Reduced basis methods for nonlinear ill-posed inverse problems

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Joint work with Bastian Harrach and Bernard Haasdonk

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Introduction

The forward problem

Consider

$$\nabla \cdot (\sigma(x)\nabla u(x)) = 1,\ x \in \Omega := (0,1)^2,\ u(x) = 0,\ x \in \partial\Omega.$$

Assume: σ is piecewise constant $\rightsquigarrow \sigma(x) = \sum_{q=1}^{p} \boldsymbol{\sigma}_q \chi_{\Omega_q}(x)$.

Forward operator

$F : \mathcal{P} \subset \mathbb{R}^p \to X \subset H_0^1(\Omega),\ \sigma \mapsto u^\sigma$, the detailed solution, solving

$$b(u^\sigma, v; \sigma) = f(v),\ \text{for all } v \in X, \text{ with} \tag{1a}$$

$$b(u, w; \sigma) := \int_\Omega \sigma \nabla u \cdot \nabla w \, dx, \quad f(v) := -\int_\Omega v \, dx. \tag{1b}$$

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Inverse Problem

For given solution $u \in X$ of (1), find corresponding parameter $\sigma^+ \in \mathcal{P}$ with $F(\sigma^+) = u$ („a-example“).

Task: Given u^δ, $\|u - u^\delta\|_X \leq \delta$, $\delta > 0$, find approximation σ^δ to σ^+.

Nonlinear Landweber iteration

- $\sigma^\delta_{n+1} := \sigma^\delta_n + \omega F'(\sigma^\delta_n)^*(u^\delta - F(\sigma^\delta_n))$
- Terminate as $\|F(\sigma^\delta_n) - u^\delta\|_X \leq \tau\delta$ (discrepancy principle)

$\rightsquigarrow$ **Many-query setting**

The reduced problem & Properties

Reduced basis (RB) space: for given *carefully* selected *snapshots* $\phi_i = F(\sigma_i)$, define $X_N := \mathrm{span}\{\Phi_N\} = \mathrm{span}\{\phi_1, \ldots, \phi_N\}$.

Reduced forward operator

$F_N : \mathcal{P} \to X_N$, $\sigma \mapsto u_N^\sigma$ with u_N^σ, the reduced solution, solving

$$b(u_N^\sigma, v; \sigma) = f(v), \quad \forall v \in X_N.$$

- Certification: $\|u^\sigma - u_N^\sigma\|_X \leq \Delta_N(\sigma) := \frac{\|v_r\|_X}{\alpha(\sigma)}$, with $\langle v_r, v\rangle_X := r(v;\sigma) := f(v) - b(u_N^\sigma, v; \sigma), \forall v \in X$
- Offline/online decomposition: rapid computation of u_N^σ
- Reproduction of solutions: $u^\sigma \in X_N \Rightarrow u_N^\sigma = u^\sigma$

RBM and Inverse problems

G./Haasdonk/Harrach, A Reduced Basis Landweber method for nonlinear inverse problems, 2016 Inverse Problems 32 (3) (doi).

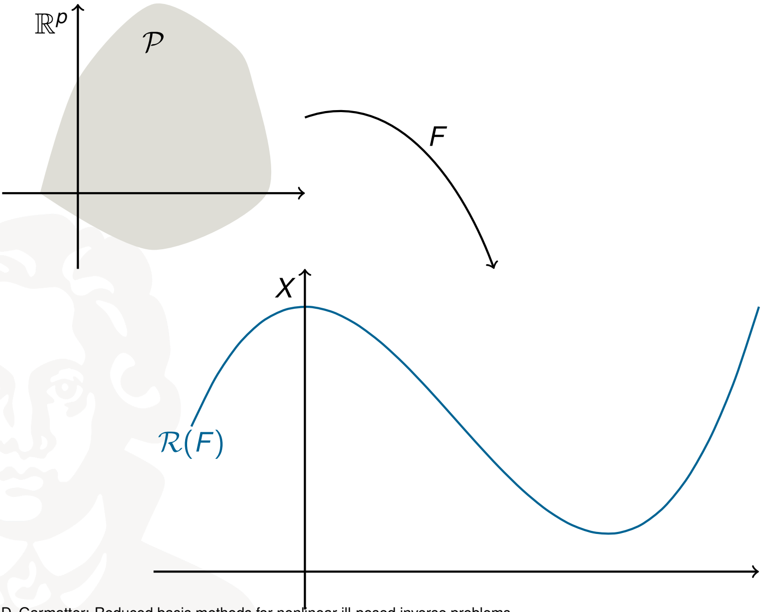
Naive approach

- Construct global X_N approximating whole $\mathcal{R}(F)$ (offline-phase)
- Rapidly compute $F_N(\sigma)$ and substitute $F(\sigma)$ for $F_N(\sigma)$ in the Landweber iteration („online-phase“)

