Reduced Basis Methods for inverse problems

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Reduced Basis Methods

Motivation

- Linear elliptic parametrized PDE of the form

$$\mathcal{L}u(\mathbf{x};\mu) = f, \text{ in } \Omega \tag{1a}$$
$$u(\mathbf{x};\mu) = g, \text{ on } \partial\Omega \tag{1b}$$

- Solution of (1) for many different parameters in a small amount of time (e.g. design optimization, optimal control, online-simulation, inverse problems)
- Detailed solution (e.g. FEM, FV, FD) is rather expensive

$\rightsquigarrow$ **model order reduction**

The detailed and reduced problem

Detailed problem:
Let X be a Hilbert space, $\mathcal{P} \subset \mathbb{R}^p$ a bounded parameter domain, $b : X \times X \times \mathcal{P} \to \mathbb{R}$ a continuous, coercive, symmetric bilinearform, $f : X \times \mathcal{P} \to \mathbb{R}$ a continuous linearform. For a given parameter $\mu \in \mathcal{P}$, we seek the **detailed solution** $u(\mu) \in X$ of

$$b(u(\mu), v; \mu) = f(v; \mu), \quad \forall v \in X. \tag{P}$$

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Reduced problem:
Let a Problem (P) be given and let $X_N \subset X$ be a reduced basis space (RB-space). For a given parameter $\mu \in \mathcal{P}$, we seek the **reduced solution** $u_N(\mu) \in X_N$ of

$$b(u_N(\mu), v; \mu) = f(v; \mu), \quad \forall v \in X_N. \tag{PN}$$

The reduced basis space X_N

Error estimator

$$\|u - u_N\|_X \leq \Delta_N(\mu) := \frac{\|v_r\|_X}{\alpha(\mu)}, \text{ with}$$

$$\langle v_r, v \rangle_X := r(v;\mu) := f(v;\mu) - b(u_N(\mu), v;\mu), \forall v \in X$$

Algorithm 1 Greedy-Procedure

1: $X_N := \{0\},\ \Phi_N := \emptyset,\ M_{train},\ \varepsilon_{tol},\ \Delta_N(\mu)$
2: **repeat**
3: $\quad \mu^\star := \arg\max_{\mu \in M_{train}} \Delta_N(\mu)$
4: $\quad \phi := u(\mu^\star),\ \Phi_N := \Phi_N \cup \phi,\ X_N := X_N + \text{span}(\phi)$
5: $\quad \varepsilon := \max_{\mu \in M_{train}} \Delta_N(\mu)$
6: **until** $\varepsilon \leq \varepsilon_{tol}$
7: **return** $\Phi_N,\ X_N$

Offline/Online Decomposition

Assumption: b and f are affine in parameter, i.e.

$$b(u, v; \mu) = \sum_{q=1}^{Q_b} \Theta_b^q(\mu) b^q(u, v),$$
$$f(v; \mu) = \sum_{q=1}^{Q_f} \Theta_f^q(\mu) f^q(v), \forall u, v \in X.$$

Offline-phase (once)

