Degenerations of Projective Space Induced by Points in an Affine Building

(work in progress)

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joint work with Annette Werner (U Frankfurt),
Dustin Cartwright and Bernd Sturmfels (UC Berkeley)

Berlin, 2009

Definition: The Bruhat-Tits Building of PGL_d

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The corresponding clique complex is called the Bruhat-Tits building of $\mathrm{PGL}_d(K)$ and denoted $\mathcal{B}_d$.

Facts about $\mathcal{B}_d$

The vertices adjacent to $[L]$ correspond to lattices between tL and L, i. e. (nontrivial) k-subvector spaces of $L/tL \cong k^d$. Simplices in the link of $[L]$ correspond to flags in L/tL.

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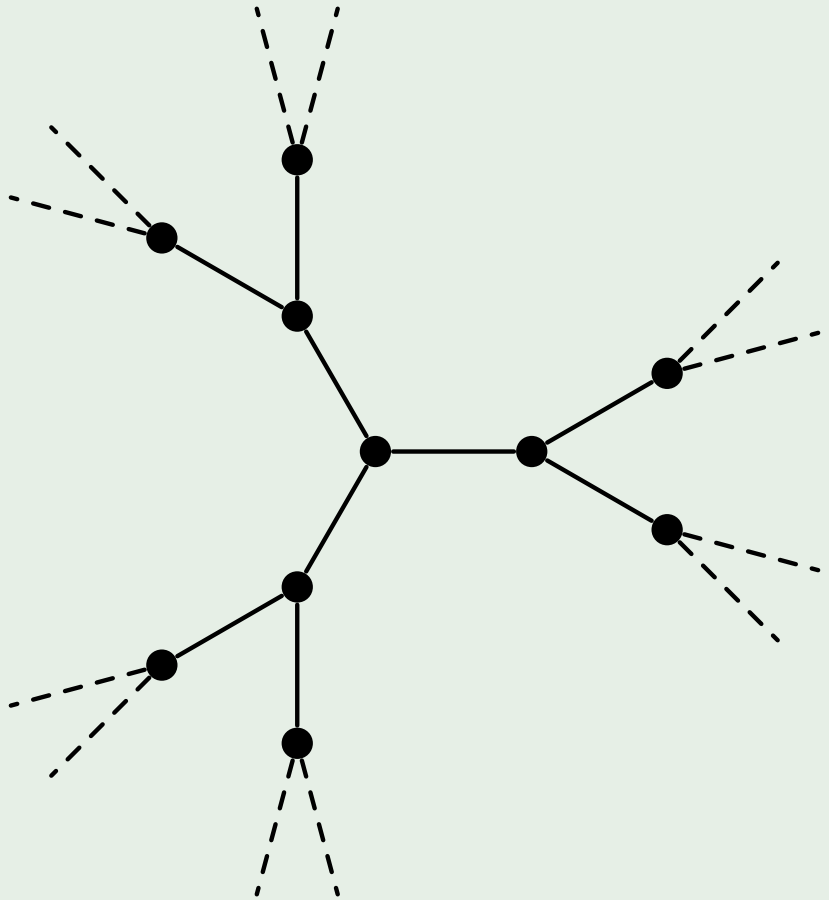
Example ($d = 2$)

If $d = 2$, nontrivial subspaces of $L/tL \cong k^2$ are lines, so
$\#\{\text{vertices adjacent to } [L]\} = \#\mathbb{P}^1 k$.
$\mathcal{B}_2$ is of dimension 1, so it is a graph, and, in fact, a tree.

$\mathcal{B}_2$

Example ($d = 2$, $k = \mathbb{F}_2$)



Apartments: Why Buildings are Tropical

$\mathcal{B}_d$ is covered by special subcomplexes called apartments:
For any basis $\{b_1, \ldots, b_d\}$ of V, let

$$\mathcal{A}_d^0(b_1, \ldots, b_d) := \left\{ \left[\sum \mathrm{R}t^{u_i} b_i \right] : \mathbf{u} \in \mathbb{Z}^d \right\},$$

and let $\mathcal{A}_d(b_1, \ldots, b_d)$ be the full subcomplex of $\mathcal{B}_d$ with vertex set $\mathcal{A}_d^0(b_1, \ldots, b_d)$.

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Since $[\sum \mathrm{Rt}^{u_i} b_i] = [\sum \mathrm{Rt}^{v_i} b_i] \iff \mathbf{u} - \mathbf{v} \in (1, \ldots, 1)\mathbb{Z}$, there is a bijection $\mathcal{A}_d^0(b_1, \ldots, b_d) \xrightarrow{\sim} \mathbb{Z}^d/(1, \ldots, 1)\mathbb{Z}$.

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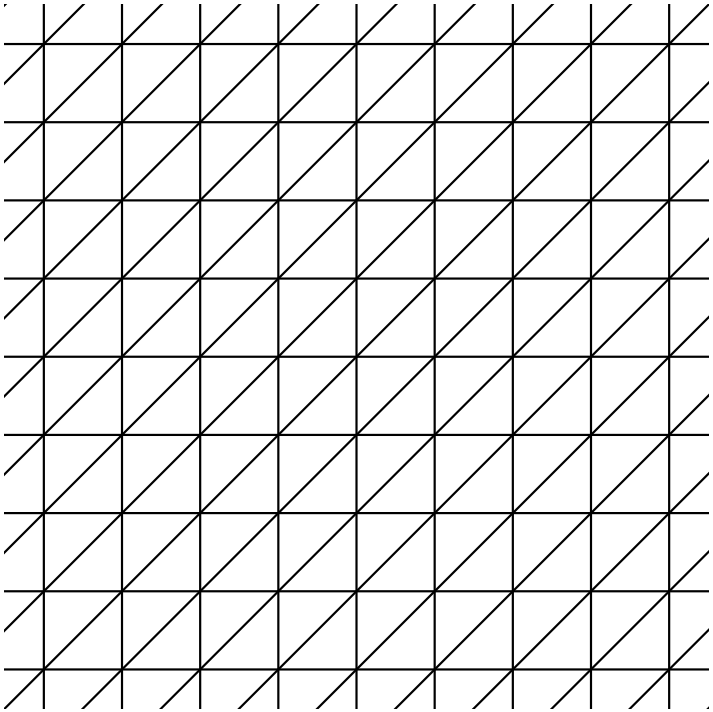
It induces an isomorphism from $\mathcal{A}_d(b_1, \ldots, b_d)$ to the standard integer triangulation of $\mathbb{TP}^{d-1} = \mathbb{R}^d/(1, \ldots, 1)\mathbb{R}$.

