Reduced Basis Methods for MREIT

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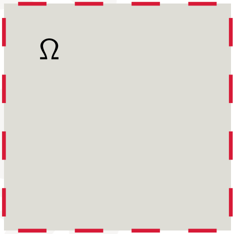
Magnetic **R**esonance **E**lectrical **I**mpedance **T**omography (MREIT)[1]

[1] Based on Seo, Woo, et al. since 2003

Motivation: Electrical Impedance Tomography (EIT)

- **Setting:** Imaging object $O \subset \mathbb{R}^3$ with electrode pairs attached
- **Aim:** reconstruct cross-sectional ($\Omega = O \cap \{z = z_0\} \subset \mathbb{R}^2$) image of electrical conductivity inside Ω

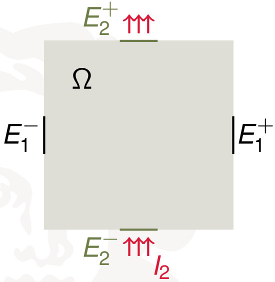


In EIT

- Data: Current-Voltage measurements on boundary
- Difficulty: highly ill-posed
  $\rightsquigarrow$ low spatial resolution

B_z-based MREIT: Setting

Aim: achieve higher resolution of conductivity σ.



- Object Ω with $E_1^{\pm}$, $E_2^{\pm}$ attached
  $\rightsquigarrow$ placed inside MRI-Scanner
- Apply current I between electrode pair
- Generates magnetic flux density B
- B_z measurable with MRI scanner
  (full internal data set)

Forward problem (shunt model)

- $\mathcal{P} = \{\sigma \in C^{1,\alpha}(\bar{\Omega}) \mid \sigma(x) > 0,\ x \in \overline{\Omega}\},\ \alpha \in (0,1)$

Detailed problem ($j \in \{1,2\}$ determines active electrode pair)

For $\sigma \in \mathcal{P}$, find u_j^σ, the detailed solution of

$$\nabla \cdot (\sigma \nabla u_j) = 0 \text{ in } \Omega, \quad I_j = \int_{E_j^+} \sigma \frac{\partial u_j}{\partial \mathbf{n}} ds = -\int_{E_j^-} \sigma \frac{\partial u_j}{\partial \mathbf{n}} ds \tag{1a}$$

$$\nabla u_j \times \mathbf{n} = 0, \text{ on } E_j^+ \cup E_j^-, \quad \sigma \frac{\partial u_j}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega \backslash \overline{\left(E_j^+ \cup E_j^-\right)} \tag{1b}$$

- Unique solution of (1) up to an additive constant

Inverse Problem

Aim: Determine $\sigma^\star$ from $B^1_{z,\star}$, $B^2_{z,\star}$.

Assume: $\sigma^\star \in \mathcal{P}$ with $\sigma^\star|_{\Omega\setminus\tilde{\Omega}} = \sigma_b$, $\sigma_b > 0$, $\tilde{\Omega} \subset\subset \Omega$, σ_b known.

- Maxwell Equations

$$-\sigma \nabla u_j^\sigma = \frac{1}{\mu_0} \nabla \times B^j \text{ (Ampère's law)} \quad \text{and} \quad \nabla \cdot B^j = 0$$

- $\nabla\times$ of Ampère's law $\rightsquigarrow$ $\nabla u_j^\sigma \times \nabla \sigma = \frac{1}{\mu_0} \nabla^2 B^j$

$\rightsquigarrow$ Relation for $\sigma^\star$ (logarithmic version, point-wise in Ω)

$$\nabla \ln \sigma^\star = \frac{1}{\mu_0} (\sigma^\star \mathbb{A}[\sigma^\star])^{-1} \begin{pmatrix} \nabla^2 B^1_{z,\star} \\ \nabla^2 B^2_{z,\star} \end{pmatrix}, \ \mathbb{A}[\sigma^\star] := \begin{pmatrix} \frac{\partial u_1^{\sigma^\star}}{\partial y} & -\frac{\partial u_1^{\sigma^\star}}{\partial x} \\ \frac{\partial u_2^{\sigma^\star}}{\partial y} & -\frac{\partial u_2^{\sigma^\star}}{\partial x} \end{pmatrix}$$

Reconstruction of $\sigma^\star$

Initial guess: $\sigma^0 = \sigma_b$.