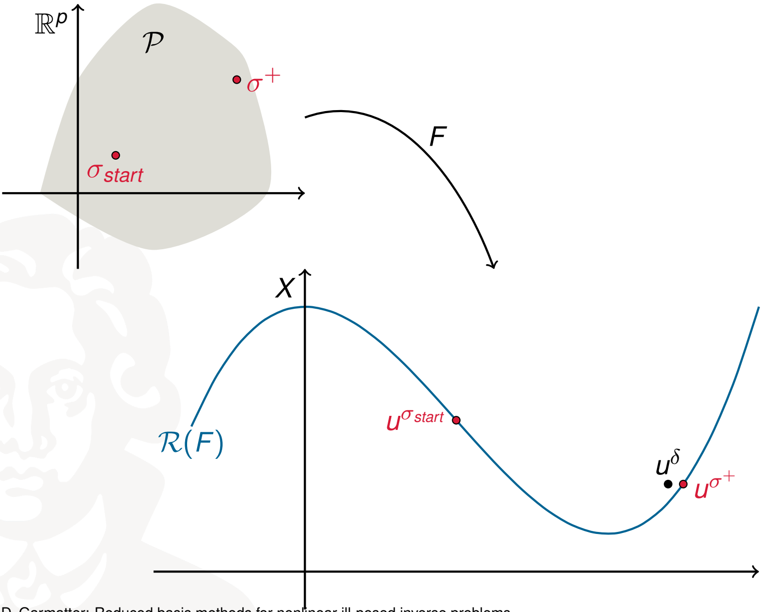
Limitation: Only feasible for low-dimensional parameter spaces, not feasible for imaging.

Our approach: Create problem-adapted RB-space by iterative enrichment (inspired by Druskin & Zaslavski 2007, Zahr & Fahrhat 2015 and Lass 2014).

Combining RBM & LW - Idea



Combining RBM & LW - Idea



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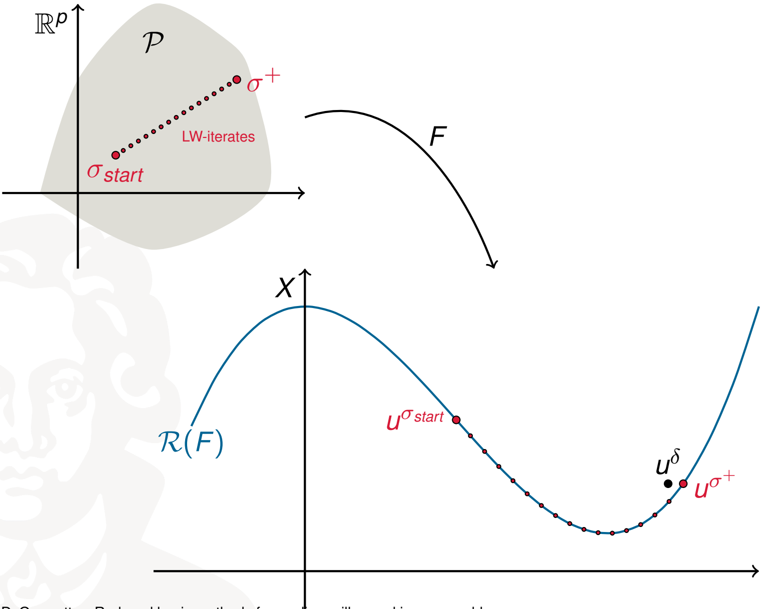

