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Reduced Basis Methods for Inverse Problems

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Joint work with Bastian Harrach

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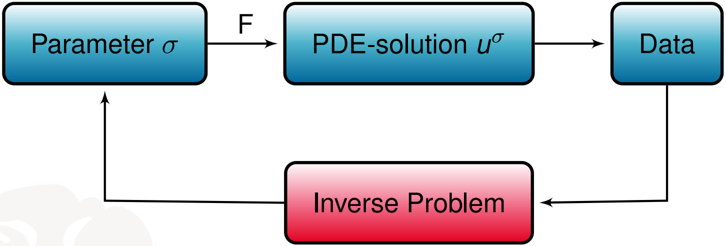
Motivation

Abstract problem formulation

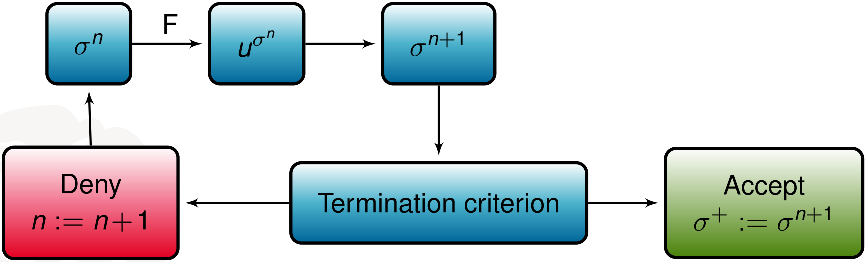


- Parameter-to-solution map: $F : \sigma \in \mathcal{P} \mapsto u^\sigma \in X$ (elliptic PDE)
- Imaging context: $\mathcal{P}$ very high-dimensional
- The (measured) data depends on u^σ

Aim: speed-up solution procedure of inverse problem.

Iterative solution of Inverse Problem

- Given initial guess σ^0

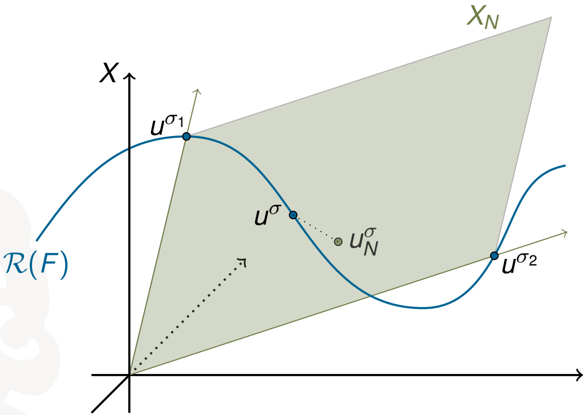


- Usually parameter-to-solution map F is expensive

$\rightsquigarrow$ Reduced Basis Methods (RBM)

RBM and Inverse Problems

Reduced Basis Methods (RBM): Idea



⇝ Construction of X_N via *carefully* chosen *snapshots* u^{σ_i}

RBM: The detailed & reduced problem

Detailed problem (e.g. fine grid FEM)

$F : \mathcal{P} \to X$, $\sigma \mapsto u^\sigma$, the detailed solution of

$$b(u^\sigma, v; \sigma) = f(v), \text{ for all } v \in X.$$

Assume: $X_N := \mathrm{span}\{u^{\sigma_1}, \ldots, u^{\sigma_N}\} \subset X$ is given.

Reduced problem (Galerkin projection)

For $\sigma \in \mathcal{P}$, find $u_N^\sigma \in X_N \subset X$, the reduced solution of

$$b(u_N^\sigma, v; \sigma) = f(v), \quad \forall v \in X_N.$$

- Reproduction of solutions: $u^\sigma \in X_N \;\Rightarrow\; u_N^\sigma = u^\sigma$
- Offline/Online-decomposition: rapid computation of u_N^σ

Certification - rigorous a-posteriori error estimator

$$\|u^\sigma - u_N^\sigma\|_X \leq \Delta_N(\sigma) := \frac{\|v_r\|_X}{\alpha(\sigma)}, \text{ with}$$

$$\langle v_r, v\rangle_X := r(v;\sigma) := f(v) - b(u_N^\sigma, v;\sigma), \forall v \in X$$

Naive approach

- Construct global X_N approximating whole $\mathcal{R}(F)$ (offline-phase)
- Use u_N^σ instead of u^σ in the solution procedure („online-phase")

Limitation: Only feasible for low-dimensional parameter spaces, not feasible for imaging.

