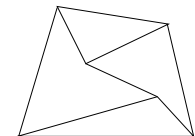

Mixed Volume Techniques for Embeddings of Laman Graphs

Reinhard Steffens, Thorsten Theobald

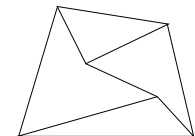


Nancy, March 18th 2008



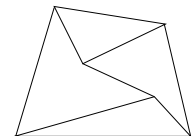
Laman Graphs

- A **Laman Graph** is a graph $G = (V, E)$ with $|E| = 2|V| - 3$ edges where each subset of k vertices spans at most $2k - 3$ edges.
- A **framework** is a tuple (G, L) where $G = (V, E)$ is a graph and $L = \{l_{i,j} : [v_i, v_j] \in E\}$ is a set of $|E|$ positive numbers interpreted as edge lengths.



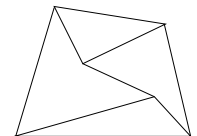
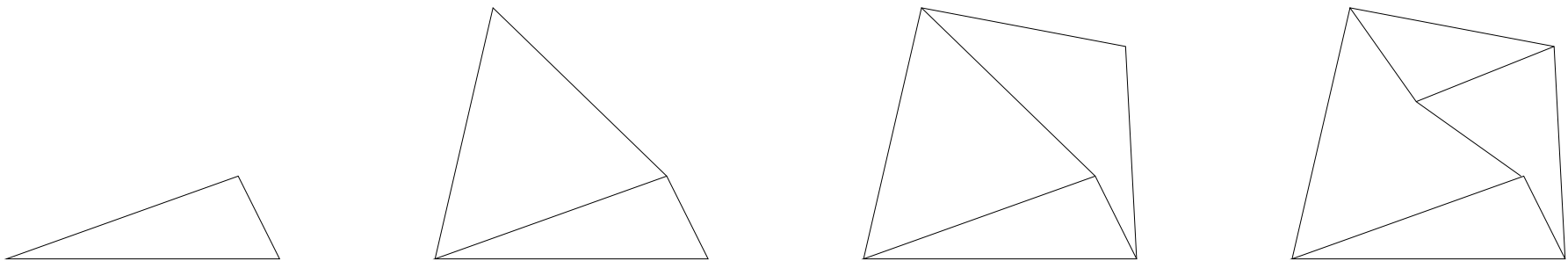
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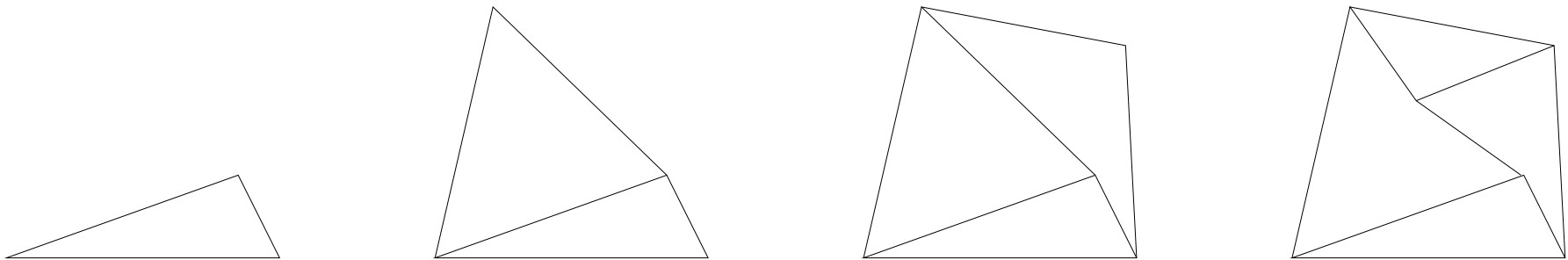
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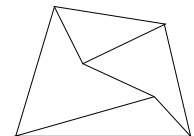