Example: An Apartment in $\mathcal{B}_3$

Every apartment in $\mathcal{B}_3$ is isomorphic to the standard integer triangulation of $\mathbb{TP}^2 = \mathbb{R}^3/(1,\ldots,1)\mathbb{R}$:



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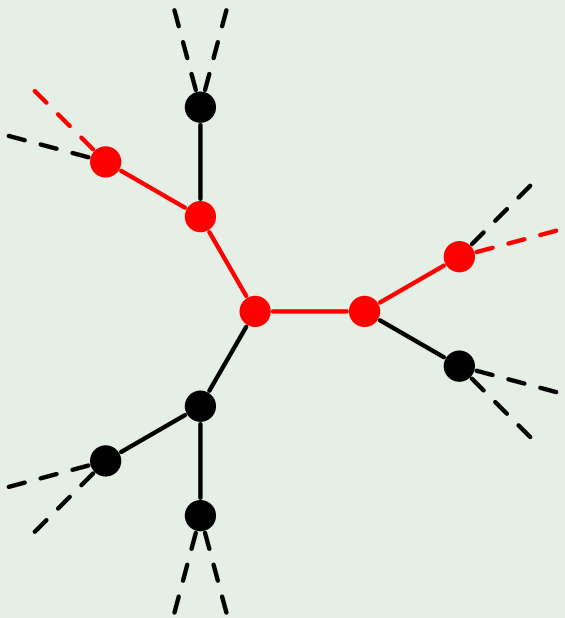

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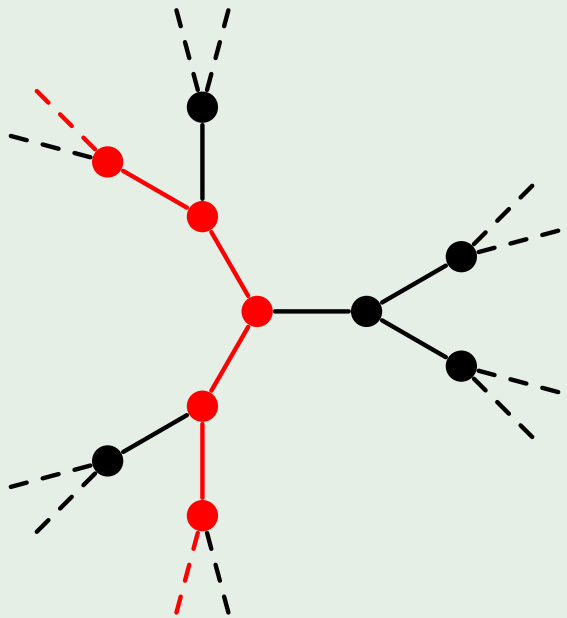

The Scheme Associated to a Vertex

For any lattice L, set $\mathcal{P}(L) := \operatorname{Proj} \operatorname{Sym} L$, where

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Since $\operatorname{Sym} L = R[b_1, \ldots, b_d]$ for any basis $\{b_1, \ldots, b_d\}$ of L, we have $\mathcal{P}(L) \cong \mathbb{P}_R^{d-1}$; and especially $\mathcal{P}(L_1) \cong \mathcal{P}(L_2)$ for any lattices L_1, L_2, but non-canonically.

