

# Mixed Volume Techniques for Embeddings of Laman Graphs

Reinhard Steffens, Thorsten Theobald\*

## Abstract

We use Bernstein's Theorem [1] to obtain combinatorial bounds for the number of embeddings of Laman graph frameworks modulo rigid motions. For this, we study the mixed volume of suitable systems of polynomial equations obtained from the edge length constraints. The bounds can easily be computed and for some classes of graphs, the bounds are tight.

## 1 Introduction

Let  $G = (V, E)$  be a graph with  $|E| = 2|V| - 3$  edges. If each subset of  $k$  vertices spans at most  $2k - 3$  edges, we say that  $G$  has the *Laman property* and call it a *Laman graph* (see [7]). For generic edge lengths, Laman graphs are minimally rigid (see [3]), i.e. they become flexible if any edge is removed.

A *Henneberg sequence* for a graph  $G$  is a sequence  $(G_i)_{3 \leq i \leq n}$  of Laman graphs such that  $G_3$  is a triangle,  $G_n = G$  and each  $G_i$  is obtained by  $G_{i-1}$  via one of the following two types of steps: A *Henneberg I step* adds one new vertex  $v_{i+1}$  and two new edges, connecting  $v_{i+1}$  to two arbitrary vertices of  $G_i$ . A *Henneberg II step* adds one new vertex  $v_{i+1}$  and three new edges, connecting  $v_{i+1}$  to three vertices of  $G_i$  such that at least two of these vertices are connected via an edge  $e$  of  $G_i$  and this certain edge  $e$  is removed (see Figure 1). Any Laman graph  $G$  can be constructed via a

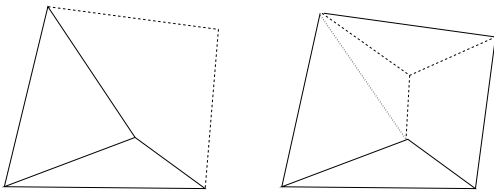


Figure 1: A Henneberg I and a Henneberg II step. New edges are dashed and the deleted edge is pointed.

Henneberg sequence and any graph constructed via a Henneberg sequence has the Laman property (see [9]). We call  $G$  a *Henneberg I graph* if it is constructable using only Henneberg I steps. Otherwise we call it *Henneberg II*.

In the following we look at *frameworks* which are tuples  $(G, L)$  where  $G = (V, E)$  is a graph and

$L = \{l_{i,j} : [v_i, v_j] \in E\}$  is a set of  $|E|$  positive numbers interpreted as edge lengths. Given a framework we want to know how many embeddings, i.e. maps  $\alpha : V \rightarrow \mathbb{R}^2$ , exist such that the Euclidean distance between two points in the image is exactly  $l_{i,j}$  for all  $[v_i, v_j] \in E$ . Since every rotation or translation of an embedding gives another one, we ask how many embeddings exist *modulo rigid motions*.

Due to the minimal rigidity property, questions about embeddings of Laman graphs arise naturally in rigidity and linkage problems (see [2] and the references therein). Graphs with less edges will have zero or infinitely many embeddings modulo rigid motions, and graphs with more edges do not have any embeddings for a generic choice of edge lengths.

Determining the maximal number of embeddings (modulo rigid motions) for a given Laman graph is an open problem. The best upper bounds are due to Borcea and Streinu [2] who show that the number of embeddings is bounded by  $\binom{2|V|-4}{|V|-2}$ . Their bounds are based on degree results of determinantal varieties, but do not seem to fully exploit the specific combinatorial structure of Laman graphs.

Here, we present an alternative, combinatorial approach to bound the number of embeddings of a Laman graph based on Bernstein's theorem for sparse polynomial systems. Since the systems of polynomial equations describing the Laman embeddings are sparse, the mixed volume of the Newton polytopes provides a simple combinatorial upper bound on the number of solutions. It is particularly interesting that for some classes of graphs, the mixed volume bound is tight (and in these cases improves the general bound in [2]).

