Reduced Basis Landweber method for nonlinear ill-posed inverse problems

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Joint work with Bastian Harrach and Bernard Haasdonk

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Introduction

The forward problem

Consider

$$\nabla \cdot (\sigma(x)\nabla u(x)) = 1,\ x \in \Omega := (0,1)^2,\ u(x) = 0,\ x \in \partial\Omega.$$

Assume: σ is piecewise constant $\rightsquigarrow \sigma(x) = \sum_{q=1}^{p} \boldsymbol{\sigma}_q \chi_{\Omega_q}(x)$.

Forward operator

$F : \mathcal{P} \subset \mathbb{R}^p \to X \subset H_0^1(\Omega),\ \sigma \mapsto u^\sigma$, the detailed solution, solving

$$b(u^\sigma, v; \sigma) = f(v),\ \text{for all } v \in X, \text{ with} \tag{1a}$$

$$b(u, w; \sigma) := \int_\Omega \sigma \nabla u \cdot \nabla w \, dx, \quad f(v) := -\int_\Omega v \, dx. \tag{1b}$$

Inverse problem and its difficulties

Inverse Problem

For given solution $u \in X$ of (1), find corresponding parameter $\sigma^+ \in \mathcal{P}$ with $F(\sigma^+) = u$ („a-example“).

- Naive inversion (solving $\sigma^+ = F^{-1}(u)$) fails due to ill-posedness of the problem (in general F^{-1} discontinuous!)

  $\rightsquigarrow$ Small errors get amplified!

- Typically only noisy data u^δ ($\|u - u^\delta\|_X < \delta$) given

  $$\rightsquigarrow F^{-1}(u^\delta) \nrightarrow F^{-1}(u) \text{ as } \delta \to 0!$$

Goal: $R_{n(u^\delta,\delta)}(u^\delta) \to F^{-1}(u)$ as $\delta \to 0$.

Landweber method

Landweber iteration

- $\sigma_{n+1}^\delta := \sigma_n^\delta + \omega F'(\sigma_n^\delta)^*(u^\delta - F(\sigma_n^\delta))$
- Terminate as $\|F(\sigma_n^\delta) - u^\delta\|_X \leq \tau\delta$ (discrepancy principle)

- Numerous evaluations of F for many different parameters.
- Detailed solution (e.g. FEM, FV, FD) is expensive.

$\rightsquigarrow$ **model order reduction**

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Assume: snapshot based reduced basis (RB) space X_N is given.

Reduced forward operator

$F_N : \mathcal{P} \to X_N$, $\sigma \mapsto u_N^\sigma$ with u_N^σ, the reduced solution, solving

$$b(u_N^\sigma, v; \sigma) = f(v), \quad \forall v \in X_N.$$

- Certification: $\|u^\sigma - u_N^\sigma\|_X \leq \Delta_N(\sigma) := \frac{\|v_r\|_X}{\alpha(\sigma)}$,
  with $\langle v_r, v \rangle_X := r(v; \sigma) := f(v) - b(u_N^\sigma, v; \sigma), \forall v \in X$
- Offline/online decomposition: rapid computation of u_N^σ
- Reproduction of solutions: $u^\sigma \in X_N \Rightarrow u_N^\sigma = u^\sigma$

RBM and Inverse problems

G./Haasdonk/Harrach, A Reduced Basis Landweber method for nonlinear inverse problems, 2016 Inverse Problems 32 (3) (doi).