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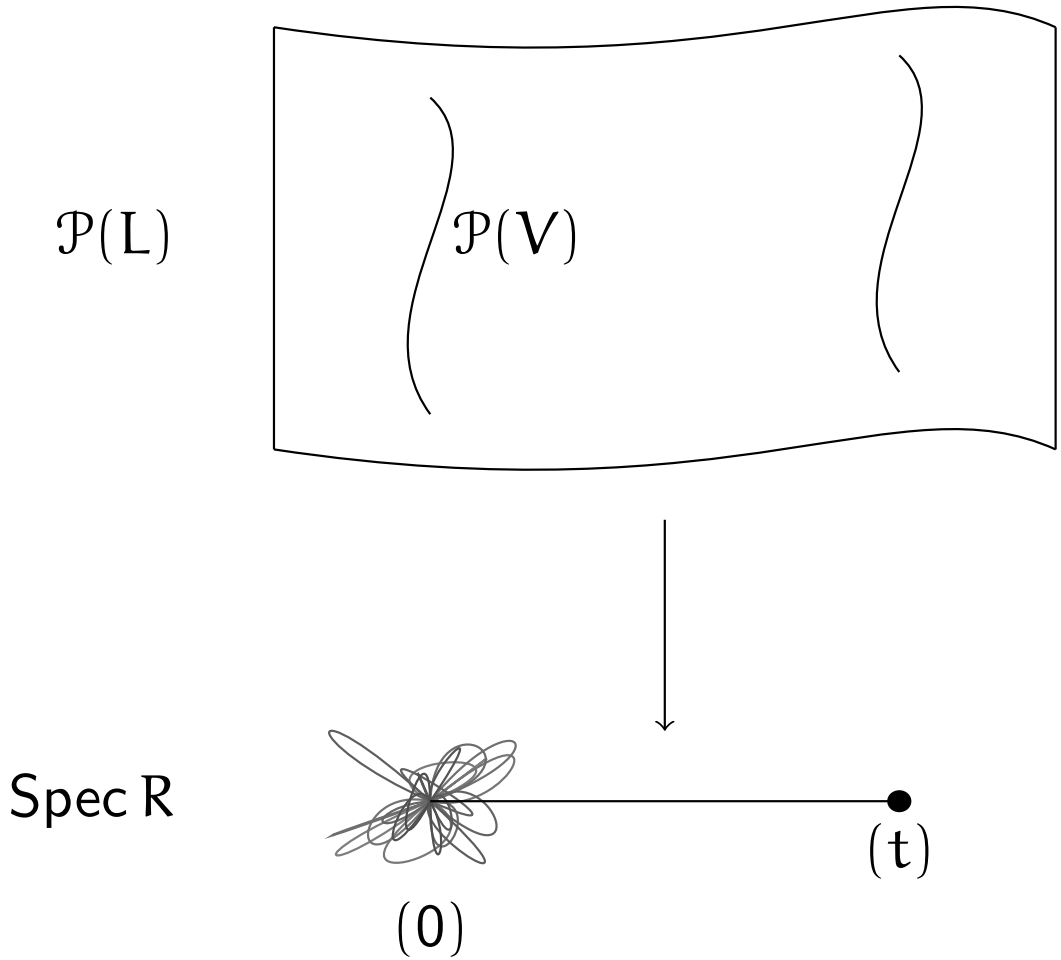
On the other hand, $L \otimes_R K = V$ canonically, so we have canonical embeddings $\mathcal{P}(V) \hookrightarrow \mathcal{P}(L)$ as generic fiber for each L.

The Scheme Associated to a Vertex (schematically, $d = 2$)



The Mustafin Variety

For a finite set of vertices $[L_1], \ldots, [L_n]$, the embeddings $\mathcal{P}(V) \hookrightarrow \mathcal{P}(L_i)$ give an embedding $\mathcal{P}(V) \hookrightarrow \prod_i \mathcal{P}(L_i)$. We call the closure of the image the Mustafin variety $\mathcal{M}([L_1], \ldots, [L_n])$.

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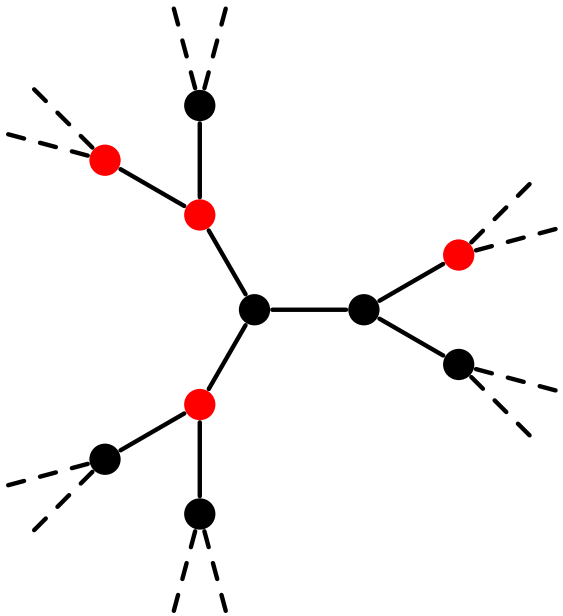
It has been studied by Mumford in the case $d = 2$, and by Mustafin in the case that the vertices form a *convex* subcomplex of $\mathcal{B}_d$.

Some Results: $d = 2$

The case $d = 2$ is now well understood.
The special fiber of $\mathcal{M}([L_1], \ldots, [L_n])$ has exactly n comonents, each isomorphic to $\mathbb{P}^1_k$, meeting transversally in such a way that they form a tree, "dual" to the convex hull of $[L_1], \ldots, [L_n]$ in $\mathcal{B}_2$:



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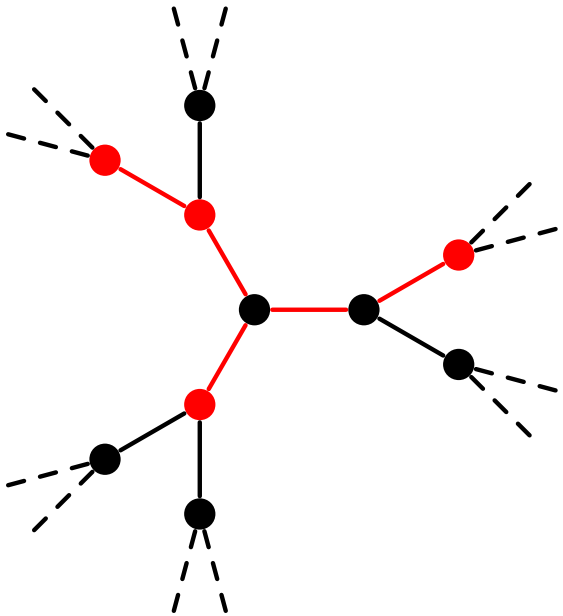