To use algebraic tools for this problem we formulate the embedding problem as a system of polynomial equations. Each prescribed edge length translates into a polynomial equation. I.e. if  $[v_i, v_j] \in E$  with length  $l_{i,j}$ , we require  $(x_i - x_j)^2 + (y_i - y_j)^2 = l_{i,j}^2$  where  $\alpha(v_i) = (x_i, y_i)$  and  $\alpha(v_j) = (x_j, y_j)$ . Thus we obtain a system of  $|E|$  quadratic equations whose solutions represent the embeddings of our framework. To get rid of translations and rotations we fix one point  $\alpha(v_1) = (x_1, y_1) = (c_1, c_2)$  and the direction of the edge  $[v_1, v_2]$  by setting  $y_2 = c_3$ . (Here we assume without loss of generality that there is an edge between  $v_1$  and  $v_2$ .) For practical reasons we choose  $c_i \neq 0$  and as well  $c_1 \neq l_{1,2}$ . Hence we want to study

\*FB 12 – Institut für Mathematik, Postfach 111932, D-60054 Frankfurt am Main, Germany

the solutions to the following system.

$$\left\{ \begin{array}{l} x_1 - c_1 = 0 \\ y_1 - c_2 = 0 \\ x_2 - (l_{1,2} - c_1) = 0 \\ y_2 - c_3 = 0 \\ (x_i - x_j)^2 + (y_i - y_j)^2 - l_{i,j}^2 = 0 \\ \forall [v_i, v_j] \in E - \{[v_1, v_2]\} \end{array} \right\} \quad (1)$$

We will give bounds on the number of solutions in  $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$  to this system where we assume that the edge lengths  $l_{i,j}$  are generically chosen such that no solutions with zero components occur. To do this we will study the mixed volume of the Newton polytopes (i.e. the convex hulls of the monomial exponent vectors, see for example [8]) of the system (1).

## 2 Bernstein's Theorem and Technical Tools

Let  $P_1, \dots, P_n$  be  $n$  polytopes in  $\mathbb{R}^n$ . For non-negative parameters  $\lambda_1, \dots, \lambda_n$ , the volume  $\text{vol}_n(\lambda_1 P_1 + \dots + \lambda_n P_n)$  is a homogeneous polynomial of degree  $n$  in  $\lambda_1, \dots, \lambda_n$  with non-negative coefficients (see [10]). The coefficient of the monomial  $\lambda_1 \dots \lambda_n$  is called the *mixed volume* of  $P_1, \dots, P_n$  and is denoted by  $MV_n(P_1, \dots, P_n)$ . We have two explicit formulas for this quantity (see [8] and [5]):

$$MV_n(P_1, \dots, P_n) = (-1)^n \sum_{(\alpha_1, \dots, \alpha_n) \in \{0,1\}^n} (-1)^{\sum_i \alpha_i} \text{vol}_n \left( \sum_i \alpha_i P_i \right) \quad (2)$$

$$= \sum_{\substack{Q \text{ mixed cell of a} \\ \text{mixed subdivision} \\ \text{of } P := \sum P_j}} \text{vol}_n(Q) \quad (3)$$

For further background on mixed subdivisions, see also [5] and [4].

The core theorem that gives a connection between solutions to systems of polynomial equations and discrete geometry is the following.

**Theorem 1 (Bernstein [1])** *Given polynomials  $f_1, \dots, f_n$  over  $\mathbb{C}$  with finitely many common zeroes in  $(\mathbb{C}^*)^n$ , let  $P_i$  denote the Newton polytope of  $f_i$  in  $\mathbb{R}^n$ . Then the number of common zeroes of the  $f_i$  in  $(\mathbb{C}^*)^n$  is bounded above by the mixed volume  $MV_n(P_1, \dots, P_n)$ . Moreover for generic choices of the coefficients in the  $f_i$ , the number of common solutions is exactly  $MV_n(P_1, \dots, P_n)$ .*

Bernstein also gives an explicit condition when a choice of coefficients is generic. We can show that the system (1) is never generic in that sense. Then the mixed volume of it will always be a strict upper bound on the number of common solutions.

In the special case of Henneberg I graphs our system (1) will be in a shape that allows to separate the mixed volume calculation into smaller pieces. Our main tool to do this is the following Lemma.

**Lemma 2** *Let  $P_1, \dots, P_k$  be polytopes in  $\mathbb{R}^{m+k}$  and  $Q_1, \dots, Q_m$  be polytopes in  $\mathbb{R}^m \subset \mathbb{R}^{m+k}$ . Then*

$$MV_{m+k}(Q_1, \dots, Q_m, P_1, \dots, P_k) = MV_m(Q_1, \dots, Q_m) * MV_k(\pi(P_1), \dots, \pi(P_k))$$

where  $\pi : \mathbb{R}^{m+k} \rightarrow \mathbb{R}^k$  denotes the projection on the last  $k$  coordinates.

