Coupling reduced basis methods and the Landweber method to solve inverse problems

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Reduced Basis Methods

Motivation & forward operator

Consider $\nabla \cdot (\sigma(x)\nabla u(x)) = 1,\ x \in \Omega := [0,1]^2,\ u(x) = 0,\ x \in \partial\Omega.$

Forward operator

$F : \mathcal{D}(F) \subset Y \longrightarrow X,\ \sigma \longmapsto u^\sigma$ between Hilbert spaces with u^σ, the detailed solution, solving

$$b(u^\sigma, v; \sigma) = f(v;\sigma),\ \forall v \in X, \text{ with} \tag{1a}$$

$$b(u, w; \sigma) := \int_\Omega \sigma \nabla u \cdot \nabla w \, dx, \quad f(v;\sigma) := -\int_\Omega v \, dx. \tag{1b}$$

- Rapid and numerous evaluation of F for many parameters, e.g. optimal control, real-time-simulation, inverse problems.
- Detailed solution (e.g. FEM, FV, FD) is expensive.

$\rightsquigarrow$ **model order reduction**

The reduced problem

Assume

- Reduced basis space (RB-space) $X_N := \text{span}\{\phi_1, \dots, \phi_N\} \subset X$, $\dim X_N = N$ ($N \ll \dim X$), is given (e.g. $\phi_i = F(\sigma_i)$ with *meaningful* parameters $\sigma_i \in \mathcal{D}(F)$, $i = 1, \dots, N$).

Reduced forward operator

For given F and $X_N \subset X$, define $F_N : \mathcal{D}(F) \subset Y \longrightarrow X_N$, $\sigma \longmapsto u_N^\sigma$ with u_N^σ, the reduced solution, solving

$$b(u_N^\sigma, v; \sigma) = f(v; \sigma), \quad \forall v \in X_N.$$

Properties

Residual error estimator

$$\|u^\sigma - u_N^\sigma\|_X \le \Delta_N(\sigma) := \frac{\|v_r\|_X}{\alpha(\sigma)}, \text{ with}$$

$$\langle v_r, v\rangle_X := r(v;\sigma) := f(v;\sigma) - b(u_N^\sigma, v;\sigma), \forall v \in X.$$

- Reproduction of solutions: for $\sigma \in \mathcal{D}(F)$ we have $F(\sigma) \in X_N \Rightarrow F_N(\sigma) = F(\sigma)$.
- Offline/online decomposition: enables efficient and cheap evaluation of $F_N(\sigma)$.

Combining RBM & Landweber method

Inverse problem & Landweber

Inverse problem

For given solution $u \in X$ of (1), find corresponding parameter $\sigma^+ \in \mathcal{D}(F)$ with $F(\sigma^+) = u$ („a-example“).

Given u^δ, $\|u - u^\delta\|_X \leq \delta$, $\delta > 0$, find approximation σ^δ to σ^+.

Algorithm 1 Landweber(σ_{start}, τ)

$n := 0,\ \sigma_0^\delta := \sigma_{start}$
while $\|F(\sigma_n^\delta) - u^\delta\|_X > \tau\delta$ **do**
    $\sigma_{n+1}^\delta := \sigma_n^\delta + \omega F'(\sigma_n^\delta)^*(u^\delta - F(\sigma_n^\delta))$
    $n := n + 1$
end while
return $\sigma_{LW} := \sigma_n^\delta$

Various approaches

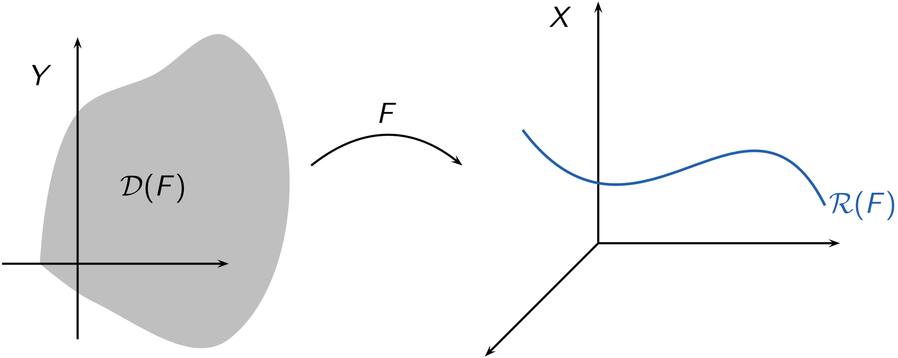
Standard approach:

- Construct global X_N approximating whole $\mathcal{R}(F)$ (i.e. providing good reduced solutions u_N^σ, $\forall \sigma \in \mathcal{D}(F)$) $\leadsto$ „offline-phase“.
- Using this space, quickly compute $F_N(\sigma)$ and substitute $F(\sigma)$ for $F_N(\sigma)$ in Algorithm 1 $\leadsto$ „online-phase“.

Problem: Only feasible for low-dimensional parameter spaces (≤ 30), not feasible for imaging.