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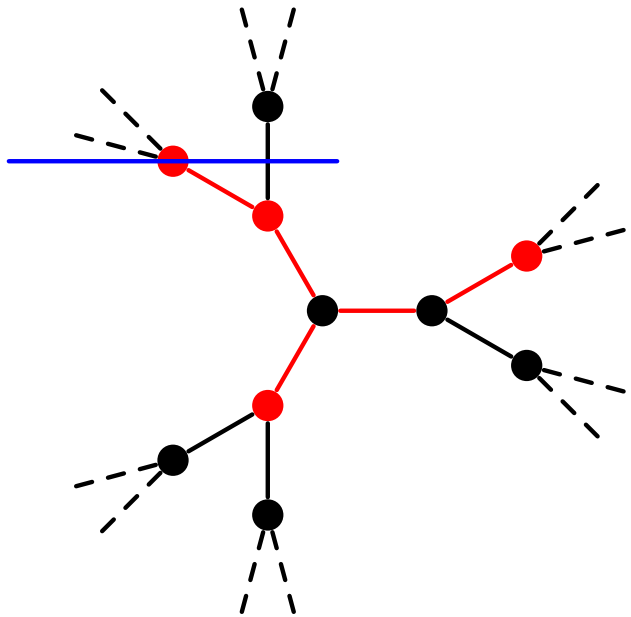

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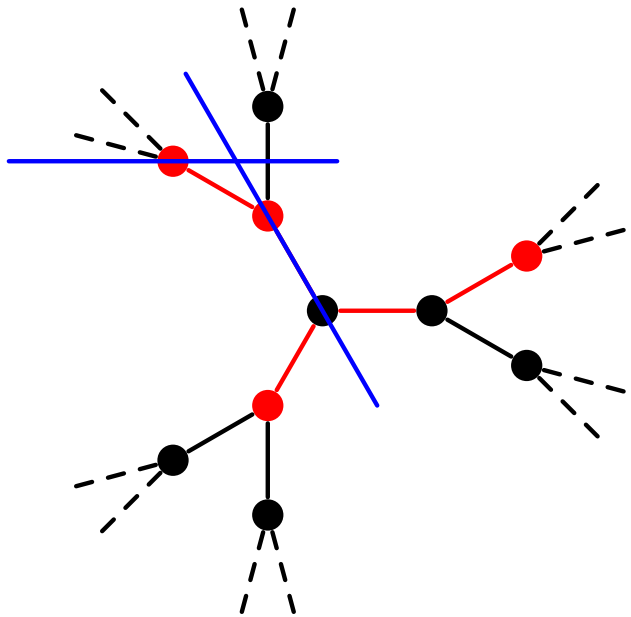

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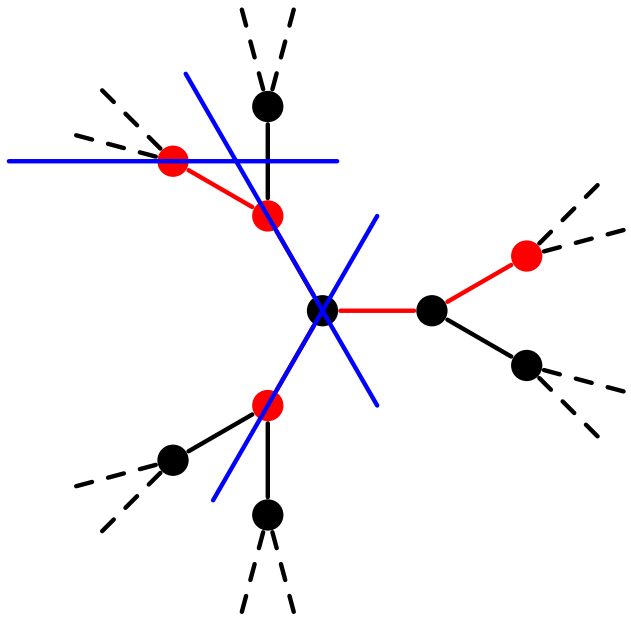

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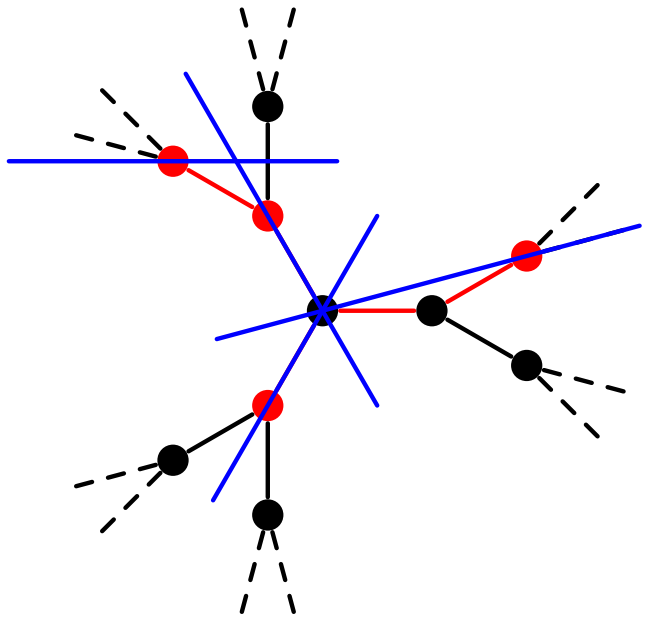