**Proof.** Using the explicit formula (2) we have:

$$\begin{aligned} MV_{m+k}(Q_1, \dots, Q_m, P_1, \dots, P_k) &= (-1)^{m+k} \sum_{\beta \in \{0,1\}^k} \sum_{\alpha \in \{0,1\}^m} (-1)^{\sum_i \alpha_i} (-1)^{\sum_j \beta_j} \\ &\quad \text{vol}_{m+k} \left( \sum_{i=1}^m \alpha_i Q_i + \sum_{j=1}^k \beta_j P_j \right). \end{aligned}$$

Since any polytopes  $P \subset \mathbb{R}^{m+k}$  and  $Q \subset \mathbb{R}^m$  satisfy  $\text{vol}_{m+k}(Q + P) = \text{vol}_m(Q) \text{vol}_k(\pi(P)) + \text{vol}_{m+k}(P)$ , this equals

$$\begin{aligned} &(-1)^{m+k} \sum_{\beta \in \{0,1\}^k} \sum_{\alpha \in \{0,1\}^m} (-1)^{\sum_i \alpha_i} \sum_j \beta_j \\ &\quad * \left[ \text{vol}_m \left( \sum_{i=1}^m \alpha_i Q_i \right) \text{vol}_k \left( \pi \left( \sum_{j=1}^k \beta_j P_j \right) \right) \right. \\ &\quad \left. + \text{vol}_{m+k} \left( \sum_{j=1}^k \beta_j P_j \right) \right]. \end{aligned}$$

Using that  $\pi(P_1 + P_2) = \pi(P_1) + \pi(P_2)$  for any polytopes  $P_1, P_2 \subset \mathbb{R}^{m+k}$ , we obtain

$$\begin{aligned} &(-1)^{m+k} \sum_{\alpha \in \{0,1\}^m} (-1)^{\sum_i \alpha_i} \left[ \sum_{\beta \in \{0,1\}^k} (-1)^{\sum_j \beta_j} \right. \\ &\quad \left. * \text{vol}_{m+k} \left( \sum_{j=1}^k \beta_j P_j \right) \right] \\ &+ (-1)^m \sum_{\alpha \in \{0,1\}^m} (-1)^{\sum_i \alpha_i} \text{vol}_m \left( \sum_{i=1}^m \alpha_i Q_i \right) \\ &\quad * \left[ (-1)^k \sum_{\beta \in \{0,1\}^k} (-1)^{\sum_j \beta_j} \text{vol}_k \left( \pi \left( \sum_{j=1}^k \beta_j P_j \right) \right) \right]. \end{aligned}$$

Now the first two lines equal 0 because we just add and subtract  $2^{m-1}$  times the term in square brackets, the third line is  $MV_m(Q_1, \dots, Q_m)$  and finally the last line equals  $MV_k(\pi(P_1), \dots, \pi(P_k))$  according to our alternating formula for the mixed volume (2).  $\square$

Another technical tool which will be needed in a subsequent proof is the following Lemma. This goes back to an idea of Emiris and Verschelde [4] to use linear programming and the formula (3) to compute the mixed volume. The proof (which we do not give here) is based on the duality theorem for linear programming.

**Lemma 3** *Given polytopes  $P_1, \dots, P_n \subset \mathbb{R}^n$  and lifting vectors  $\mu_1, \dots, \mu_n \in \mathbb{R}_{\geq 0}^n$ . Denote the vertices of  $P_i$  by  $v_1^{(i)}, \dots, v_{m_i}^{(i)}$  and choose one edge  $e_i = [v_{k_i}^{(i)}, v_{l_i}^{(i)}]$  from each  $P_i$ . Then  $\sum_{i=1}^n e_i$  is a mixed cell of the mixed subdivision induced by the liftings  $\mu_i$  if and only if*

i) *The edge matrix  $E := V_a - V_b$  is non-singular (where  $V_a := (v_{k_1}^{(1)}, \dots, v_{k_n}^{(n)})$  and  $V_b := (v_{l_1}^{(1)}, \dots, v_{l_n}^{(n)})$ ) and*

ii) *For all polytopes  $P_i$  and all vertices  $v_s^{(i)}$  of  $P_i$  which are not in  $e_i$  we have:*

$$\left( \text{diag}(\mu^T E)^T E^{-1} - \mu_i^T \right) \cdot (v_{l_i}^{(i)} - v_s^{(i)}) \geq 0 \quad (4)$$

where  $\mu := (\mu_1, \dots, \mu_n)$  and where  $\text{diag}(V)$  denotes the vector of the diagonal entries of  $V$ .

