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Reduced Basis Methods for Inverse Problems

Dominik Garmatter

garmatter@math.uni-frankfurt.de

Group for Numerics of Partial Differential Equations, Goethe University Frankfurt, Germany

Joint work with Bastian Harrach and Bernard Haasdonk

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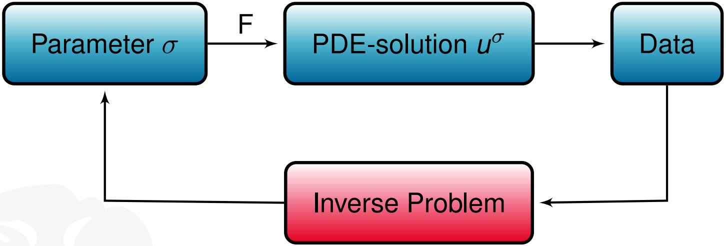
Introduction

Abstract problem formulation

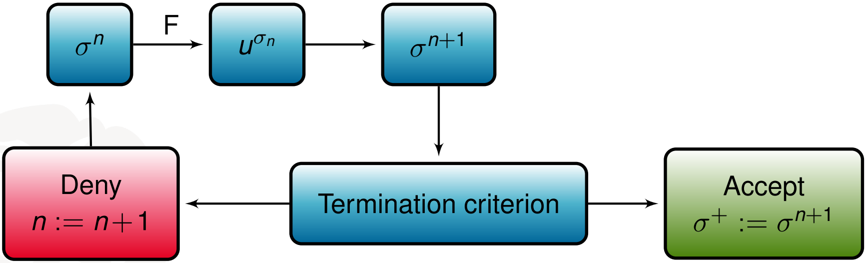


- Parameter-to-solution map: $F : \sigma \in \mathcal{P} \mapsto u^\sigma \in X$ (elliptic PDE)
- Imaging context: $\mathcal{P}$ very high-dimensional
- The (measured) data can be u^σ or just depend on u^σ

Aim: speed-up solution procedure of inverse problem.

Iterative solution of Inverse Problem

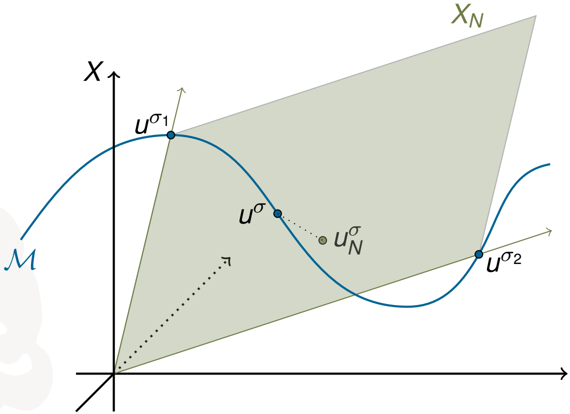
- Given initial guess $\sigma^0 := \sigma^{start}$



- Usually parameter-to-solution map F is expensive

$\rightsquigarrow$ Reduced Basis Methods

Reduced Basis Methods (RBM): Idea



- Solution manifold $\mathcal{M} := \{u^\sigma \mid \sigma \in \mathcal{P}\}$
- Construction of X_N via *carefully* chosen *snapshots* u^{σ_i}

RBM: The detailed & reduced problem

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Detailed problem (e.g. fine grid FEM)

For $\sigma \in \mathcal{P}$, find $u^\sigma \in X$, the detailed solution, of

$$b(u^\sigma, v; \sigma) = f(v), \text{ for all } v \in X.$$

Assume: snapshot-based reduced basis (RB) space given:

$$X_N := \mathrm{span}\{\phi_1, \ldots, \phi_N\} \text{ with } \phi_i = u^{\sigma_i}.$$

Reduced problem (Galerkin projection)

For $\sigma \in \mathcal{P}$, find $u_N^\sigma \in X_N \subset X$, the reduced solution, of

$$b(u_N^\sigma, v; \sigma) = f(v), \quad \forall v \in X_N.$$

RBM: Properties

- Reproduction of solutions: $u^\sigma \in X_N \;\Rightarrow\; u_N^\sigma = u^\sigma$
- Offline/Online-decomposition: rapid computation of u_N^σ

Certification - rigorous a-posteriori error estimator

$$\|u^\sigma - u_N^\sigma\|_X \leq \Delta_N(\sigma) := \frac{\|v_r\|_X}{\alpha(\sigma)}, \text{ with}$$

$$\langle v_r, v\rangle_X := r(v;\sigma) := f(v) - b(u_N^\sigma, v;\sigma), \forall v \in X$$