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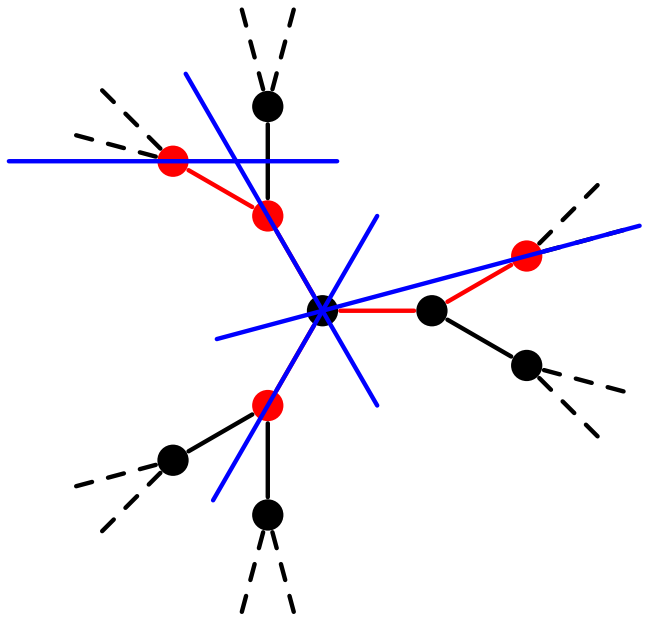

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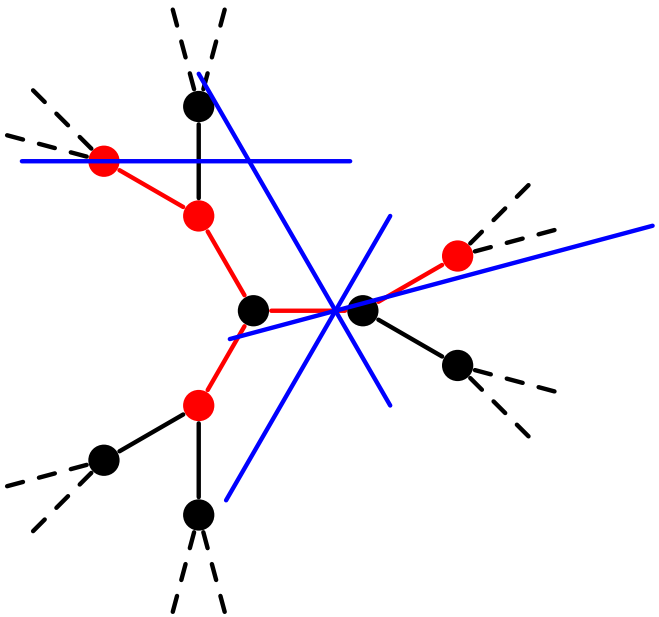

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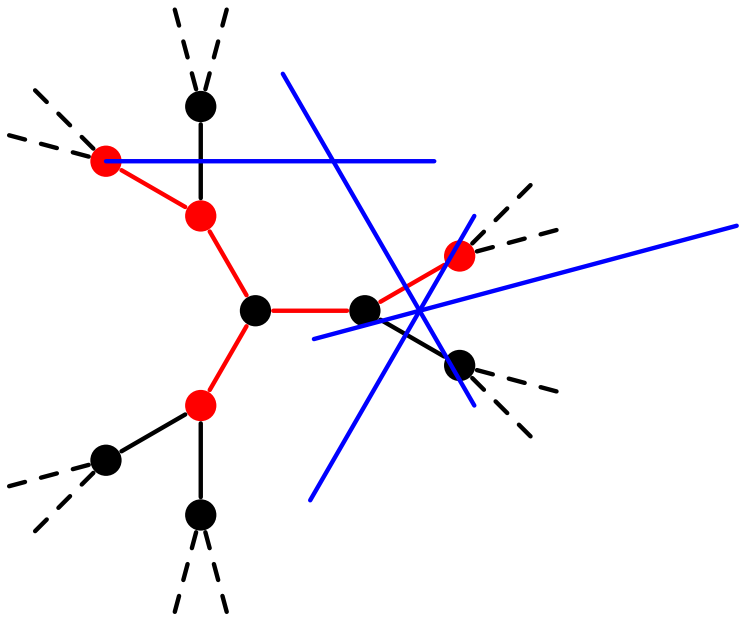

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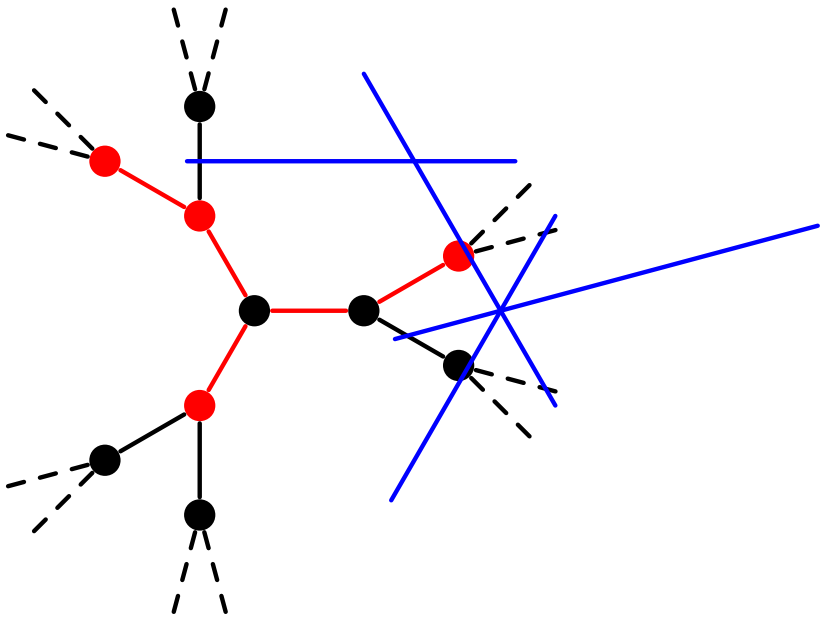