Iteration sequence ($\rightsquigarrow$ Harmonic B_z Algorithm[2])

- calculate $\mathcal{V}^{n+1}(\mathbf{r}) = \frac{1}{\mu_0}(\sigma^n \mathbb{A}[\sigma^n])^{-1} \begin{pmatrix} \nabla^2 B^1_{z,\star} \\ \nabla^2 B^2_{z,\star} \end{pmatrix}(\mathbf{r}), \; \forall \mathbf{r} \in \Omega$
- define $\ln \sigma^{n+1}$ as the solution of

$$\nabla^2 \ln \sigma^{n+1} = \nabla \cdot \mathcal{V}^{n+1} \quad \text{in } \Omega, \quad \ln \sigma^{n+1} = \ln \sigma^\star \quad \text{on } \partial\Omega$$

- $\sigma^{n+1} = \exp(\ln \sigma^{n+1})$ (ensures positivity)

[2] Seo, Woo 2003, et al.

Approximative approach

Given: for $\sigma \in \mathcal{P}$ let $u^{\sigma}_{j,N}$ be an (unspecified) approximation to u^{σ}_{j}.

Approximative iteration sequence

- calculate $\mathcal{V}_N^{n+1}(\mathbf{r}) = \frac{1}{\mu_0}(\sigma^n \mathbb{A}_N[\sigma^n])^{-1}\begin{pmatrix} \nabla^2 B^1_{z,\star} \\ \nabla^2 B^2_{z,\star} \end{pmatrix}(\mathbf{r}), \ \forall \mathbf{r} \in \Omega,$

  with $\mathbb{A}_N[\sigma^n] := \begin{pmatrix} \frac{\partial u^{\sigma^n}_{1,N}}{\partial y} & -\frac{\partial u^{\sigma^n}_{1,N}}{\partial x} \\ \frac{\partial u^{\sigma^n}_{2,N}}{\partial y} & -\frac{\partial u^{\sigma^n}_{2,N}}{\partial x} \end{pmatrix}$
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- $\sigma^{n+1} = \exp(\ln \sigma^{n+1})$

Preliminary result[3]

There exists an $\epsilon > 0$, such that if $\|\nabla \ln \sigma^\star\|_{C^{0,\alpha}(\Omega)} < \epsilon$ and the approximations $u_{j,N}^{\sigma^n}$ fulfill

- $u_{j,N}^{\sigma^n} \in C^{1,\alpha}(\tilde{\tilde{\Omega}}), \qquad \tilde{\Omega} \subset\subset \tilde{\tilde{\Omega}} \subset\subset \Omega$
- $\|\nabla u_{j,N}^{\sigma^n} - \nabla u_j^{\sigma^n}\|_{C^{0,\alpha}(\tilde{\Omega})} \leq C_1 \epsilon^{n+1}$

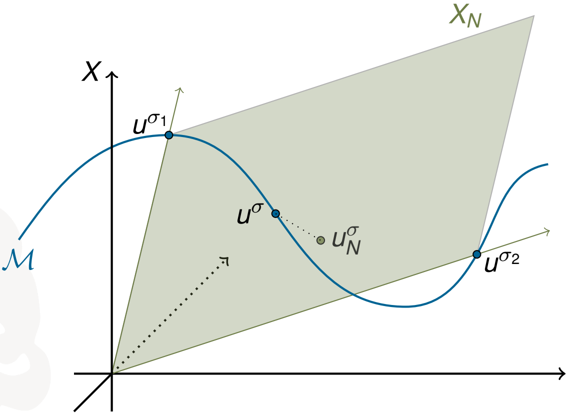
throughout the approximative iteration, the resulting sequence of iterates σ^n, $n = 1, 2, \dots$, with initial guess $\sigma^0 = \sigma_b$ satisfies

$$\|\ln \sigma^n - \ln \sigma^\star\|_{C^{1,\alpha}(\tilde{\Omega})} \leq C_2 \left(\frac{1}{2}\right)^n \epsilon, \quad n = 1, 2, \dots$$

[3] Extension of Seo, Woo, Liu, 2010. Preprint available soon.

