Linkage Problems and Real Algebraic Geometry

Reinhard Steffens
Thorsten Theobald

JOHANN WOLFGANG GOETHE
UNIVERSITÄT
FRANKFURT AM MAIN

Frankfurt, Germany

February 2007

- **Framework:** A *framework* is a graph $G = (V, E)$ together with a set $L = \{L_{i\,j} : ij \in E\}$ of non-negative real numbers $L_{i\,j}$ interpreted as edge lengths.

- **Linkage:** A *linkage* is a framework whose underlying graph is a simple polygonal path.

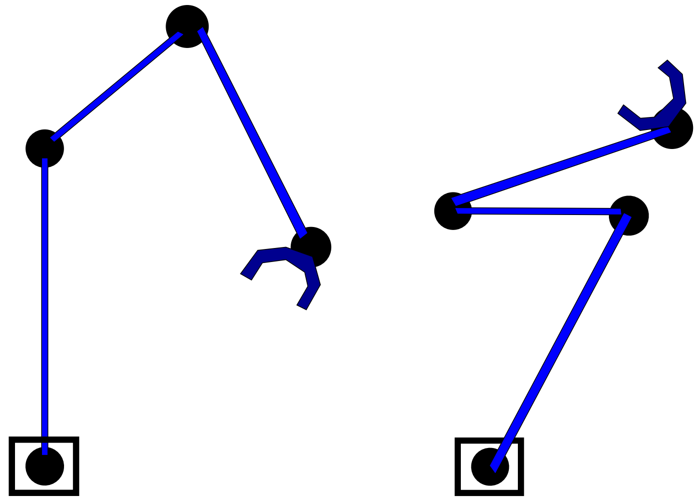
- **Embedding:** An *embedding* $G(P)$ of (G, L) in $\mathbb{R}^d$ (here we consider only $d = 2$) is given by a map $\alpha : V \to \mathbb{R}^d$ such that $L_{i\,j}$ equals the Euclidean distance between the two points $\alpha(v_i) = p_i$ and $\alpha(v_j) = p_j$.

- **Laman Graph:** Let G be a graph with n vertices and $m = 2n - 3$ edges. If each subset of k vertices spans at most $2k - 3$ edges, we say that G has the *Laman property* and call it a *Laman graph*.

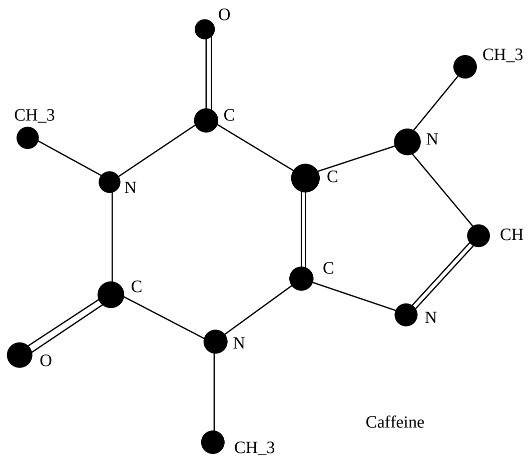
- **1DOF Mechanism:** A one-degree-of-freedom (1DOF) mechanism is a framework whose configuration space is a one-dimensional curve.

These concepts appear naturally in the following scenarios.

- Robot Kinematics
  - Planar non-colliding robot arm motion planning

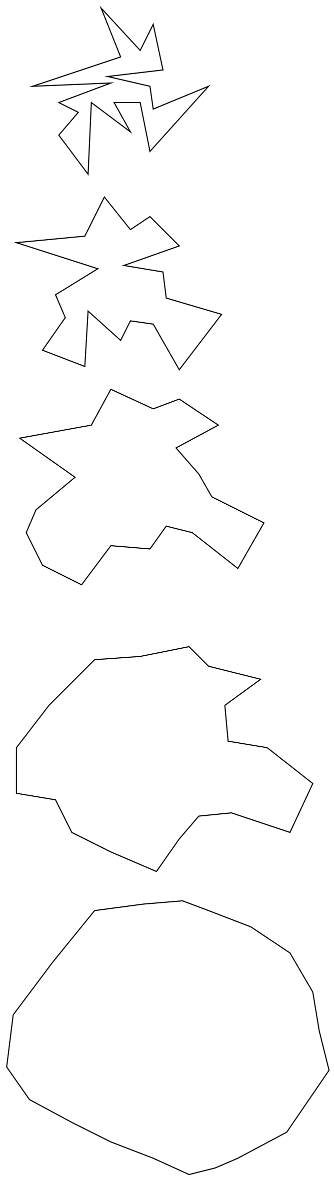


- Molecular biology
  - Efficient computer simulation of molecule conformation.