Our approach: Create problem-adapted RB-space by iterative enrichment (inspired by Druskin & Zaslavski, 2007).

Motivation - Combining RBM & LW



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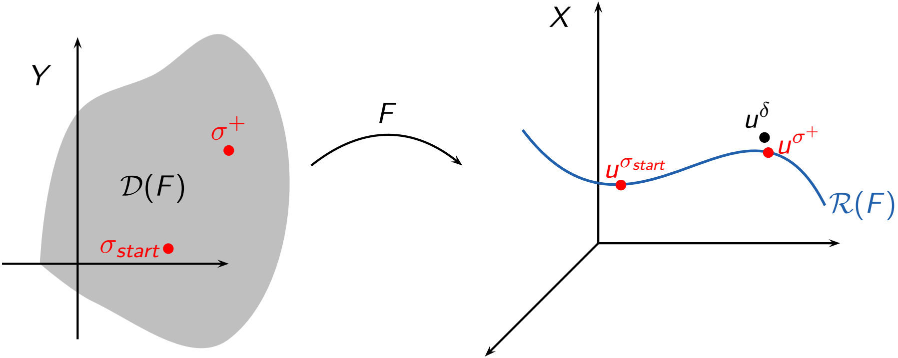

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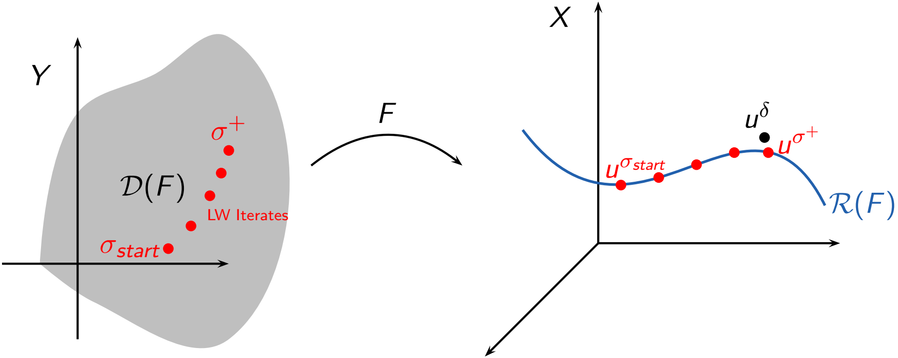

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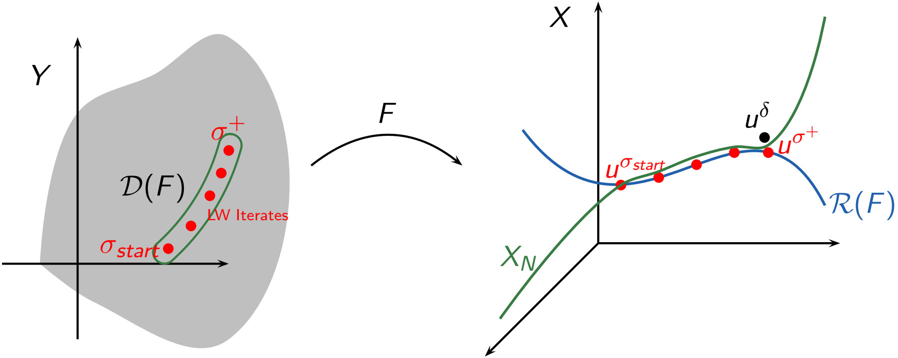

Procedure

1. Start with initial guess σ_{start} and initial RB-spaces $X_{N,1}$, $X_{N,2}$.
2. Update $X_{N,1}$, $X_{N,2}$ using current iterate (first update via σ_{start}).
3. Solve the inverse problem up to a certain accuracy using the nonlinear Landweber method projected onto $X_{N,1}$ and $X_{N,2}$.
4. If resulting iterate is accepted by the discrepancy principle, terminate, else go to step 2.

Termination of step 3

Let σ_n^δ be the current iterate.

- detailed discrepancy principle $(\|F(\sigma_n^\delta) - u^\delta\|_X \leq \tau\delta)$?

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  - $\|F(\sigma_n^\delta) - u^\delta\|_X \leq \|F(\sigma_n^\delta) - F_N(\sigma_n^\delta)\|_X + \|F_N(\sigma_n^\delta) - u^\delta\|_X$

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- Heuristically don't allow $\|F_N(\sigma) - F(\sigma)\|_X > (\tau - 2)\delta$.