Reduced **B**asis **M**ethods (RBM) and MREIT

Reduced Basis Methods (RBM): Idea



- Solution manifold $\mathcal{M} := \{u^\sigma \mid \sigma \in \mathcal{P}\}$
- Construction of X_N via *carefully* chosen *snapshots* u^{σ_i}

RBM: The detailed & reduced problem

Detailed problem (shunt model)

$F : \mathcal{P} \to X$, $\sigma \mapsto u^\sigma$, the detailed solution of

$$b(u^\sigma, v; \sigma) = f(v) \text{ for all } v \in X \quad b, f \text{ associated to (1).}$$

Assume: $X_N := \mathrm{span}\{u^{\sigma_1}, \dots, u^{\sigma_N}\} \subset X$ is given.

Reduced problem (Galerkin projection)

For $\sigma \in \mathcal{P}$, find $u_N^\sigma \in X_N \subset X$, the reduced solution of

$$b(u_N^\sigma, v; \sigma) = f(v), \quad \forall v \in X_N.$$

(one detailed/reduced problem/RB-space per $j = 1, 2$)

RBM: Properties

- Reproduction of solutions: $u^\sigma \in X_N \Rightarrow u_N^\sigma = u^\sigma$
- Offline/Online-decomposition: rapid computation of u_N^σ

Certification - rigorous a-posteriori error estimator

$$\|u^\sigma - u_N^\sigma\|_X \leq \Delta_N(\sigma) := \frac{\|v_r\|_X}{\alpha(\sigma)}, \text{ with}$$

$$\langle v_r, v\rangle_X := r(v;\sigma) := f(v) - b(u_N^\sigma, v;\sigma), \forall v \in X$$

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Naive/direct approach

- Construct global X_N (greedy, POD,...) approximating whole $\mathcal{M}$
- Use u_N^σ instead of u^σ in the inversion scheme

Limitation: Only feasible for low-dimensional parameter spaces, not feasible for imaging.

Our approach[4]: Create problem-adapted RB-space by iterative enrichment (inspired by Druskin & Zaslavski 2007, Zahr & Fahrhat 2015 and Lass 2014).

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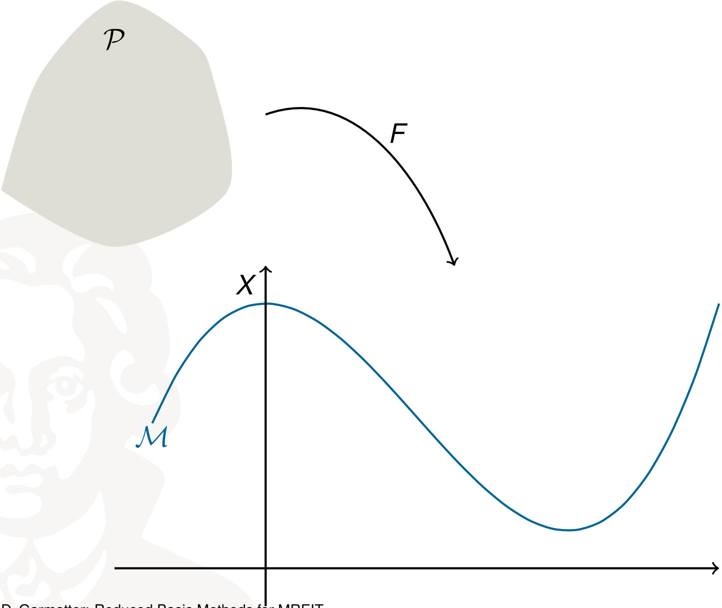
[4] **G.**/Haasdonk/Harrach, A Reduced Basis Landweber method for nonlinear inverse problems, 2016 Inverse Problems 32 (3) (doi).