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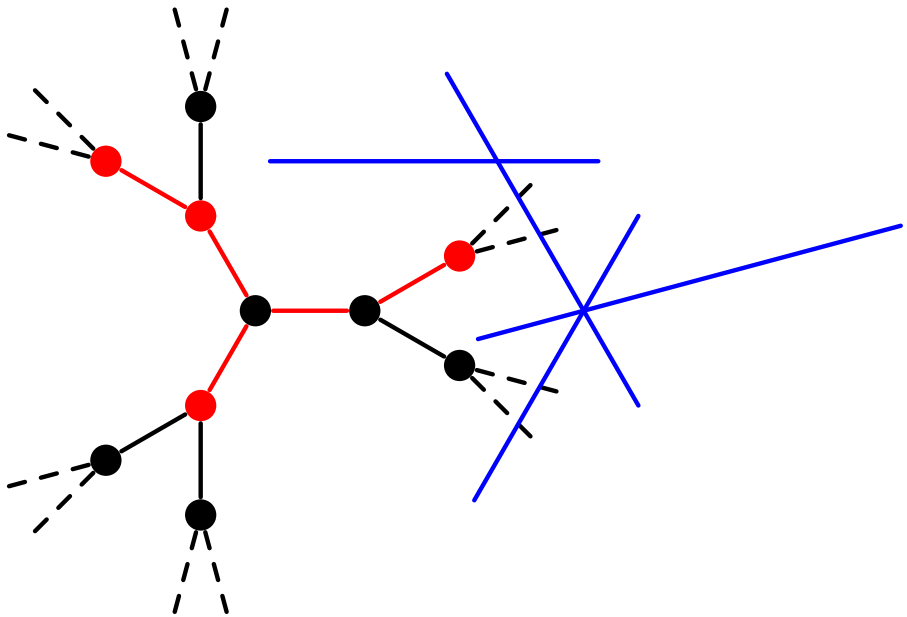

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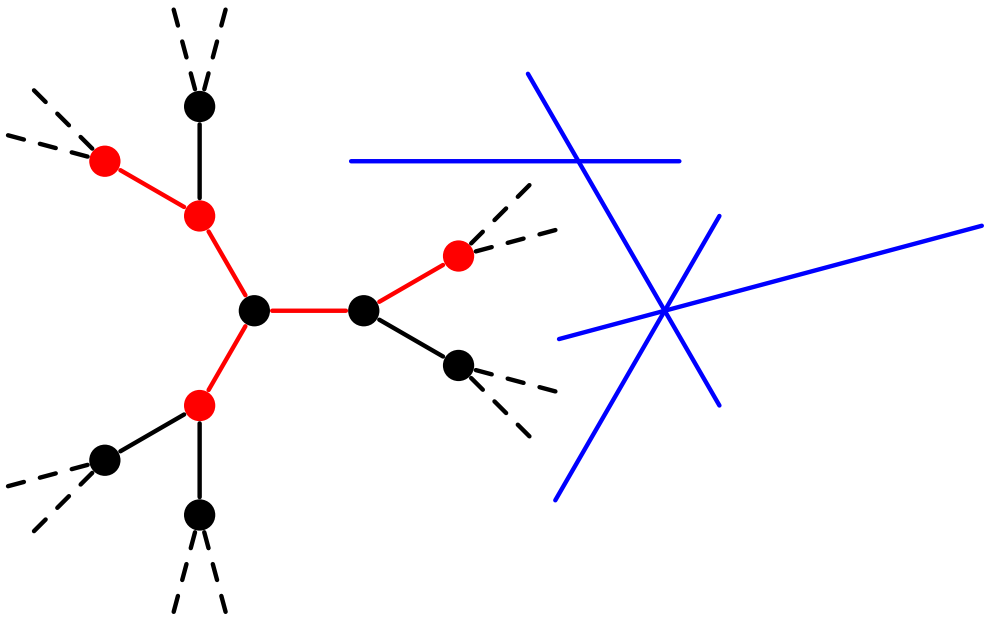

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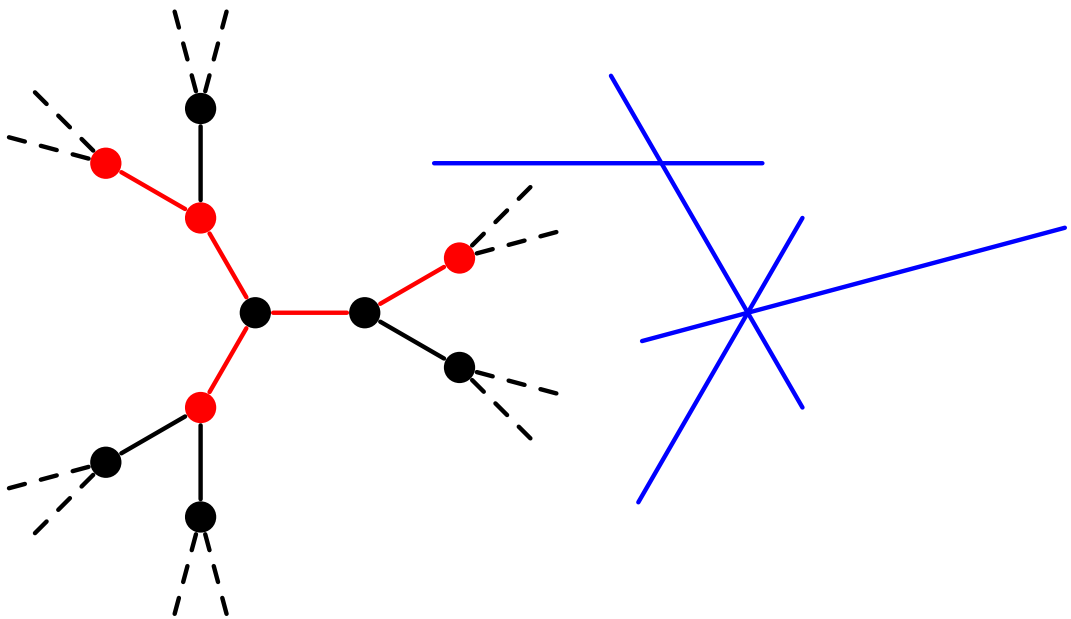

