Some New Techniques for Explicit Mixed Volume Computation

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Let $P_1, \ldots, P_n$ be n polytopes in $\mathbb{R}^n$ and $\lambda_1, \ldots, \lambda_n$ non-negative real parameters. Then $\mathrm{vol}_n(\lambda_1 P_1 + \ldots + \lambda_n P_n)$ is a homogeneous polynomial of degree n in $\lambda_1, \ldots, \lambda_n$ with non-negative coefficients. The coefficient of $\lambda_1 \cdots \lambda_n$ is called the mixed volume of $P_1, \ldots, P_n$ and is denoted by $\mathrm{MV}_n(P_1, \ldots, P_n)$.

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- Symmetry: $\mathrm{MV}_n(P_1, \ldots, P_n) = \mathrm{MV}_n(P_{\sigma(1)}, \ldots, P_{\sigma(n)})$ for a permutation σ.
- Linearity: $\mathrm{MV}_n(\alpha P_1 + \beta P_1', \ldots, P_n) = \alpha\, \mathrm{MV}_n(P_1, \ldots, P_n) + \beta\, \mathrm{MV}_n(P_1', \ldots, P_n)$
- Generalizes the Volume: $\mathrm{MV}_n(P, \ldots, P) = n!\, \mathrm{vol}_n(P)$
- Invariance under rigid motions: $\mathrm{MV}_n(P_1, \ldots, P_n) = \mathrm{MV}_n(\alpha(P_1), \ldots, \alpha(P_n))$ for any volume preserving α.

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- Choose a linear lifting function $\mu_i : \mathbb{R}^n \to \mathbb{R}$ for each P_i.
- Lift the polytopes P_i to $\hat{P}_i := \{(q, \mu_i(q)) \, : \, q \in P_i\} \subset \mathbb{R}^{n+1}$.
- Compute the lower hull of $\hat{P} := \hat{P}_1 + \ldots + \hat{P}_n$, i.e. those facets of $\hat{P}$ which have an inner normal with positive $(n+1)$-coordinate.
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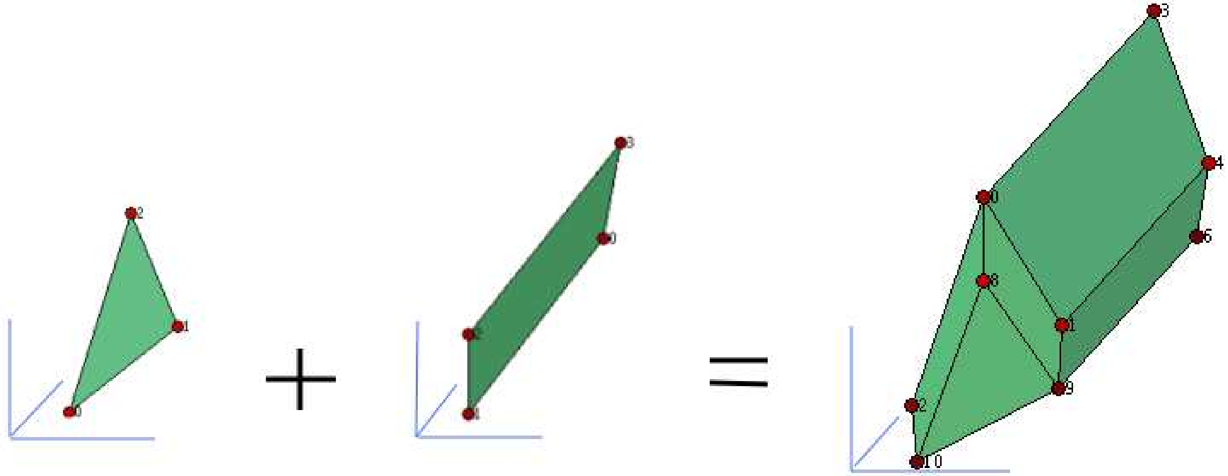
We call such a subdivision coherent and we will say it is induced by $\mu_1, \ldots, \mu_n$.