RBM & Inverse Problems: Various approaches

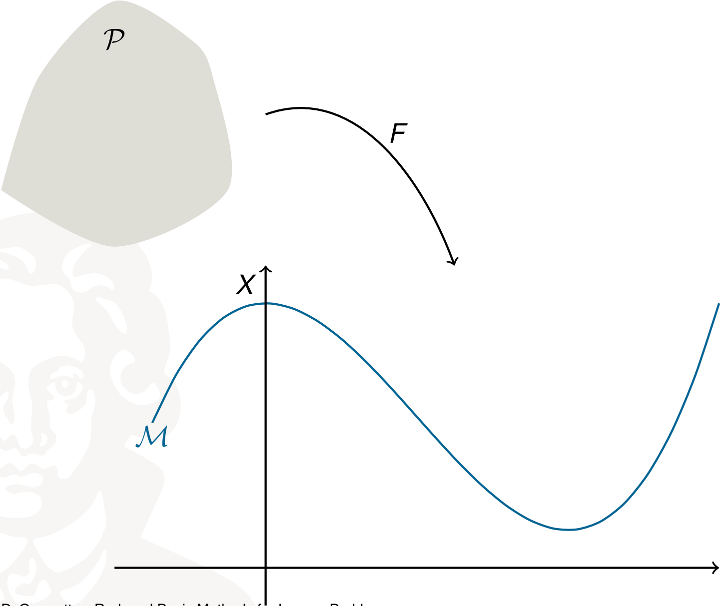
Naive approach

- Construct global X_N approximating whole $\mathcal{M} \rightsquigarrow$ offline-phase
- Use u_N^σ instead of u^σ in the solution procedure $\rightsquigarrow$ „online-phase“

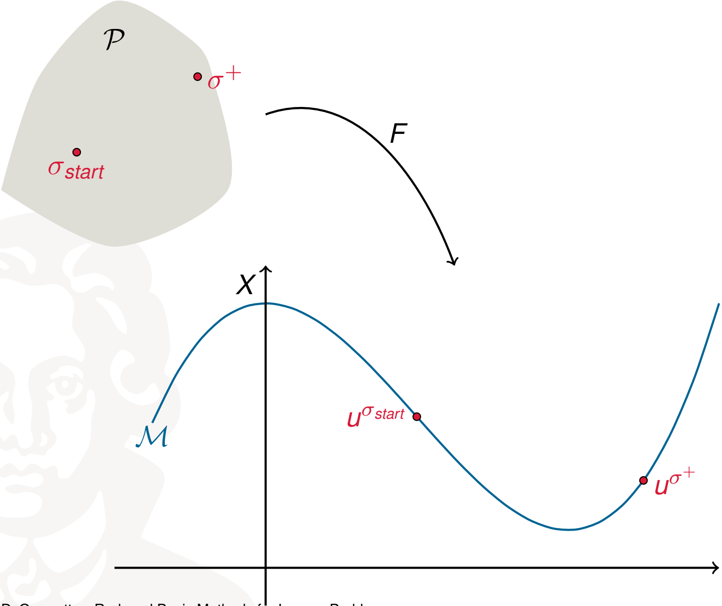
Limitation: Only feasible for low-dimensional parameter spaces, not feasible for imaging.

Our approach: Create problem-adapted RB-space by iterative enrichment (inspired by Druskin & Zaslavski 2007, Zahr & Fahrhat 2015 and Lass 2014).