The **Carpenter's Rule Problem**: Consider a simple planar polygonal chain with fixed edge lengths. The edges (bars) are allowed to move freely around the vertices (joints). Is it possible to move this closed linkage continuosly from an arbitrary initial configuration to any final configuration avoiding collisions between the bars?

- Always possible (Connelly, Demain and Rote (2000))
- Algebraic-Combinatoric algorithm by Streinu (2000)



- How many embeddings has a given framework of a minimally rigid graph?

  → Best bound so far, *Borcea '02*: For a generic choice of edge lengths, a Laman Graph with n vertices has at most $\binom{2n-4}{n-2}$ distinct embeddings in $\mathbb{R}^2$, up to rigid motions.

  → This is a bound for the complex solutions.

  → Even over $\mathbb{C}$ it is in general not sharp.

- Given a 1-degree-of-freedom framework, how can one characterize and compute the trajectory of the vertices?

A framework (G, L) can be formulated by the following system of 4 linear and $|E| - 1$ quadratic equations in $2|V|$ unknowns.

$$\begin{aligned} x_1 - c_1 &= 0 \\ y_1 - c_2 &= 0 \\ x_2 - (L_{1\,2} + c_1) &= 0 \\ y_2 - c_2 &= 0 \\ (x_i - x_j)^2 + (y_i - y_j)^2 - L_{i\,j}^2 &= 0 \quad \forall \{i, j\} \neq \{1, 2\} \in E \end{aligned}$$

Now the questions concerning the framework can be translated in questions concerning the common real solution to this system.

Our main tool to study these systems will be Bernstein's theorem.

Definition (Mixed Volume)**.** Let $P_1, \ldots, P_n$ be n polytopes in $\mathbb{R}^n$. Then the *mixed volume of* $P_1, \ldots, P_n$ is defined as

$$\mathrm{MV}_n(P_1, \ldots, P_n) = \sum_{\substack{Q \text{ mixed cell of a} \\ \text{mixed subdivision} \\ \text{of } \sum P_j}} \mathrm{vol}_n\,(Q)$$

Definition (Newton Polytope)**.** For $f = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} c_\alpha x^\alpha \in \mathbb{C}[x_1, \ldots, x_n]$ the *Newton polytope* of f is defined as

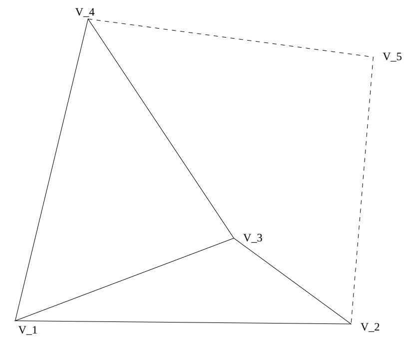
$$\mathrm{conv}\left(\{\alpha \in \mathbb{Z}^n_{\geq 0} : c_\alpha \neq 0\}\right) .$$

Theorem (Bernstein '75)**.** *Let $f_1, \ldots, f_n \in \mathbb{C}[x_1, \ldots, x_n]$ be polynomials with finitely many common zeros in $(\mathbb{C}^*)^n$, and let $P_1, \ldots, P_n$ be the Newton Polytopes of the f_i's. Then the number of common zeros in $(\mathbb{C}^*)^n$ is less than or equal the mixed volume $MV_n(P_1, \ldots, P_n)$. Equality holds if the coefficients of the f_i are generic.*

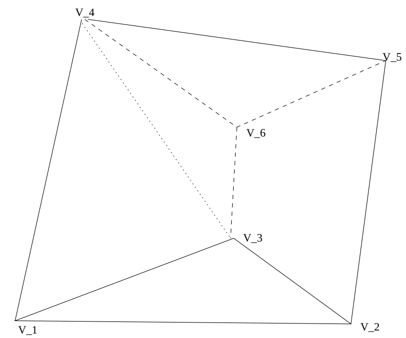
A *Henneberg sequence* for a graph G is a sequence $G_3, G_4, \ldots, G_n$ of Laman graphs on $3, 4, \ldots, n$ vertices, such that G_3 is a triangle, $G_n = G$ and each G_i is obtained by G_{i-1} via one of the following two types of steps:

- Henneberg I step (HI): Add one new vertex v_{i+1} and two new edges, connecting v_{i+1} to two arbitrary vertices of G_i
- Henneberg II step (HII): Add one new vertex v_{i+1} and three new edges, connecting v_{i+1} to three vertices of G_i such that at least two of these vertices are connected via an edge e of G_i and this edge e is removed.