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Terminate step 3 via

$$\|F_N(\sigma_n^\delta) - u^\delta\|_X \leq \tau\delta \quad \text{or} \quad \Delta_N(\sigma_n^\delta) > (\tau - 2)\delta.$$

Reduced Basis Landweber (RBL) method

Algorithm 2 RBL$(\sigma_{start}, \tau, X_{N,1}, X_{N,2})$

$n := 0,\ \sigma_0^\delta := \sigma_{start}$
while $\|F(\sigma_n^\delta) - u^\delta\|_X > \tau\delta$ **do**
  enrich $X_{N,1},\ X_{N,2}$
  $i := 1,\ \sigma_i^\delta := \sigma_n^\delta$
  repeat
    $\sigma_{i+1}^\delta := \sigma_i^\delta + \omega F_N'(\sigma_i^\delta)^*(u^\delta - F_N(\sigma_i^\delta))$
    $i := i + 1$
  until $\|F_N(\sigma_i^\delta) - u^\delta\|_X \leq \tau\delta$ **or** $\Delta_N(\sigma_i^\delta) > (\tau - 2)\delta$
  $\sigma_{n+1}^\delta := \sigma_i^\delta$
  $n := n + 1$
end while
return $\sigma_{RBL} := \sigma_n^\delta$

Dual problem

For $\sigma, \kappa \in \mathcal{D}(F)$ and $l \in X$, one can show

$$\langle \kappa, F'(\sigma)^* \, l \rangle_Y = \int_\Omega \kappa \nabla u^\sigma \cdot \nabla u_l^\sigma \, dx, \tag{2}$$

with $u_l^\sigma \in X$ the unique solution of the dual problem

$$b(u, v; \sigma) = m(v; l), \forall v \in X, \quad m(v) := -\int_\Omega l \, v \, dx. \tag{3}$$

In Algorithm 2

- enrich $X_{N,1}$ with $F(\sigma_n^\delta)$ and $X_{N,2}$ with $u_l^{\sigma_n^\delta}$ solving (3) for $l := u^\delta - F(\sigma_n^\delta)$.
- evaluate $F_N'(\sigma_i^\delta)^*(u^\delta - F_N(\sigma_i^\delta))$ using (2) and associated reduced solutions.

Numerics - compare reconstructions

Setting: 900 pixels, $\tau = 2.5$, $\delta \approx 0.08\%$ and $\omega = \frac{1}{2}(\|F'(\sigma_{start})\|)^{-1}$.

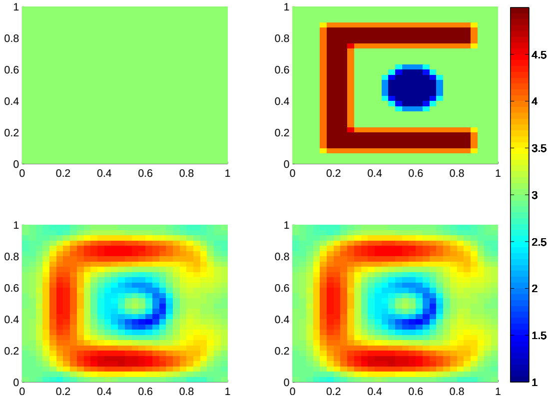

Figure: Exact solution (top right), σ_{start} (top left). Reconstruction via Algorithm 2 (bottom left) and Algorithm 1 (bottom right).

Numerics - time comparison

- Outer iteration: space enrichment, projection („offline“).
- Inner iteration: one iteration of repeat loop („online“).

Algorithm	Landweber	RBL	
time (s)	238059	20083	
# Iterations	769034	outer	23
		inner	769043
time per Iteration (s)	0.309	outer	4.663
		inner	0.026
# forward solves	1538068	46	

Table: Time comparison of Algorithms 1 & 2.

Numerics - algorithmic behaviour

Update error $\|s_{n,RBL} - s_{n,LW}\|_Y$

$s_{n,RBL} := F_N'(\sigma_n^\delta)^*(u^\delta - F_N(\sigma_n^\delta))$, $s_{n,LW} := F'(\sigma_n^\delta)^*(u^\delta - F(\sigma_n^\delta))$

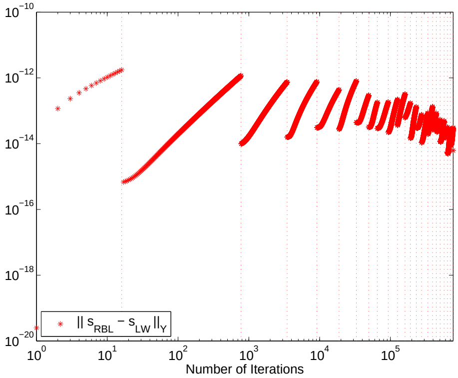

Figure: Update error $\|s_{n,RBL} - s_{n,LW}\|_Y$ over the course of the iteration.

Conclusion & Outlook

Conclusion

- Solving non-linear inverse coefficient problems typically requires many PDE solutions.
- Reduced basis (RB) approaches can speed up PDE solutions.
- But standard RB methods are only applicable for low dimensional parameter spaces.
- Using problem-specific RB enrichment, we can handle high-dimensional parameter spaces, e.g. for imaging problems.

$\rightsquigarrow$ RBL-method outperforms standard Landweber by an order without loss of accuracy.

Outlook

- Convergence theory for Algorithm 2.
- Apply methodology to other inverse problems and iterative regularization algorithms of Gauß-Newton type.

Thank you for your attention!