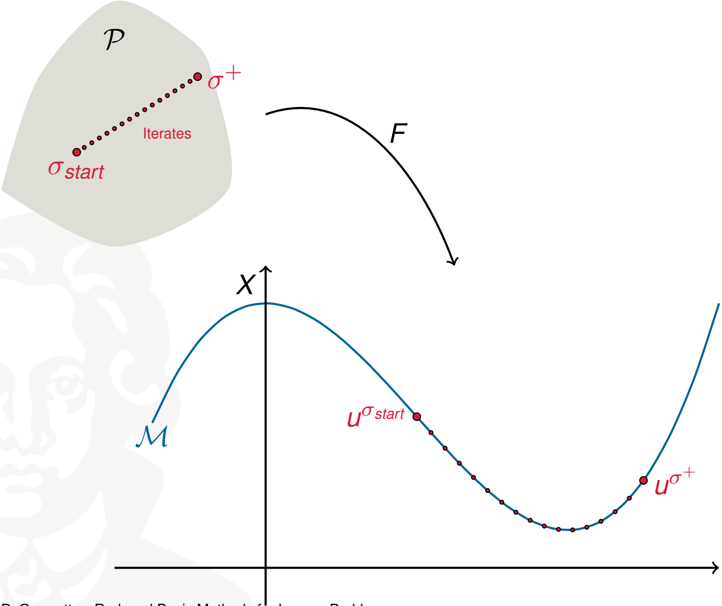
Adaptive RBM & IP: Idea



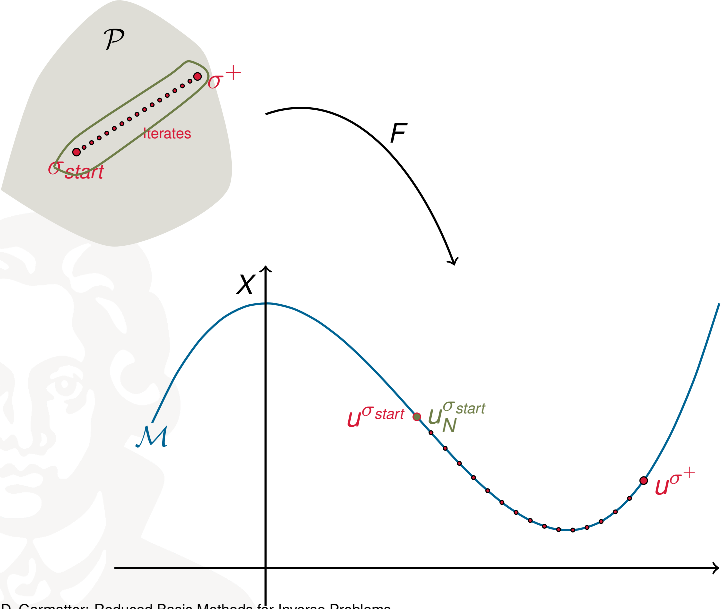
Adaptive RBM & IP: Idea



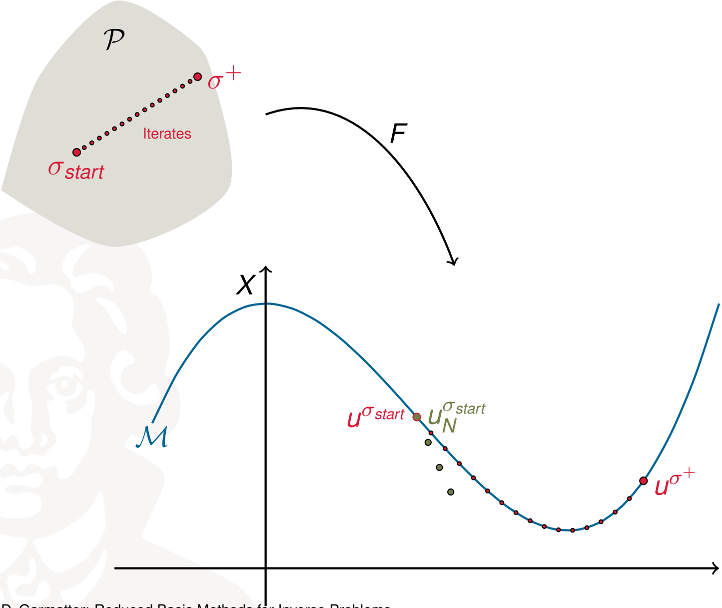
Adaptive RBM & IP: Idea



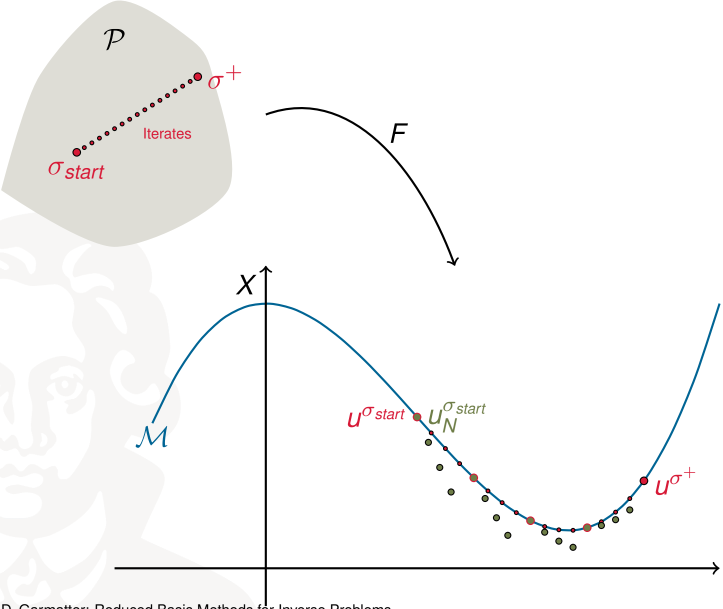
Adaptive RBM & IP: Idea



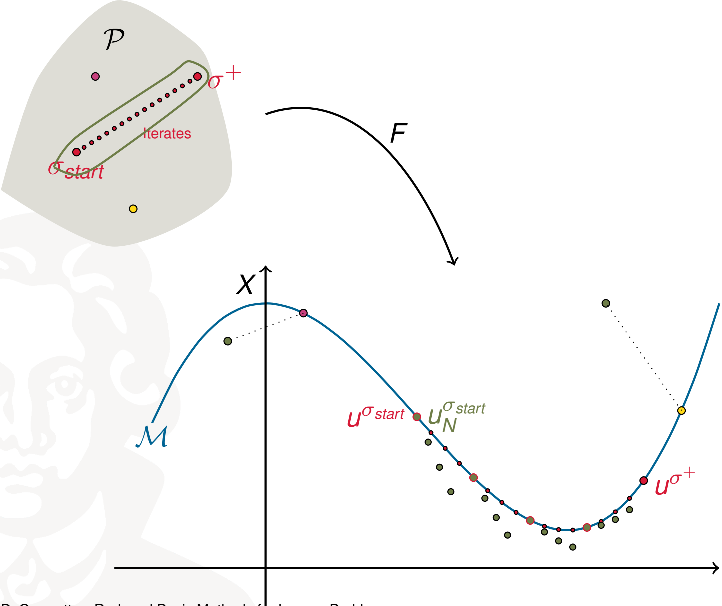
Adaptive RBM & IP: Idea



Adaptive RBM & IP: Idea



Adaptive RBM & IP: Idea



Problem I: Academic Example

G./Haasdonk/Harrach, A Reduced Basis Landweber method for nonlinear inverse problems, 2016 Inverse Problems 32 (3) (doi).

Forward problem

- Consider
  $\nabla \cdot (\sigma(x)\nabla u(x)) = 1,\ x \in \Omega := (0,1)^2,\ u(x) = 0,\ x \in \partial\Omega.$
- Assume: σ is piecewise constant $\rightsquigarrow \sigma(x) = \sum_{q=1}^{p} \boldsymbol{\sigma}_q \chi_{\Omega_q}(x)$.

Forward operator

$F : \mathcal{P} \to X := H_0^1(\Omega),\ \sigma \mapsto u^\sigma,\ u^\sigma$ the detailed solution solving

$$b(u^\sigma, v; \sigma) = f(v),\ \text{for all } v \in X, \text{ with} \tag{1a}$$

$$b(u, w; \sigma) := \int_\Omega \sigma \nabla u \cdot \nabla w \, dx, \quad f(v) := -\int_\Omega v \, dx. \tag{1b}$$

- For given $X_N \subset X$: associated $F_N : \mathcal{P} \to X_N,\ \sigma \mapsto u_N^\sigma$.

Inverse problem and its difficulties

Inverse Problem

For given solution $u \in X$ of (1), find corresponding parameter $\sigma^+ \in \mathcal{P}$ with $F(\sigma^+) = u$.

- Ill-posedness: solving $\sigma^+ = F^{-1}(u)$ fails (gen. F^{-1} discont.)

  $\rightsquigarrow$ Small errors get amplified

- Noisy data: u^δ with $\|u - u^\delta\|_X < \delta$ given (δ known)