Adaptive RBM & IP: Idea



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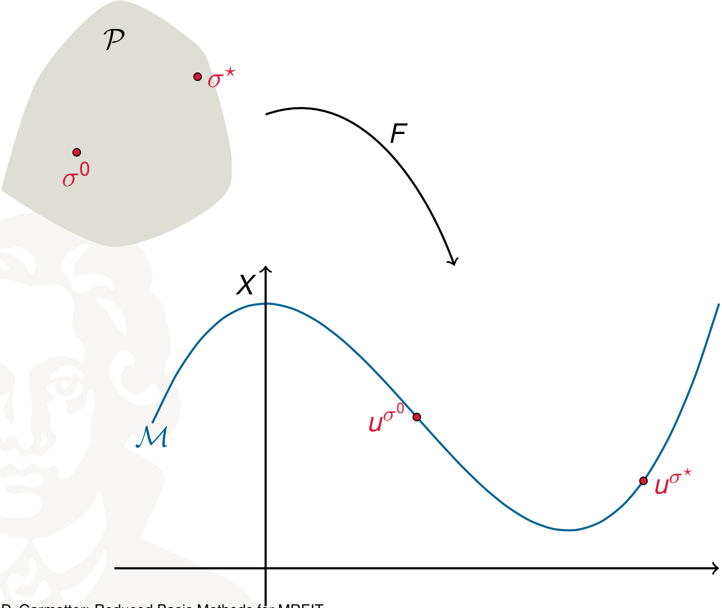
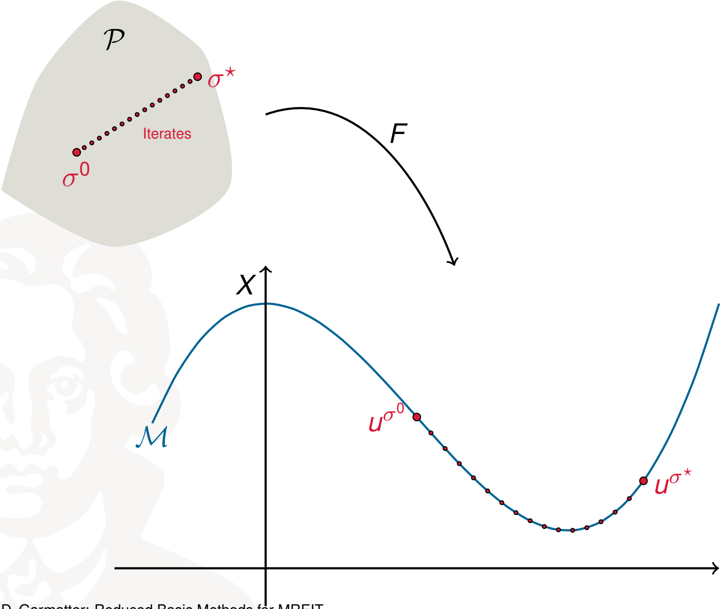
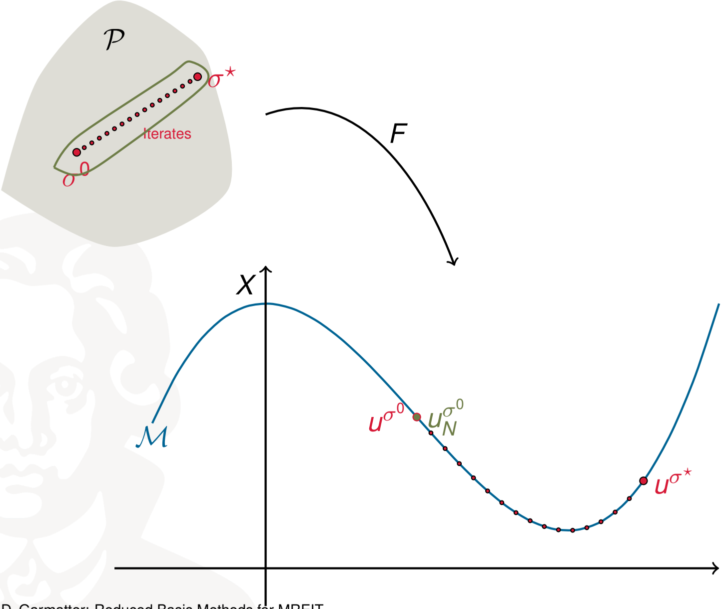