Our approach[1]**:** Create problem-adapted RB-space by iterative enrichment (inspired by Druskin & Zaslavski 2007, Zahr & Fahrhat 2015 and Lass 2014).

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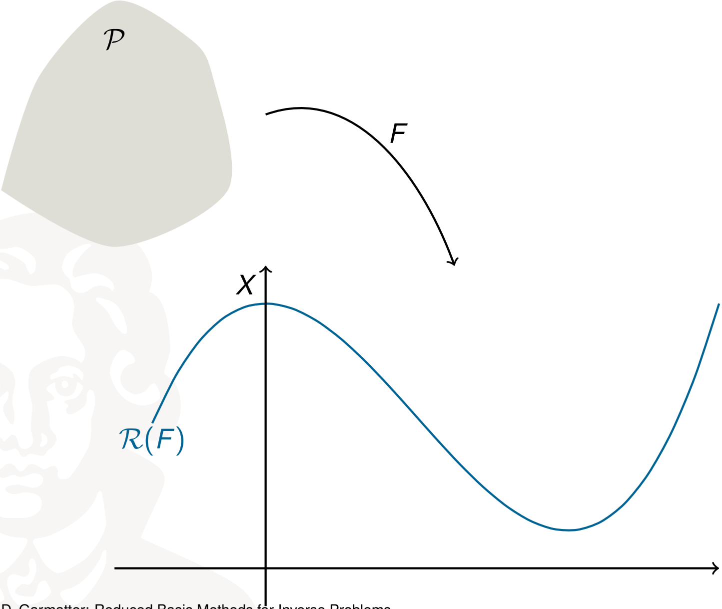
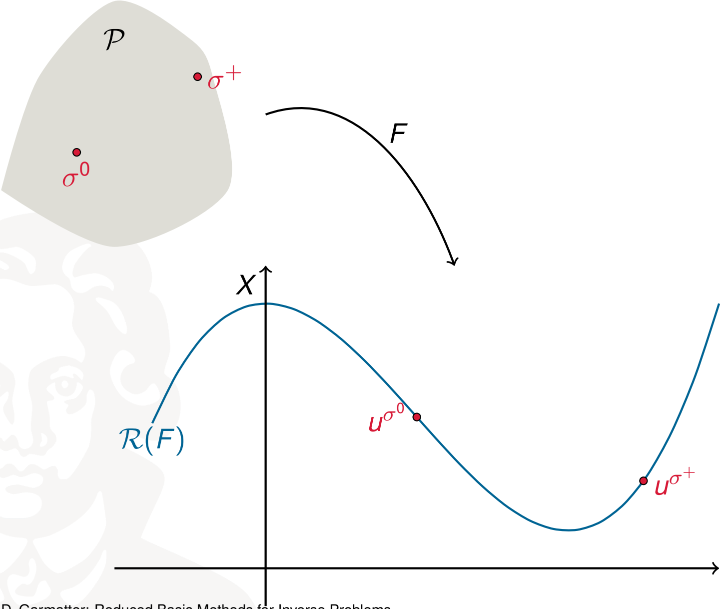
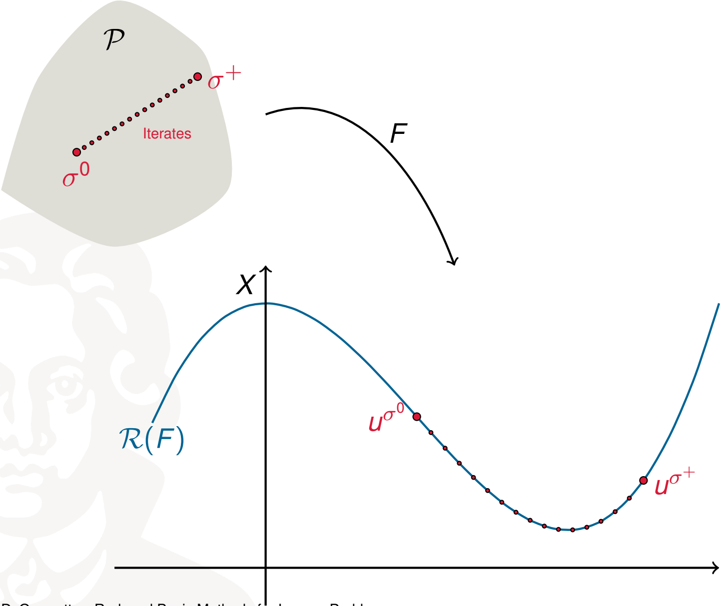
[1] **G.**/Haasdonk/Harrach, A Reduced Basis Landweber method for nonlinear inverse problems, 2016 Inverse Problems 32 (3) (doi).