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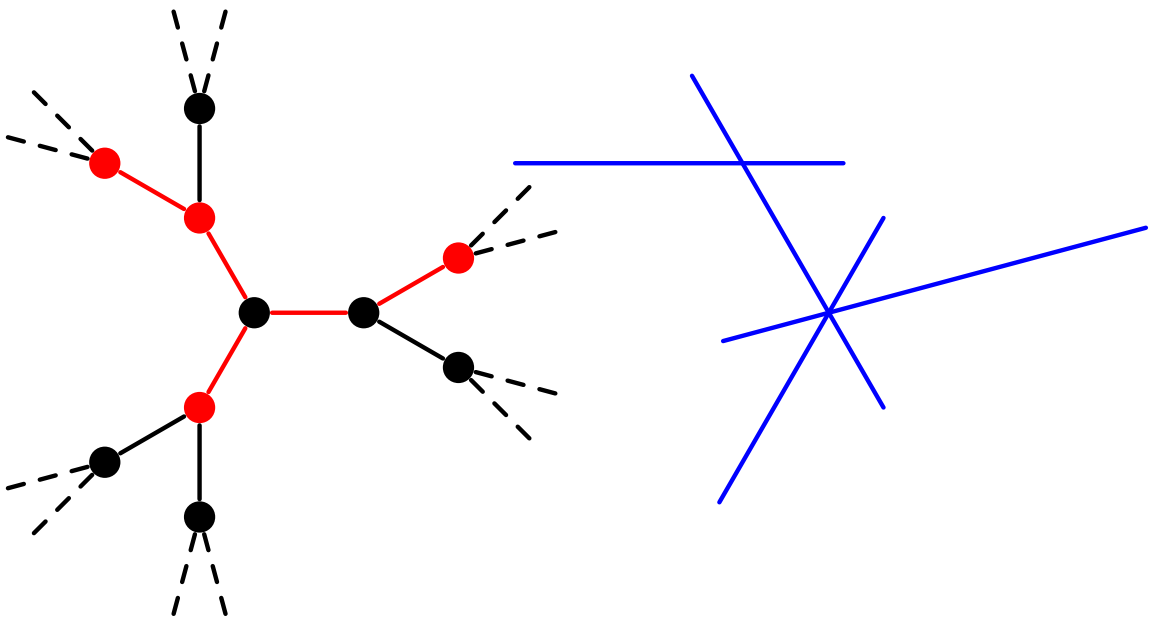

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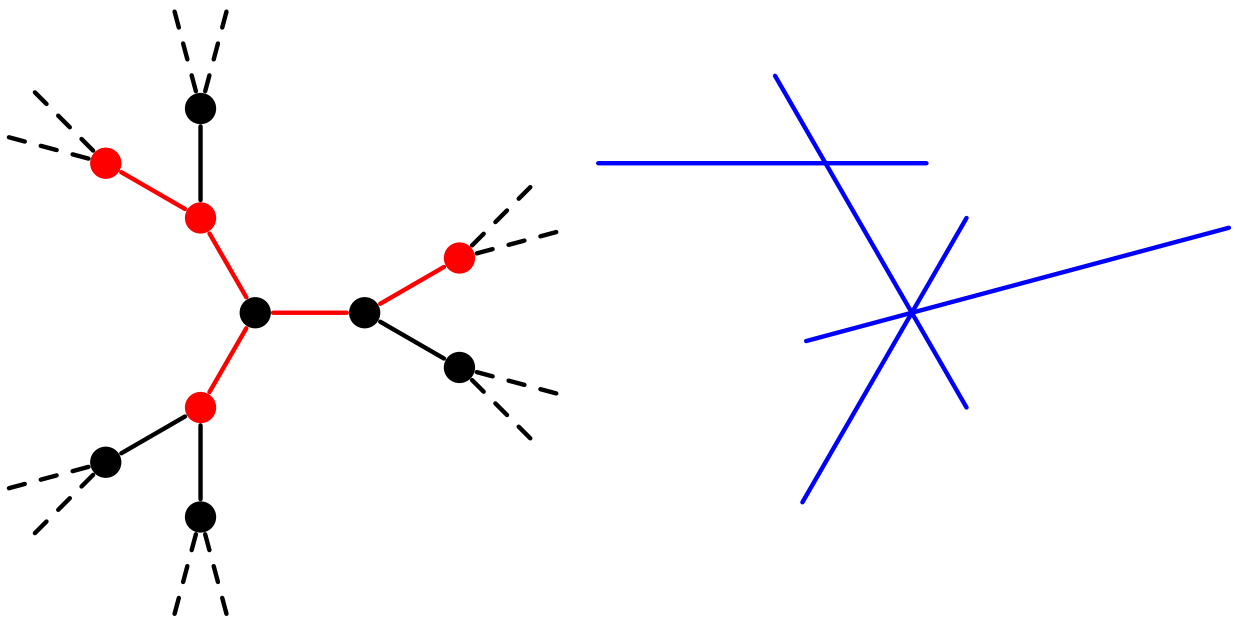
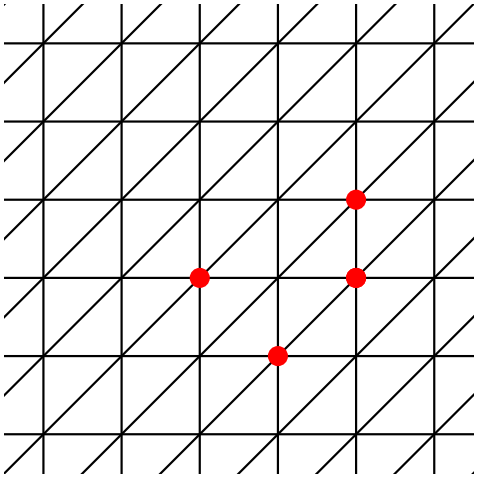