Note that (4) is linear in the  $\mu_j$ . Hence given a choice of edges we can explicitly calculate  $\sum_{i=1}^n m_i$  normal vectors defining a cone in  $\mathbb{R}^{n^2}$ . The interior of this cone consists of all liftings  $(\mu_1^t, \dots, \mu_n^t)$  which induce a mixed subdivision that contains our chosen cell as a mixed cell.

### 3 Henneberg I graphs

For this simple class of Laman graphs the mixed volume bound is tight as we will demonstrate below. Our proof exploits the inductive structure of Henneberg I graphs which is why it cannot be used for Henneberg II graphs.

**Theorem 4** *A Henneberg I step at most doubles the number of embeddings of the framework and there is always a choice of edge lengths such that the number of embeddings is doubled.*

**Proof.** In a Henneberg I step we add one vertex  $v_{|V|+1}$  and two edges  $[v_r, v_{|V|+1}], [v_s, v_{|V|+1}]$  with lengths  $l_{r,|V|+1}$  and  $l_{s,|V|+1}$ . So our system of equations (1) gets two new equations, namely

$$(x_r - x_{|V|+1})^2 + (y_r - y_{|V|+1})^2 - l_{r,|V|+1}^2 = 0 \quad (5)$$

$$(x_s - x_{|V|+1})^2 + (y_s - y_{|V|+1})^2 - l_{s,|V|+1}^2 = 0. \quad (6)$$

In our new system of equations these two are the only polynomials involving  $x_{|V|+1}$  and  $y_{|V|+1}$ , so we can

use Lemma 2 to calculate the mixed volume separately. Unfortunately, the mixed volume of the projection of the Newton polytopes of these equations equals 4 which would imply that the number of embeddings is at most quadrupled. But the following simple trick (which we will refer to as the *truncation trick*) solves this problem immediately. The set of solutions of a system of polynomial equations is not changed when we subtract one equation from another. So instead of adding equation (6) we add the equation (6)-(5) which equals

$$x_s^2 - x_r^2 + 2x_{|V|+1}(x_r - x_s) + y_s^2 - y_r^2 + 2y_{|V|+1}(y_r - y_s) - l_{s,|V|+1}^2 + l_{r,|V|+1}^2 = 0. \quad (7)$$

Now the projections of the two new Newton polytopes corresponding to (5) and (7) to their last two coordinates have mixed volume 2 which proves the first part of our theorem. To get two new embeddings for each previous one we choose our new edge lengths to be almost equal to each other and much larger than all previous edges lengths (larger than the sum of all previous is certainly enough). This leads to the desired new embeddings.  $\square$

Each Henneberg sequence starts with a triangle which has obviously at most 2 embeddings up to rigid motions (we count reflections separately). Hence using our Theorem inductively we get the following corollary.

**Corollary 5** *The number of embeddings of Henneberg I graphs is less than or equal  $2^{|V|-2}$  and this bound is sharp.*

### 4 Laman graphs on 6 Vertices

For Laman graphs on 6 vertices, the general bound in [2] on the number of embeddings is 70. From the Henneberg constructions and simple combinatorial considerations, it follows that the only Henneberg II Laman graphs on 6 vertices are the Desargues graph and  $K_{3,3}$  (see figure 2). For the Desargues graph, an

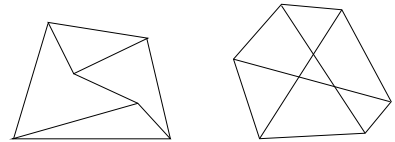


Figure 2: Left: Desargues graph. Right:  $K_{3,3}$ .

explicit analysis is given in [2] which shows that the correct number is only 24, and that there is a choice of edge lengths giving 24 different embeddings. For the  $K_{3,3}$ , Manfred Husty found a construction with 32 embeddings [6].

When we set up the system (1) and use the truncation trick like in the proof of Theorem 4 several times, our mixed volume approach yields a bound of 32 for both graph classes on 6 vertices. So in the case of 6 vertices our bound is tight. By glueing several copies of  $K_{3,3}$  together and using Lemma 2 to calculate the mixed volume we get an infinite class of graphs where our bound is tight as well.

## 5 General Case

For the classes discussed above (Henneberg I, graphs on six vertices) as well as some other special cases, our bound on the number of embeddings improves the known general bounds. We were not able to generalize the truncation trick to arbitrary Henneberg II graphs. For the general case, our mixed volume approach for the untruncated system (1) provides a simple, but very weak bound. However, it may be of independent interest, that for this class of problems, it is possible to determine the mixed volume exactly.

