POSITIVITY OF HIGHER-CODIMENSIONAL SUBVARIETIES

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1. INTRODUCTION

Positivity of divisors/line bundles (or more generally, of vector bundles or coherent sheaves) on varieties is one of the central aspects of higher-dimensional algebraic geometry. Main applications that come to mind include the minimal model program and the construction of moduli spaces of varieties.

Roughly speaking, positive line bundles are characterized by an abundance of global sections, which then results in other good properties of numerical, cohomological or geometric nature. From a geometric point of view, a good starting point are very ample bundles, i.e. bundles whose associated Kodaira map yields a closed embedding into projective space. Ampleness, the main notion of positivity, comes to life in that we define a line bundle $\mathscr{L}$ to be ample if some power $\mathscr{L}^{\otimes m}$ becomes very ample. Ampleness for line bundles (or, equivalently, for Cartier divisors) admits characterizations in terms of vanishing of cohomology groups and intersection numbers, and, in addition, possesses good formal properties.

However, for the purposes of many applications — for instance the birational classification of projective varieties — ampleness is too restrictive, and various weakenings are of great interest. In the course of these lectures we will encounter too such notions: nefness (incorporating the idea that positive line bundles should intersect curves non-negatively), and partial or q-ampleness (building on the observation that ample line bundles tend to kill higher cohomology).

The purpose of this seminar is to study generalizations of ampleness and nefness to subvarieties of higher codimension. Ampleness for subvarieties in general builds on q-ampleness, while nefness in higher codimension falls apart into as many as four inequivalent definitions. At the same time, the intimate relationship between ample and nef divisors becomes much less tangible in the higher-codimension world, and unexpected things happen: there exist nef subvarieties that are not limits of effective cycles, and ampleness is no longer invariant under rational equivalence.

Although the foundations have been laid, the area has just been born and full of open questions. One does not expect a general understanding of ample/nef subvarieties of a given variety; not the least since these questions seem to be out of the reach of our current techniques even in dimension two (a good example is the cone of ample divisors on Hilbert modular surfaces). On the other hand, it is a feasible project to study positive subvarieties in varieties with lots of extra structure (for instance Noether–Lefschetz loci in orthogonal Shimura varieties).

About the structure of the program: the first three pairs of talks are devoted to the classical theory of positive divisors while mostly ignoring the birational aspects. This is established material with the almost canonical reference [PAG]. The fourth pair of lectures is about partial cohomological ampleness, here the references are [DPS, S, T]. The two final occasions contain the actual higher-codimensional material. The source for ample subschemes is the beautifully written article [O1], while our study of numerical positivity of higher-codimensional cycles will be studied based on [FL2]. Note that the references include papers that are not needed directly for the talks, but provide extra information to the interested reader.

We will work over a base field that is algebraically closed and of characteristic zero; at points this field is going to be taken to be $\mathbb{C}$.

2. PROGRAM

The lectures are divided into pairs, the distribution of the material between the individual talks should follow as it is seen fit by the speakers. In my opinion it helps a lot in digesting such a seminar program if one has a guiding question that can be used as a compass in the area. My suggestion goes as follows.

What do we know about ample line bundles on Hilbert modular surfaces?

Lectures 1 & 2 (Fundamentals of positivity). Line bundles and Cartier divisors, intersection numbers á la Kleiman–Snapper, numerical equivalence of divisors, Néron–Severi space, the Riemann–Roch theorem and its asymptotic version, the theorem of Cartan–Grothendieck–Serre, first formal properties (restriction to irreducible components, behaviour under pullbacks, amplitude in families, etc.). Examples: $\mathbb{P}^n$, $\mathbb{P}^2$ blown up at a point.
Literature: [PAG, Sections 1.1 and 1.2], for Kleiman–Snapper see [B], [D], or [K2] for instance.

Lectures 3 & 4 (Numerical positivity). Nakai–Moishezon–Kleiman criterion, numerical nature of ampleness, $\mathbb{Q}$- and $\mathbb{R}$-Cartier divisors and their amplitude, openness of amplitude in the Néron–Severi space, nef line bundles and divisors, formal properties of nefness, Kleiman's theorem, cones of ample and nef divisors with simple examples.
Literature: [PAG, Sections 1.2 through 1.4], [KM]

Lectures 5 & 6 (Topological consequences of ampleness and vanishing theorems). Metric characterization of ampleness, Kodaira's embedding theorem (without proof), topology of affine varieties (review), cohomological dimension, Lefschetz hyperplane theorem, Kodaira's and Fujita's vanishing theorems (if time permits, comparison with Serre).
Literature: [PAG, Sections 1.2.C, 1.4.D, 3.1.A, 3.1.B, 4.2], [K]

Lectures 7 & 8 (Partial cohomological ampleness). The various definitions of q-ampleness and their equivalence (outline of the proof), formal properties, upper-semicontunuity, homogeneity, numerical invariance, examples in dimension two and three, geometric q-ampleness.
Literature: The main source is Totaro's paper [T] where he proves the equivalence of the three competing definitions. Other possibilities include the original articles [DPS, S] and the survey [GK].