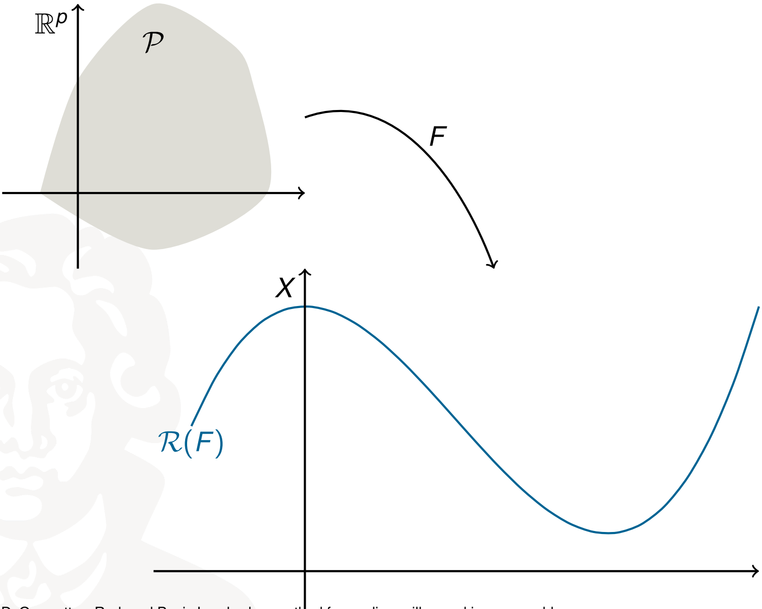
Naive approach

- Construct global X_N approximating whole $\mathcal{R}(F)$ (offline-phase)
- Rapidly compute $F_N(\sigma)$ and substitute $F(\sigma)$ for $F_N(\sigma)$ in the Landweber iteration („online-phase“)

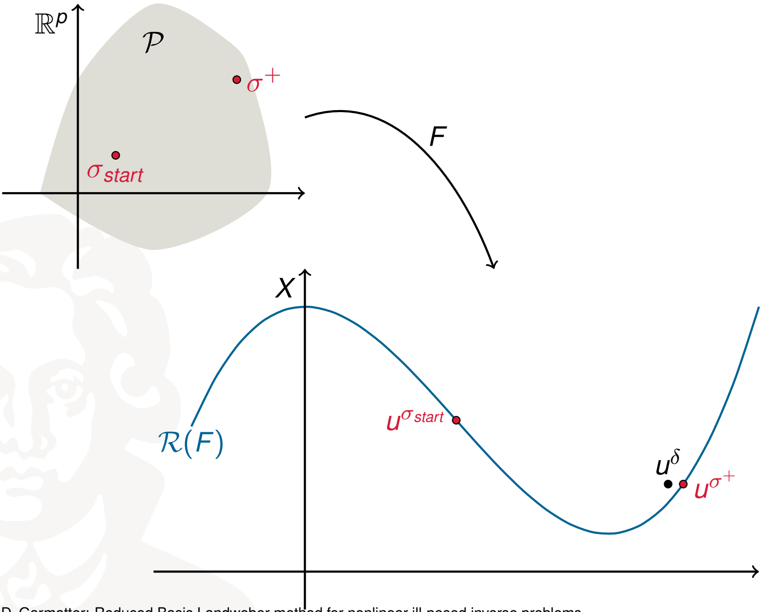
Limitation: Only feasible for low-dimensional parameter spaces, not feasible for imaging.

Our approach: Create problem-adapted RB-space by iterative enrichment (inspired by Druskin & Zaslavski 2007, Zahr & Fahrhat 2015 and Lass 2014).

Combining RBM & LW - Idea



Combining RBM & LW - Idea



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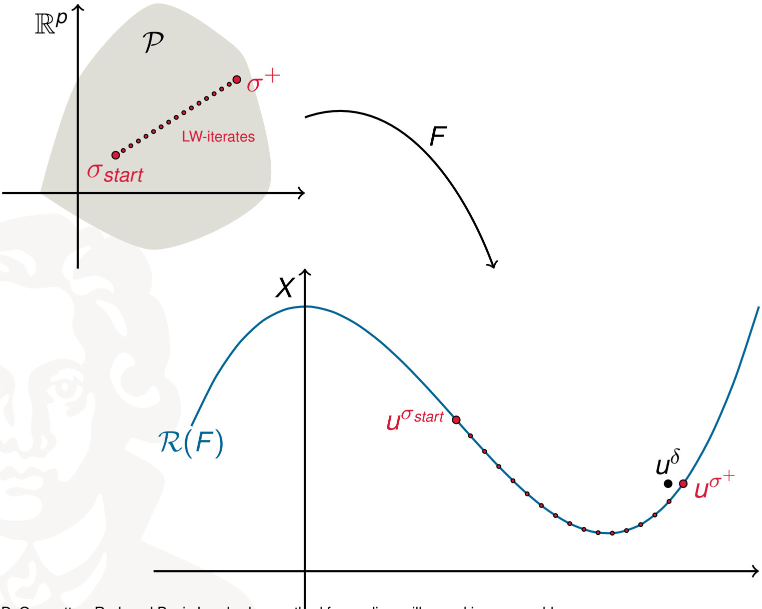