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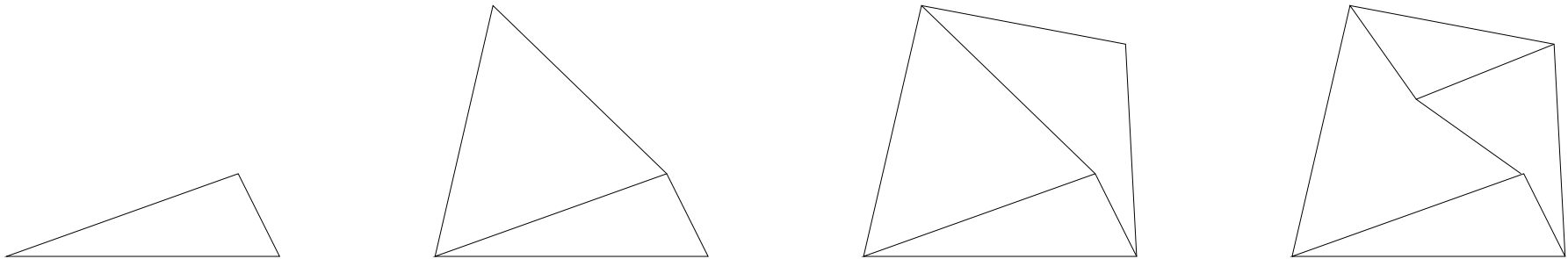


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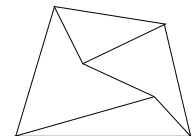
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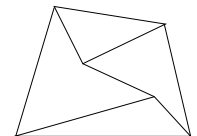
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A **Henneberg II step** adds one new vertex v and three new edges, connecting v to three previous vertices such that at least two of these vertices are connected via an edge e and this certain edge is removed.



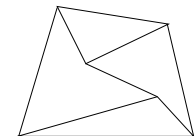
The Problem and its History

Question: Given a Laman graph on $|V|$ vertices with generic edge lengths, how many embeddings does it have modulo rigid motions (translations and rotation)?



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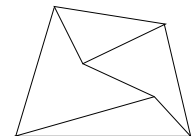
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- Describe point configurations by their squared pairwise distances.
- Valid distance sets lie on a complex projective variety (the Cayley-Menger variety).
- The Hyperplane obtained by the prescribed (generic) distances cuts this variety in $\frac{1}{2} \binom{2|V|-4}{|V|-2}$ points.

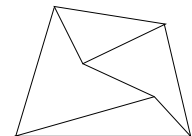


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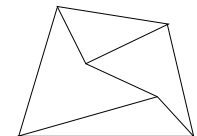
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We have an upper bound of order $4^{|V|}$ and a lower bound of order $2.28^{|V|}$.



Formulation as a System of Equations

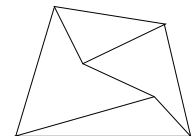
Each prescribed edge length translates into a polynomial equation. I.e. if $[v_i, v_j] \in E$ with length $L_{i,j}$, we require $(x_i - x_j)^2 + (y_i - y_j)^2 = L_{i,j}^2$ where (x_i, y_i) is the embedding of v_i .



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To get rid of translations and rotations we fix one point $(x_1, y_1) = (c_1, c_2)$ and the direction of the edge $[v_1, v_2]$ by setting $y_2 = c_3$.



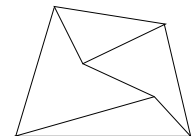
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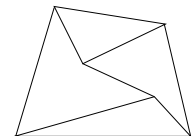
Hence we want to study the solutions to the following system.

$$\left\{ \begin{array}{l} x_1 - c_1 = 0 \\ y_1 - c_2 = 0 \\ x_2 - (L_{1,2} - c_1) = 0 \\ y_2 - c_3 = 0 \\ (x_i - x_j)^2 + (y_i - y_j)^2 - L_{i,j}^2 = 0 \\ \forall [v_i, v_j] \in E - \{[v_1, v_2]\} \end{array} \right. \quad (1)$$



Mixed Volumes and Bernstein's Theorem

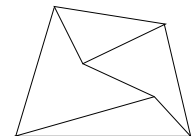
Let P_1, \dots, P_n be n polytopes in \mathbb{R}^n . For non-negative parameters $\lambda_1, \dots, \lambda_n$, the volume $\text{vol}_n(\lambda_1 P_1 + \dots + \lambda_n P_n)$ is a homogeneous polynomial of degree n in $\lambda_1, \dots, \lambda_n$ with non-negative coefficients. The coefficient of the monomial $\lambda_1 \cdots \lambda_n$ is called the *mixed volume* of P_1, \dots, P_n and is denoted by $MV_n(P_1, \dots, P_n)$.



