Reduced Basis Landweber method for nonlinear ill-posed inverse problems

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Introduction

Forward problem

Consider $\nabla \cdot (\sigma(x)\nabla u(x)) = 1,\ x \in \Omega := (0,1)^2,\ u(x) = 0,\ x \in \partial\Omega.$

Forward operator

$F : \mathcal{D}(F) \subset Y \longrightarrow X,\ \sigma \longmapsto u^\sigma$ between Hilbert spaces with u^σ, the detailed solution, solving

$$b(u^\sigma, v; \sigma) = f(v; \sigma),\ \text{ for all } v \in X, \text{ with} \tag{1a}$$

$$b(u, w; \sigma) := \int_\Omega \sigma \nabla u \cdot \nabla w \, dx, \quad f(v;\sigma) := -\int_\Omega v \, dx. \tag{1b}$$

Inverse problem and its difficulties

Inverse problem

For given solution $u \in X$ of (1), find corresponding parameter $\sigma^+ \in \mathcal{D}(F)$ with $F(\sigma^+) = u$ (e.g. groundwater hydrology).

- Naive inversion, i.e. solving $\sigma^+ = F^{-1}(u)$ fails due to ill-posedness of the problem (F^{-1} is discontinuous!)

  $\leadsto$ Small errors get amplified!

- Typically only noisy data u^δ with $\|u - u^\delta\| < \delta$ given

  $\leadsto F^{-1}(u^\delta) \nrightarrow F^{-1}(u)$ for $\delta \to 0$.

$\leadsto$ Remedy: Regularization methods, e.g. Landweber method, still provide stable approximative solutions σ^δ to σ^+.

Landweber method

- Solve $\min_{\sigma \in \mathcal{D}(F)} \|F(\sigma) - u^\delta\|_X^2$, where $F'(\sigma)^*(F(\sigma) - u^\delta)$ is the negative Gradient of the functional.

Landweber iteration

$\sigma_{n+1}^\delta := \sigma_n^\delta + \omega F'(\sigma_n^\delta)^*(u^\delta - F(\sigma_n^\delta))$

Terminate as soon as $\|F(\sigma_n^\delta) - u^\delta\|_X \leq \tau\delta$ (discrepancy principle)

- Numerous evaluations of F for many different parameters.
- Detailed solution (e.g. FEM, FV, FD) is expensive.

$\rightsquigarrow$ **model order reduction**

Reduced basis method

Reduced basis space (RB-space) $X_N := \text{span}\{\phi_1, \ldots, \phi_N\} \subset X$ ($\dim X_N \ll \dim X$), with $\phi_i = F(\sigma_i)$ for *meaningful* $\sigma_i \in \mathcal{D}(F)$, $i = 1, \ldots, N$, assumed to be given.

Reduced forward operator

For given F and $X_N \subset X$, define $F_N : \mathcal{D}(F) \subset Y \longrightarrow X_N$, $\sigma \longmapsto u_N^\sigma$ with u_N^σ, the reduced solution, solving

$$b(u_N^\sigma, v; \sigma) = f(v; \sigma), \quad \text{for all } v \in X_N.$$

- Residual error estimator[1]: $\|u^\sigma - u_N^\sigma\|_X \leq \Delta_N(\sigma) := \frac{\|v_r\|_X}{\alpha(\sigma)}$.
- Offline/online decomposition: enables efficient and cheap evaluation of $F_N(\sigma)$.

[1] e.g. Patera & Rozza 2006

Combining RBM & Landweber method

Various approaches

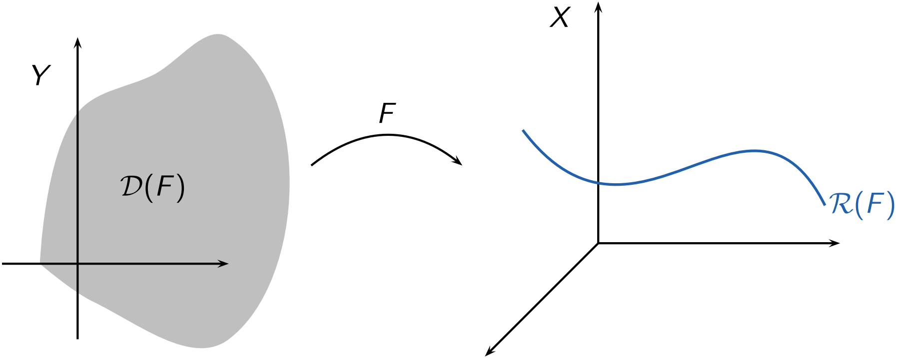
Standard approach:

- Construct global X_N approximating whole $\mathcal{R}(F)$ (providing good reduced solutions u_N^σ for all $\sigma \in \mathcal{D}(F)$) $\rightsquigarrow$ „offline-phase“.
- Using this space, quickly compute $F_N(\sigma)$ and substitute $F(\sigma)$ for $F_N(\sigma)$ in the Landweber iteration $\rightsquigarrow$ „online-phase“.