$\mathcal{P}$
F
X
$\mathcal{R}(F)$

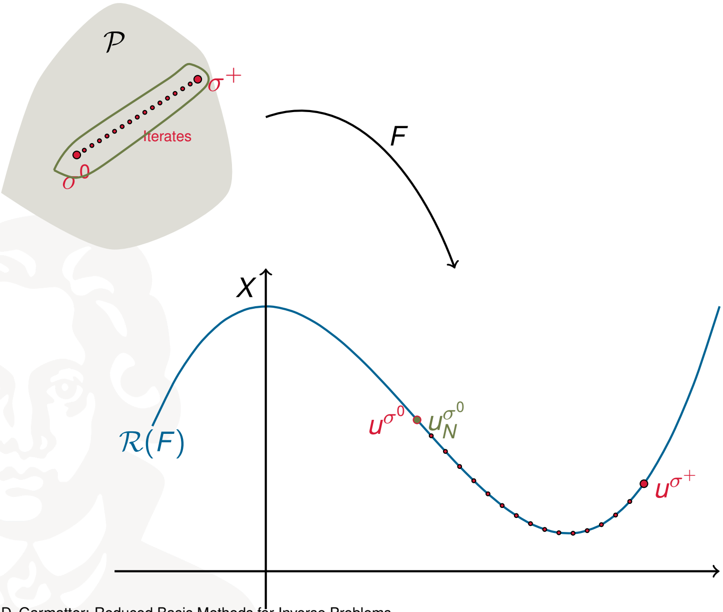
Adaptive RBM & IP: Idea



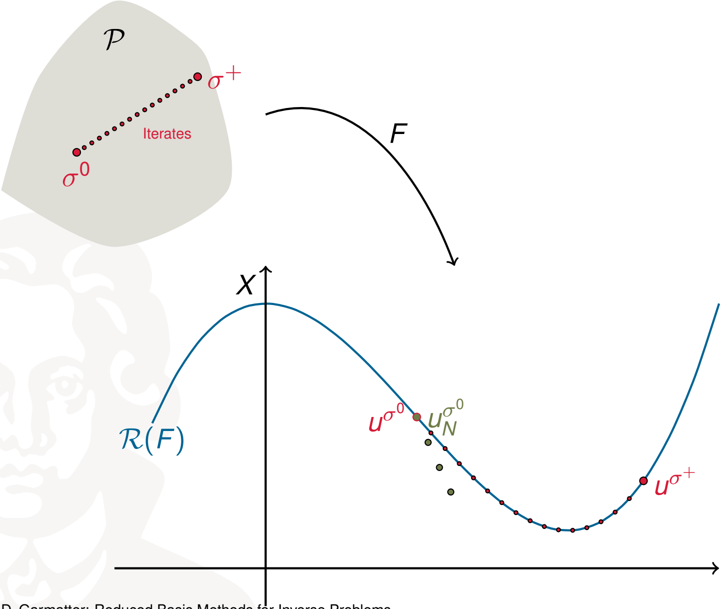
Adaptive RBM & IP: Idea



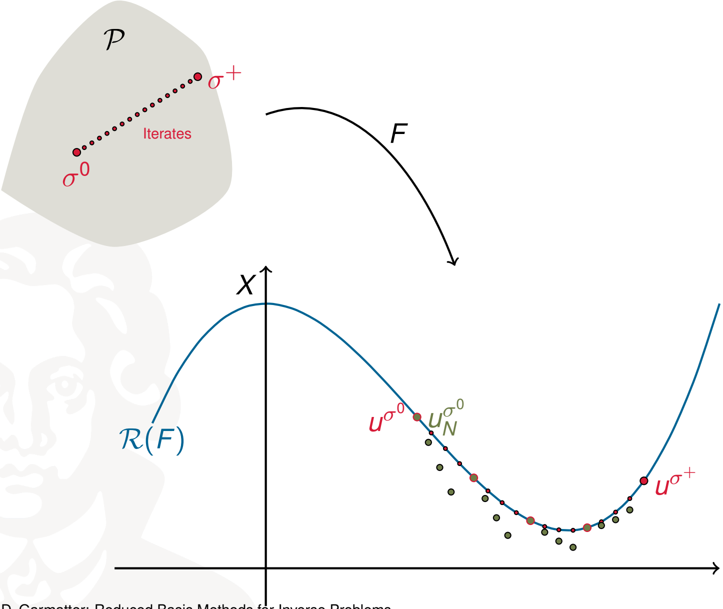
Adaptive RBM & IP: Idea



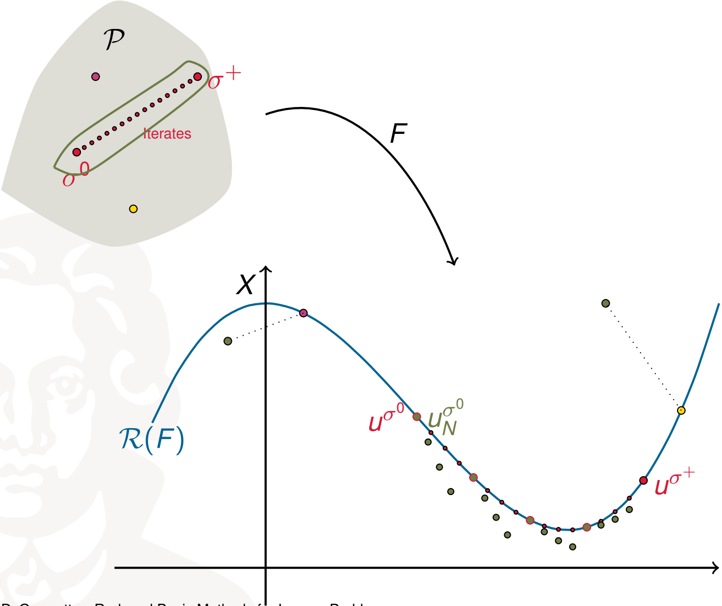
Adaptive RBM & IP: Idea



Adaptive RBM & IP: Idea



Adaptive RBM & IP: Idea

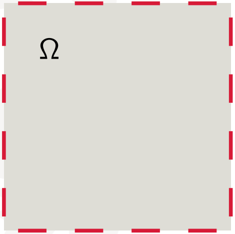


Application: **M**agnetic **R**esonance **E**lectrical **I**mpedance **T**omography[2]

[2]Based on Seo, Woo, et al. since 2003

Motivation: Electrical Impedance Tomography (EIT)

