Magnet Resonance Electrical Impedance Tomography (MREIT): convergence of the Harmonic B_z Algorithm

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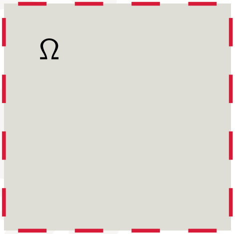
MREIT: the Harmonic B_z Algorithm[1]

[1] Based on Seo, Woo, et al. since 2003

Motivation: Electrical Impedance Tomography (EIT)

- **Setting:** Imaging object $O \subset \mathbb{R}^3$ with electrode pairs attached
- **Aim:** reconstruct cross-sectional ($\Omega = O \cap \{z = z_0\} \subset \mathbb{R}^2$) image of electrical conductivity inside Ω

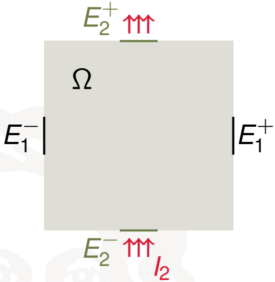


In EIT

- Data: Current-Voltage measurements on boundary
- Difficulty: highly ill-posed
  $\rightsquigarrow$ low spatial resolution

MREIT: Setting

Aim: achieve higher resolution of conductivity σ.



- Object Ω with two electrode pairs $E_j^{\pm}$ attached
- Place object inside MRI-Scanner
- Apply current I between electrode pair
- Generates magnetic flux density B (z-comp. measurable with MRI scanner)

$\rightsquigarrow$ Full internal data set to overcome ill-posedness

Forward problem

Consider ($j = 1, 2$ determines active electrode pair)

$$\nabla \cdot (\sigma \nabla u_j) = 0 \text{ in } \Omega, \quad I_j = \int_{E_j^+} \sigma \frac{\partial u_j}{\partial \mathbf{n}} ds = -\int_{E_j^-} \sigma \frac{\partial u_j}{\partial \mathbf{n}} ds \tag{1a}$$

$$\nabla u_j \times \mathbf{n} = 0, \text{ on } E_j^+ \cup E_j^-, \quad \sigma \frac{\partial u_j}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega \setminus \overline{\left(E_j^+ \cup E_j^-\right)} \tag{1b}$$

- $\mathcal{P} \equiv C^1_{\pm}(\bar{\Omega}) := \{\sigma \in C^1(\bar{\Omega}) \mid 0 < \underline{\sigma} \leq \sigma \leq \overline{\sigma} < \infty\}$
- Unique solution of (1) up to an additive constant

Inverse Problem

Aim: Determine σ from B.

- Maxwell Equations

$$-\sigma \nabla u_j = \frac{1}{\mu_0} \nabla \times B^j \text{ (Ampère's law)} \quad \text{and} \quad \nabla \cdot B^j = 0$$

- $\nabla \times$ of Ampère's law $\rightsquigarrow \nabla u_j \times \nabla \sigma = \frac{1}{\mu_0} \nabla^2 B^j$
- Only B_z^j measurable and two different electrode pairs/currents

Core-relation (point-wise in Ω)

$$\begin{pmatrix} \frac{\partial \sigma}{\partial x} \\ \frac{\partial \sigma}{\partial y} \end{pmatrix} = \frac{1}{\mu_0} \mathbb{A}[\sigma]^{-1} \begin{pmatrix} \nabla^2 B_z^1 \\ \nabla^2 B_z^2 \end{pmatrix}, \; \mathbb{A}[\sigma] := \begin{pmatrix} \frac{\partial u_1}{\partial y} & -\frac{\partial u_1}{\partial x} \\ \frac{\partial u_2}{\partial y} & -\frac{\partial u_2}{\partial x} \end{pmatrix} \quad \textbf{(CR)}$$

Recovery of σ from $\nabla\sigma$

So far: $\nabla\sigma$ obtainable in Ω from $\nabla^2 B_z^j$ via (**CR**). How to get σ?

- Fundamental solution: for $\mathbf{r} = (x, y), \mathbf{r}' = (x', y') \in \Omega \subset \mathbb{R}^2$

$$\Phi(\mathbf{r} - \mathbf{r}') := \frac{1}{2\pi} \ln |\mathbf{r} - \mathbf{r}'| \quad \text{fulfilling} \quad \nabla^2 \Phi(\mathbf{r} - \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}')$$

- For $\sigma \in C^1(\overline{\Omega})$ one can show

$$\sigma(\mathbf{r}) = -\int_\Omega \nabla_{\mathbf{r}'} \Phi(\mathbf{r} - \mathbf{r}') \cdot \nabla\sigma(\mathbf{r}') \mathrm{d}\mathbf{r}' + \int_{\partial\Omega} \nu(\mathbf{r}') \cdot \nabla_{\mathbf{r}'} \Phi(\mathbf{r} - \mathbf{r}') \sigma(\mathbf{r}') \mathrm{d} l_{\mathbf{r}'} \tag{2}$$

Harmonic B_z Algorithm[2]

