and to determine d_l and d_m . We may project this expression to $\widehat{\Gamma}_{0,[6]}$, sending α_i to τ_i for $i = 1, \dots, 5$. More precisely the image lies in $p^{-1}(S_4)$, where $p : \widehat{\Gamma}_{0,[6]} \rightarrow S_6$ is induced from $p : B_6 \rightarrow S_6$ and where $S_4 \subset S_6$ fixes the first and the last marked point. We may hence reduce mod $\langle \tau_1^2, \tau_5^2 \rangle$ (and shift indices by -1). Using $\varepsilon = w_3^2$ we obtain in $\widehat{\Gamma}_{0,[4]}$

$$f(\tau_2, \tau_1^4) \tau_2^{2d_l} f(\tau_1 \tau_3, \tau_2) \in \text{Centr}_{\widehat{\Gamma}_{0,[4]}}(\tau_1 \tau_3).$$

This is a consequence of relation (IV) in \widehat{B}_3 . Indeed the elements $\tau_1 \tau_3$ and τ_2 satisfy the defining relation $[\tau_1 \tau_3, \tau_2 \tau_1 \tau_3 \tau_2] = 1$. Hence applying (IV) we obtain in $\widehat{\Gamma}_{0,[4]}$

$$f(\tau_2, \tau_1^4) \tau_2^{-2\rho_2(\sigma)} f(\tau_1 \tau_3, \tau_2) (\tau_2 \tau_1 \tau_3)^{2\rho_2(\sigma)} (\tau_1 \tau_3)^{-2\rho_2(\sigma)} = 1 \quad (1)$$

and in particular $d_l = -\rho_2(\sigma)$.

7.3. Some lemmas on centralizers. We now prove some lemmas that will be applied in the next section. The author is grateful to W. Herfort for these results.

Lemma 7.2. *Let the profinite group G act continuously and freely on a profinite space X . Then for the induced action on $\widehat{F}(X)$ the equation $\varphi f = f$ for some $\varphi \in G$ and $f \in \widehat{F}(X)$ yields either $\varphi = 1$ or $f = 1$.*

Proof: Suppose, on the contrary, there exist non-trivial elements $\varphi \in G$ and $f \in \widehat{F}(X)$ with $\varphi f = f$. We first want to show that G and X can be assumed to be finite. Since G acts freely on X , using Lemma 5.6.5 (a) in [19] we find a continuous section $\sigma : G \backslash X \rightarrow X$. Hence we may identify X as a G -space with $G \times \Lambda$, where, for short, we have put $\Lambda := G \backslash X$, with the left regular action of G . Let N be a normal open subgroup of G and R a clopen relation on Λ , such that $g \notin N$ and f has a non-trivial image under canonical projection from $\widehat{F}(G \times \Lambda)$ onto $\widehat{F}(G/N \times \Lambda/R)$. Such N and R exist as Proposition 1.7 in [1] shows. Since G/N acts freely on $G/N \times \Lambda/R$ we have shown that indeed it suffices to assume G and $X = G \times \Lambda$ both to be finite.

Let $\Gamma := G * \widehat{F}(\Lambda)$ be the free product and define a map η from $G \cup \widehat{F}(\Lambda)$ to the holomorph $H := G \rtimes \widehat{F}(X)$ as follows. We send $\varphi \in G$ to $\varphi \in G$ (as a subgroup of the holomorph) and we extend the map that sends $\lambda \in \Lambda$ to $(1, \lambda) \in G \times \Lambda$, to a continuous homomorphism from all of $\widehat{F}(\Lambda)$ to H by using the universality of the free group $\widehat{F}(\Lambda)$. Use the universal property of Γ being a free product in order to extend $\eta : G \cup \widehat{F}(\Lambda) \rightarrow H$ to a continuous epimorphism $\omega : \Gamma \rightarrow H$.