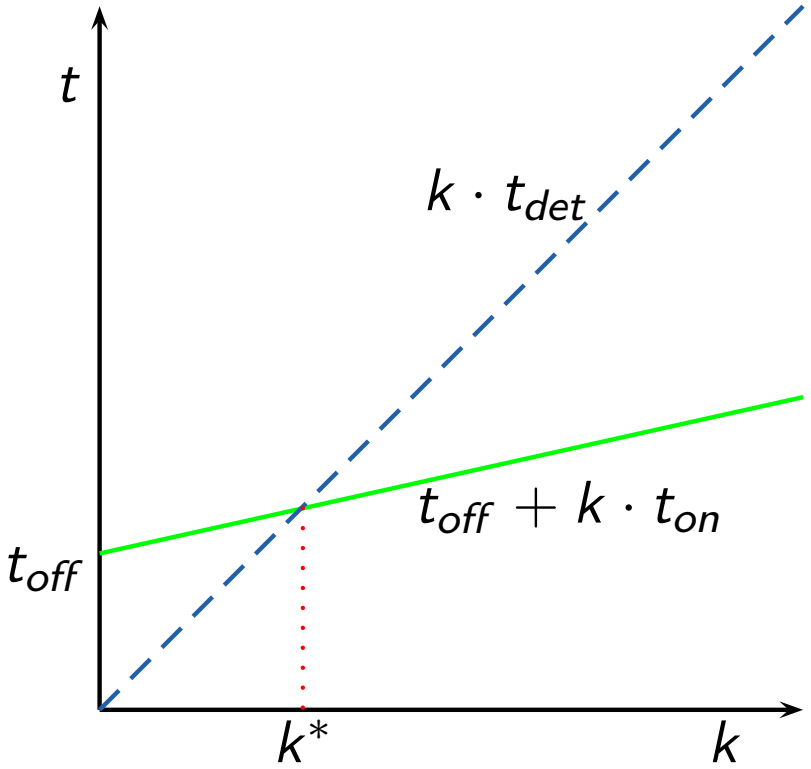
- Compute RB $\Phi_N = \{\phi_1, \dots, \phi_N\}$ and RB-space X_N
- Compute **parameter-independant** components $\underline{B}_N^q := (b^q(\phi_i, \phi_j))_{i,j=1}^N \in \mathbb{R}^{N \times N}$ and $\underline{f}_N^q := (f^q(\phi_i))_{i=1}^N \in \mathbb{R}^N$

Online-phase (for each new μ)

- Evaluate **parameter-dependant** coefficients $\Theta_b^q(\mu)$, $\Theta_f^q(\mu)$
- Assemble $\underline{B}_N(\mu)$, $\underline{f}_N(\mu)$ and solve $\underline{B}_N \underline{u}_N = \underline{f}_N$
- Reconstruct solution of (PN) via $u_N = \sum_i \underline{u}_{N,i} \phi_i$

Conclusion RBM

- Can accelerate forward solver of a PDE in a multi-query context due to efficient offline/online decomposition
- Greedy-scheme to construct RB-space X_N
- Approximates high-dimensional solution u (FEM, FD, FV)



RBM for inverse problems

Model problem

Forward problem:
Non-linear operator $F : \mathcal{D}(F) \subset Y \to X$ between Hilbert spaces maps a parameter $\sigma \in \mathcal{D}(F)$ to a solution u of a given PDE:

$$F(\sigma) = u.$$

Inverse problem:
For a given PDE-solution $u \in X$ find the corresponding parameter $\sigma \in \mathcal{D}(F)$ with $F(\sigma) = u$.

Numerical approach

- For given noisy data u^δ ($\|u - u^\delta\| \leq \delta$) and exact solution σ^+ ($F(\sigma^+) = u$) consider the iteration $\sigma^\delta_{n+1} = \sigma^\delta_n + s^\delta_n$ with starting value $\sigma^\delta_0 \in \mathcal{D}(F)$
- Want s^δ_n to approximate $s^e_n := \sigma^+ - \sigma^\delta_n$
- If F is Fréchet-differentiable s^e_n solves the linear system

$$F'(\sigma^\delta_n)s^e_n = u - F(\sigma^\delta_n) - E(\sigma^+, \sigma^\delta_n)$$

- Compute s^δ_n as a solution of $F'(\sigma^\delta_n)s = u^\delta - F(\sigma^\delta_n)$

REGINN(REGularisation based on INexact Newtoniteration)[1]

Algorithm 2 REGINN($\sigma_{start}, \tau, \{\Theta_n\}$)