A subdivision induced by $\mu_1, \ldots, \mu_n$ is mixed if every vertex of the lower hull can be expressed uniquely as a Minkowski sum. Such a set of liftings will be called (sufficiently) generic.

Lifting the rectangle and the triangle by two different linear functions yields the following 3 dimensional objects.



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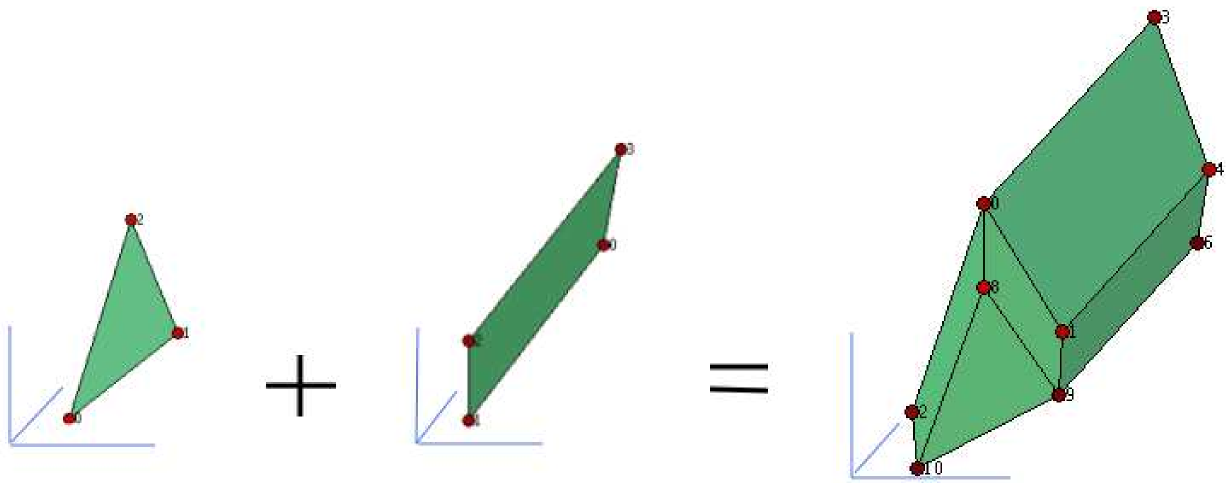
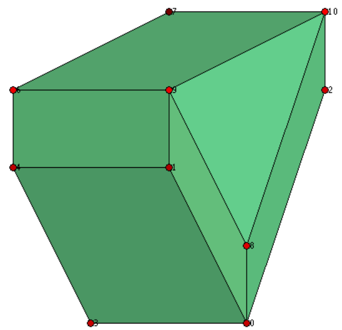


The lower hull of this Minkowski sum looks like this.



New Technical Tools I: The Seperation Lemma

Lemma. *Let* $P_1, \ldots, P_k$ *be polytopes in* $\mathbb{R}^{m+k}$ *and* $Q_1, \ldots, Q_m$ *be polytopes in* $\mathbb{R}^m \subset \mathbb{R}^{m+k}$ *. Then*

$$\mathrm{MV}_{m+k}(Q_1, ..., Q_m, P_1, ..., P_k) = \mathrm{MV}_m(Q_1, ..., Q_m) * \mathrm{MV}_k(\pi(P_1), ..., \pi(P_k))$$

where $\pi : \mathbb{R}^{m+k} \to \mathbb{R}^k$ *denotes the projection on the last* k *coordinates.*

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Proof: Assume that $P_1 = \ldots = P_k = P$ and $Q_1 = \ldots = Q_m = Q$. Then we have to show that

$$\mathrm{MV}_{m+k}(Q, \ldots, Q, P, \ldots, P) = m!\,k! * \mathrm{vol}_m(Q) * \mathrm{vol}_k(\pi(P)) \ .$$

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We define g_1 and g_2 via

$$\begin{aligned} g_1(Q_1, \ldots, Q_m, P_1, \ldots, P_k) &:= \mathrm{MV}_{m+k}(Q_1, \ldots, Q_m, P_1, \ldots, P_k) \\ g_2(Q_1, \ldots, Q_m, P_1, \ldots, P_k) &:= \mathrm{MV}_m(Q_1, ..., Q_m) * \mathrm{MV}_k(\pi(P_1), ..., \pi(P_k)) \ . \end{aligned}$$

Both are invariant under reordering the Q_i or the P_j and linear in each argument.