**Theorem 6** *The mixed volume of our initial system (1) is exactly  $4^{|V|-2}$ .*

**Proof.** The mixed volume of (1) is at most the product of the degrees of the polynomial equations because it is less than or equal to the Bézout bound (see [8]). To show that the mixed volume is at least this number we will use Lemma 3 to give a lifting that induces a mixed cell of volume  $4^{|V|-2}$ .

The first 4 equations of (1) give rise to a single edge as a Newton polytope which is part of any mixed cell. Now we claim that we can order the Newton polytopes  $P_i$  in such a way that, for  $i \geq 5$ ,  $P_i$  contains the edge  $[0, 2\xi_i]$  where  $\xi_i$  denotes the  $i^{\text{th}}$  unit vector. To see this, note first that every equation in (1) has a non vanishing constant term and therefore its Newton polytope contains the point 0. To see that  $P_i$  contains  $2\xi_i$  it is enough to show there is a labeling of the edges of our graph with a direction such that each vertex has exactly two incoming edges. Figure 3 sketches how to choose the edge di-

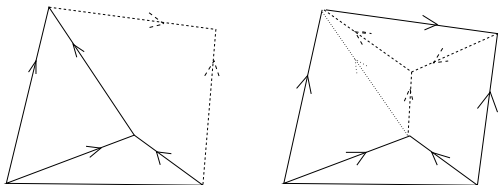


Figure 3: A Henneberg I and a Henneberg II step with directed edges.

rections in the Henneberg steps to satisfy this. Now using Lemma 3 we describe a lifting that induces a subdivision that has  $\sum_i [0, 2\xi_i]$  as a mixed cell. In the

notation of Lemma 3 our chosen edges give rise to the edge matrix  $E = \begin{pmatrix} \mathbb{E}_4 & \mathbf{0} \\ \mathbf{0} & 2\mathbb{E}_{2^{|V|-4}} \end{pmatrix}$ . Substituting this into the second condition (4) we get that for each Newton polytope  $P_i$  all vertices  $v_s^{(i)}$  of  $P_i$  which are not 0 or  $2\xi_i$  have to satisfy

$$\left( (\mu_1^{(1)}, \dots, \mu_{2^{|V|}}^{(2^{|V|})}) - \mu^{(i)} \right) \cdot v_s^{(i)} \leq 0,$$

where we denote by  $\mu^{(j)} \in \mathbb{Q}^{2^{|V|}}$  the lifting vector for  $P_j$ . Since all the entries of each  $v_s^{(i)}$  are non-negative this can easily be done by choosing the vectors  $\mu^{(j)}$  such that their  $j^{\text{th}}$  entry is relatively small and all other entries are relatively large.  $\square$

**Corollary 7** *The number of embeddings of a Laman graph framework with generic edge lengths is strictly less than  $4^{|V|-2}$ .*

## References

- [1] D.N. Bernstein. The number of roots of a system of equations. *Funkcional. Anal. i Priložen.*,9(3):1-4, 1975.
- [2] C. Borcea and I. Streinu. The number of embeddings of minimally rigid graphs, *Discrete Comput. Geom.*, 31(2):287-303, 2004.
- [3] R. Connelly. Rigidity. In *Handbook of Convex Geometry, Vol. A*, pages 223-271. North Holland, Amsterdam, 1993.
- [4] I.Z. Emiris and J. Verschelde. How to count efficiently all affine roots of a polynomial system. *Discrete Appl. Math.*, 93(1):21-32, 1999.
- [5] B. Huber and B. Sturmfels. A polyhedral method for solving sparse polynomial systems. *Math. Comp.*, 64(212):1541-1555, 1995.
- [6] M. Husty. Talk given at the IMA workshop 'Applications in Biology, Dynamics and Statistics', May 2007.
- [7] G. Laman. On graphs and rigidity of plane skeletal structures. *J. Engrg. Math.*, 4:331-340, 1970.
- [8] B. Sturmfels. *Solving systems of polynomial equations*, volume 97 of *CBMS Regional Conference Series in Mathematics*, 2002.
- [9] T.-S. Tay and Walter Whiteley. Generating isostatic graphs. *Structural Topology*, (11):21-69, 1985.
- [10] R. Webster. *Convexity*, Oxford University Press, New York, 1994.