We claim ω to be an isomorphism. Indeed, since ω induces the identity on $\Gamma/(\widehat{F}(\Lambda))_\Gamma$, we conclude $\text{Ker } \omega \leq (\widehat{F}(\Lambda))_\Gamma$. (For a profinite group G and a subset A of G let $(A)_G$ denote the normal closure, i.e., the smallest closed normal subgroup of G containing A). Use the Kurosh Subgroup Theorem (e.g. Thm. 9.1.9 in [19]) by applying it to the normal open subgroup $(\widehat{F}(\Lambda))_\Gamma$ in order to see that its rank equals $|G| \times |\Lambda|$. Since $(\widehat{F}(\Lambda))_\Gamma$ is hopfian and it goes onto $\widehat{F}(X)$ (viewed as a subgroup of H) conclude $\text{Ker } \omega = \{1\}$.

Let us point out that the action of φ on X becomes conjugation in the holomorph H . Now use Theorem 9.1.12 in [19] in order to see that $C_\Gamma(g) \leq G$, so that applying ω one finds $C_H(g) \leq G$. This however contradicts the choice of φ and f . \square

We apply this lemma in the following two cases:

Lemma 7.3. *Let $\widehat{F}_3 = \langle x, y, z \rangle$ be the profinite free group on three generators and φ the following automorphism of \widehat{F}_3 : $\varphi(x) = x$, $\varphi(y) = xyx^{-1}$ and $\varphi(z) = x^{-1}zx$. Then the fixed group of φ in \widehat{F}_3 is the profinite free group generated by x .*

Proof: Let N denote the normal subgroup generated by y and z . By specialising Thm. 8.1.3 in [19] to the finite case, we deduce that N is the profinite free group on the generators $X = \{x^{-l}yx^l, x^{-l}zx^l \mid l \in \widehat{\mathbb{Z}}\}$. φ acts freely on X and Lemma 7.2 yields $\text{Fix}(\varphi) \cap N = \{1\}$. Hence the isomorphism $\widehat{F}_3/N \rightarrow \langle x \rangle$ induces an isomorphism $\text{Fix}(\varphi) \rightarrow \langle x \rangle$. \square

Lemma 7.4. *Let $\widehat{F}_4 = \langle w, x, y, z \rangle$ be the profinite free group on 4 generators and φ the following automorphisms of \widehat{F}_4 : $\varphi(w) = w$, $\varphi(x) = x$, $\varphi(y) = xyx^{-1}$ and $\varphi(z) = w^{-1}x^{-1}zxw$. Then the fixed group of φ in \widehat{F}_4 is the profinite free group generated by w and x .*

Proof: Let N denote the normal subgroup generated by z . As above, N is free on the generators $X = \{u^{-1}zu \mid u \in \langle w, x, y \rangle\}$. We check that φ acts freely on X .

Suppose for $l \in \widehat{\mathbb{Z}}$ we have $\varphi^l(u^{-1}zu) = u^{-1}zu$. By definition of φ this implies

$$(xw)^l \varphi^l(u) u^{-1} \in \text{Centr}_{\widehat{F}_4}(z).$$

This centralizer equals $\langle z \rangle$ and has trivial intersection with $\langle w, x, y \rangle$. Thus $(xw)^l \varphi^l(u) = u$. If we consider this modulo the normal subgroup N_y generated by y , the action of φ is trivial, hence $(xw)^l \in N_y$. This is only possible for $l = 0$.

By Lemma 7.2 hence $\text{Fix}(\varphi) \cap N = \{1\}$. Thus $\text{Fix}(\varphi)$ injects into $\widehat{F}_4/N \cong \widehat{F}_3 = \langle w, x, y \rangle$. Let $N_y \subset \widehat{F}_3$ be the normal subgroup generated by y . With the same arguments we conclude that $\text{Fix}(\varphi) \cap N_y = \emptyset$ and hence $\text{Fix}(\varphi) = \langle w, x \rangle$. \square

