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A reduced basis method for linear evolution equations

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Motivation

- Parabolic parametrized partial differential equation of the form

$$\partial_t u(\mathbf{x}, t; \mu) + \mathcal{L}u(\mathbf{x}, t; \mu) = f, \tag{1a}$$

$$u(\mathbf{x}, 0; \mu) = u_0, \tag{1b}$$

- Solution of (1) for many different parameters in a small amount of time (i.e. design optimization, optimal control, online-simulation, financial markets)
- Computation of a detailed solution (i.e. FEM, FV, FD) is rather expensive

$\rightsquigarrow$ **model order reduction**

The detailed (semi-discretized) evolution problem

Let $X \subset Y := L^2(\Omega)$ be a Hilbert space and $\mu \in \mathcal{P} \subset \mathbb{R}^p$. We want the solution sequence $u = \left(u^k\right)_{k=0}^{M} \in (X)^{M+1}$ of the **detailed evolution scheme**

$$u^0 = P_X(u_0) \tag{2a}$$

$$\mathfrak{L}^I(\mu, t^k)u^{k+1} = \mathfrak{L}^E(\mu, t^k)u^k + b(\mu, t^k) \tag{2b}$$

$$s^k(\mu) = l(u^k, \mu), \tag{2c}$$

with $\mathfrak{L}^I, \mathfrak{L}^E \in L(X)$, $P_X : Y \to X$ the continuous projection, $l : X \times \mathcal{P} \to \mathbb{R}$ a linear and continuous functional, $u_0 \in Y$ the initial values and $b \in X$ the inhomogeniety, so that

$$u^k(x;\mu) \approx u(x, t^k; \mu),\ k = 0, \ldots, M.$$

The reduced (semi-discretized) evolution problem

Let a problem (2) be given. Let $X_N \subset X$ be a reduced basis space, $\mu \in \mathcal{P}$. We want the solution sequence $u_N = \left(u_N^k\right)_{k=0}^M \in (X_N)^{M+1}$ of the **reduced evolution scheme**

$$u_N^0 = P_{X_N}(P_X(u_0)) \tag{3a}$$

$$\mathfrak{L}_N^I(\mu, t^k)u_N^{k+1} = \mathfrak{L}_N^E(\mu, t^k)u_N^k + b_N(\mu, t^k) \tag{3b}$$

$$s_N^k(\mu) = l(u_N^k, \mu), \tag{3c}$$

with $P_{X_N} : X \to X_N$ the orthogonal projection related to the scalar product $\langle \cdot, \cdot \rangle_X$ and with the operators

$$\mathfrak{L}_N^I := P_{X_N} \circ \mathfrak{L}^I$$

$$\mathfrak{L}_N^E := P_{X_N} \circ \mathfrak{L}^E$$

$$b_N := P_{X_N}(b).$$

Offline/Online decomposition

Offline (high-dimensional quantities)

- Compute the reduced basis Φ_N and the reduced basis space X_N
- Compute the matrices $K := (\langle \psi_i, \psi_j \rangle_X)_{i,j=1}^H$ and $K_N = \Phi_N^t K \Phi_N$
- Compute the *parameter-independant* components

$$
\begin{aligned}
L_N^{I,q} &= \Phi_N^t K L^{I,q} \Phi_N \\
L_N^{E,q} &= \Phi_N^t K L^{E,q} \Phi_N \\
b_N^q &= \Phi_N^t K b^q
\end{aligned}
$$

Online (low-dimensional quantities)

- For new μ and all time steps evaluate the *parameterdependant* coefficients $\Theta_I^q(\mu, t^k)$, $\Theta_E^q(\mu, t^k)$, $\Theta_b^q(\mu, t^k)$
- Assemble scheme components $L_N^I(\mu, t^k)$, $L_N^E(\mu, t^k)$, $b_N(\mu, t^k)$
- Solve the small linear system $L_N^I u_N^{k+1} = L_N^E u_N^k + b_N$ in (3)

Regarding the reduced space X_N

Demands on X_N

- Small approximation error

$$\|u^k(\mu) - u_N^k(\mu)\|_X, \quad \forall \mu \in \mathcal{P},\ k = 0, \ldots, M$$

- "Small" dimension N of X_N
- Reduced basis Φ_N should be ONB (numerical stability)
- "Rich" training set of parameters $P_{train} \subset \mathcal{P}$

Idea of the POD-Greedy-procedure

$$X_N := \underset{\substack{Y \subset X \\ \dim Y = N}}{\arg\min} \max_{\mu \in P_{train}} \frac{1}{M+1} \sum_{k=0}^{M} \|u^k(\mu) - P_Y u^k(\mu)\|^2$$

Definition of the residual

$$R^k := \frac{1}{\Delta t}\left(\mathfrak{L}^E u_N^{k-1} - \mathfrak{L}^I u_N^k + b\right) \in X.$$