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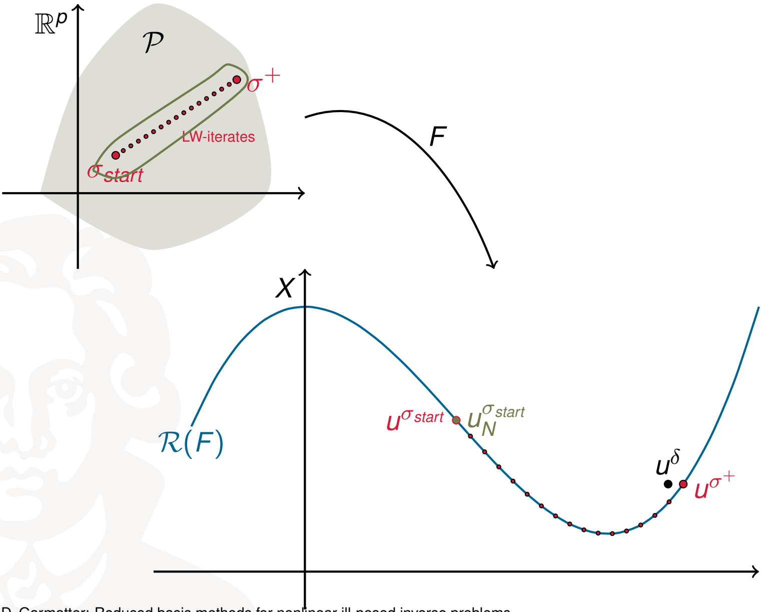

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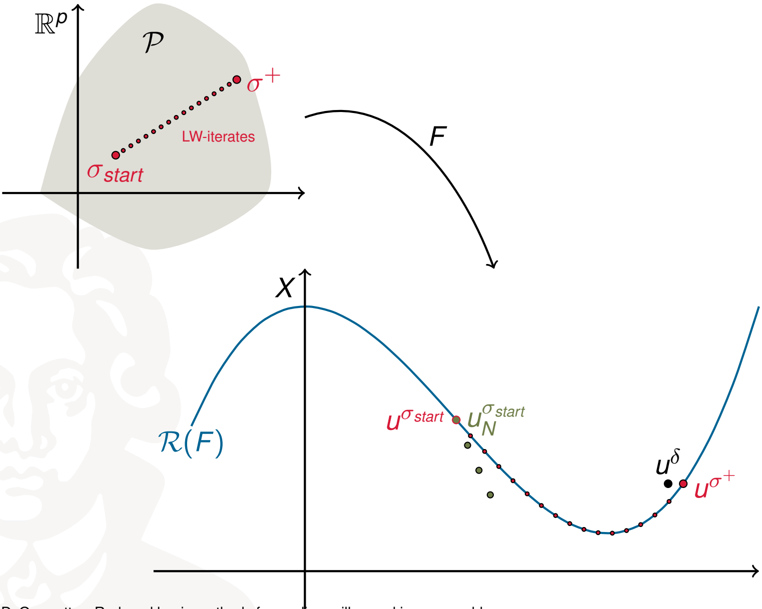

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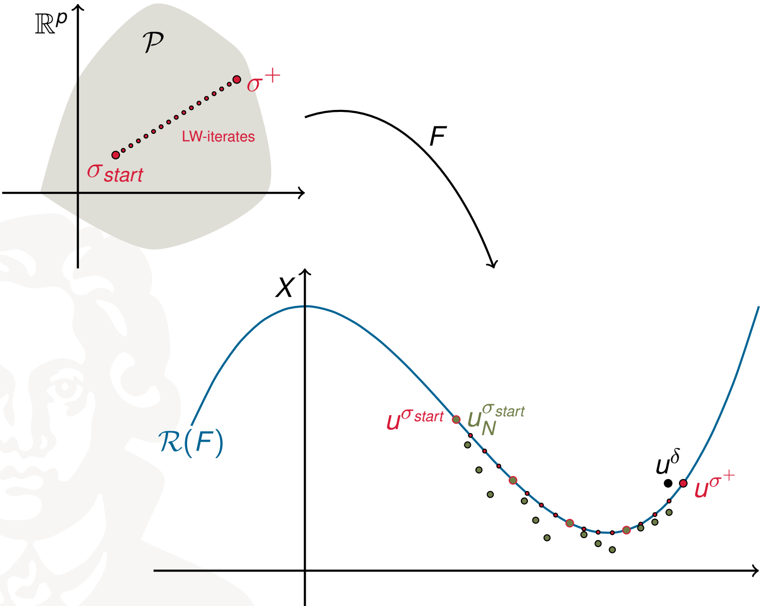

