Laman Graph Embeddings

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Definitions and Questions

Definition (Framework, Embedding)

- Let $G = (V, E)$ be a graph with n vertices $v_1, \ldots, v_n$ and m edges. A *framework* (G, L) is a graph together with a set $L = \{L_{ij} : ij \in E\}$ of non-negative real numbers L_{ij} interpreted as edge lengths.
- An *embedding* $G(P)$ of (G, L) in $\mathbb{R}^d$ (here we consider only $d = 2$) is given by a map $\alpha : V \rightarrow \mathbb{R}^d$ such that L_{ij} equals the Euclidean distance between the two points $\alpha(v_i) = p_i$ and $\alpha(v_j) = p_j$.

Note that edges may cross and vertices may coincide.

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Question

- Given a framework (G, L), how many embeddings are possible?
- How many solutions $(p_1, \ldots, p_n)$ in $(\mathbb{R}^d)^n$ are there for the following system of $|E|$ quadratic equations

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Definition (Configuration Space)

- The *configuration space* of (G, L) in $\mathbb{R}^d$ is the space of all possible embeddins, modulo the group of rigid motions (in $\mathbb{R}^2$ this group of rigid motions consists of translations and rotations, but not reflections).

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Definition (Rigidity)

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Definition (Laman Graph)

- Let G be a graph with n vertices and $m = 2n - 3$ edges. If each subset of k vertices spans at most $2k - 3$ edges, we say that G has the *Laman property* and call it a *Laman graph*.

Definition (Henneberg Sequence)

- A *Henneberg sequence* for a graph G is a sequence $G_3, G_4, \ldots, G_n$ of Laman graphs on $3, 4, \ldots, n$ vertices, such that G_3 is a triangle, $G_n = G$ and each G_i is obtained by G_{i-1} via one of the following two types of steps:
  - Henneberg I step (HI): Add one new vertex v_{i+1} and two new edges, connecting v_{i+1} to two arbitrary vertices of G_i

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  - Henneberg II step (HII): Add one new vertex v_{i+1} and three new egdes,connecting v_{i+1} to three vertices of G_i such that at least two of these vertices are connected via an edge e of G_i and this certain edge e is removed.

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Question

How many embeddings has a Laman Graph for a generic choice of edge lengths modulo rigid motions?

Distance-Sets and Point Configuratoins

$$D(p_0, \ldots, p_{n-1}) = \begin{pmatrix} 0 & 1 & 1 & 1 & \ldots & 1 \\ 1 & 0 & d_{0\,1} & d_{0\,2} & \ldots & d_{0\,n-1} \\ 1 & d_{0\,1} & 0 & d_{1\,2} & \ldots & d_{1\,n-1} \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ 1 & d_{0\,n-2} & & & 0 & d_{n-2\,n-1} \\ 1 & d_{0\,n-1} & d_{1\,n-1} & \ldots & d_{n-2\,n-1} & 0 \end{pmatrix}$$

Theorem

$d_{ij} \in \mathbb{R}_{\geq 0}$ $(0 \leq i < j \leq n-1)$ describe the squared pairwise distances between n points $p_0 = 0, p_2, \ldots, p_{n-1}$ in $\mathbb{R}^k$ if and only if

i) sign det $D(p_0, \ldots, p_j) = (-1)^{j+1}$ *or* 0 *for* $j = 1, \ldots, n-1$

ii) rank $D(p_0, \ldots, p_{n-1}) \leq k+2$

where $D(p_0, \ldots, p_j)$ denotes the Distance-Matrix:

$$D(p_0, \ldots, p_j) = \begin{pmatrix} 0 & 1 & 1 & 1 & \ldots & 1 \\ 1 & 0 & d_{0\,1} & d_{0\,2} & \ldots & d_{0\,j} \\ 1 & d_{0\,1} & 0 & d_{1\,2} & \ldots & d_{1\,j} \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ 1 & d_{0\,j-1} & & & 0 & d_{j-1\,j} \\ 1 & d_{0\,j} & d_{1\,j} & \ldots & d_{j-1\,j} & 0 \end{pmatrix}.$$