Error estimator

Let u and u_N be solutions of (2) and (3), then the error $e^k := \|u^k - u_N^k\|_X$ is bounded by

$$\|u^k - u_N^k\|_X \leq \Delta_N(\mu, t^k)$$

with

$$\Delta_N(\mu, t^k) := \sum_{i=1}^{k} \left(\frac{\gamma^E}{\alpha^I}\right)^{k-i} \frac{\Delta t}{\alpha^I} \|R^i\|_X + \left(\frac{\gamma^E}{\alpha^I}\right)^{k} \|e^0\|.$$

Benefits of a dual problem

- Improved reduced output

$$s_N^*(\mu) = \tilde{l}(u_N^{pr}(\mu)) - r^{pr}(u_N^{du}(\mu);\mu)$$

with *correction term* $r^{pr}(u_N^{du}(\mu);\mu)$

- Improved output error estimator having *quadratic effect*

$$|s(\mu) - s_N^*(\mu)| \leq \Delta_N^{s,*}(\mu) := \frac{\|v_r^{pr}\|\,\|v_r^{du}\|}{\tilde{\alpha}_{LB}(\mu)}$$

with v_r^{pr}, $v_r^{du} \in \tilde{X}'$ the riesz-representants of the respective residuals

Application - The Black-Scholes-Equation

Pricing of a vanilla European put option, i.e. searching the solution $P(S_1, S_2, t, \mu)$ of the parametrized parabolic PDE

$$\frac{\partial P}{\partial t} - \frac{1}{2} \sum_{k,l=1}^{2} \Xi(k,l) S_k S_l \frac{\partial^2 P}{\partial S_k \partial S_l} - \sum_{k=1}^{2} r S_k \frac{\partial P}{\partial S_k} + rP = 0$$

with the parameter matrix

$$\Xi = \begin{pmatrix} \sigma_1^2 & \frac{2\rho}{1+\rho^2}\sigma_1\sigma_2 \\ \frac{2\rho}{1+\rho^2}\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

and the parameter vector $\mu = (r,\ \rho,\ \sigma_1,\ \sigma_2)$.

Numerical results - error estimator

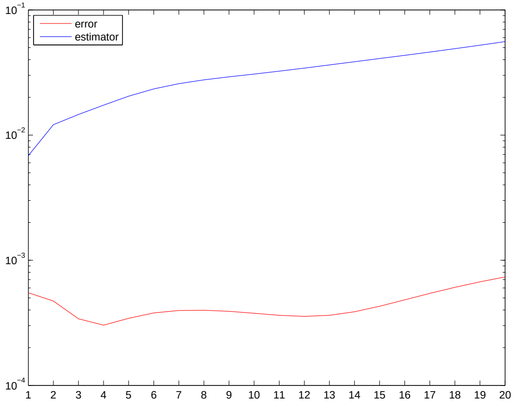

Figure: Sequences of the error $e^k := \|u^k - u_N^k\|_X$ (red) and the error estimator $\Delta_N(\mu, t^k)$ (blue) over the time steps t^k for a reduced basis with 200 basis vectors and $\mu = (0.05, 0.4, 2, 0.5)$.

- Average time for a detailed solution: ~ 21.5 seconds

Reduced Basis with 100 **basis vectors**

- Offline-time: 2581 seconds
- Average online-time: ~ 0.02 seconds
- Error estimator ranges between 10^{-2} and 10^{-1}

$\Rightarrow$ Approximation for ~ 121 different parameters required until the reduced basis method pays off.

Reduced Basis with 200 **basis vectors**

- Offline-time: 5928 seconds
- Average online-time: ~ 0.04 seconds
- Error estimator ranges between 10^{-3} and 10^{-2}

$\Rightarrow$ Approximation for ~ 277 different parameters required until the reduced basis method pays off.

functional l	$\frac{1}{\lvert G_0\rvert}\sum_{x_{ij}\in G_0} P_{i,j}^k$	$\frac{1}{\lvert G_0\rvert}\sum_{x_{ij}\in G_0} \frac{P_{i+1,j}^k - P_{i-1,j}^k}{2h_1}$
$s^{20}(\mu)$	13.8331045579815	-0.248507988692125
$s_N^{20}(\mu)$	13.8324367556314	-0.248461984173756
$s_N^*(\mu)$	13.8324367558403	-0.248461984698320
Δ_N^s	$2.975 \cdot 10^4$	$2.661 \cdot 10^4$
$\Delta_N^{s,*}$	0.062	0.018

Table: Table including detailed output $s^{20}(\mu)$, reduced output $s_N^{20}(\mu)$, reduced improved output $s_N^*(\mu)$ and the difference between the output error estimator Δ_N^s and improved output error estimator $\Delta_N^{s,*}$ for various output functionals and $\mu = (0.05, 0.4, 2, 0.5)$.

Conclusion

- Reduced basis methods can accelerate forward solvers when solution is required for a high number of parameters.
- Additional dual problem improves accuracy of output

Outlook

On March 1st, 2013, I have started to work on my PhD thesis supervised by Bastian Harrach (University of Stuttgart) on the subject of reduced basis methods for optimization and inverse problems.