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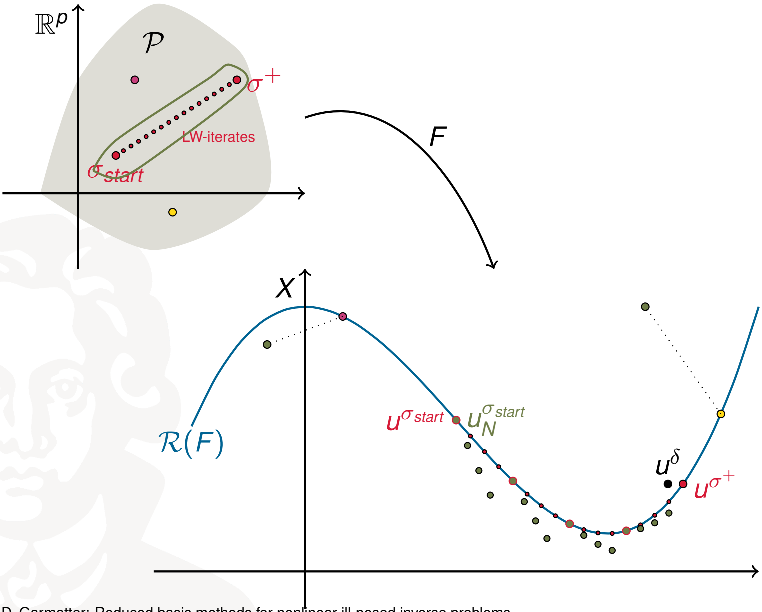


Reduced Basis Landweber (RBL) method

Algorithm 1 RBL($\sigma_{start}, \tau, \Phi_N$)

$n := 0,\ \sigma_0^\delta := \sigma_{start}$
while $\|F(\sigma_n^\delta) - u^\delta\|_X > \tau\delta$ **do**
    enrich RB using σ_n^δ
    $i := 1,\ \sigma_i^\delta := \sigma_n^\delta$
    repeat
        calculate reduced Landweber update $s_{n,i}$
        $\sigma_{i+1}^\delta := \sigma_i^\delta + \omega s_{n,i}$
        $i := i + 1$
    until $\|F_N(\sigma_i^\delta) - u^\delta\|_X \leq \tau\delta$ **or** $\Delta_N(\sigma_i^\delta) > (\tau - 2)\delta$
    $\sigma_{n+1}^\delta := \sigma_i^\delta$
    $n := n + 1$
end while
return $\sigma_{RBL} := \sigma_n^\delta$

Numerics - compare reconstructions

Setting: $p = 900$, $\tau = 2.5$, $\delta = 1\%$ and $\omega = \frac{1}{2}(\|F'(\sigma_{start})\|)^{-1}$.

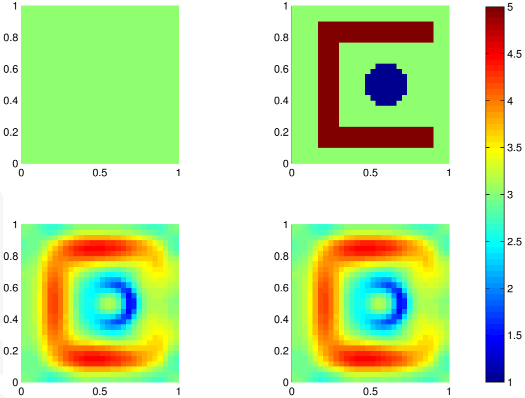


Figure: σ_{start} (top left), exact solution σ^+ (top right). Reconstruction via RBL method (bottom left) and Landweber method (bottom right).