Corollary

A necessary condition for $\binom{n}{2}$ *numbers* $d_{ij} \in \mathbb{R}_{\geq 0}$ $(0 \leq i < j \leq n-1)$ *to describe the squared pairwise distances between* n *points* $p_0 = 0, p_2, \dots, p_{n-1}$ *in* $\mathbb{R}^2$ *is that*

$$\operatorname{rank} D(p_0, \dots, p_{n-1}) \leq 4$$

Cayley-Menger Bound

- $\mathbb{R}^{2n} / \begin{matrix} \text{rigid motions,reflections} \\ \text{rescaling} \\ \text{not all equal} \end{matrix}$
- Identify $\mathbb{R}^2$ with $\mathbb{C}$: $\mathbb{C}^{n} / \begin{matrix} \text{rigid motions,reflections} \\ \text{rescaling} \\ \text{not all equal} \end{matrix}$

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We have an injective map

$$\begin{array}{rcl} \varphi : \mathbb{P}^{n-2}(\mathbb{C})/\text{conj.} & \rightarrow & \mathbb{P}^{\binom{n}{2}-1}(\mathbb{C}) \\ [p_1 : \ldots : p_{n-1}] & \mapsto & (d_{k\,m} = |p_k - p_m|^2)_{1 \leq k < m \leq n-1} \end{array}$$

Cayley-Menger variety:

$$\mathrm{CM}^{2,n}(\mathbb{C}) = \text{'the smallest variety containing im}(\varphi)\text{'}$$

$$= \mathbb{V}\left(5 \times 5 \text{ minors of } \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & d_{0\,1} & \cdots & d_{0\,n-1} \\ \vdots & & \ddots & \ddots & \vdots \\ 1 & & & 0 & d_{n-2\,n-1} \\ 1 & & & & 0 \end{pmatrix}\right)$$

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$$\deg \mathrm{CM}^{2,n}(\mathbb{C}) = \frac{1}{2}\binom{2n-4}{n-2}.$$

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Setting the graph edges to prescribed lengths corresponds to the following system of homogenous equations.

$$\frac{d_{i\,j}}{d_{k\,m}} = \frac{L_{i\,j}^2}{L_{k\,m}^2} \quad \forall ij, km \in E$$

This gives at most $|E| - 1$ independent equations each defining a linear section of codimension 1.

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$$\mathcal{L}^E = \left\{ D \in \mathbb{P}^{\binom{n}{2}-1}(\mathbb{C}) \;\middle|\; d_{i\,j}L_{k\,m}^2 = d_{k\,m}L_{i\,j}^2 \; \forall ij, km \in E \right\}$$

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So by definition of the degree we have

$$\sharp\left(\mathcal{L}^E \cap \mathrm{CM}^{2,n}(\mathbb{C})\right) \leq \deg \mathrm{CM}^{2,n}(\mathbb{C}) = \frac{1}{2}\binom{2n-4}{n-2} .$$

Theorem (Borcea '02)

For a generic choice of edge lengths, a Laman Graph with n vertices has at most $\binom{2n-4}{n-2}$ distinct embeddings in $\mathbb{R}^2$, up to rigid motions.

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Mixed Volume Bound

Question

How many embeddings has a Laman Graph for a generic choice of edge lengths modulo rigid motions?

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How many solutions has the following system of $2n$ equations in $2n$ unknowns for a generic choice of the $L_{i\,j}$?

$$\begin{aligned} x_1 - c_1 &= 0 \\ y_1 - c_2 &= 0 \\ x_2 - (L_{1\,2} + c_1) &= 0 \\ y_2 - c_2 &= 0 \\ (x_i - x_j)^2 + (y_i - y_j)^2 - L_{i\,j}^2 &= 0 \quad \forall ij \neq 12 \in E \end{aligned}$$

Definition (Mixed Volume)

Let $P_1, \ldots, P_r$ be r polytopes in $\mathbb{R}^n$ and let $\lambda_1, \ldots, \lambda_r$ be r non-negative real numbers. Then $\mathrm{vol}_n(\lambda_1 P_1 + \ldots + \lambda_r P_r)$ is a homogeneous polynomial of degree n in $\lambda_1, \ldots, \lambda_r$ with non-negative coefficients. The coefficient of the monomial $\lambda_1 \ldots \lambda_r$ in $\mathrm{vol}_n(\lambda_1 P_1 + \ldots + \lambda_r P_r)$ is called the *mixed volume of* $P_1, \ldots, P_r$ and denoted by $\mathrm{MV}_n(P_1, \ldots, P_r)$.