Let $f : A \times \ldots \times A \to B$ be a symmetric multilinear function, where A and B are semigroups. By expanding the right hand side it can be seen that

$$f(a_1, ..., a_n) = \frac{1}{n!} \sum_{1 \le i_1 < ... < i_q \le n} (-1)^{n-q} f(a_{i_1} + ... + a_{i_q}, ..., a_{i_1} + ... + a_{i_q}) \ .$$

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The functions

$$\begin{aligned} \tilde{g}_i^{(P_1, \ldots, P_k)}(Q_1, \ldots, Q_m) &:= g_i(Q_1, \ldots, Q_m, P_1, \ldots, P_k) \text{ and} \\ \bar{g}_i^{(Q)}(P_1, \ldots, P_k) &:= g_i(Q, \ldots, Q, P_1, \ldots, P_k) \text{ for } i = 1, 2 \end{aligned}$$

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Hence for $i = 1, 2$ we can expand $g_i(Q_1, \ldots, Q_m, P_1, \ldots, P_k)$ twice and we see that both functions g_1 and g_2 are fully determined by their images on tuples of polytopes where $Q_1 = \ldots = Q_m = Q$ and $P_1 = \ldots = P_k = P$. This proves the Lemma.

Remark: Equiv. result probably know: V.P. Fedotov. The sum of p-th surface functions. (Ukrain. Geom. Sb.)

Lemma. *Given polytopes* $P_1, \ldots, P_n \subset \mathbb{R}^n$ *and lifting vectors* $\mu_1, \ldots, \mu_n \in \mathbb{R}^n_{\geq 0}$. *Denote the vertices of* P_i *by* $v_1^{(i)}, \ldots, v_{r_i}^{(i)}$ *and choose one edge* $e_i = \{v_{k_i}^{(i)}, v_{l_i}^{(i)}\}$ *from each* P_i. *Then* $C := e_1 + \ldots + e_n$ *is a mixed cell of the mixed subdivision induced by the liftings* μ_i *if and only if*

i) The edge matrix $E := V_a - V_b$ *is non-singular (where* $V_a := (v_{k_1}^{(1)}, \ldots, v_{k_n}^{(n)})$ *and* $V_b := (v_{l_1}^{(1)}, \ldots, v_{l_n}^{(n)})$*) and*

ii) For all polytopes P_i *and all vertices* $v_s^{(i)}$ *of* P_i *which are not in* e_i *we have:*

$$\left(\operatorname{diag}\left(\mu^T E\right)^T E^{-1} - \mu_i^T\right) \cdot \left(v_{l_i}^{(i)} - v_s^{(i)}\right) \geq 0$$

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Proof (Sketch)**.** Write the condition that C is a mixed cell as a linear program and then use linear programming duality to get explicit conditions for the optimality and hence for C to be a mixed cell.

We demonstrate this with a 2-dimensional example. Which liftings μ_1 and μ_2 induce a subdivision that has the red cell as a mixed cell?