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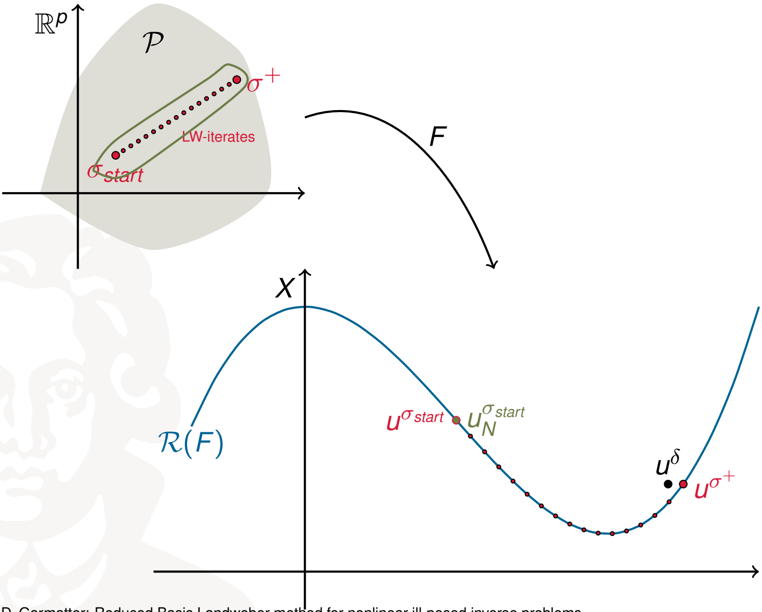

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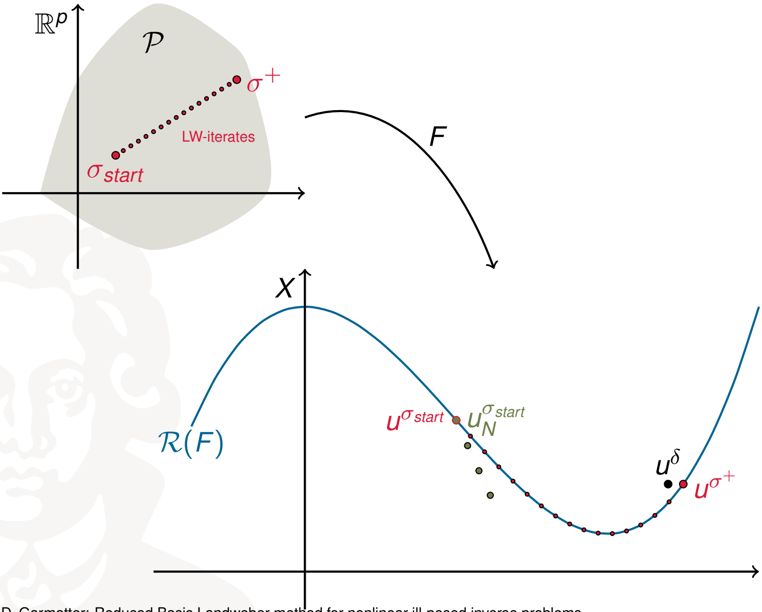

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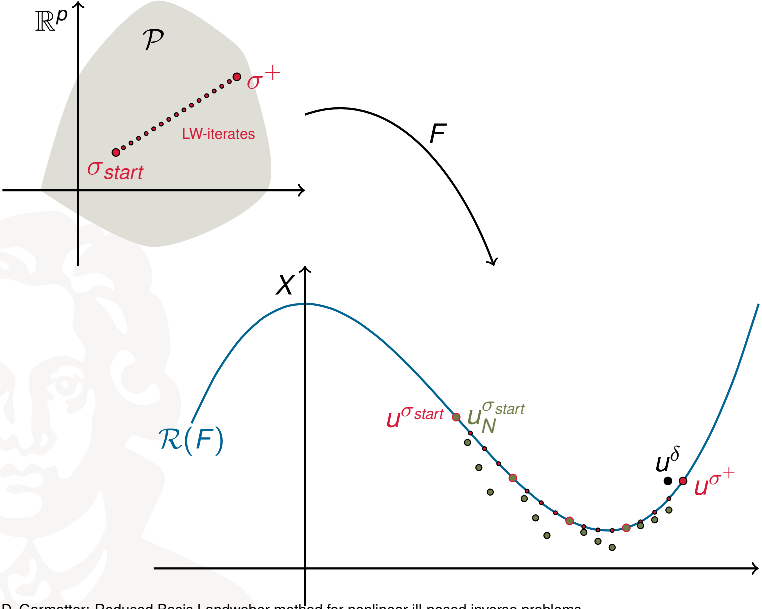

