Exploiting Symmetries in Mixed Volume Computation

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1 Mixed volumes and mixed subdivisions

Let $P_1, \ldots, P_n$ be n polytopes in $\mathbb{R}^n$. For non-negative parameters $\lambda_1, \ldots, \lambda_n$, the function $\mathrm{vol}_n(\lambda_1 P_1 + \ldots + \lambda_n P_n)$ is a homogeneous polynomial of degree n in $\lambda_1, \ldots, \lambda_n$ with non-negative coefficients. The coefficient of the monomial $\lambda_1 \cdots \lambda_n$ is called the *mixed volume of* $P_1, \ldots, P_n$ and is denoted by $\mathrm{MV}_n(P_1, \ldots, P_n)$.

We denote by $\mathrm{MV}_n(P_1, k_1; \ldots; P_r, k_r)$ the mixed volume where P_i is taken k_i times and $\sum_{i=1}^{r} k_i = n$.

Let $\mathcal{S} = (S^{(1)}, \ldots, S^{(m)})$ be a sequence of finite point sets in $\mathbb{R}^n$ that affinely spans the full space. A sequence $C = (C^{(1)}, \ldots, C^{(m)})$ of subsets $C^{(i)} \subseteq S^{(i)}$ is called a *cell* of $\mathcal{S}$. A *subdivision* of $\mathcal{S}$ is a collection $\Gamma = \{C_1, \ldots, C_k\}$ of cells such that

i) $\dim(\mathrm{conv}(C_i)) = n$ for all cells C_i,

ii) $\mathrm{conv}(C_i) \cap \mathrm{conv}(C_j)$ is a face of both convex hulls and

iii) $\bigcup_{i=1}^{k} \mathrm{conv}(C_i) = \mathrm{conv}(\mathcal{S})$

where $\mathrm{conv}(A) := \mathrm{conv}(A^{(1)} + \ldots + A^{(m)})$ for a sequence of point sets A. A subdivision is called *mixed* if additionally

iv) $\sum_{i=1}^{m} \dim(\mathrm{conv}(C_j^{(i)})) = n$ for all cells C_j in Γ

and it is called *fine mixed* if additionally

v) $\sum_{i=1}^{m} (\#C_j^{(i)} - 1)) = n$ for all cells C_j in Γ

where $\#A$ denotes the number of points in a finite set $A \subset \mathbb{R}^n$. The *type* of a cell is defined as

$$\mathit{type}(C) = (\dim(\mathrm{conv}(C^{(1)})), \ldots, \dim(\mathrm{conv}(C^{(m)})))$$

and cells of type $(1, 1, \ldots, 1)$ will be called *mixed cells*. These definitions extend naturally to sequences of polytopes by considering their vertices as the point sets above. If all cells of a subdivision are simplices we will call the subdivision a *triangulation*.

With this terminoligy we can state an explicit formula to calculate the mixed volume (see [2]).

$$\mathrm{MV}_n(P_1,k_1;\ldots;P_r,k_r) = \sum_{\substack{C \text{ cell type } (k_1,\ldots,k_r) \\ \text{of a mixed subdivision} \\ \text{of } (P_1,\ldots,P_r)}} k_1!\cdots k_r!\ \mathrm{vol}_n(C) \tag{1}$$

For a cell $C = (C^{(1)},\ldots,C^{(r)})$ of type $(k_1,\ldots,k_r)$ in a mixed subdivion with $C^{(i)} = \{p_0^{(i)},\ldots,p_{k_i}^{(i)}\}$ we define the matrix $M(C)$ to be the $n \times n$ matrix whose rows are $p_j^{(i)} - p_0^{(i)}$ for $1 \le i \le r$ and $1 \le j \le k_i$. We have (see [2]) that

$$|\det(M(C))| = k_1!\cdots k_r! \cdot \mathrm{vol}(C) \tag{2}$$

which simplifies the computation of (1).

To construct mixed subdivisions we proceed as in [2]. For each of the point sets $S^{(i)}$ from $\mathcal{S}$ we choose a real valued lifting function $\mu_i:\ S^{(i)} \to \mathbb{R}$. For a point set A we denote by $\hat{A}$ the lifted point set $\{(q,\mu_i(q)) : q \in A\} \in \mathbb{R}^{n+1}$.

The set of those facets of $\mathrm{conv}(\hat{S}^{(1)} + \ldots + \hat{S}^{(m)})$ which have an inward pointing normal with a positive last coordinate is called the lower hull of the Minkowski sum. If we project down this lower hull back to $\mathbb{R}^n$ by forgetting the last coordinate we get a subdivision of $(S^{(1)},\ldots,S^{(m)})$. We call such a subdivision *coherent* and will say it is *induced by* $\mu = (\mu_1,\ldots,\mu_m)$.

Not all coherent subdivsions are mixed but there are conditions on liftings which guarantee that the induced subdivision is mixed.

Lemma 1 (See [2]) *If for each n-dimensional cell C in the subdivision of $(S^{(1)},\ldots,S^{(m)})$ induced by μ we have that $M(\hat{C})$ has maximal rank, i.e. $\min(n+1,\#C-m)$, then the subdivision is fine mixed.*

A lifting μ that satisfies the condition of Lemma 1 is called *sufficiently generic*. The maximal minors of $M(\hat{C})$ give linear conditions on the values $\mu(q)$ for $q \in S^{(i)}$.