7.4. Lifting the relation. We now want to lift the equation (1) successively to $\widehat{\Gamma}_{0,[5]}$, $\widehat{\Gamma}_{0,[6]}$ and to $\widehat{\Gamma}_{2,0}$. Let $S_4 \subset S_5$ be the permutation group of the last 4 strings and $p : \widehat{\Gamma}_{0,[5]} \rightarrow S_5$ the permutation representation. Reducing mod $\langle \tau_1^2 \rangle$ gives a morphism $p_1 : p^{-1}(S_4) \rightarrow \widehat{\Gamma}_{0,[4]}$. Undoing the shift of indices, we know that

$$\begin{aligned} f(\tau_3, (\tau_1^2 \tau_2)^4) f(\tau_1^2, \tau_2^2) \tau_1^{2d_1} \tau_3^{-2\rho_2(\sigma)} f(\tau_2 \tau_4, \tau_1^2 \tau_3) (\tau_3 \tau_2 \tau_4)^{2\rho_2(\sigma)} (\tau_2 \tau_4)^{-2\rho_2(\sigma)} \\ \in \text{Ker}(p_1) \cap \text{Centr}_{\widehat{\Gamma}_{0,[5]}}(\tau_2 \tau_4). \end{aligned} \quad (2)$$

$\text{Ker}(p_1)$ is the free profinite group on the three generators $x_{12}, x_{12}x_{13}$ and x_{15} and the conjugate action of the square of $\tau_2 \tau_4$ on $\text{Ker}(p_1)$ is given by

$$\begin{aligned} (\tau_2 \tau_4)^{-2} x_{12} x_{13} (\tau_2 \tau_4)^2 &= x_{12} x_{13} \\ (\tau_2 \tau_4)^{-2} x_{12} (\tau_2 \tau_4)^2 &= (x_{12} x_{13})^{-1} x_{12} (x_{12} x_{13})^{-1} \\ (\tau_2 \tau_4)^{-2} x_{15} (\tau_2 \tau_4)^2 &= (x_{12} x_{13})^{-1} x_{15} (x_{12} x_{13}) \end{aligned}$$

Using Lemma 7.3 we conclude that the above expression (2) equals a power of $x_{12}x_{13}$. To determine the exponent, we cannot simply abelianize the subgroup $p^{-1}((24)(35))$ of $\widehat{\Gamma}_{0,5}$ because $x_{12}x_{13}$ vanishes. Therefore we first shift indices by -1 and use the natural embedding $\widehat{B}_4/\langle w_4 \rangle \rightarrow \widehat{\Gamma}_{0,[5]}$, $\tau_i \mapsto \tau_i$ for $1 \leq i \leq 4$. The equation then becomes

$$f(\tau_2, x_{34}^2) f(z_3, \tau_1^2) z_3^{d_1} x_{23}^{-\rho_2(\sigma)} f(\tau_1 \tau_3, z_3 \tau_2) (\tau_2 \tau_1 \tau_3)^{2\rho_2(\sigma)} (\tau_1 \tau_3)^{-2\rho_2(\sigma)} = (z_3 \tau_1 z_3 \tau_1^{-1})^a \quad (3)$$

where $z_3 = (\tau_2 \tau_3)^3$. Note that in $\widehat{\Gamma}_{0,5}$ we have $x_{12} = (\tau_3 \tau_4)^3$. We now let $k_5 := [\tau_1 \tau_3, \tau_2 z_3 \tau_1 \tau_3 \tau_2 z_3]$ and map equation (2) to $\widehat{G} = \widehat{B}_4/\langle w_4, k_5 \rangle$ in order to apply relation (IV) to $f(\tau_1 \tau_3, z_3 \tau_2)$.

The abelianization of \widehat{G} is generated by the x_{ij} for $1 \leq i < j \leq 4$ with the relations

$$x_{12} \equiv (x_{13} x_{14} x_{23} x_{24} x_{34})^{-1} \quad \text{and} \quad x_{13} \equiv x_{24}$$

due to the factorization mod w_4 and k_5 .