$n := 0,\ \sigma_0^\delta := \sigma_{start}$
while $\|F(\sigma_n^\delta) - u^\delta\|_{H^1} > \tau\delta$ **do**
  $i := 0$
  repeat
    $i := i + 1$
    $s_{n,i} := \left(F'(\sigma_n^\delta)^* F'(\sigma_n^\delta) + \frac{1}{i}\,\mathrm{Id}\right)^{-1} F'(\sigma_n^\delta)^*(u^\delta - F(\sigma_n^\delta))$
  until $\|F'(\sigma_n^\delta)s_{n,i} + F(\sigma_n^\delta) - u^\delta\| < \Theta_n \|F(\sigma_n^\delta) - u^\delta\|$
  $\sigma_{n+1}^\delta := \sigma_n^\delta + s_{n,i}$
  $n := n + 1$
end while
$\sigma_{REGINN} := \sigma_n^\delta$

[1] see A.Rieder: Keine Probleme mit Inversen Problemen, Seite 251.

REGINN utilizing RB

Algorithm 3 REGINN_utilizing_RB$(\sigma_{start}, \tau, \{\Theta_n\}, X_N)$

$n := 0,\ \sigma_0^\delta := \sigma_{start}$
while $\|F_N(\sigma_n^\delta) - u^\delta\| > \tau\delta$ **do**
  $i := 0$
  repeat
    $i := i + 1$
    $s_{n,i} := \left(F_N'(\sigma_n^\delta)^* F_N'(\sigma_n^\delta) + \frac{1}{i}\,\mathrm{Id}\right)^{-1} F_N'(\sigma_n^\delta)^*(u^\delta - F_N(\sigma_n^\delta))$
  until $\|F_N'(\sigma_n^\delta)s_{n,i} + F_N(\sigma_n^\delta) - u^\delta\| < \Theta_n\|F_N(\sigma_n^\delta) - u^\delta\|$
  $\sigma_{n+1}^\delta := \sigma_n^\delta + s_{n,i}$
  $n := n + 1$
end while
$\sigma_{REGINN,RB} := \sigma_n^\delta$

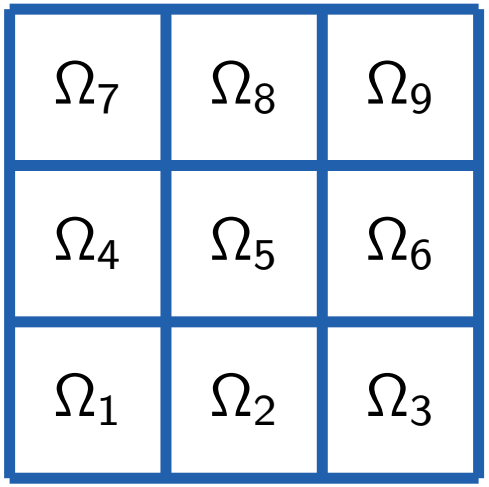
Numerical toy problem

Find a solution $u \in H_0^1(\Omega)$ of

$$\operatorname{div}(\bar{\sigma}\nabla u) = 1, \qquad \text{in } \Omega = [0,1]^2,$$

with $\bar{\sigma} \in L_+^\infty(\Omega)$.

To make current RB-approach (Greedy-procedure) applicable ($\mathcal{P} \subset \mathbb{R}^p$) restrict to $\sigma = \sum_{i=1}^{9} \sigma_i \Omega_i$, $\sigma_i \in (0.1, 10]$, $\forall i$.



Inverse problem

For a given δ and u^δ with $\|u - u^\delta\| \leq \delta$ reconstruct the target parameter $\sigma^+ \in (0.1, 10]^9$, s.t. u is the corresponding PDE solution.

Numerical results - How good is RB?