Lectures 9 & 10 (Ample subschemes). Definition of ample subschemes, examples, the fact that ampleness does not respect rational equivalence, ampleness of the normal bundle, ample versus local complete intersection, regular section of ample vector bundles, intersections with divisors, cohomology of the complement and the Lefschetz hyperplane theorem, fundamental groups of ample subvarieties, the question of Kodaira vanishing for q-ample line bundles.
Literature: The original and main source for this material is [O1].

Lectures 11 & 12 (Nefness and its relatives in higher codimension). Cycles and dual cycles, Chern classes of vector bundles (quick review), Schur classes, pseudo-effective/nef/pliant/base point free/universally pseudo-effective cones, all of these cones are full-dimensional, strictly convex, and contain complete intersection classes in their interiors, no two cones are equal in general.
Literature: [FL2]

References

[B] Lucian Bădescu, *Algebraic Surfaces*, Universitext, Springer-Verlag, New York, 2001. ↑2

[D] Olivier Debarre, *Higher-Dimensional Algebraic Geometry*, Universitext, Springer-Verlag, New York, 2001. ↑2

[DELV] Olivier Debarre, Lawrence Ein, Robert Lazarsfeld, and Claire Voisin, *Pseudoeffective and nef classes on abelian varieties*, Compos. Math. **147** (2011), no. 6, 1793–1818. ↑

[DPS] Jean-Pierre Demailly, Thomas Peternell, and Michael Schneider, *Holomorphic line bundles with partially vanishing cohomology*, Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993), Israel Math. Conf. Proc., vol. 9, Bar-Ilan Univ., Ramat Gan, 1996, pp. 165–198. ↑1, 2

[F] William E. Fulton, *Intersection Theory*, Springer-Verlag, New York, 1998. ↑

[FL1] William E. Fulton and Robert Lazarsfeld, *Positive polynomials for ample vector bundles*, Ann. of Math. (2) **118** (1983), no. 1, 35–60. ↑

[FL2] Mihai Fulger and Brian Lehmann, *Positive cones of dual cycle classes*, Algebraic Geometry **4** (2017), no. 1, 1–28. arXiv:1408.5154. ↑1, 2

[GK] Daniel Greb and Alex Küronya, *Partial positivity: geometry and cohomology of q-ample line bundles*, Recent advances in algebraic geometry, London Math. Soc. Lecture Note Ser., vol. 417, Cambridge Univ. Press, Cambridge, 2015, pp. 207–239. ↑2

[H] Robin Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 52, Springer-Verlag, New York, 1977. ↑

[K1] János Kollár, *Shafarevich Maps and Automorphic Forms*, Princeton University Press, 2014. ↑2

[K2] ———, *Rational Curves on Algebraic Varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., vol. 32, Springer-Verlag, Berlin, 1996. ↑2

[KM] János Kollár and Shigefumi Mori, *Birational Geometry of Algebraic Varieties*, Cambridge University Press, 1998. ↑2

[KL] Alex Küronya and Victor Lozovanu, *Geometric aspects of Newton–Okounkov bodies* (2017). arxiv:1703.09980. ↑

[PAG] Robert Lazarsfeld, *Positivity in algebraic geometry. I.*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., vol. 48, Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series. ↑1, 2, 2, 2

[O1] John Christian Ottem, *Ample subvarieties and q-ample divisors*, Adv. Math. **229** (2012), no. 5, 2868–2887. ↑1, 2

[O2] ———, *Nef cycles on some hyperkahler fourfolds* (2015). arXiv:1505.01477. ↑

[S] Andrew John Sommese, *Submanifolds of Abelian varieties*, Math. Ann. **233** (1978), no. 3, 229–256. ↑1, 2

[T] Burt Totaro, *Line bundles with partially vanishing cohomology*, J. Eur. Math. Soc. (JEMS) **15** (2013), no. 3, 731–754. ↑1, 2