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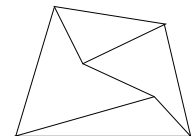
$$MV_n(P_1, \dots, P_n) = \sum_{\substack{Q \text{ mixed cell of a} \\ \text{mixed subdivision} \\ \text{of } P := \sum P_j}} \text{vol}_n(Q)$$



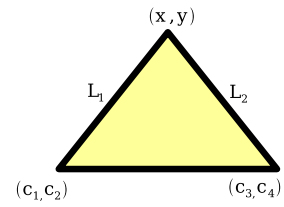
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Theorem (Bernstein '75). *Given polynomials f_1, \dots, f_n over \mathbb{C} with finitely many common zeroes in $(\mathbb{C}^*)^n$, let P_i denote the Newton polytope of f_i in \mathbb{R}^n . Then the number of common zeroes of the f_i in $(\mathbb{C}^*)^n$ is bounded above by the mixed volume $MV_n(P_1, \dots, P_n)$. Moreover for generic choices of the coefficients in the f_i , the number of common solutions is exactly $MV_n(P_1, \dots, P_n)$.*

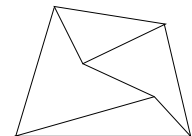


An Example

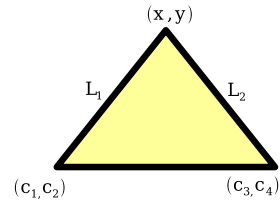


$$f_1(x, y) = (x - c_1)^2 + (y - c_2)^2 - L_1^2 = 0$$

$$f_2(x, y) = (x - c_3)^2 + (y - c_4)^2 - L_2^2 = 0$$

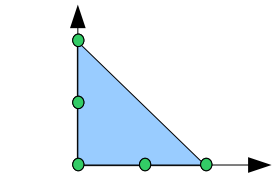


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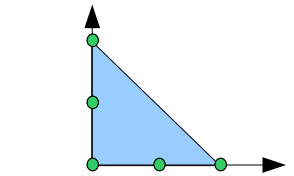


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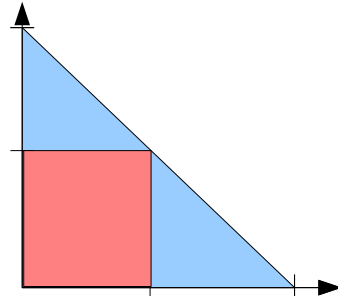
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The Newton Polytope of the first equation.

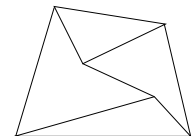


The Newton Polytope of the second equation.

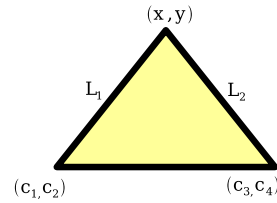


A mixed subdivision of their Minkowski sum.

The only mixed cell in this subdivision is the red square and this has volume 4.

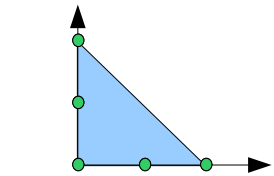


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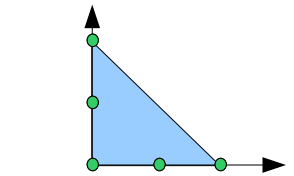


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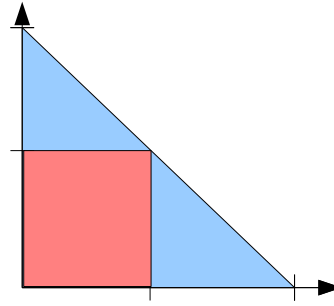
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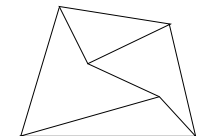
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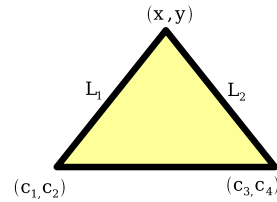
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To repair this defect we use the equation $f_2 - f_1 = 0$ instead of $f_2 = 0$.