- Assume: $\sigma^\star \in \mathcal{P}$ with $\sigma^\star |_{\Omega \setminus \tilde{\Omega}} = \sigma_b$, $\tilde{\Omega} \subset \Omega$, $\sigma_b > 0$ known
- Idea: iteration for (2) + (**CR**) + ln-formulation for positivity

Algorithm 1 BZ(σ_b, μ_0, tol)

1: $n := 0$, $\sigma^0 := \sigma_b$
2: Calculate $\mathrm{BI}(\mathbf{r}) := \int_{\partial\Omega} \nu(\mathbf{r}') \cdot \nabla_{\mathbf{r}'} \Phi(\mathbf{r} - \mathbf{r}') \ln \sigma^\star(\mathbf{r}') \mathrm{d}l_{\mathbf{r}'}$
3: **repeat**
4: $F^{n+1}(\mathbf{r}) := \frac{1}{\mu_0} \frac{1}{\sigma^n} \mathbb{A}[\sigma^n]^{-1} \begin{pmatrix} \nabla^2 B^1_{z,\star} \\ \nabla^2 B^2_{z,\star} \end{pmatrix} (\mathbf{r})$
5: $\ln \sigma^{n+1}(\mathbf{r}) := \mathrm{BI}(\mathbf{r}) - \int_\Omega \nabla_{\mathbf{r}'} \Phi(\mathbf{r} - \mathbf{r}') \cdot F^{n+1}(\mathbf{r}') \mathrm{d}\mathbf{r}'$
6: $n := n + 1$
7: **until** $\| \ln \sigma^n - \ln \sigma^{n-1} \|_{\mathcal{P}} \leq tol$
8: **return** $\sigma_{BZ} := \sigma^n$

[2] Seo, Woo 2003

Harmonic B_z Algorithm: convergence

- Define $\Xi(\sigma_b, \epsilon_0) := \{\sigma \in \mathcal{P} \mid \sigma|_{\Omega\setminus\tilde{\Omega}} = \sigma_b, \ \|\nabla \ln \sigma\|_{C(\Omega)} < \epsilon_0\}$.

Existing result[3]

There exists $0 < \epsilon < \epsilon_0$, such that for each $\sigma^\star \in \Xi(\sigma_b, \epsilon_0)$ with $\|\nabla \ln \sigma^\star\|_{C(\Omega)} \leq \epsilon$ the sequence $\{\sigma^n\}$ generated by the Harmonic B_z Algorithm with initial guess σ_b satisfies

$$\sigma^n \equiv \sigma_b \text{ in } \Omega \setminus \tilde{\Omega}, \quad \|\ln \sigma^n - \ln \sigma^\star\|_{C^1(\tilde{\Omega})} \leq K \left(\frac{1}{2}\right)^n \epsilon,$$

with $K := diam(\Omega) + 1$.

[3] Seo, Woo, Liu, 2010

Harmonic B_z Algorithm: our approach

- Idea: Replace $u_j^{\sigma_n}$ (see $\mathbb{A}[\sigma^n]$) in Algo. 1 by approximations $u_{N,j}^{\sigma_n}$
- Assumptions (for all $n = 0, 1, 2, \dots$)
  - $u_{N,j}^{\sigma_n} \in C^1(\tilde{\Omega})$
  - $\|\nabla u_{N,j}^{\sigma_n} - \nabla u_j^{\sigma_n}\|_{C(\tilde{\Omega})} \leq \epsilon^{n+1} C, \quad \epsilon$ from Theorem and $C > 0$

Assumptions fulfilled $\Rightarrow$ Existing result is replicated

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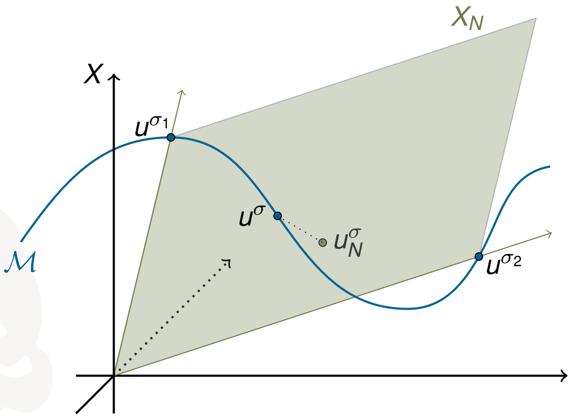
Applications

- Fineness of FEM-mesh (actual numerical convergence)
- Our scope: approximations $u_{N,j}^{\sigma_n}$ via Reduced Basis Methods
  $\rightsquigarrow$ also speed-up the method

Reduced Basis Methods & MREIT[4]

[4]Based on **G.**/Haasdonk/Harrach, A Reduced Basis Landweber method for nonlinear inverse problems, 2016 Inverse Problems 32 (3) (doi).

Reduced Basis Methods (RBM): Idea



- Solution manifold $\mathcal{M} := \{u^\sigma \mid \sigma \in \mathcal{P}\}$
- Construction of X_N via *carefully* chosen *snapshots* u^{σ_i}

RBM: problem formulation & properties (for j=1,2)

Assume: $X_{N,j} := \mathrm{span}\{u_j^{\sigma_1}, \ldots, u_j^{\sigma_N}\}$ is given.