  $\rightsquigarrow F^{-1}(u^\delta) \nrightarrow F^{-1}(u)$ as $\delta \to 0$

Goal: $R_{n(u^\delta,\delta)}(u^\delta) \to F^{-1}(u)$ as $\delta \to 0$.

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Idea (linear F)

- solve $F\sigma = u$ for $\sigma \rightsquigarrow$ Gaussian normal equation
- Fixed-point formulation (damping parameter ω)

$$\sigma = \sigma - \omega(F^* F\sigma - F^* u) = \sigma + \omega F^*(u - F\sigma)$$

- Iteration (for $u^\delta \in X$): $\sigma_{n+1}^\delta := \sigma_n^\delta + \omega F^*(u^\delta - F\sigma_n^\delta)$

Landweber iteration (nonlinear: $F(\sigma) = u$)

- $\sigma_{n+1}^\delta := \sigma_n^\delta + \omega F'(\sigma_n^\delta)^*(u^\delta - F(\sigma_n^\delta))$
- Terminate as $\|F(\sigma_{n+1}^\delta) - u^\delta\|_X \leq \tau\delta$ (discrepancy principle)

Reduced Basis Landweber (RBL) method

Algorithm 1 RBL$(\sigma_{start}, \tau, \Phi_N)$

$n := 0,\ \sigma_0^\delta := \sigma_{start}$
while $\|F(\sigma_n^\delta) - u^\delta\|_X > \tau\delta$ **do**
  enrich RB Φ_N using σ_n^δ
  $i := 1,\ \sigma_i^\delta := \sigma_n^\delta$
  repeat
    calculate reduced LW update $s_{n,i}$ (dual problem + $F_N(\sigma_i^\delta)$)
    $\sigma_{i+1}^\delta := \sigma_i^\delta + \omega s_{n,i}$
    $i := i + 1$
  until $\|F_N(\sigma_i^\delta) - u^\delta\|_X \leq \tau\delta$ **or** $\Delta_N(\sigma_i^\delta) > (\tau - 2)\delta$
  $\sigma_{n+1}^\delta := \sigma_i^\delta$
  $n := n + 1$
end while
return $\sigma_{RBL} := \sigma_n^\delta$

Numerics: compare reconstructions

Setting: $p = 900$, $\tau = 2.5$, $\delta = 1\%$ and $\omega = \frac{1}{2}(\|F'(\sigma_{start})\|)^{-1}$.