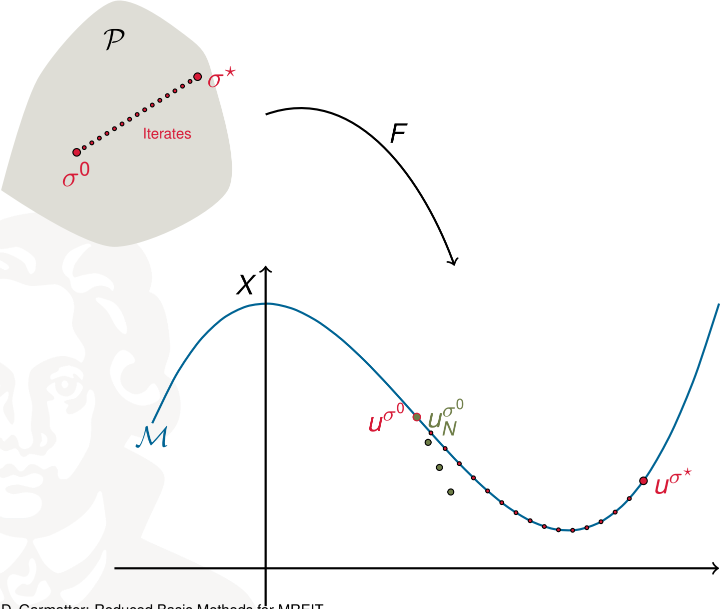
Adaptive RBM & IP: Idea



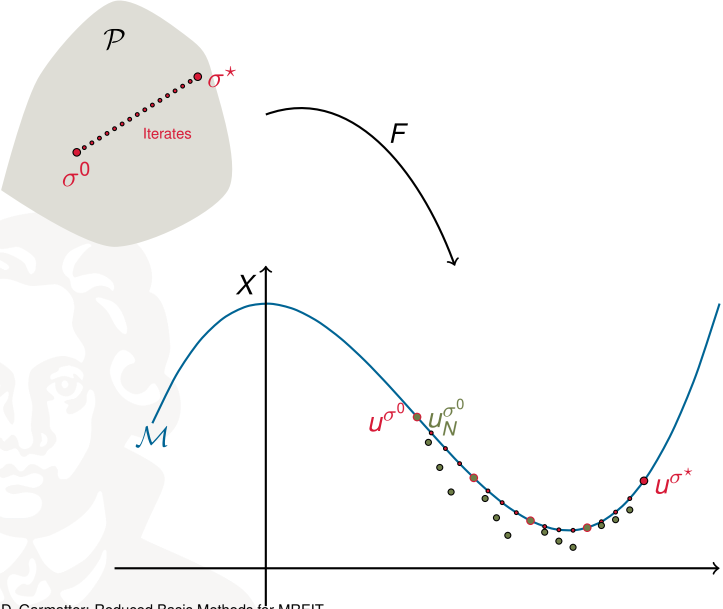
Adaptive RBM & IP: Idea



Adaptive RBM & IP: Idea



Adaptive RBM & IP: Idea



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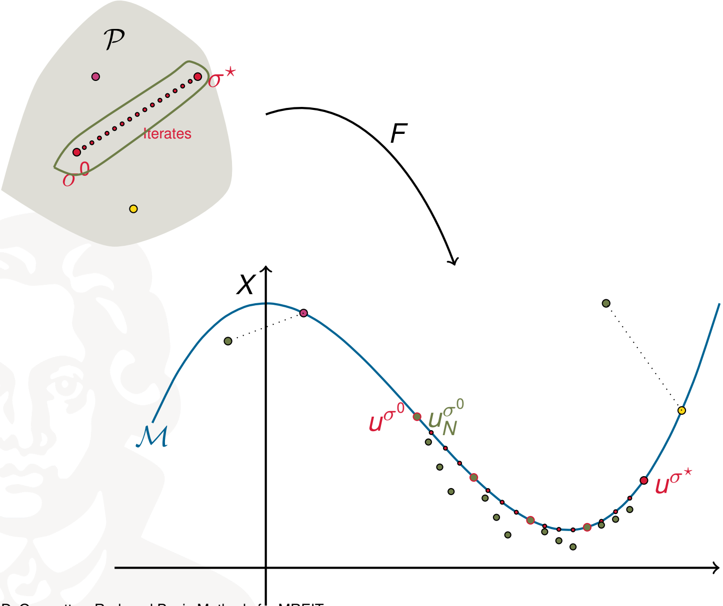

Recall: approximative approach for MREIT

Given: RB-spaces $X_{N,1},\ X_{N,2} \rightsquigarrow u^{\sigma}_{j,N}$ is respective RB-approximation.