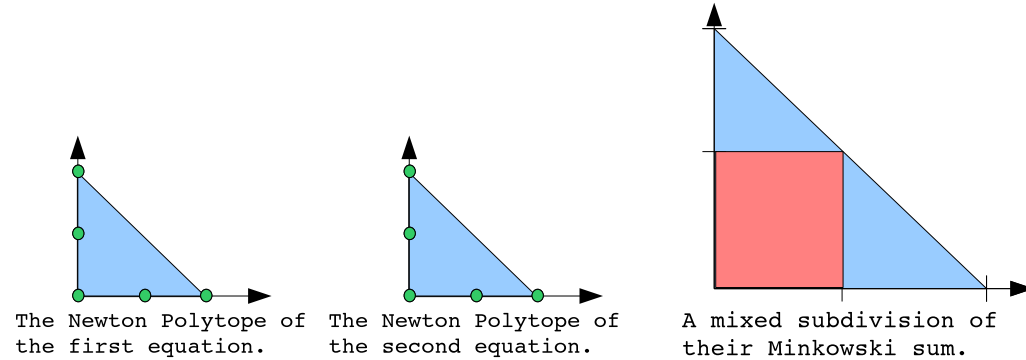


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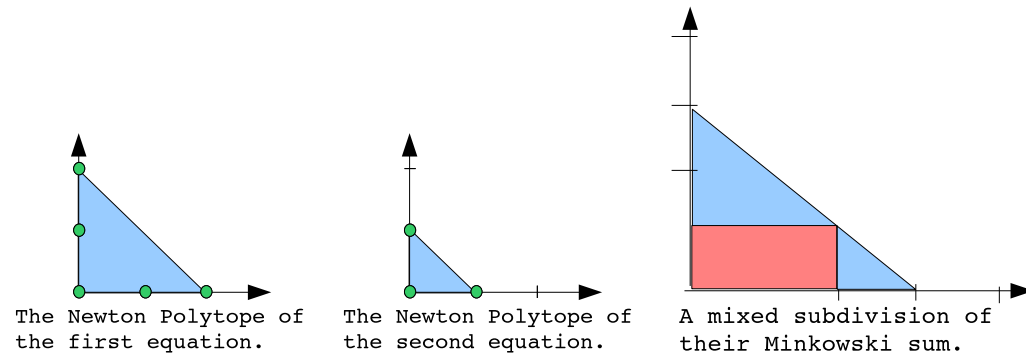
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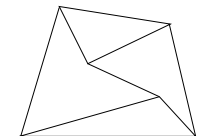


The only mixed cell in this subdivision is the red square and this has volume 4.

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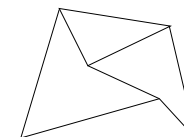
The only mixed cell in this subdivision is the red rectangle and this has volume 2.



Lemma. *Let P_1, \dots, P_k be polytopes in \mathbb{R}^{m+k} and Q_1, \dots, Q_m be polytopes in $\mathbb{R}^m \subset \mathbb{R}^{m+k}$. Then*

$$\text{MV}_{m+k}(Q_1, \dots, Q_m, P_1, \dots, P_k) = \text{MV}_m(Q_1, \dots, Q_m) * \text{MV}_k(\pi(P_1), \dots, \pi(P_k)) ,$$

where $\pi : \mathbb{R}^{m+k} \rightarrow \mathbb{R}^k$ denotes the projection on the last k coordinates.



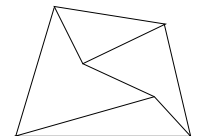
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Proof (Sketch for the case of integer polytopes). • Choose generic polynomials f_1, \dots, f_m with Newton polytopes Q_1, \dots, Q_m and g_1, \dots, g_k with Newton polytopes P_1, \dots, P_k .

- The system $f_1 = \dots = f_m = g_1 = \dots = g_k = 0$ has $\text{MV}_{m+k}(Q_1, \dots, Q_m, P_1, \dots, P_k)$ solutions in $(\mathbb{C}^*)^{m+k}$.



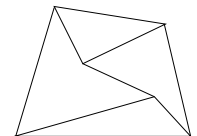
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- On the other hand the system $f_1 = \dots = f_m = 0$ has $\text{MV}_m(Q_1, \dots, Q_m)$ solutions in $(\mathbb{C}^*)^m$.
- Each solution plugged into the g_i 's gives a system with Newton polytopes $\pi(P_1), \dots, \pi(P_k)$ which has $\text{MV}_k(\pi(P_1), \dots, \pi(P_k))$ solutions in $(\mathbb{C}^*)^k$.