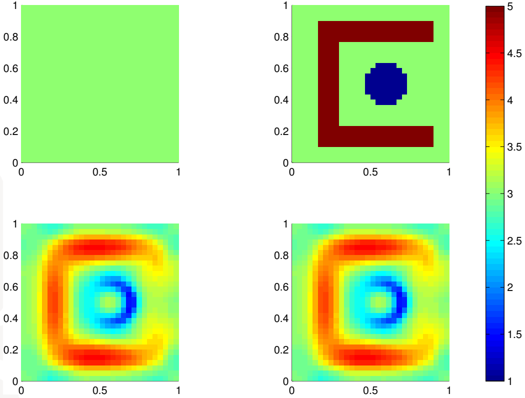

Figure: σ_{start} (top left), σ^{+} (top right), σ_{RBL} (bottom left), σ_{LW} (bottom right).

Numerics: time comparison

- Outer iteration: space enrichment, projection („offline")
- Inner iteration: one iteration of repeat loop („online")

Algorithm	Landweber	RBL	
time (s)	187189	14661	
# Iterations	608067	outer	20
		inner	608083
time per Iteration (s)	0.308	outer	3.705
		inner	0.024
# forward solves	1216134	40	

$$\|\sigma_{RBL} - \sigma_{LW}\|_{\mathcal{P}} \approx 1.118 \cdot 10^{-5}$$

Indermediate Conclusion

- Solving inverse coefficient problem requires many PDE solves
- Reduced basis (RB) approach can speed up PDE solution
- But standard RB approach is only applicable for low dimensional parameter spaces
- Using adaptive problem-specific RB enrichment, we can handle high-dimensional parameter spaces, e.g. for imaging problems

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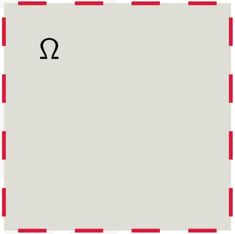
Question: Performance and applicability of the methodology in **modern** algorithms?

Problem II: **M**agnetic **R**esonance **E**lectrical **I**mpedance **T**omography[1]

[1] Based on Seo, Woo, et al. since 2003

Motivation: Electrical Impedance Tomography (EIT)

- **Setting:** Imaging object $O \subset \mathbb{R}^3$ with electrode pairs attached
- **Aim:** reconstruct cross-sectional ($\Omega = O \cap \{z = z_0\} \subset \mathbb{R}^2$) image of electrical conductivity inside Ω

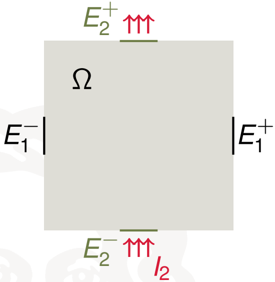


In EIT

- Data: Current-Voltage measurements on boundary
- Difficulty: highly ill-posed
  $\rightsquigarrow$ low spatial resolution

MREIT: Setting

Aim: achieve higher resolution of conductivity σ.



- Object Ω with two electrode pairs $E_j^\pm$ attached
- Place object inside MRI-Scanner
- Apply current I between electrode pair
- Generates magnetic flux density B (z-comp. measurable with MRI scanner)

$\rightsquigarrow$ Full internal data set to overcome ill-posedness

Forward problem

Consider ($j = 1, 2$ determines active electrode pair)

$$\nabla \cdot (\sigma \nabla u_j) = 0 \text{ in } \Omega, \quad I_j = \int_{E_j^+} \sigma \frac{\partial u_j}{\partial \mathbf{n}} ds = -\int_{E_j^-} \sigma \frac{\partial u_j}{\partial \mathbf{n}} ds \qquad \text{(2a)}$$

$$\nabla u_j \times \mathbf{n} = 0, \text{ on } E_j^+ \cup E_j^-, \quad \sigma \frac{\partial u_j}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega \backslash \overline{\left(E_j^+ \cup E_j^-\right)} \qquad \text{(2b)}$$

- $\mathcal{P} \equiv C^1_{\pm}(\bar{\Omega}) := \{\sigma \in C^1(\bar{\Omega}) \mid 0 < \underline{\sigma} \leq \sigma \leq \overline{\sigma} < \infty\}$
- Unique solution of (2) up to an additive constant

Inverse Problem

Aim: Determine σ from B.