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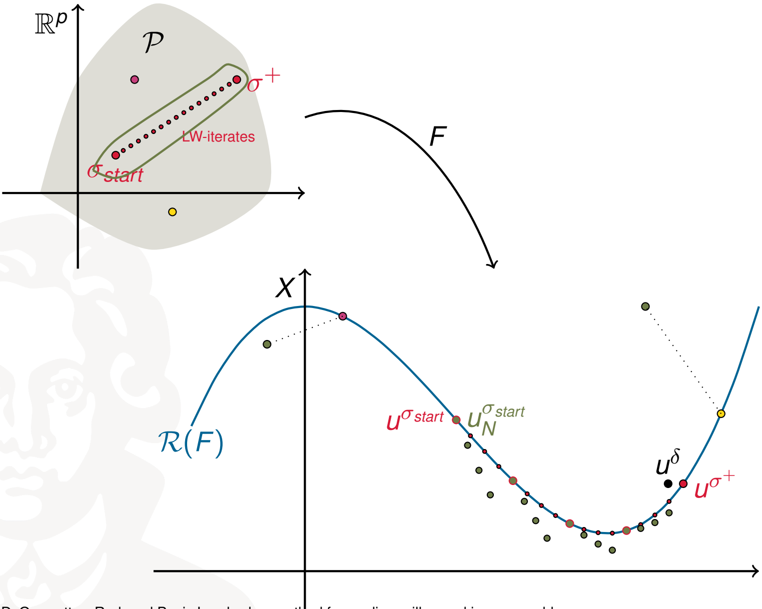

Reduced Basis Landweber (RBL) method

Algorithm 1 RBL$(\sigma_{start}, \tau, \Phi_N)$

$n := 0,\ \sigma_0^\delta := \sigma_{start}$
while $\|F(\sigma_n^\delta) - u^\delta\|_X > \tau\delta$ **do**
  enrich RB using σ_n^δ
  $i := 1,\ \sigma_i^\delta := \sigma_n^\delta$
  repeat
    calculate reduced Landweber update $s_{n,i}$
    $\sigma_{i+1}^\delta := \sigma_i^\delta + \omega s_{n,i}$
    $i := i + 1$
  until $\|F_N(\sigma_i^\delta) - u^\delta\|_X \leq \tau\delta$ **or** $\Delta_N(\sigma_i^\delta) > (\tau - 2)\delta$
  $\sigma_{n+1}^\delta := \sigma_i^\delta$
  $n := n + 1$
end while
return $\sigma_{RBL} := \sigma_n^\delta$

The dual problem

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Recall $\sigma_{n+1}^\delta := \sigma_n^\delta + \omega F'(\sigma_n^\delta)^*(u^\delta - F(\sigma_n^\delta))$

For $\sigma, \kappa \in \mathcal{P}$ and $l \in X$, one can show

$$\langle \kappa, F'(\sigma)^* l \rangle_{\mathcal{P}} = \int_\Omega \kappa \nabla u^\sigma \cdot \nabla u_l^\sigma \, dx, \tag{2}$$

with $u_l^\sigma \in X$ the unique solution of the dual problem

$$b(u, v; \sigma) = m(v; l), \text{ for all } v \in X, \quad m(v; l) := -\int_\Omega l \, v \, dx.$$

In Algorithm 1 $\rightsquigarrow$ two RB spaces $X_{N,1}$, $X_{N,2}$

- enrich $X_{N,1}$ via $F(\sigma_n^\delta)$ and $X_{N,2}$ via $u_l^{\sigma_n^\delta}$ with $l := u^\delta - F(\sigma_n^\delta)$
- calculate $s_{n,i}$ using (2) and associated reduced solutions

Numerics - compare reconstructions

Setting: $p = 900$, $\tau = 2.5$, $\delta = 1\%$ and $\omega = \frac{1}{2}(\|F'(\sigma_{start})\|)^{-1}$.