Any Laman graph G can be constructed via a Henneberg sequence and any graph constructed via a Henneberg sequence has the Laman property. We call G a *Henneberg I graph* if it is constructable using only Henneberg I steps. Otherwise we call it *Henneberg II*.



HI step



HII step

Theorem. *The number of 2-dimensional embeddings, up to rigid motions, of a Henneberg I Laman graph G with n vertices and given edge lengths is at most 2^{n-2}. This bound is sharp.*

Using sparse elimination theory, this result can be seen as a consequence of the following statement on mixed volumes.

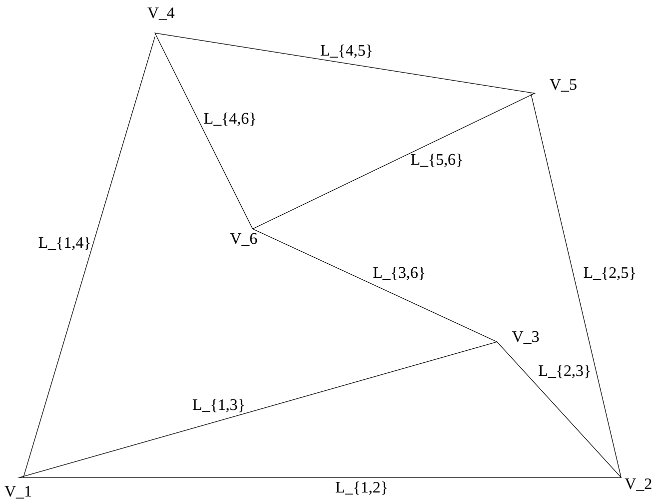
Lemma. *Let $\Gamma_1, \ldots, \Gamma_k$ be polytopes in $\mathbb{R}^{m+k}$ and $\Delta_1, \ldots, \Delta_m$ be polytopes in $\mathbb{R}^m \subset \mathbb{R}^{m+k}$. Then*

$$\mathrm{MV}_{m+k}(\Delta_1, \ldots, \Delta_m, \Gamma_1, \ldots, \Gamma_k) = \mathrm{MV}_m(\Delta_1, \ldots, \Delta_m) * \mathrm{MV}_k(\pi(\Gamma_1), \ldots, \pi(\Gamma_k)) \ ,$$

where $\pi : \mathbb{R}^{m+k} \to \mathbb{R}^k$ denotes the projection on the last k coordinates.

Corollary. *Each Henneberg I step increases the number of embeddings by at most 2.*

The smallest example of a Henneberg II graph is the following graph on 6 vertices:



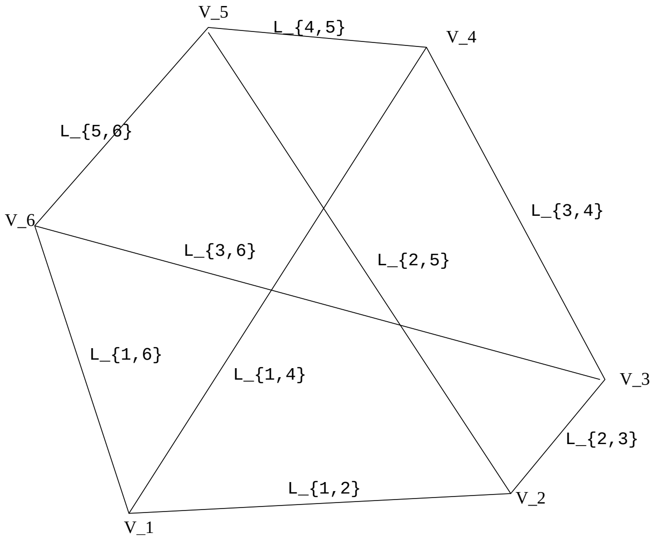
There is a choice of edge lengths giving **24** distinct real embeddings of this framework. Borcea's theorem tells us, that there are at most $\binom{8}{4} = 70$. With our approach we get a better bound.