- Maxwell Equations

$$-\sigma \nabla u_j = \frac{1}{\mu_0} \nabla \times B^j \text{ (Ampère's law)} \quad \text{and} \quad \nabla \cdot B^j = 0$$

- $\nabla \times$ of Ampère's law $\rightsquigarrow \nabla u_j \times \nabla \sigma = \frac{1}{\mu_0} \nabla^2 B^j$
- Only B_z^j measurable and two different electrode pairs/currents

Core-relation (point-wise in Ω)

$$\begin{pmatrix} \frac{\partial \sigma}{\partial x} \\ \frac{\partial \sigma}{\partial y} \end{pmatrix} = \frac{1}{\mu_0} \mathbb{A}[\sigma]^{-1} \begin{pmatrix} \nabla^2 B_z^1 \\ \nabla^2 B_z^2 \end{pmatrix}, \quad \mathbb{A}[\sigma] := \begin{pmatrix} \frac{\partial u_1}{\partial y} & -\frac{\partial u_1}{\partial x} \\ \frac{\partial u_2}{\partial y} & -\frac{\partial u_2}{\partial x} \end{pmatrix} \quad \textbf{(CR)}$$

Recovery of σ from $\nabla\sigma$

So far: $\nabla\sigma$ obtainable in Ω from $\nabla^2 B_z^j$ via (**CR**). How to get σ?

- Fundamental solution: for $\mathbf{r} = (x, y), \mathbf{r}' = (x', y') \in \Omega \subset \mathbb{R}^2$

$$\Phi(\mathbf{r} - \mathbf{r}') := \frac{1}{2\pi} \ln |\mathbf{r} - \mathbf{r}'| \quad \text{fulfilling} \quad \nabla^2 \Phi(\mathbf{r} - \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}')$$

- For $\sigma \in C^1(\overline{\Omega})$ one can show

$$\begin{aligned} \sigma(\mathbf{r}) = &- \int_\Omega \nabla_{\mathbf{r}'} \Phi(\mathbf{r} - \mathbf{r}') \cdot \nabla\sigma(\mathbf{r}') \mathrm{d}\mathbf{r}' \\ &+ \int_{\partial\Omega} \nu(\mathbf{r}') \cdot \nabla_{\mathbf{r}'} \Phi(\mathbf{r} - \mathbf{r}') \sigma(\mathbf{r}') \mathrm{d}l_{\mathbf{r}'} \end{aligned} \quad (3)$$

Harmonic B_z Algorithm[2]

- **Idea:** fixed-point iteration of (3) + (**CR**) for $\nabla\sigma$ in Ω
- **Assume:** $\sigma^+ \in \mathcal{P}$ with $\sigma^+ |_{\Omega\setminus\tilde{\Omega}} = \sigma_b$, $\sigma_b > 0$, $\tilde{\Omega} \subset \Omega$, σ_b known

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Algorithm 2 BZ(σ_b, μ_0, tol)

1: $n := 0$, $\sigma^0 := \sigma_b$

2: Calculate $\mathrm{BI}(\mathbf{r}) := \int_{\partial\Omega} \nu(\mathbf{r}') \cdot \nabla_{\mathbf{r}'} \Phi(\mathbf{r} - \mathbf{r}') \ln \sigma^+(\mathbf{r}') \mathrm{d}l_{\mathbf{r}'}$

3: **repeat**

4: $\quad \mathrm{GU}(\mathbf{r}) := \frac{1}{\mu_0} \sigma^n \mathbb{A}[\sigma^n]^{-1} \begin{pmatrix} \nabla^2 B^1_{z,+} \\ \nabla^2 B^2_{z,+} \end{pmatrix} (\mathbf{r})$

5: $\quad \ln \sigma^{n+1}(\mathbf{r}) := \mathrm{BI}(\mathbf{r}) - \int_\Omega \nabla_{\mathbf{r}'} \Phi(\mathbf{r} - \mathbf{r}') \cdot \mathrm{GU}(\mathbf{r}') \mathrm{d}\mathbf{r}'$

6: $\quad n := n + 1$

7: **until** $\| \ln \sigma^n - \ln \sigma^{n-1} \|_{\mathcal{P}} \leq tol$

8: **return** $\sigma_{BZ} := \sigma^n$

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[2] Seo, Woo 2003

Harmonic B_z Algorithm: convergence

Define $\Xi(\sigma_b, \epsilon_0) := \{\sigma \in \mathcal{P} \mid \sigma|_{\Omega\setminus\tilde{\Omega}} = \sigma_b,\ \|\nabla \ln \sigma\|_{C(\Omega)} < \epsilon_0\}$.