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Two explicit formulas for the Mixed Volume:

$$\mathrm{MV}_n(P_1,\ldots,P_n) = \sum_{J\subseteq\{1,\ldots,n\}} (-1)^{n-\#J}\,\mathrm{vol}_n\left(\sum_{j\in J} P_j\right)$$

$$= \sum_{\substack{Q \text{ mixed cell of a} \\ \text{mixed subdivision} \\ \text{of } \sum P_j}} \mathrm{vol}_n\left(Q\right)$$

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Definition (Newton Polytope)

For

$$f = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} c_\alpha x^\alpha \quad \in \mathbb{C}[x_1, \ldots, x_n]$$

the *Newton polytope* of f is defined as

$$\operatorname{conv}\left(\{\alpha \in \mathbb{Z}^n_{\geq 0} \ : \ c_\alpha \neq 0\}\right) .$$

Theorem (Bernstein '75)

Let $f_1, \ldots, f_n \in \mathbb{C}[x_1, \ldots, x_n]$ be polynomials with finitely many common zeros in $(\mathbb{C}^*)^n$, and let $P_1, \ldots, P_n$ be the Newton Polytopes of the f_i's. Then the number of common zeros in $(\mathbb{C}^*)^n$ is less or equal the mixed volume $MV_n(P_1, \ldots, P_n)$. Equality holds if the coefficients of the f_i are generic.

Theorem

The number of 2-dimensional embeddings, up to rigid motions, of a Henneberg I constructable Laman graph G with n vertices and given edge lengths is at most 2^{n-2}*. This bound is sharp.*

Proof: We use induction on the number of vertices n.
So consider a graph with 3 vertices and 3 edges. So we want to know the number of solutions $(x_1, y_1, x_2, y_2, x_3, y_3)$ in $\mathbb{R}^6$ of the following system of equations.

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$$
\begin{aligned}
x_1 - 1 &= 0\\
y_1 - 1 &= 0\\
x_2 - (L_{1\,2} + 1) &= 0\\
y_2 - 1 &= 0\\
(x_1 - x_3)^2 + (y_1 - y_3)^2 - L_{1\,3}^2 &= 0\\
(x_2 - x_3)^2 + (y_2 - y_3)^2 - L_{2\,3}^2 &= 0
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Unfortunately: $\mathrm{MV}_6(P_1, \ldots, P_6) = 4.$

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Unfortunately: $\mathrm{MV}_6(P_1, \ldots, P_6) = 4.$

Truncation-Trick:
Substract the fifth equation from the sixth to get rid of x_3^2 and y_3^2 in the sixth equation. So our new system is:

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Now we have, as disired: $\mathrm{MV}_6(P_1, \ldots, P_6) = 2$.

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Now we have, as disired: $\mathrm{MV}_6(P_1, \ldots, P_6) = 2$.

Assume we are given a Laman Graph framework with n vertices. Now add another vertex v_{n+1} and two new edges with lengths $L_{r\,n+1}$ and $L_{s\,n+1}$ via a Henneberg I step. This corresponds to adding the two equations

$$\begin{aligned}(x_r - x_{n+1})^2 + (y_r - y_{n+1})^2 - L_{r\,n+1}^2 &= 0\\(x_s - x_{n+1})^2 + (y_s - y_{n+1})^2 - L_{s\,n+1}^2 &= 0\end{aligned}$$

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Before we calculate the Mixed Volume of this new system we apply the truncation-trick to the new equations and obatain

$$
\begin{aligned}
(x_r - x_{n+1})^2 + (y_r - y_{n+1})^2 - L_{r\,n+1}^2 &= 0 \\
x_s^2 - x_r^2 - 2x_s x_{n+1} + 2x_r x_{n+1} + y_s^2 - y_r^2 & \\
-2y_s y_{n+1} + 2y_r y_{n+1} - L_{s\,n+1}^2 + L_{r\,n+1}^2 &= 0\,.
\end{aligned}
$$

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Lemma

Let $\Gamma_1, \ldots, \Gamma_k$ be polytopes in $\mathbb{R}^{m+k}$ and $\Delta_1, \ldots, \Delta_m$ be polytopes in $\mathbb{R}^m \subset \mathbb{R}^{m+k}$. Then

$$\mathrm{MV}_{m+k}(\Delta_1, \ldots, \Delta_m, \Gamma_1, \ldots, \Gamma_k) = \mathrm{MV}_m(\Delta_1, \ldots, \Delta_m) * \mathrm{MV}_k(\pi(\Gamma_1), \ldots, \pi(\Gamma_k)) ,$$

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Perspective

- The mixed volume of an untruncated system corresponding to a Laman Graph framework with n vertices is (less or) equal 4^{n-2}. The truncation lowers the mixed volume and with that the bound. **How much?**
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Danke!