Reduced forward problem (Galerkin projection)

For $\sigma \in \mathcal{P}$, find $u_{N,j}^{\sigma} \in X_{N,j} \subset X$, the reduced solution, of

$$b(u_{N,j}^{\sigma}, v; \sigma) = f(v), \quad \forall v \in X_{N,j}.$$

- Reproduction of solutions: $u_j^{\sigma} \in X_{N,j} \Rightarrow u_{N,j}^{\sigma} = u_j^{\sigma}$
- Offline/Online-decomposition: rapid computation of $u_{N,j}^{\sigma}$
- Certification: $\|u_j^{\sigma} - u_{N,j}^{\sigma}\|_X \leq \Delta_{N,j}(\sigma) := \frac{\|v_{r,j}\|_X}{\alpha(\sigma)}$

Reduced Basis Harmonic B_z Algorithm

Algorithm 2 RBZ$(\sigma_b, \mu_0, tol_1, tol_2, X_{N,1}, X_{N,2})$

$n := 0$, $\sigma^0 := \sigma_b$
Calculate $\mathrm{BI}(\mathbf{r}) := \int_{\partial\Omega} \nu(\mathbf{r}') \cdot \nabla_{\mathbf{r}'} \Phi(\mathbf{r} - \mathbf{r}') \ln \sigma^\star(\mathbf{r}') \mathrm{d}l_{\mathbf{r}'}$
repeat
    enrich $X_{N,1}$, $X_{N,2}$ using $u_1^{\sigma^n}$, $u_2^{\sigma^n}$
    $i := 1, \sigma^i := \sigma^n$
    repeat
        $F_N^{n+1}(\mathbf{r}) := \frac{1}{\mu_0} \frac{1}{\sigma^i} \mathbb{A}_N[\sigma^i]^{-1} \begin{pmatrix} \nabla^2 B_{z,\star}^1 \\ \nabla^2 B_{z,\star}^2 \end{pmatrix} (\mathbf{r})$
        $\ln \sigma^{i+1}(\mathbf{r}) := \mathrm{BI}(\mathbf{r}) - \int_\Omega \nabla_{\mathbf{r}'} \Phi(\mathbf{r} - \mathbf{r}') \cdot F_N^{n+1}(\mathbf{r}') \mathrm{d}\mathbf{r}'$
        $i := i + 1$
    until $\|\sigma^i - \sigma^{i-1}\|_{\mathcal{P}} \leq tol_1$ **or** $\Delta_{N,1}(\sigma_i) > tol_2$ **or** $\Delta_{N,2}(\sigma_i) > tol_2$
    $\sigma^{n+1} := \sigma^i$, $n := n + 1$
until $\|\sigma^i - \sigma^{i-1}\|_{\mathcal{P}} \leq tol_1$
return $\sigma_{RBZ} := \sigma^n$

Numerics - Setting

- $\Omega := [-1, 1] \times [-2, 2]$
- $E_1^{\pm} := \{(\pm 1, y) \mid |y| < 0.1\}$, $E_2^{\pm} := \{(x, \pm 2) \mid |x| < 0.1\}$
- For $r = \sqrt{x^2 + y^2}$, $x, y \in \Omega$

$$\sigma^\star \approx \sigma(r) := \begin{cases} 10\left(\cos(r) - \frac{\sqrt{3}}{2}\right) + 2, \; 0 \leq r \leq \pi/6 \\ 2, \; \text{otherwise} \end{cases}$$

  using 40×80 rectangles (piecewise constant approximation)
- $\sigma_b = 2$, $\mu_0 = 1$, $tol_1 = 10^{-5}$, $tol_2 = \frac{1}{100}$, no noise

Numerics - Comparison

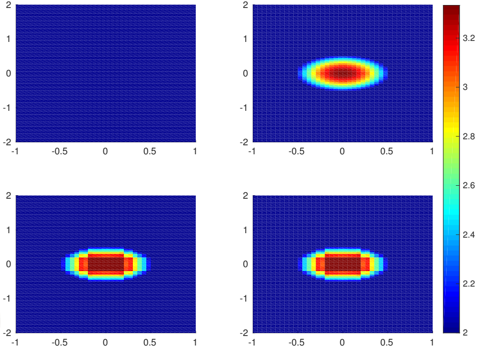


Figure: σ_b (top left), $\sigma^\star$ (top right), σ_{BZ} (bottom left), σ_{RBZ} (bottom right).

- BZ: 3.45s and 16 PDE solves | RBZ: 2.21s and 6 PDE solves
- $\|\sigma^\star - \sigma_{BZ}\|_{\mathcal{P}} \approx 0.05$, $\|\sigma_{BZ} - \sigma_{RBZ}\|_{\mathcal{P}} \approx 6.760 \cdot 10^{-6}$

Conclusion

- Summary:
  - Presented Harmonic B_z Algorithm to solve MREIT imaging problem
  - Extended existing convergence result to numerical applicability
  - Applied adaptive RB approach to also speed-up the algorithm
- Ongoing work:
  - Improve theoretical and numerical results
  - Publish paper

Thank you for your attention!