- **Setting:** Imaging object $O \subset \mathbb{R}^3$ with electrode pairs attached
- **Aim:** reconstruct cross-sectional $(\Omega = O \cap \{z = z_0\} \subset \mathbb{R}^2)$ image of electrical conductivity inside Ω

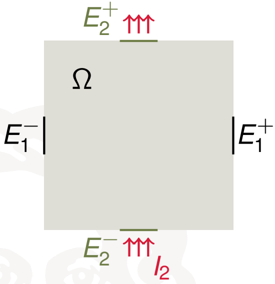


In EIT

- Data: Current-Voltage measurements on boundary
- Difficulty: highly ill-posed
  $\rightsquigarrow$ low spatial resolution

MREIT: Setting

Aim: achieve higher resolution of conductivity σ.



- Object Ω with two electrode pairs $E_j^{\pm}$ attached
- Place object inside MRI-Scanner
- Apply current I between electrode pair
- Generates magnetic flux density B (z-comp. measurable with MRI scanner)

$\rightsquigarrow$ Full internal data set to overcome ill-posedness

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Forward modelling

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- $\mathcal{P} \equiv \{\sigma \in C^{1,\alpha}(\bar{\Omega}) \mid \frac{1}{\lambda} \leq \sigma \leq \lambda,\ \lambda > 0\}$ $\qquad \alpha \in (0,1)$

Forward problem ($j \in \{1,2\}$ determines active electrode pair)

For $\sigma \in \mathcal{P}$, find u_j^σ solving

$$\nabla \cdot (\sigma \nabla u_j^\sigma) = 0 \text{ in } \Omega, \quad I_j = \int_{E_j^+} \sigma \frac{\partial u_j^\sigma}{\partial \mathbf{n}} ds = -\int_{E_j^-} \sigma \frac{\partial u_j^\sigma}{\partial \mathbf{n}} ds \tag{1a}$$

$$\nabla u_j^\sigma \times \mathbf{n} = 0, \text{ on } E_j^+ \cup E_j^-, \quad \sigma \frac{\partial u_j^\sigma}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega \backslash \overline{\left(E_j^+ \cup E_j^-\right)} \tag{1b}$$

- Unique solution of (1) up to an additive constant

Inverse Problem

Aim: Determine σ from B.

- Maxwell Equations

$$-\sigma \nabla u_j^\sigma = \frac{1}{\mu_0} \nabla \times B^j \text{ (Ampère's law)} \quad \text{and} \quad \nabla \cdot B^j = 0$$

- $\nabla \times$ of Ampère's law $\rightsquigarrow \nabla u_j^\sigma \times \nabla \sigma = \frac{1}{\mu_0} \nabla^2 B^j$
- Only B_z^j measurable and two different electrode pairs/currents

Core-relation (logarithmic version, point-wise in Ω)

$$\nabla \ln \sigma = \frac{1}{\mu_0} (\sigma \mathbb{A}[\sigma])^{-1} \begin{pmatrix} \nabla^2 B_z^1 \\ \nabla^2 B_z^2 \end{pmatrix}, \mathbb{A}[\sigma] := \begin{pmatrix} \frac{\partial u_1^\sigma}{\partial y} & -\frac{\partial u_1^\sigma}{\partial x} \\ \frac{\partial u_2^\sigma}{\partial y} & -\frac{\partial u_2^\sigma}{\partial x} \end{pmatrix} \quad \text{(CR)}$$

Recovery of σ

Assume: $\sigma^+ \in \mathcal{P}$ with $\sigma^+|_{\Omega\setminus\tilde{\Omega}} = \sigma_b$, $\sigma_b > 0$, $\tilde{\Omega} \subset\subset \Omega$, σ_b known

Iteration sequence ($\rightsquigarrow$ Harmonic B_z Algorithm[3])

For $\sigma^0 = \sigma_b$ and $\nabla^2 B^j_{z,+}, j \in \{1,2\}$, via (**CR**) using σ^+