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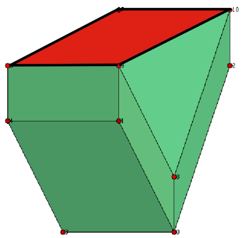
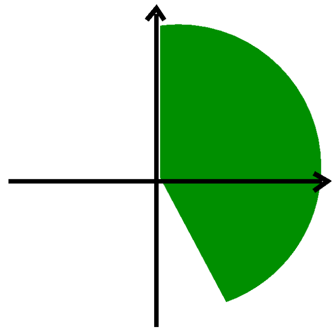


$$C = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\} + \left\{ \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}$$



$$\xrightarrow{Lemma} \begin{pmatrix} 2 \\ 1 \end{pmatrix} (\mu_2 - \mu_1) \geq 0 \text{ and } \begin{pmatrix} 1 \\ 0 \end{pmatrix} (\mu_2 - \mu_1) \geq 0$$

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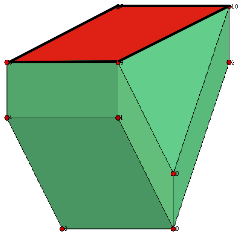

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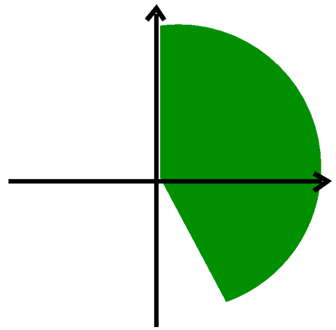

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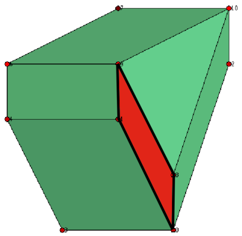

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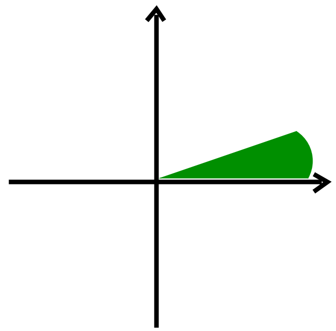

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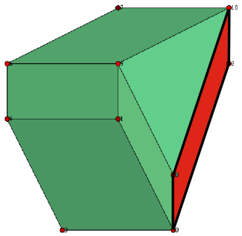
The Complete Fan of Mixed Subdivisions II



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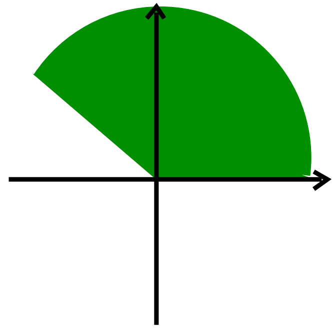

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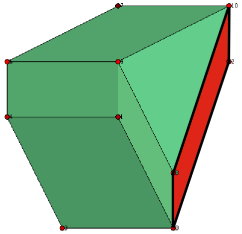

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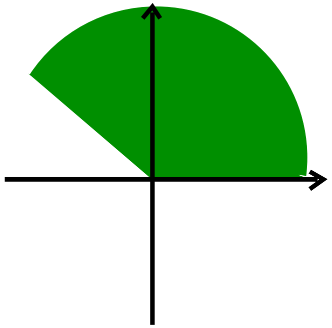

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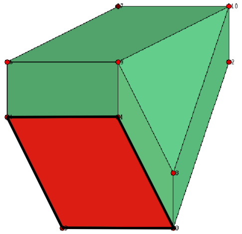

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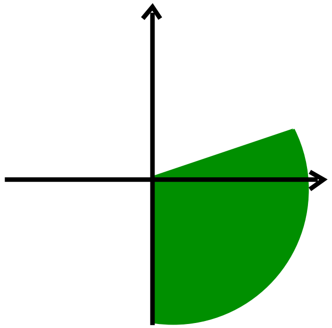