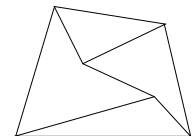
Lemma. *Given polytopes $P_1, \dots, P_n \subset \mathbb{R}^n$ and lifting vectors $\mu_1, \dots, \mu_n \in \mathbb{R}_{\geq 0}^n$. Denote the vertices of P_i by $v_1^{(i)}, \dots, v_{m_i}^{(i)}$ and choose one edge $e_i = [v_{k_i}^{(i)}, v_{l_i}^{(i)}]$ from each P_i . Then $\sum_{i=1}^n e_i$ is a mixed cell of the mixed subdivision induced by the liftings μ_i if and only if*

i) The edge matrix $E := V_a - V_b$ is non-singular (where $V_a := (v_{k_1}^{(1)}, \dots, v_{k_n}^{(n)})$ and $V_b := (v_{l_1}^{(1)}, \dots, v_{l_n}^{(n)})$) and

ii) For all polytopes P_i and all vertices $v_s^{(i)}$ of P_i which are not in e_i we have:

$$\left(\text{diag}(\mu^T E)^T E^{-1} - \mu_i^T \right) \cdot \left(v_{l_i}^{(i)} - v_s^{(i)} \right) \geq 0$$

where $\mu := (\mu_1, \dots, \mu_n)$ and where $\text{diag}(V)$ denotes the vector of the diagonal entries of V .



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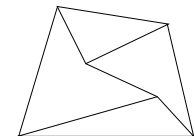
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ii) For all polytopes P_i and all vertices $v_s^{(i)}$ of P_i which are not in e_i we have:

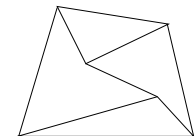
$$\left(\text{diag}(\mu^T E)^T E^{-1} - \mu_i^T \right) \cdot \left(v_{l_i}^{(i)} - v_s^{(i)} \right) \geq 0$$

where $\mu := (\mu_1, \dots, \mu_n)$ and where $\text{diag}(V)$ denotes the vector of the diagonal entries of V .

Proof (Sketch). Write the condition that a sum of edges is a mixed cell as a linear program and then use linear programming duality to get explicit conditions for the optimality and hence for the sum to be a mixed cell.



Theorem. *A Henneberg I step **at most doubles** the number of embeddings of the framework and there is always a choice of edge lengths such that the number of embeddings is doubled.*

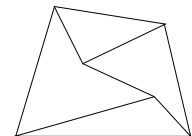


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Proof. In a Henneberg I step we add two new equations, namely

$$(x_r - x_{|V|+1})^2 + (y_r - y_{|V|+1})^2 - L_{r,|V|+1}^2 = 0 \quad (2)$$

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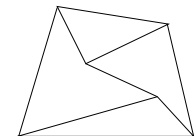
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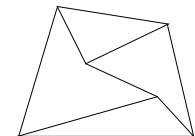
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To get two new embeddings for each previous one we choose our new edge lengths to be "large" and almost equal to each other. □



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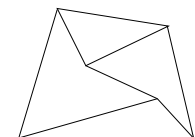
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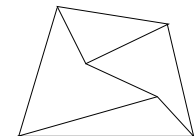
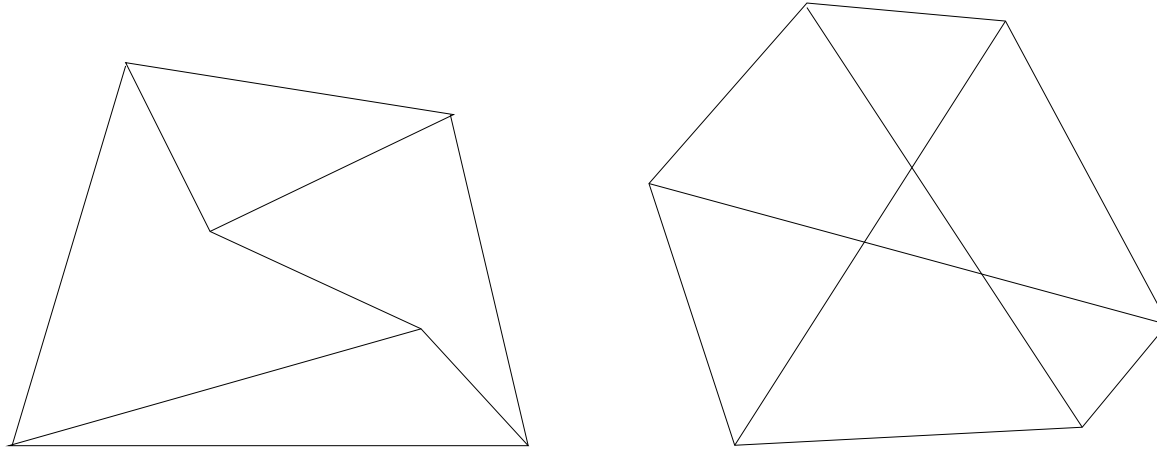
To get two new embeddings for each previous one we choose our new edge lengths to be "large" and almost equal to each other. □