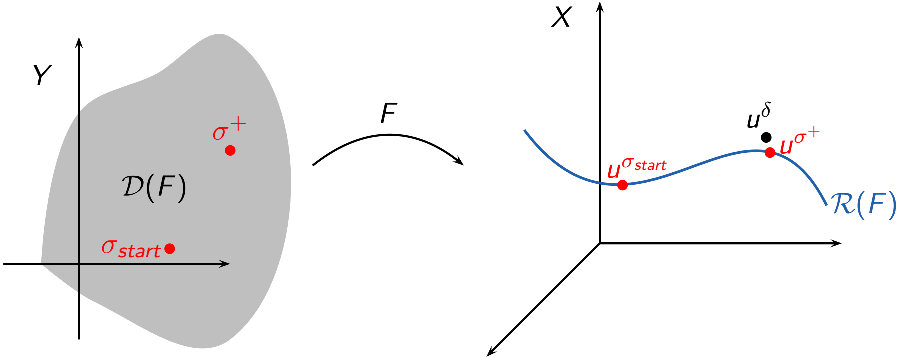
Problem: Only feasible for low-dimensional parameter spaces (≤ 30), not feasible for imaging.

Our approach: Create problem-adapted RB-space by iterative enrichment (inspired by Druskin & Zaslavski 2007, Zahr & Fahrhat 2015 and Lass 2014).

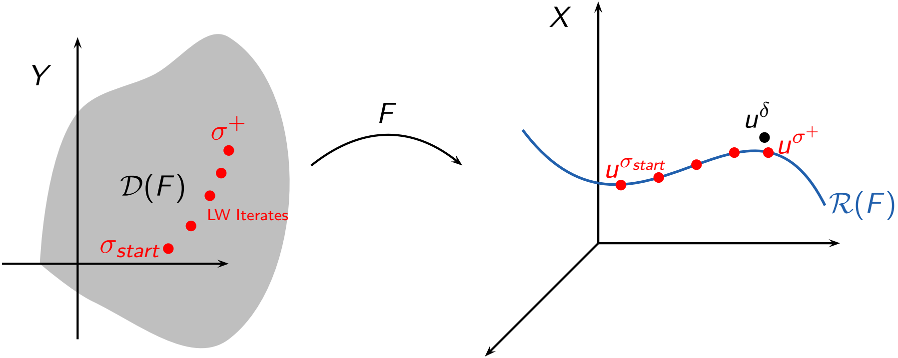
Motivation - Combining RBM & LW



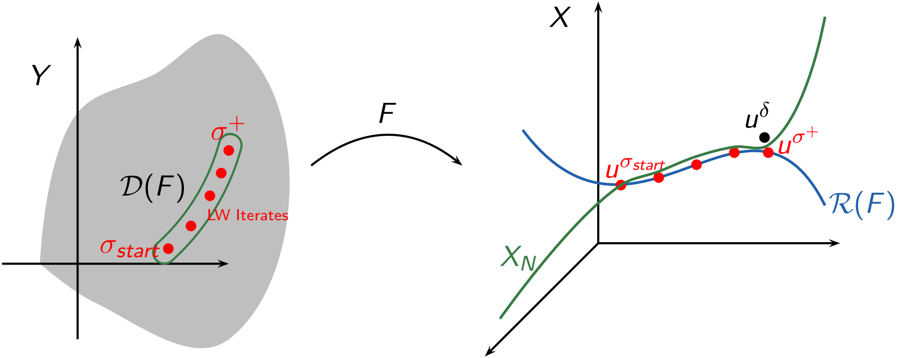
Motivation - Combining RBM & LW



Motivation - Combining RBM & LW



Motivation - Combining RBM & LW



Procedure - adaptive space enrichment

1. Start with initial guess σ_{start} and initial RB-spaces $X_{N,1}$, $X_{N,2}$ (Landweber method requires adjoint of the derivative!).
2. Update $X_{N,1}$, $X_{N,2}$ using current iterate (first update via σ_{start}).
3. Solve the inverse problem up to a certain accuracy using the Landweber method projected onto $X_{N,1}$ and $X_{N,2}$.
4. If resulting iterate is accepted by the discrepancy principle, terminate, else go to step 2.