Some Results: $d > 2$

For $d > 2$, $\mathcal{M}([L_1], \ldots, [L_n])$ might have more than n components, and they need not be isomorphic to $\mathbb{P}_k^{d-1}$.
But they are rational varieties, and if $[L_1], \ldots, [L_n]$ lie in a common apartment, they are toric varieties corresponding to the cells in the Newton polytope of a tropical hyperplane arrangement:



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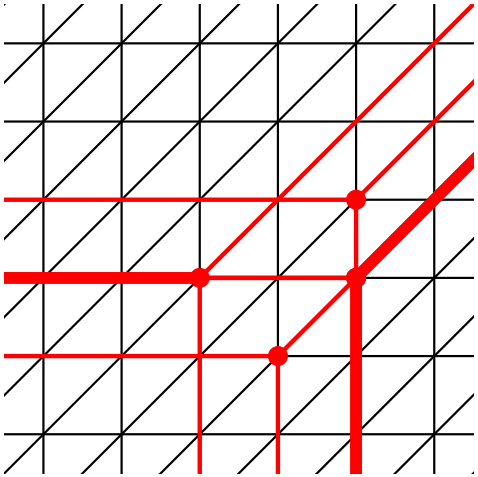

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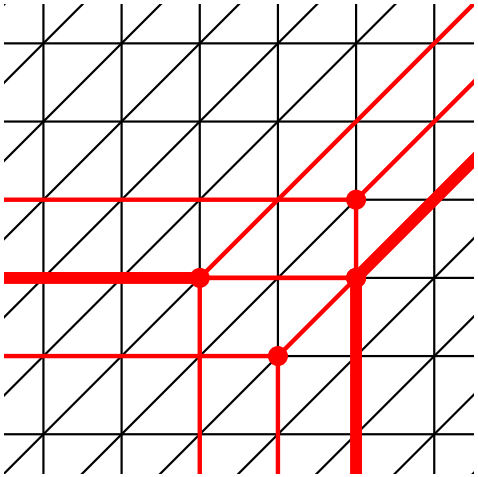

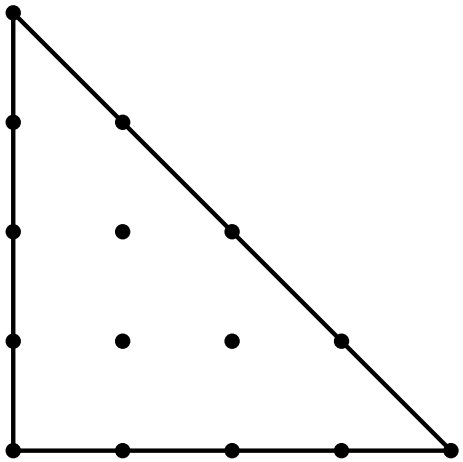

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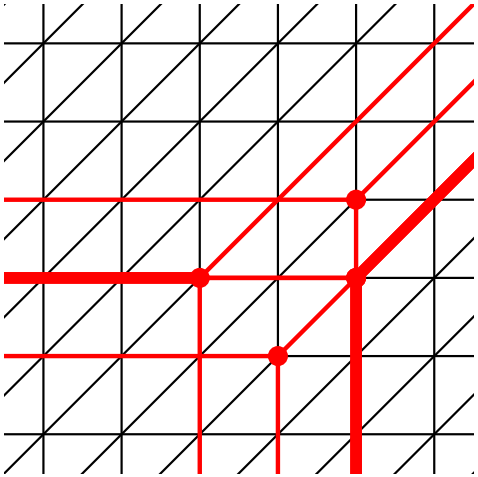

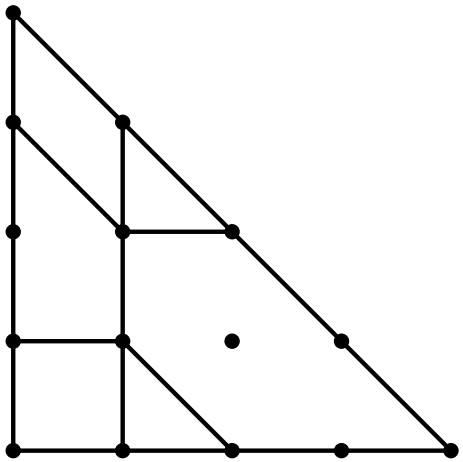

Some Open Questions

Some questions that are still open:

- What happens for vertices not contained in one apartment?
- Which algebro-geometric invariant of the Mustafin variety describes the distance of vertices in the building?
  (This is known for $d = n = 2$.)
- Does it make sense to build the Mustafin variety for an infinite number of vertices?
  (It is known that it does in special cases.)

Further reading:


D. Mumford: *An analytic construction of degenerating curves over complete local rings.* Compos. Math. **24** (1972) pp. 129–174


G. A. Mustafin: *Nonarchimedian uniformization.* Math. USSR Sbornik (1978)


M. Joswig, B. Sturmfels, J. Yu: *Affine buildings and tropical convexity.* Albanian J. of Math. (2007)

Thank you!