- calculate $\mathcal{F}^{n+1}(\mathbf{r}) = \frac{1}{\mu_0}(\sigma^n \mathbb{A}[\sigma^n])^{-1} \begin{pmatrix} \nabla^2 B^1_{z,+} \\ \nabla^2 B^2_{z,+} \end{pmatrix} (\mathbf{r}), \forall \mathbf{r} \in \Omega$
- define $\ln \sigma^{n+1}$ as the solution of

$$\nabla^2 \ln \sigma^{n+1} = \nabla \cdot \mathcal{F}^{n+1} \quad \text{in } \Omega, \quad \ln \sigma^{n+1} = \ln \sigma^+ \quad \text{on } \partial\Omega$$

- $\sigma^{n+1} = \exp(\ln \sigma^{n+1}) \rightsquigarrow$ ensures positivity

[3] Seo, Woo, et al., 2003

Reduced Basis Approach

Given $X_{N,1}$, $X_{N,2}$ $\rightsquigarrow$ for $\sigma \in \mathcal{P}$ RB-approximations $u_{N,j}^{\sigma}$ available.

Reduced Iteration: For $\sigma^0 = \sigma_b$, $\nabla^2 B_{z,+}^j$ via (**CR**) using σ^+

- calculate $\mathcal{F}_N^{n+1}(\mathbf{r}) = \frac{1}{\mu_0}(\sigma^n \mathbb{A}_N[\sigma^n])^{-1} \begin{pmatrix} \nabla^2 B_{z,+}^1 \\ \nabla^2 B_{z,+}^2 \end{pmatrix} (\mathbf{r})$, $\forall \mathbf{r} \in \Omega$,

  with $\mathbb{A}_N[\sigma^n] := \begin{pmatrix} \frac{\partial u_{N,1}^{\sigma^n}}{\partial y} & -\frac{\partial u_{N,1}^{\sigma^n}}{\partial x} \\ \frac{\partial u_{N,2}^{\sigma^n}}{\partial y} & -\frac{\partial u_{N,2}^{\sigma^n}}{\partial x} \end{pmatrix}$

- define $\ln \sigma^{n+1}$ as the solution of

$$\nabla^2 \ln \sigma^{n+1} = \nabla \cdot \mathcal{F}_N^{n+1} \quad \text{in } \Omega, \quad \ln \sigma^{n+1} = \ln \sigma^+ \quad \text{on } \partial\Omega$$

- $\sigma^{n+1} = \exp(\ln \sigma^{n+1})$

Reduced iteration: convergence

- $\Xi(\sigma_b, \epsilon_0) := \{\sigma \in \mathcal{P} \mid \sigma|_{\Omega \setminus \tilde{\Omega}} = \sigma_b,\ \|\nabla \ln \sigma\|_{C^{0,\alpha}(\Omega)} < \epsilon_0\}$

Preliminary result[4]

There exists $0 < \epsilon < \epsilon_0$, such that for each $\sigma^+ \in \Xi(\sigma_b, \epsilon_0)$ with $\|\nabla \ln \sigma^+\|_{C^{0,\alpha}(\Omega)} \leq \epsilon$, and as long as

- $u_{N,j}^{\sigma^n} \in C^{1,\alpha}(\tilde{\tilde{\Omega}})$, $\tilde{\Omega} \subset\subset \tilde{\tilde{\Omega}} \subset\subset \Omega$
- $\|\nabla u_{N,j}^{\sigma^n} - \nabla u_j^{\sigma^n}\|_{C^{0,\alpha}(\tilde{\Omega})} \leq \epsilon^{n+1} C$

hold throughout the Reduced Iteration, the sequence $\{\sigma^n\}$ generated by Reduced Iteration with initial guess σ_b satisfies

$$\|\ln \sigma^n - \ln \sigma^+\|_{C^{1,\alpha}(\tilde{\Omega})} \leq K_\Omega \left(\frac{1}{2}\right)^n \epsilon.$$

[4] based on Seo, Woo, Liu, 2010

Reduced Basis Harmonic B_z Algorithm (RBZ)