Reduced Basis Landweber (RBL) method

Algorithm 1 RBL$(\sigma_{start}, \tau, X_{N,1}, X_{N,2})$

$n := 0,\ \sigma_0^\delta := \sigma_{start}$
while $\|F(\sigma_n^\delta) - u^\delta\|_X > \tau\delta$ **do**
  enrich $X_{N,1},\ X_{N,2}$
  $i := 1,\ \sigma_i^\delta := \sigma_n^\delta$
  repeat
    $\sigma_{i+1}^\delta := \sigma_i^\delta + \omega F_N'(\sigma_i^\delta)^*(u^\delta - F_N(\sigma_i^\delta))$
    $i := i + 1$
  until $\|F_N(\sigma_i^\delta) - u^\delta\|_X \leq \tau\delta$ **or** $\Delta_N(\sigma_i^\delta) > (\tau - 2)\delta$
  $\sigma_{n+1}^\delta := \sigma_i^\delta$
  $n := n + 1$
end while
return $\sigma_{RBL} := \sigma_n^\delta$

Dual problem

For $\sigma, \kappa \in \mathcal{D}(F)$ and $l \in X$, one can show

$$\langle \kappa, F'(\sigma)^* \, l \rangle_Y = \int_\Omega \kappa \nabla u^\sigma \cdot \nabla u_l^\sigma \, dx, \tag{2}$$

with $u_l^\sigma \in X$ the unique solution of the dual problem

$$b(u, v; \sigma) = m(v; l), \text{ for all } v \in X, \quad m(v) := -\int_\Omega l \, v \, dx. \tag{3}$$

In Algorithm 1

- enrich $X_{N,1}$ with $F(\sigma_n^\delta)$ and $X_{N,2}$ with $u_l^{\sigma_n^\delta}$ solving (3) for $l := u^\delta - F(\sigma_n^\delta)$.
- evaluate $F_N'(\sigma_i^\delta)^*(u^\delta - F_N(\sigma_i^\delta))$ using (2) and associated reduced solutions.

Numerics - compare reconstructions

Setting: 900 pixels, $\tau = 2.5$, $\delta \approx 0.8\%$ and $\omega = \frac{1}{2}(\|F'(\sigma_{start})\|)^{-1}$.

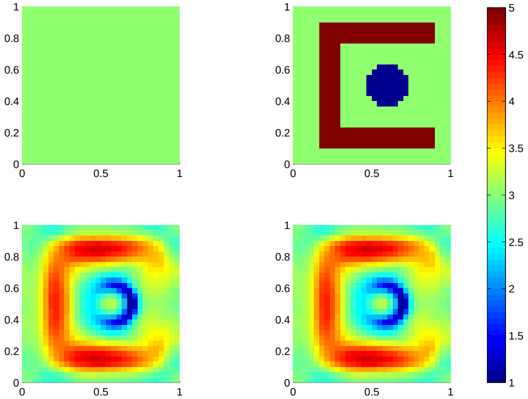


Figure: σ_{start} (top left), exact solution (top right). Reconstruction via RBL method (bottom left) and Landweber method (bottom right).

Numerics - time comparison

- Outer iteration: space enrichment, projection („offline").
- Inner iteration: one iteration of repeat loop („online").

Algorithm	Landweber	RBL	
time (s)	266958	22014	
# Iterations	789304	outer	25
		inner	789318
time per Iteration (s)	0.338	outer	8.961
		inner	0.028
# forward solves	1578608	50	

Table: Time comparison of Landweber method & RBL method.

Conclusion & Outlook

- Solving inverse coefficient problem requires many PDE solves.
- Reduced basis (RB) approach can speed up PDE solution.
- But standard RB approach is only applicable for low dimensional parameter spaces.
- Using adaptive problem-specific RB enrichment, we can handle high-dimensional parameter spaces, e.g. for imaging problems.

$\rightsquigarrow$ RBL method outperforms standard Landweber by an order without loss of accuracy.

Outlook

- Convergence theory for RBL method.
- Apply methodology to other inverse problems and iterative regularization algorithms of Gauß-Newton type.

Thank you for your attention!

Preprint available:
A Reduced Basis Landweber method for nonlinear inverse problems
(arXiv $\rightsquigarrow$ 1507.05434).