The truncated system of equations corresponding to this framework is:

$$\begin{aligned}
x_1 - 1 &= 0 \\
y_1 - 1 &= 0 \\
x_2 - (L_{1,2} + 1) &= 0 \\
y_2 - 1 &= 0 \\
(x_1 - x_3)^2 + (y_1 - y_3)^2 - L_{1,3}^2 &= 0 \\
x_2^2 - x_1^2 - 2x_2x_3 + 2x_1x_3 + y_2^2 - y_1^2 - & \\
2y_2y_3 + 2y_1y_3 - L_{2,3}^2 + L_{1,3}^2 &= 0 \\
(x_1 - x_4)^2 + (y_1 - y_4)^2 - L_{1,4}^2 &= 0 \\
2x_4 + 2x_5 - 2x_4x_5 + 2y_4 + 2y_5 - 2y_4y_5 - & \\
L_{4,5}^2 + L_{1,4}^2 + L_{2,5}^2 &= 0 \\
(x_2 - x_5)^2 + (y_2 - y_5)^2 - L_{2,5}^2 &= 0 \\
(x_3 - x_6)^2 + (y_3 - y_6)^2 - L_{3,6}^2 &= 0 \\
2x_4 + 2x_6 - 2x_4x_6 + 2y_4 + 2y_6 - 2y_4y_6 - & \\
L_{4,6}^2 + L_{1,4}^2 + L_{3,6}^2 &= 0 \\
2x_5 + 2x_6 - 2x_5x_6 + 2y_5 + 2y_6 - 2y_5y_6 - & \\
L_{5,6}^2 + L_{2,5}^2 + L_{3,6}^2 &= 0\,.
\end{aligned}$$

The Mixed Volume of this system is **32**.

Another example of a Henneberg II graph is $K_{3,3}$:



Again, Borcea's theorem gives 70 as a bound on the number of possible embeddings in $\mathbb{R}^2$, and as above we can improve that bound using Berstein's theorem.

The truncated system of equations corresponding to this framework is:

$$\begin{aligned}
x_1 - 1 &= 0 \\
y_1 - 1 &= 0 \\
x_2 - (L_{1,2} + 1) &= 0 \\
y_2 - 1 &= 0 \\
(x_1 - x_4)^2 + (y_1 - y_4)^2 - L_{1,4}^2 &= 0 \\
(x_2 - x_3)^2 + (y_2 - y_3)^2 - L_{2,3}^2 &= 0 \\
(x_2 - x_5)^2 + (y_2 - y_5)^2 - L_{2,5}^2 &= 0 \\
(x_1 - x_6)^2 + (y_1 - y_6)^2 - L_{1,6}^2 &= 0 \\
-2x_3x_4 - 2y_3y_4 - x_1^2 + 2x_1x_4 - y_1^2 + 2y_1y_4 & \\
-x_2^2 + 2x_2x_3 - y_2^2 + 2y_2y_3 - L_{3,4}^2 + L_{1,4}^2 + L_{2,3}^2 &= 0 \\
-2x_3x_6 - 2y_3y_6 - x_1^2 + 2x_1x_6 - y_1^2 + 2y_1y_6 & \\
-x_2^2 + 2x_2x_3 - y_2^2 + 2y_2y_3 - L_{3,6}^2 + L_{1,6}^2 + L_{2,3}^2 &= 0 \\
-2x_4x_5 - 2y_4y_5 - x_1^2 + 2x_1x_4 - y_1^2 + 2y_1y_4 & \\
-x_2^2 + 2x_2x_5 - y_2^2 + 2y_2y_5 - L_{4,5}^2 + L_{1,4}^2 + L_{2,5}^2 &= 0 \\
-2x_5x_6 - 2y_5y_6 - x_1^2 + 2x_1x_6 - y_1^2 + 2y_1y_6 & \\
-x_2^2 + 2x_2x_5 - y_2^2 + 2y_2y_5 - L_{5,6}^2 + L_{1,6}^2 + L_{2,5}^2 &= 0
\end{aligned}$$

The Mixed Volume of this system is **32**.

Open Questions:

- The mixed volume of an untruncated system corresponding to a Laman Graph framework with n vertices is (less or) equal 4^{n-2}. The truncation lowers the mixed volume and with that the bound.

  → How much?

  → Is there an algorithm that transforms any given system to an equivalent one possesing the minimum mixed volume?

- Obtain a bound that respects the structure of the graph. (i.e. That takes into account the number of Henneberg I and Henneberg II steps needed to construct it.)

- To split the mixed volume calculation of a large framework in two smaller ones, identify the largest Laman-Sub-Graph (or largest HI-Sub-Graph) in a given Laman Graph.