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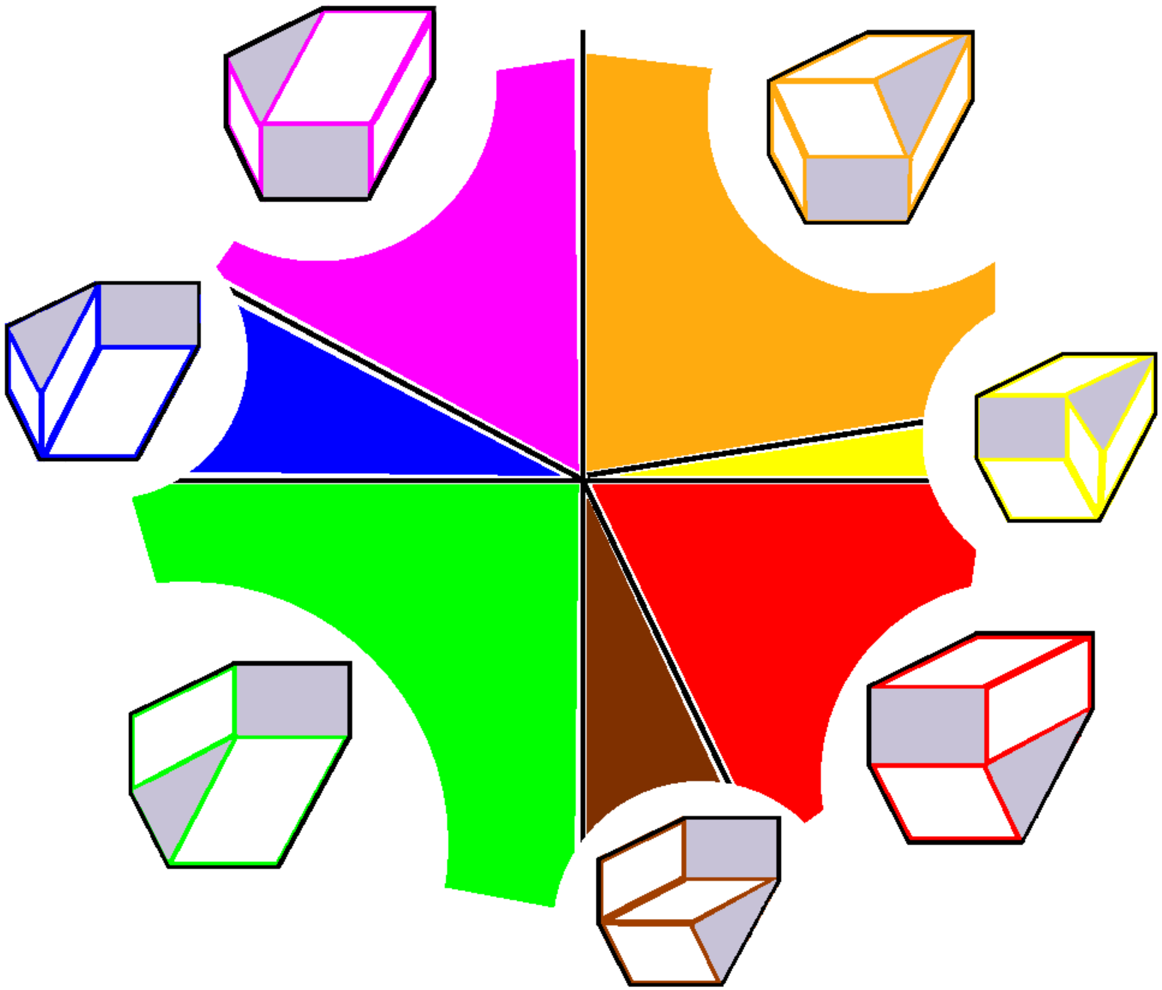

Theorem (Bernstein). *Given polynomials* $f_1, \ldots, f_n$ *over* $\mathbb{C}$ *with finitely many common zeroes in* $(\mathbb{C}^*)^n$, *let* P_i *denote the Newton polytope of* f_i *in* $\mathbb{R}^n$.

Then the number of common zeroes of the f_i *in* $(\mathbb{C}^*)^n$ *is bounded above by the mixed volume* $MV_n(P_1, \ldots, P_n)$.

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Moreover for generic choices of the coefficients in the f_i, the number of common solutions is exactly $MV_n(P_1, \ldots, P_n)$.

- Bernstein also gives explicit conditions when coefficients are generic or not.
- The bound on the number of solutions obtained from the mixed volume is called the BKK-bound due to the work of Bernstein, Kushnirenko and Khonvanskii.
- An upper bound for the common solutions in $\mathbb{C}$ for the system above is $\mathrm{MV}_n(\mathrm{conv}(P_1 \cup 0), \ldots, \mathrm{conv}(P_n \cup 0))$ (Li, Wang).
- The BKK-bound generalizes the Bézout bound.