A simple calculation in \widehat{G}^{ab} yields $[z_3 \tau_2, (\tau_1 \tau_3)^2] \equiv x_{12}^{-1} x_{13} x_{24} x_{34}^{-1}$. On this commutator $z_3 \tau_2$ acts by (-1) and $(\tau_1 \tau_3)^2$ acts trivially. Hence Ihara's Blanchfield-Lyndon calculus (see [17] Lemma 2.3) implies

$$f(z_3 \tau_2, (\tau_1 \tau_3)^2) \equiv (x_{12}^{-1} x_{13} x_{24} x_{34}^{-1})^{-\rho_2(\sigma)}.$$

For the same reason

$$f(\tau_2, x_{34}^2) \equiv (x_{24}^2 x_{34}^{-2})^{-\rho_2(\sigma)}.$$

In \widehat{G}^{ab} the right hand side of equation (3) equals

$$(z_3\tau_1 z_3\tau_1^{-1})^a \equiv (x_{14}x_{23}x_{24}^2x_{34}^2)^a.$$

The left hand side, using relation (IV) and the above formula (note that $f(z_3, \tau_1^2)$ vanishes), equals

$$LHS(2) \equiv x_{14}^{2\rho_2(\sigma)} x_{23}^{d_l+3\rho_2(\sigma)} (x_{24}x_{34})^{d_l+4\rho_2(\sigma)}.$$

We conclude that $d_l = -2\rho_2(\sigma)$ and $a = \rho_2(\sigma)$.

To go on to $\widehat{\Gamma}_{0,[6]}$ let $p^{-1}(S_4) \subset \widehat{\Gamma}_{0,[6]}$ be the subgroup that fixes the first and last marked point. Denote by $p_6 : p^{-1}(S_4) \rightarrow \widehat{\Gamma}_{0,[5]}$ the reduction mod $\langle \tau_5^2 \rangle$.

If we undo the indexshift and multiply the expression by $(x_{46}x_{56})^{-\rho_2(\sigma)}$ to obtain an expression symmetric with respect to $\tau_i \leftrightarrow \tau_{6-i}$ (note that $x_{12}x_{13} \leftrightarrow x_{46}x_{56}$) we conclude

$$\begin{aligned} & f(\tau_3, (\tau_1^2\tau_2)^4) f(\tau_1^2, \tau_2^2) f(\tau_5^2, \tau_4^2) \tau_1^{-4\rho_2(\sigma)} \tau_3^{-2\rho_2(\sigma)} \tau_5^{-4\rho_2(\sigma)} f(\tau_2\tau_4, \tau_1^2\tau_3\tau_5^2) \\ & \cdot (\tau_3\tau_2\tau_4)^{2\rho_2(\sigma)} (\tau_2\tau_4)^{-2\rho_2(\sigma)} (x_{12}x_{13})^{-\rho_2(\sigma)} (x_{46}x_{56})^{-\rho_2(\sigma)} \in \text{Ker}(p_6) \cap \text{Centr}_{\widehat{\Gamma}_{0,[6]}}(\tau_2\tau_4). \end{aligned} \quad (4)$$

$\text{Ker}(p_6)$ is the free profinite group on 4 generators x_{16} , $x_{46}x_{56}$, x_{56} and x_{26} and the conjugate action of $(\tau_2\tau_4)^2$ (we only use the action of the square) on $\text{Ker}(p_6)$ is given by

$$\begin{aligned} (\tau_2\tau_4)^{-2} x_{16} (\tau_2\tau_4)^2 &= x_{16} \\ (\tau_2\tau_4)^{-2} x_{46}x_{56} (\tau_2\tau_4)^2 &= x_{46}x_{56} \\ (\tau_2\tau_4)^{-2} x_{56} (\tau_2\tau_4)^2 &= (x_{46}x_{56}) x_{56} (x_{46}x_{56})^{-1} \\ (\tau_2\tau_4)^{-2} x_{26} (\tau_2\tau_4)^2 &= x_{16}^{-1} (x_{46}x_{56})^{-1} x_{26} (x_{46}x_{56}) x_{16} \end{aligned}$$