Algorithm 1 RBZ($\sigma_b, \mu_0, tol_1, tol_2, X_{N,1}, X_{N,2}$)

$\sigma^0 := \sigma_b$, $n := 0$
repeat
  enrich spaces $X_{N,1}, X_{N,2}$ using $u_1^{\sigma^n}, u_2^{\sigma^n}$
  $i := 1, \sigma^i := \sigma^n$
  repeat
    $\mathcal{F}_N^{i+1}(\mathbf{r}) := \frac{1}{\mu_0}(\sigma^i \mathbb{A}_N[\sigma^i])^{-1} \begin{pmatrix} \nabla^2 B_{z,+}^1 \\ \nabla^2 B_{z,+}^2 \end{pmatrix}(\mathbf{r}), \ \forall \mathbf{r} \in \Omega$
    define $\ln \sigma^{i+1}$ as solution of
    $\nabla^2 \ln \sigma^{i+1} = \nabla \cdot \mathcal{F}_N^{i+1}$ in Ω, $\quad \ln \sigma^{i+1} = \ln \sigma^+$ on $\partial\Omega$
    $i := i + 1$
  until $\|\sigma^i - \sigma^{i-1}\|_{\mathcal{P}} \leq tol_1$ **or** $\Delta_{N,1}(\sigma_i) > tol_2$ **or** $\Delta_{N,2}(\sigma_i) > tol_2$
  $n := n + 1, \sigma^n := \sigma^i$
until $\|\sigma^i - \sigma^{i-1}\|_{\mathcal{P}} \leq tol_1$
return $\sigma_{RBZ} := \sigma^n$

Numerics - Setting

- $\Omega := [-1, 1] \times [-2, 2]$
- $E_1^{\pm} := \{(\pm 1, y) \mid |y| < 0.1\}$, $E_2^{\pm} := \{(x, \pm 2) \mid |x| < 0.1\}$
- For $r = \sqrt{x^2 + y^2}$, $x, y \in \Omega$

$$\sigma^+ \approx \sigma(r) := \begin{cases} 10\left(\cos(r) - \frac{\sqrt{3}}{2}\right) + 2, & 0 \leq r \leq \pi/6 \\ 2, & \text{otherwise} \end{cases} \in \mathcal{P}$$

  using 40×80 rectangles (piecewise constant approximation)
- $\sigma_b = 2$, $\mu_0 = 1$, $tol_1 = 10^{-5}$, $tol_2 = \frac{1}{100}$, no noise

Numerics - Comparison

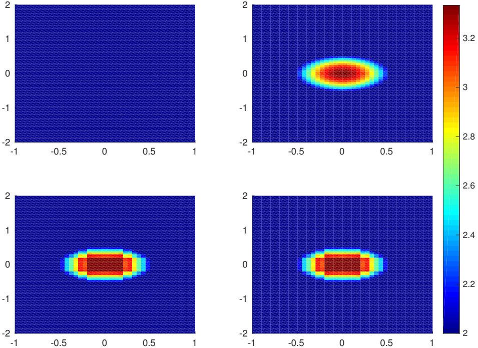

Figure: σ_b (top left), σ^+ (top right), σ_{BZ} (bottom left), σ_{RBZ} (bottom right).

- BZ: 3.45s and 16 PDE solves | RBZ: 2.21s and 6 PDE solves
- $\|\sigma^+ - \sigma_{BZ}\|_\mathcal{P} \approx 0.05$, $\|\sigma_{BZ} - \sigma_{RBZ}\|_\mathcal{P} \approx 6.760 \cdot 10^{-6}$

Conclusion

Summary

- Reduced basis (RB) approaches can speed-up the solution procedure of inverse coefficient problems
- Standard RB approach not feasible in imaging context
- Adaptive RB approach for high-dimensional parameter spaces
- Presented RBZ-Algorithm for MREIT including preliminary convergence result

Future work

- Finalize convergence result and numerics

Thank you for your attention!