RB iteration sequence

- calculate $\mathcal{V}_N^{n+1}(\mathbf{r}) = \frac{1}{\mu_0}(\sigma^n \mathbb{A}_N[\sigma^n])^{-1}\begin{pmatrix} \nabla^2 B^1_{z,\star} \\ \nabla^2 B^2_{z,\star} \end{pmatrix}(\mathbf{r}),\ \forall \mathbf{r} \in \Omega,$

  with $\mathbb{A}_N[\sigma^n] := \begin{pmatrix} \frac{\partial u^{\sigma^n}_{1,N}}{\partial y} & -\frac{\partial u^{\sigma^n}_{1,N}}{\partial x} \\ \frac{\partial u^{\sigma^n}_{2,N}}{\partial y} & -\frac{\partial u^{\sigma^n}_{2,N}}{\partial x} \end{pmatrix}$
- define $\ln \sigma^{n+1}$ as the solution of

$$\nabla^2 \ln \sigma^{n+1} = \nabla \cdot \mathcal{V}_N^{n+1} \quad \text{in } \Omega, \quad \ln \sigma^{n+1} = \ln \sigma^\star \quad \text{on } \partial\Omega$$

- $\sigma^{n+1} = \exp(\ln \sigma^{n+1})$

Reduced Basis Harmonic B_z Algorithm (RBZ)

Algorithm 1 RBZ$(\sigma_b, \mu_0, \varepsilon_1, \varepsilon_2, X_{N,1}, X_{N,2})$

$\sigma^0 := \sigma_b$, $n := 0$

repeat

  Enrich spaces $X_{N,1}$, $X_{N,2}$ using $u_1^{\sigma^n}$, $u_2^{\sigma^n}$.

  repeat

    $\mathcal{V}_N^{n+1}(\mathbf{r}) := \frac{1}{\mu_0}(\sigma^n \mathbb{A}_N[\sigma^n])^{-1} \begin{pmatrix} \nabla^2 B_{z,\star}^1 \\ \nabla^2 B_{z,\star}^2 \end{pmatrix} (\mathbf{r}),\ \forall \mathbf{r} \in \Omega$

    Calculate $\ln \sigma^{n+1}$ as the solution of

    $\nabla^2 \ln \sigma^{n+1} = \nabla \cdot \mathcal{V}_N^{n+1}$ in Ω, $\quad \ln \sigma^{n+1} = \ln \sigma^\star$ on $\partial\Omega$

    $\sigma^{n+1} := \exp(\ln \sigma^{n+1}) \quad n := n+1$

  until $\| \ln \sigma^n - \ln \sigma^{n-1} \|_{C(\Omega)} \leq \varepsilon_1$ **or** $\min_{j=1,2} \{\Delta_{N,j}(\sigma^n)\} > \varepsilon_2$

until $\| \ln \sigma^n - \ln \sigma^{n-1} \|_{C(\Omega)} \leq \varepsilon_1$

return $\sigma_{RBZ} := \sigma^n$

Numerics (idealized)

Setting: 260×260 image, $\varepsilon_1 = 10^{-6}$, $\varepsilon_2 = 10^{-3}$, no noise.

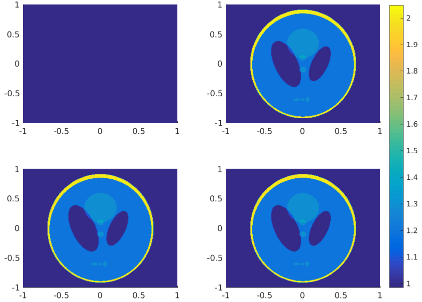

Figure: σ_b (top left), $\sigma^\star$ (top right), σ_{BZ} (bottom left), σ_{RBZ} (bottom right).

- BZ: 10.10s and 28 PDE solves | RBZ: 8.18s and 8 PDE solves
- $\frac{\|\sigma^\star - \sigma_{BZ}\|_{C(\Omega)}}{\|\sigma_{BZ}\|_{C(\Omega)}} \approx 2 \cdot 10^{-3}$, $\frac{\|\sigma_{RBZ} - \sigma_{BZ}\|_{C(\Omega)}}{\|\sigma_{BZ}\|_{C(\Omega)}} \approx 4 \cdot 10^{-4}$

Conclusion

Summary

- Inverse problem of MREIT & Harmonic B_z Algorithm as solution algorithm
- Any (sufficient) approximative forward solution $\rightsquigarrow$ convergence
- Reduced Basis Method (adaptive approach - high-dimensional parameter space) to speed-up existing algorithm
- Novel RBZ-Algorithm, including (idealized) numerics

Future work

- Finalize and publish results

Thank you for your attention!