- A Laman Graph is a graph $G = (V, E)$ with $|E| = 2|V| - 3$ edges where each subset of k vertices spans at most $2k - 3$ edges.
- A framework is a tuple (G, L) where $G = (V, E)$ is a graph and $L = \{l_{i,j} : [v_i, v_j] \in E\}$ is a set of $|E|$ positive numbers interpreted as edge lengths.

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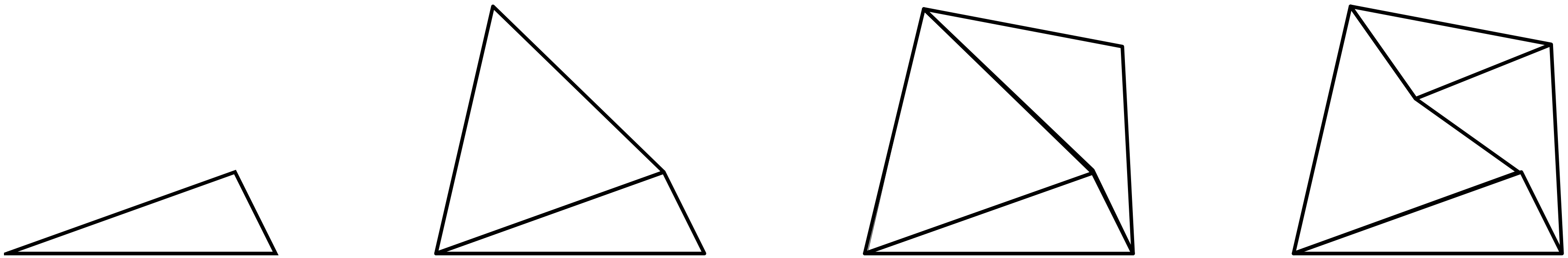

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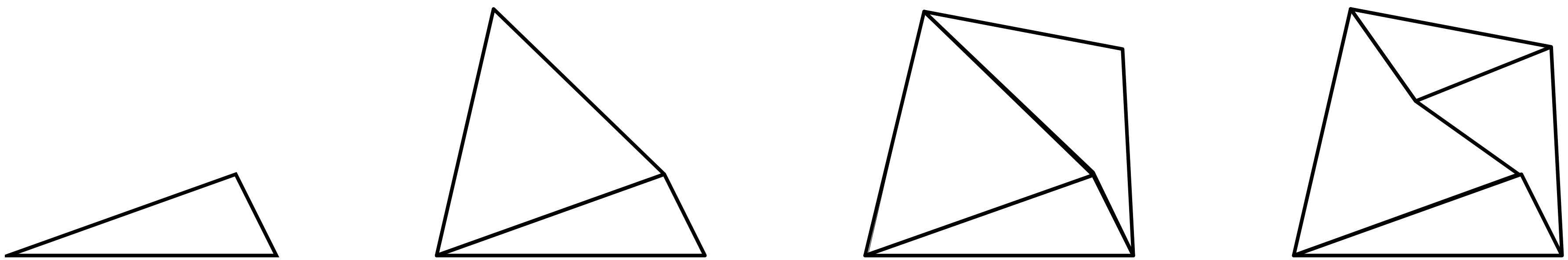