Theorem[3]

There exists $0 < \epsilon < \epsilon_0$, such that for each $\sigma^+ \in \Xi(\sigma_b, \epsilon_0)$ with $\|\nabla \ln \sigma^+\|_{C(\Omega)} \leq \epsilon$ the sequence $\{\sigma^n\}$ generated by the Harmonic B_z Algorithm with initial guess σ_b satisfies

$$\sigma^n \equiv \sigma_b \text{ in } \Omega \setminus \tilde{\Omega}, \quad \|\ln \sigma^n - \ln \sigma^+\|_{C^1(\tilde{\Omega})} \leq K \left(\frac{1}{2}\right)^n \epsilon,$$

with $K := diam(\Omega) + 1$.

[3] Seo, Woo, Liu, 2010

Harmonic B_z Algorithm and RBM

- Idea: Replace $u_j^{\sigma_n}$ (see $\mathbb{A}[\sigma^n]$) in Algo. 2 by approximations $u_{j,N}^{\sigma_n}$
- Assumptions (for all $n = 0, 1, 2, \dots$)
  - $u_{j,N}^{\sigma_n} \in C^1(\tilde{\Omega})$
  - $\|\nabla u_{j,N}^{\sigma_n} - \nabla u_j^{\sigma_n}\|_{C(\tilde{\Omega})} \leq \epsilon^{n+1} C, \quad \epsilon$ from Theorem and $C > 0$

Assumptions fulfilled $\Rightarrow$ Theorem is replicated

Applications

- Fineness of FEM-mesh (actual numerical convergence)
- For RB-version: how to control $\|\nabla u_{j,N}^{\sigma_n} - \nabla u_j^n\|_{C(\tilde{\Omega})}$?

  $\rightsquigarrow$ work in progress (proof $W^{1,\infty}$ version of Theorem)

Reduced Basis Harmonic B_z Algorithm

Algorithm 3 RBZ$(\sigma_b, \mu_0, tol_1, tol_2, \Psi_{n,1}, \Psi_{n,2})$

$n := 0$, $\sigma_0 := \sigma_b$

Calculate $\mathrm{BI}(\mathbf{r}) := \int_{\partial\Omega} \nu(\mathbf{r}') \cdot \nabla_{\mathbf{r}'} \Phi(\mathbf{r} - \mathbf{r}') \ln \sigma^+(\mathbf{r}') \mathrm{d}l_{\mathbf{r}'}$

repeat

  enrich RBs $\Psi_{n,1}, \Psi_{n,2}$ using $u_1^{\sigma_n}, u_2^{\sigma_n}$

  $i := 1, \sigma^i := \sigma^n$

  repeat

    $\mathrm{GU}(\mathbf{r}) := \frac{1}{\mu_0} \sigma^i \mathbb{A}_N[\sigma^i]^{-1} \begin{pmatrix} \nabla^2 B^1_{z,+} \\ \nabla^2 B^2_{z,+} \end{pmatrix} (\mathbf{r})$

    $\ln \sigma^{i+1}(\mathbf{r}) := \mathrm{BI}(\mathbf{r}) - \int_\Omega \nabla_{\mathbf{r}'} \Phi(\mathbf{r} - \mathbf{r}') \cdot \mathrm{GU}(\mathbf{r}') \mathrm{d}\mathbf{r}'$

    $i = i + 1$

  until $\|\sigma^i - \sigma^{i-1}\|_{\mathcal{P}} \leq tol_1$ **or** $\Delta_{N,1}(\sigma_i) > tol_2$ **or** $\Delta_{N,2}(\sigma_i) > tol_2$

  $\sigma^{n+1} := \sigma^i$, $n := n + 1$

until $\|\sigma^i - \sigma^{i-1}\|_{\mathcal{P}} \leq tol_1$

return $\sigma_{RBZ} := \sigma^n$

Numerics - Setting

- $\Omega := [-1,1] \times [-2,2]$
- $E_1^{\pm} := \{(\pm 1, y) \mid |y| < 0.1\}$, $E_2^{\pm} := \{(x, \pm 2) \mid |x| < 0.1\}$
- For $r = \sqrt{x^2 + y^2}$, $x, y \in \Omega$

$$\sigma^{+} \approx \sigma(r) := \begin{cases} 10\left(\cos(r) - \frac{\sqrt{3}}{2}\right) + 2, & 0 \leq r \leq \pi/6 \\ 2, & \text{otherwise} \end{cases} \in \mathcal{P}$$

  using 40×80 rectangles (piecewise constant approximation)
- $\sigma_b = 2$, $\mu_0 = 1$, $tol_1 = 10^{-5}$, $tol_2 = \frac{1}{100}$, no noise

