Let $D$ be a divisor on a variety $X$ and recall the different definitions of the complete linear system

$$|D| = \mathbb{P}^0(X, \mathcal{O}_X(D)) = \{D' \in \text{Div}(X) : D' \sim D \text{ and } D \geq 0\} = \mathbb{P}L(D) := \{f \text{ meromorphic function on } X : (f) + D \geq 0\}.$$

Assume that $D$ is effective and write $D = (s_D)$ for a section $s_D \in H^0(X, \mathcal{O}_X(D))$. The equivalences between the definitions above are given by

$$[s] = [\lambda s] \quad \iff \quad \{D' \sim D \text{ and } D \geq 0\} \quad \iff \quad \mathbb{P}L(D)$$

$$[f \cdot s_D] \quad \iff \quad D' = D + (f) \quad \iff \quad [f] = [\lambda \cdot f]$$

More generally, any linear subspace $V \subset |D|$ is called a linear system. We define the base locus of $V$ as

$$\text{Bs}(V) = \bigcap_{D' \in V} \{x \in X : s(x) = 0 \text{ for all } s \in V \subset \mathbb{P}H^0(X, \mathcal{O}_X(D))\}.$$

### Aufgabe 1 (5 Punkte)

Let $X \subset \mathbb{P}^2$ be the projective curve given by the equation $y^3z^2 = x^5 - z^5$ and write $\pi : X \to \mathbb{P}^1$ for the map $(x : y : z) \mapsto (x : z)$. Set $p = \pi^{-1}(\infty)$ and $r_\alpha = (e^{2\pi i/5} : 0 : 1)$ for $\alpha = 0, \ldots, 4$.

(a) Find the ramification divisor of $\pi$ and compute the genus of $X$.

**Hint:** The ramification divisor of a map $\pi$ is $\sum_{p \in X}(e_p - 1) \cdot [p]$, where $e_p$ is the ramification index of $\pi$ at $p$.

(b) Establish the linear equivalences $3p \sim 3r_\alpha$, for $\alpha = 0, \ldots, 4$, and $\sum_{\alpha=0}^{4} r_\alpha \sim 5p$.

(c) Determine the space $H^0(X, K_X)$ of holomorphic differentials on $X$.

**Hint:** Find $|D|$ where $D = (dx)$.

(d) Describe the canonical map $\phi_{K_X} : X \to \mathbb{P}^3$ and determine the equations of its image.

(e) Using part (c) show that for $D = \sum_{\alpha=0}^{4} r_\alpha$ one has $h^0(X, K_X(-D)) = 1$.

Use Riemann-Roch to conclude that $h^0(X, \mathcal{O}(D)) = 3$. 

Aufgabe 2 (3 Punkte)

Let $D$ be a divisor on a curve $X$.

(a) Prove that $D$ is base-point-free if and only if $h^0(X, D - [p]) = h^0(X, D) - 1$ for all points $p \in X$.

(b) Prove that $D$ is very ample if and only if $h^0(X, D - [p] - [q]) = h^0(X, D) - 2$ for all points $p, q \in X$.

Aufgabe 3 (4 Punkte)

Let $D$ be a divisor on $\mathbb{P}^2$ and $V \subset |D|$ a linear system. Write $\varphi_V : \mathbb{P}^2 \to \mathbb{P}^N$ for its associated map and assume that the base locus $\text{Bs}(V)$ consists of a single point $P$. Consider the blow-up $\pi : \text{Bl}_P(\mathbb{P}^2) \to \mathbb{P}^2$ and prove that there exists a morphism $\Phi : \text{Bl}_P(\mathbb{P}^2) \to \mathbb{P}^N$ everywhere defined such that $\Phi = \varphi_V \circ \pi$.

Hint: What can you say about the linear systems

$\pi^*V - kE = \{\pi^*D' - kE : D' \in V\}$

for different $k \in \mathbb{Z}$, where $E = \pi^{-1}(P)$ is the exceptional divisor?

Aufgabe 4 (4 Punkte)

In this exercise we construct maps $\phi_L$ associated to certain line bundles $L$ on projective spaces.

(a) Let $L$ be a very ample line bundle on a variety $X$. Prove that $L^\otimes m$ is very ample for every $m \geq 1$.

(b) Consider the line bundle $\mathcal{O}(d)$ on $\mathbb{P}^n$. Recall that its global sections can be identified with homogeneous polynomials of degree $d$. Describe the map $\phi_{\mathcal{O}(d)}$, called the Veronese embedding. Why is it an embedding?

(c) Let $X = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$ and write $p_i : X \to \mathbb{P}^{n_i}$ for the two natural projections. Describe the map $\phi_L$ associated to the line bundle $L = p_1^*\mathcal{O}(1) \otimes p_2^*\mathcal{O}(1)$ on $X$, called the Segre embedding. What is the base locus $\text{Bs}(|L|)$?

Abgabe Zu Beginn der Übung um 14:00 am Mittwoch, den 8. Mai.