Corollary. *The number of embeddings of Henneberg I graphs is less than or equal $2^{|V|-2}$ and this bound is sharp.*



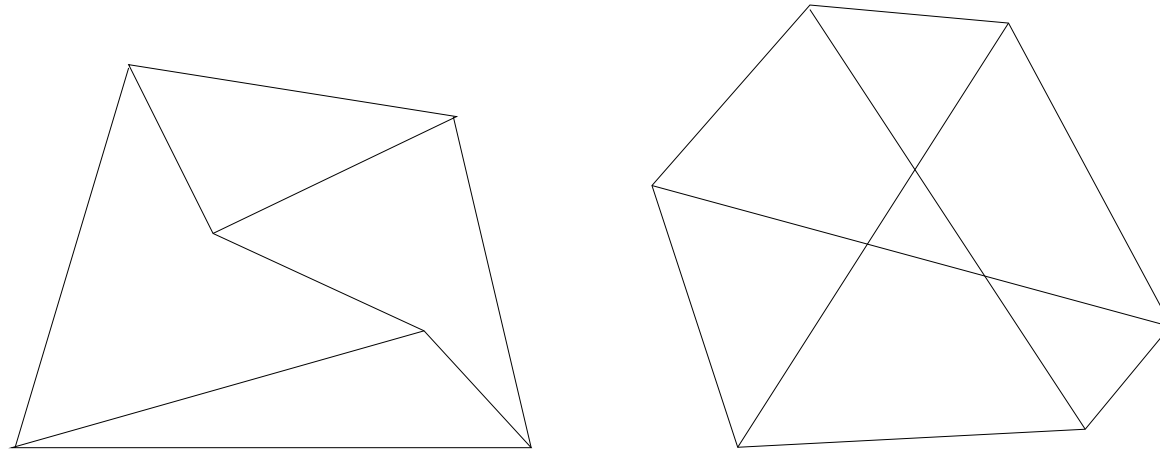
Laman Graphs an 6 Vertices

The only Henneberg II Laman graphs on 6 vertices are the Desargues graph and $K_{3,3}$.

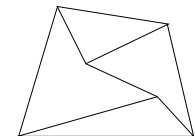


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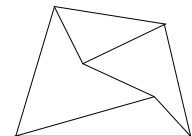
- The Mixed Volume bound is **32**. (Compared to 70 from the Borcea-Streinu bound.)
- For the Desargues graph Borcea and Streinu show that the number of embeddings is exactly 24.
- For $K_{3,3}$ Husty showed that there are at least 32 embeddings.



Theorem. *The mixed volume of our initial system*

$$\left\{ \begin{array}{l} x_1 - c_1 = 0, \quad y_1 - c_2 = 0 \\ x_2 - (L_{1,2} - c_1) = 0, \quad y_2 - c_3 = 0 \\ (x_i - x_j)^2 + (y_i - y_j)^2 - L_{i,j}^2 = 0 \\ \forall [v_i, v_j] \in E - \{[v_1, v_2]\} \end{array} \right\}$$

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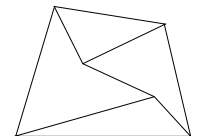


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 - Using the second Lemma we find lifting vectors that induce "large" mixed cells such that the volume adds up to this value.



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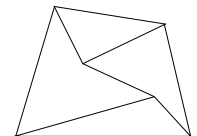
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- Using the second Lemma we find lifting vectors that induce "large" mixed cells such that the volume adds up to this value.

Corollary. *The number of embeddings of a Laman graph framework with generic edge lengths is strictly less than $4^{|V|-2}$.*



The End

Merci Beaucoup!