Numerics - Comparison

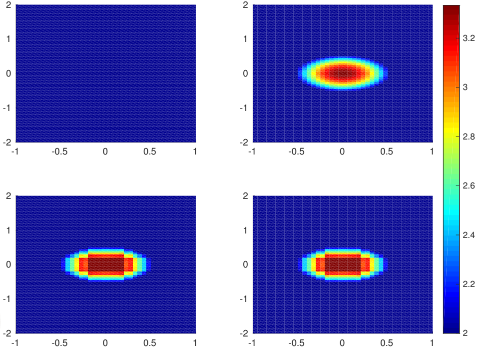

Figure: σ_b (top left), σ^+ (top right), σ_{BZ} (bottom left), σ_{RBZ} (bottom right).

- BZ: 3.45s and 16 PDE solves | RBZ: 2.21s and 6 PDE solves
- $\|\sigma^+ - \sigma_{BZ}\|_{\mathcal{P}} \approx 0.05$, $\|\sigma_{BZ} - \sigma_{RBZ}\|_{\mathcal{P}} \approx 6.760 \cdot 10^{-6}$

Conclusion

- Summary:
  - Standard RB approach can not be applied to speed-up inverse problems in imaging context
  - Adaptive RB approach works very well, both in academic and complex real-world problems
- Interesting observation: type of approximation does not affect convergence Theorem for RB-MREIT
- Future work:
  - Improve theoretical and numerical results in MREIT
  - Publish MREIT paper

Thank you for your attention!

Appendix

The dual problem

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Recall $\sigma_{n+1}^\delta := \sigma_n^\delta + \omega F'(\sigma_n^\delta)^*(u^\delta - F(\sigma_n^\delta))$

For $\sigma, \kappa \in \mathcal{P}$ and $l \in X$, one can show

$$\langle \kappa, F'(\sigma)^* \, l \rangle_{\mathcal{P}} = \int_\Omega \kappa \nabla u^\sigma \cdot \nabla u_l^\sigma \, dx, \tag{4}$$

with $u_l^\sigma \in X$ the unique solution of the dual problem

$$b(u, v; \sigma) = m(v; l), \text{ for all } v \in X, \quad m(v; l) := -\int_\Omega l \, v \, dx.$$

In Algorithm 1: two RB spaces $X_{N,1}$, $X_{N,2}$

- enrich $X_{N,1}$ via $F(\sigma_n^\delta)$ and $X_{N,2}$ via $u_l^{\sigma_n^\delta}$ with $l := u^\delta - F(\sigma_n^\delta)$
- calculate $s_{n,i}$ using (4) and associated reduced solutions

Numerics - algorithmic behaviour

Update error - $\|s_{n,RBL} - s_{n,LW}\|_{\mathcal{P}}$

$$s_{n,RBL} := \sigma_{n+1,RBL} - \sigma_{n,RBL}, \quad s_{n,LW} := \sigma_{n+1,LW} - \sigma_{n,LW}$$

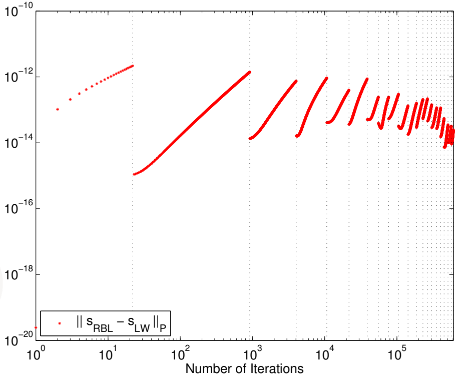

Figure: Update error $\|s_{n,RBL} - s_{n,LW}\|_{\mathcal{P}}$ over the course of the iteration.

Numerics - convergence

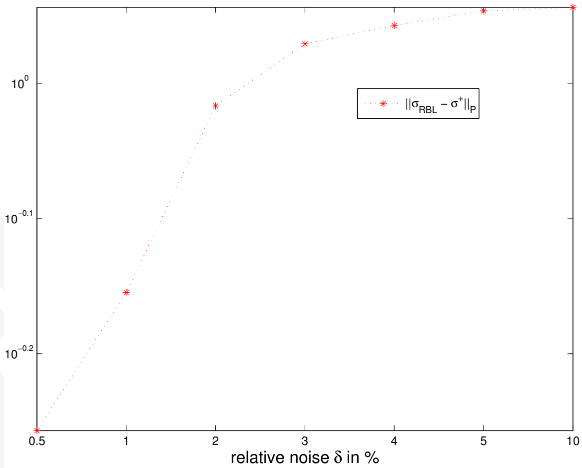

Figure: Error $\|\sigma_{RBL} - \sigma^+\|_{\mathcal{P}}$ over the decreasing relative noise level δ.