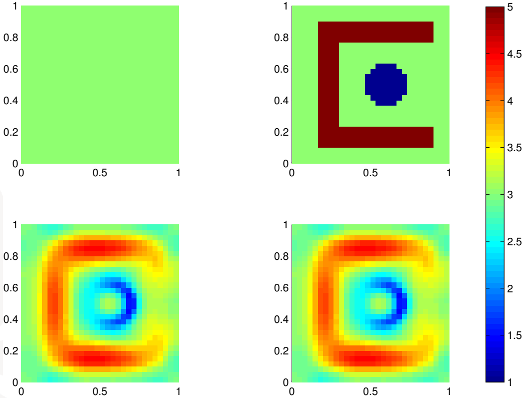


Figure: σ_{start} (top left), exact solution σ^+ (top right). Reconstruction via RBL method (bottom left) and Landweber method (bottom right).

Numerics - time comparison

- **Outer iteration:** space enrichment, projection („offline“)
- **Inner iteration:** one iteration of repeat loop („online“)

Algorithm	Landweber	RBL	
time (s)	187189	14661	
# Iterations	608067	outer	20
		inner	608083
time per Iteration (s)	0.308	outer	3.705
		inner	0.024
# forward solves	1216134	40	

$$\|\sigma_{RBL} - \sigma_{LW}\|_{\mathcal{P}} \approx 1.118 \cdot 10^{-5}$$

Numerics - regularization property

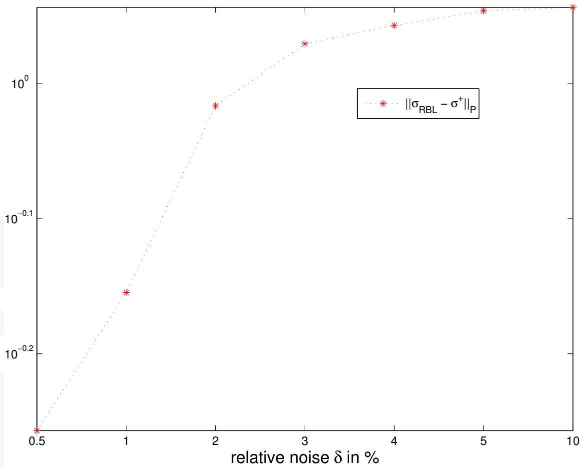
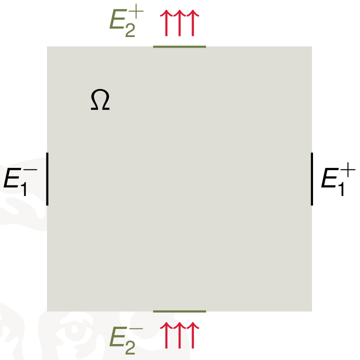


Figure: Error $\|\sigma_{RBL} - \sigma^{+}\|_{\mathcal{P}}$ over the decreasing relative noise level δ.

Current Work



Inverse Problem

- Object Ω with electrode pairs $E_j^\pm$ attached to it
- Apply current between electrode pair
- generates magnetic flux density B (measurable with MRI scanner)
- Use this data to reconstruct conductivity of object

$\rightsquigarrow$ ∇^2-Bz-Algorithm (Seo, Woo, et al. 2003)

MREIT - first numerics (10000 pixels, no noise)

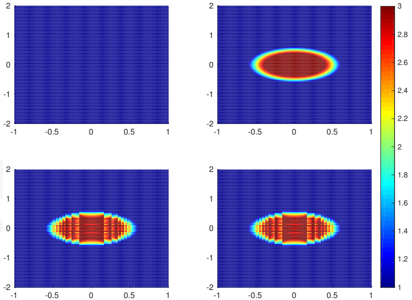


Figure: initial guess (top left), exact solution (top right), reconstruction via ∇^2-Bz-Algorithm (bottom left) and it's RB-variant (bottom right).

- ∇^2-Bz Algorithm
  14 forward solves
- RB-variant
  10 forward solves
- no time reduction
  (err. est., ...)

speed-up $\rightsquigarrow$ work in progress!

Conclusion

- Solving inverse coefficient problem requires many PDE solves
- Reduced basis (RB) approach can speed up PDE solution
- But standard RB approach is only applicable for low dimensional parameter spaces
- Using adaptive problem-specific RB enrichment, we can handle high-dimensional parameter spaces, e.g. for imaging problems

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$\rightsquigarrow$ RBL method outperforms standard Landweber (exp.: 13 times faster without loss of accuracy)

- Further improve RB-variant of ∇^2-Bz Algorithm

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Thank you for your attention!