2 Symmetric Newton polytopes and lifting functions

Let $\mathcal{S} = (S^{(1)},\ldots,S^{(n)})$ be a sequence of point sets $S^{(i)} \in \mathbb{R}^n$ and let G be a finite group, e.g. a subgroup of the symmetric group $\mathbb{S}_n$. We define a left and a right action of G on n-tuples of point sets by

$$\begin{aligned} g\,.\mathcal{S} = \ (S^{(g(1))},\ldots,S^{(g(n))}) &:= D_1(g)\,(S^{(1)},\ldots,S^{(n)})^T \\ (x_1,\ldots,x_n)\circ g &:= D_2(g)\,(x_1,\ldots,x_n)^T \end{aligned}$$

where $D_1, D_2 : G \to \mathrm{GL}_n(\mathbb{Z}_2)$ are matrix representations of G. We say that $\mathcal{S}$ is *G-symmetric* if

$$g\ .\ \mathcal{S} = \mathcal{S}\circ g \quad \forall g \in G \tag{3}$$

where the action of G on the right hand side is applied on all points of all point sets in $\mathcal{S}$.

Furthermore we call a lifting $\mu = (\mu_1, \dots, \mu_n)$ *G-symmetric* if

$$g \ . \ \hat{\mathcal{S}} = \hat{\mathcal{S}} \circ \hat{g} \quad \forall g \in G \tag{4}$$

where $\hat{g}$ acts like g on the first n coordiantes and leaves the $(n+1)$ coordinate fixed. Clearly we have that if a support set or a lifting is G-symmetric then it is as well G'-symmetric for every subgroup G' of G.

The problem of finding a symmetric lifting that is still generic is not fully understood. We will now investigate conditions on lifting values that arise from the symmetries. For a point $q \in S^{(i)}$ we will define the *point orbit* of the tuple $(q, S^{(i)})$ as

$$\mathcal{O}_{q,i} := \{(q \circ g, g(i)) \,|\, g \in G\} \ .$$

Lemma 2 (see [1]) *A lifting μ for a G-symmetric support set is G-symmetric if and only if in each point orbit $\mathcal{O}_{q,i}$ every point has the same lifting value.*

3 Exploiting symmetries in mixed volume calculation

The symmetries of a support set and a lifting function imply the following properties of the cell structure.

Proposition 3 (See [1]) *Let $\mathcal{S}$ and μ be G-symmetric and let C be a cell of the μ-induced subdivision of $\mathcal{S}$ such that $\hat{C}$ has inner normal $(\gamma, 1)$. Then we have for all $g \in G$ that*

$$D := g^{-1} \ . \ C \circ g$$

is a cell of the μ-induced subdivision as well and $\hat{D}$ has inner normal $(\gamma \circ g, 1)$.

We call $D = g^{-1} \ . \ C \circ g$ the *conjugate* of C.

Proof. Let $C = (C^{(1)}, \dots, C^{(n)})$ be a cell with inner normal $(\gamma, 1)$. Then $(\gamma, 1)\begin{pmatrix} q \\ \mu_i(q) \end{pmatrix}$ attains its minimum over $S^{(i)}$ for $q \in C^{(i)}$ for all $1 \leq i \leq n$. Since $\mathcal{S}$ is G-symmetric we have $S^{(i)} \circ g = S^{(g(i))} = g \ . \ S^{(i)}$ and $C^{(i)} \circ g \subset S^{(g(i))}$. So we can write $D = (C^{(g^{-1}(1))} \circ g, \dots, C^{(g^{-1}(n))} \circ g)$.

To prove the proposition we have to show that $(\gamma \circ g, 1)\begin{pmatrix} q \\ \mu_j(q) \end{pmatrix}$ attains its minimum over $S^{(j)}$ for $q \in C^{(g^{-1}(j))} \circ g$. This is equivalent to $(\gamma \circ g, 1)\begin{pmatrix} \bar{q} \\ \mu_{g(i)}(\bar{q}) \end{pmatrix}$ attaining its minimum over $S^{(g(i))}$ for $\bar{q} \in C^{(i)} \circ g$. Since μ is G-symmetric and since (q, i) and $(\bar{q}, g(i))$ lie in the same point orbit we have that $\mu_i(q) = \mu_{g(i)}(\bar{q})$ and that shows the result. □

For a cell C we define the *cell orbit* of C under G by

$$\mathcal{O}_C := \{g^{-1} \ . \ C \circ g \,|\, g \in G\} \ .$$

Then we have the following theorem which simplifies the calculation of mixed volumes for symmetric Newton polytopes.

Theorem 4 (See [1]) *Let $\mathcal{S} = (S^{(1)}, \ldots, S^{(r)})$ and μ be G-symmetric such that μ induces a fine mixed subdivision on $\mathcal{S}$. Then*

$$\mathrm{MV}_n(\mathrm{conv}(S^{(1)}), k_1; \ldots; \mathrm{conv}(S^{(r)}), k_r) = \sum_{\mathcal{O}_C} k_1! \cdots k_r! \, \#\mathcal{O}_c \cdot \mathrm{vol}_n(C) \qquad (5)$$

where C is a cell in the μ induced subdivision of type $(k_1, \ldots, k_r)$ that generates the orbit $\mathcal{O}_C$.

Proof. The type and volume of a cell don't change under conjugation and Proposition 3 tells us that a cell on the lower hull has all its conjugates on the lower hull as well. Formula (1) gives the result. □

4 Research ideas (participants welcome)

1. Combine the conditions for generic and symmetric liftings.
2. For given support sets $\mathcal{S}$, find conditions on the group G that allows a generic and G-symmetric lifting μ.
3. For given support sets $\mathcal{S}$, find an algorithm that computes the maximal group G that allows a lifting μ that is G-symmetric and generic.

References

[1] K. Gatermann and J. Verschelde. Symmetric newton nolytopes for solving sparse polynomial systems. (Adv. Appl. Math.) 16(1) (1995) 95-127

[2] B. Huber and B. Sturmfels. A polyhedral method for solving sparse polynomial systems. (Math. Comp.) 64(212) (1995) 1541-1555.