Numerics - time comparison

- **Outer iteration:** space enrichment, projection („offline“)
- **Inner iteration:** one iteration of repeat loop („online“)

Algorithm	Landweber	RBL	
time (s)	187189	14661	
# Iterations	608067	outer	20
		inner	608083
time per Iteration (s)	0.308	outer	3.705
		inner	0.024
# forward solves	1216134	40	

$$\|\sigma_{RBL} - \sigma_{LW}\|_{\mathcal{P}} \approx 1.118 \cdot 10^{-5}$$

Numerics - regularization property

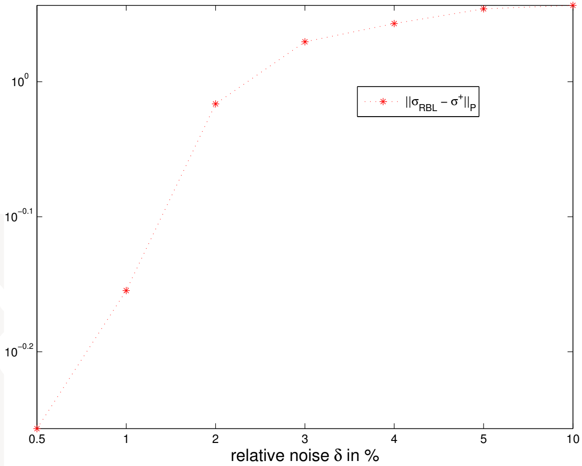
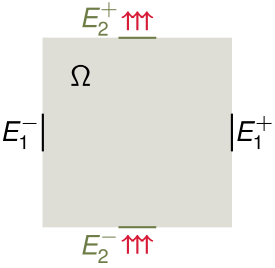


Figure: Error $\|\sigma_{RBL} - \sigma^{+}\|_{\mathcal{P}}$ over the decreasing relative noise level δ.

Current Work

Goal: Obtain high-resolution spatial images of conductivity σ of Ω

- Object Ω with electrode pairs $E_1^{\pm}$, $E_2^{\pm}$ attached to it
- Apply current between electrode pair generates magnetic flux density B
  $\rightsquigarrow$ obtainable with MRI scanner
- Use this *internal* data to overcome ill-posedness and reconstruct σ



Harmonic-Bz-Algorithm (Seo, Woo, et al. 2003)

MREIT - first numerics

Setting: 3200 pixels, $\approx$ 50000 FEM-dofs, no noise.

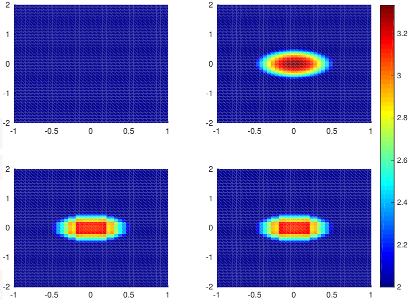


Figure: σ_{start} (top left), σ^{+} (top right), reconstruction via Bz-Algorithm (bottom left) and RBz-Algorithm (bottom right).

- Bz Algorithm: 3.01 s
- RBz-Algorithm: 1.79 s
- Normdifference: $\approx 8.33 \cdot 10^{-4}$

Conclusion

- Solving inverse coefficient problem requires many PDE solves
- Reduced basis (RB) approach can speed up PDE solution
- But standard RB approach is only applicable for low dimensional parameter spaces
- Using adaptive problem-specific RB enrichment, we can handle high-dimensional parameter spaces, e.g. for imaging problems

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$\rightsquigarrow$ RBL method outperforms standard Landweber (exp.: 13 times faster without loss of accuracy)

- Finalize theory and numerics for MREIT-project

Thank you for your attention!

Appendix

Appendix - The dual problem

Recall $\sigma_{n+1}^{\delta} := \sigma_n^{\delta} + \omega F'(\sigma_n^{\delta})^*(u^{\delta} - F(\sigma_n^{\delta}))$

For $\sigma, \kappa \in \mathcal{P}$ and $l \in X$, one can show

$$\langle \kappa, F'(\sigma)^* \, l \rangle_{\mathcal{P}} = \int_{\Omega} \kappa \nabla u^{\sigma} \cdot \nabla u_l^{\sigma} \, dx, \tag{2}$$

with $u_l^{\sigma} \in X$ the unique solution of the dual problem

$$b(u, v; \sigma) = m(v; l), \text{ for all } v \in X, \quad m(v; l) := -\int_{\Omega} l \, v \, dx.$$

In Algorithm 1 $\rightsquigarrow$ two RB spaces $X_{N,1}, X_{N,2}$

- enrich $X_{N,1}$ via $F(\sigma_n^{\delta})$ and $X_{N,2}$ via $u_l^{\sigma_n^{\delta}}$ with $l := u^{\delta} - F(\sigma_n^{\delta})$
- calculate $s_{n,i}$ using (2) and associated reduced solutions