By Lemma 7.4 and the symmetry with respect to $\tau_i \leftrightarrow \tau_{6-i}$ the expression (4) equals a power of x_{16} . To determine the exponent, we use the same technique as above and apply relation (IV) twice. In order to be able to do so, we consider the equation in $\widehat{G} = \widehat{\Gamma}_{0,6}/\langle k_6 \rangle$, where $k_6 = [\tau_2\tau_4, \tau_1^2\tau_3\tau_5 \cdot \tau_2\tau_4 \cdot \tau_1^2\tau_3\tau_5]$. The abelianization of $\widehat{\Gamma}_{0,6}$ is generated by $x_{12}, x_{13}, x_{14}, x_{15}, x_{23}, x_{24}, x_{25}, x_{34}, x_{35}$. In \widehat{G}^{ab} we have the supplementary relation $x_{35}^2 \equiv (x_{12}^{-1}x_{13}x_{14}^{-1}x_{15}x_{24}^{-1})^{-2}$. We use the Blanchfield-Lyndon calculus again to obtain

$$f(\tau_2^2\tau_4^2, \tau_1^2\tau_3\tau_5^2) \equiv (x_{12}x_{13}x_{14}x_{24}^2x_{34}x_{15}x_{25}x_{35}^2)^{\rho_2(\sigma)}$$

and

$$f(\tau_3, (\tau_1^2\tau_2)^4) \equiv (x_{12}x_{14}^{-1}x_{24}^{-1}x_{23})^{2\rho_2(\sigma)}.$$

In \widehat{G}^{ab} we can apply relation (IV) to $f(\tau_2\tau_4, \tau_1^2\tau_3\tau_5^2)$ and use to above formula. By direct calculation one verifies that

$$(\tau_2\tau_4\tau_1^2\tau_3\tau_5^2)^{-2\rho_2(\sigma)} (\tau_3\tau_2\tau_4)^{2\rho_2(\sigma)} \equiv (x_{13}^2x_{15}^2x_{24}^{-1}x_{35})^{-\rho_2(\sigma)}.$$

Hence the left hand side of equation (4) sums up to

$$LHS(3) \equiv (x_{12}x_{13}x_{14}x_{15})^{-2\rho_2(\sigma)} \equiv x_{16}^{2\rho_2(\sigma)} = RHS(3).$$

Finally we want to lift the equation to $\widehat{\Gamma}_{2,0}$, which is an extension of $\widehat{\Gamma}_{0,[6]}$ by the hyperelliptic involution generated by the central element w_5 . Note that equation (4) involves only even powers of α_1 and α_5 . We can reduce modulo $\langle \alpha_1^2, \alpha_5^2 \rangle$ (and shift indices by (-1)) to obtain in the sphere braid group $\widehat{H}_4 = \widehat{B}_4/\langle y_4 \rangle$

$$f(\tau_2, \tau_1^4) \tau_2^{-2\rho_2(\sigma)} f(\tau_1\tau_3, \tau_2) (\tau_2\tau_1\tau_3)^{2\rho_2(\sigma)} (\tau_1\tau_3)^{-2\rho_2(\sigma)} \in \text{center}(\widehat{H}_4).$$

We had noticed above that this expression equals 1 in $\widehat{\Gamma}_{0,[4]}$ by using relation (IV). Denote by $w_3 = (\tau_1 \tau_2)^3$ the center of \widehat{H}_4 . Relation (IV) now tells us that

$$\begin{aligned} & f(\tau_2, \tau_1^4) \tau_2^{-2\rho_2(\sigma)} f(\tau_1 \tau_3, \tau_2) (\tau_2 \tau_1 \tau_3)^{2\rho_2(\sigma)} (\tau_1 \tau_3)^{-2\rho_2(\sigma)} = \\ & = f(\tau_2, \tau_1^4) f(\tau_1^2 \tau_3^2, \tau_2) = f(\tau_2, \tau_1^4) f(\tau_1^4 w_3, \tau_2) = 1. \end{aligned}$$

This completes the proof of the theorem. \square

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