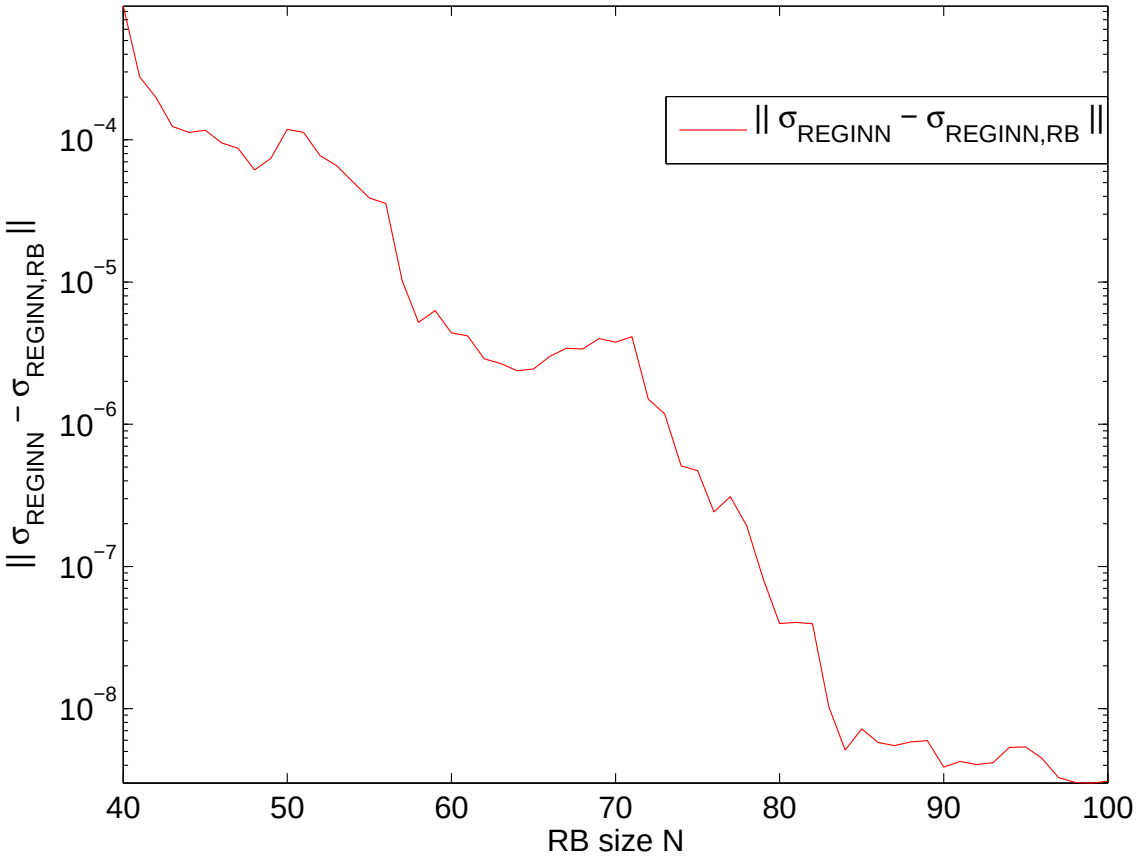


Figure : Comparison of Algorithm 2 and 3 with $\delta = 0.01$, depending on the size N of the used RB. Target parameter was $\sigma^+ = (2, 2, 2, 2, 5, 2, 2, 2, 2)$, $\sigma_{start} = (1, 1, 1, 1, 1, 1, 1, 1, 1)$.

Numerical results - When does RB pay off?

Average time of REGINN without RB (Alg. 2): ~ 45.3 s

size RB	REGINN_RB (s)	Offline-time RB (s)	querys required
40	7.1307	8908	230
50	7.4205	11674	303
60	7.5825	14717	370
70	7.8344	18659	490
80	8.7067	24257	653
90	9.2116	28765	785
100	9.4329	35390	971

Table : Average time of Algorithm 3 (using 100 random parameters with same starting value and $\delta = 0.01$).

Outlook

- Improve results using more suitable RB construction algorithms
- Extend to more realistic examples
  - drastically increase fineness of discretization of $\mathcal{P}$
  - problems with an actual application (e.g. EIT)
- Extend to parameter functions
- Provide theoretical background

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Thank you for your attention!