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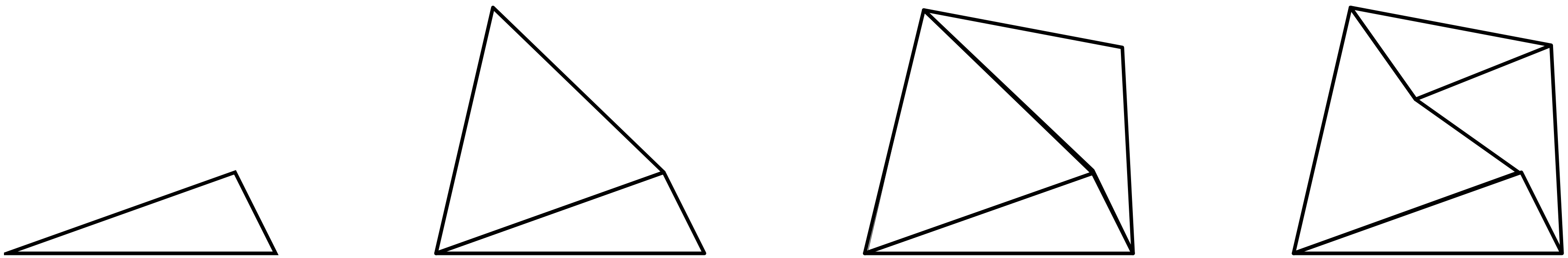

A Henneberg I step adds one new vertex v and two new edges, connecting v to two arbitrary previous vertices .

A Henneberg II step adds one new vertex v and three new edges, connecting v to three previous vertices such that at least two of these vertices are connected via an edge e and this certain edge is removed.

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Each prescribed edge length translates into a polynomial equation. I.e. if $[v_i, v_j] \in E$ with length $L_{i,j}$, we require $(x_i - x_j)^2 + (y_i - y_j)^2 = L_{i,j}^2$ where (x_i, y_i) is the embedding of v_i.

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Hence we want to study the solutions to the following system.

$$\left\{\begin{array}{l} x_1 - c_1 = 0,\ y_1 - c_2 = 0,\ x_2 - (L_{1,2} - c_1) = 0,\ y_2 - c_3 = 0 \\ (x_i - x_j)^2 + (y_i - y_j)^2 - L_{i,j}^2 = 0 \quad \forall [v_i, v_j] \in E - \{[v_1, v_2]\} \end{array}\right\}$$

Theorem. *The mixed volume of our initial system*

$$\left\{\begin{array}{l} x_1 - c_1 = 0, \quad y_1 - c_2 = 0 \\ x_2 - (L_{1,2} - c_1) = 0, \quad y_2 - c_3 = 0 \\ (x_i - x_j)^2 + (y_i - y_j)^2 - L_{i,j}^2 = 0 \\ \qquad \forall [v_i, v_j] \in E - \{[v_1, v_2]\} \end{array}\right\}$$

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Proof (Sketch)**.**

- Bound the mixed volume from above with the product of the degrees.
- Using the second Lemma we find lifting vectors that induce "large" mixed cells such that the volume adds up to this value.

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- Bound the mixed volume from above with the product of the degrees.
- Using the second Lemma we find lifting vectors that induce "large" mixed cells such that the volume adds up to this value.

Corollary. *The number of embeddings of a Laman graph framework with generic edge lengths is strictly less then* $4^{|V|-2}$.

Special Cases

Theorem. *A Henneberg I step at most doubles the number of embeddings of the framework and there is always a choice of edge lengths such that the number of embeddings is doubled.*

Corollary. *The number of embeddings of Henneberg I graphs is less than or equal* $2^{|V|-2}$ *and this bound is sharp.*

The only Henneberg II Laman graphs on 6 vertices are the Desargues graph and $K_{3,3}$.

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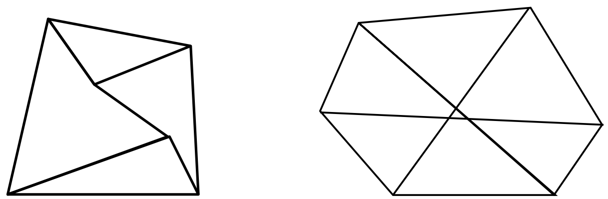

- The Mixed Volume bound is 32. (Compared to 70 from the Borcea-Streinu bound.)
- For the Desargues graph Borcea and Streinu show that the number of embeddings is exactly 24.
- For $K_{3,3}$ Husty showed that there are at least 32 embeddings.

Danke!