

Linkage Problems and Real Algebraic Geometry

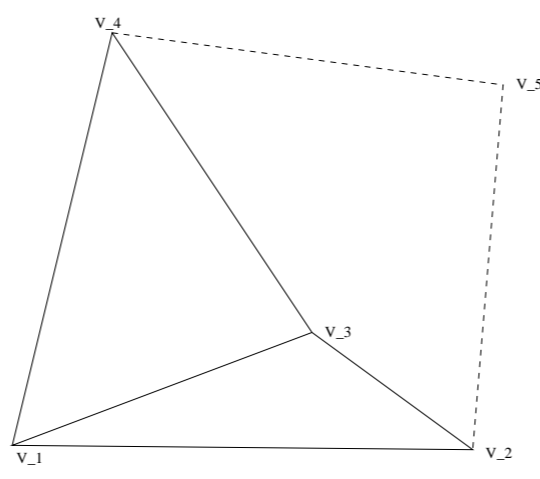
Definitions

Definition (Framework). A *framework* is a graph $G = (V, E)$ together with a set $L = \{L_{ij} : ij \in E\}$ of non-negative real numbers L_{ij} interpreted as edge lengths.

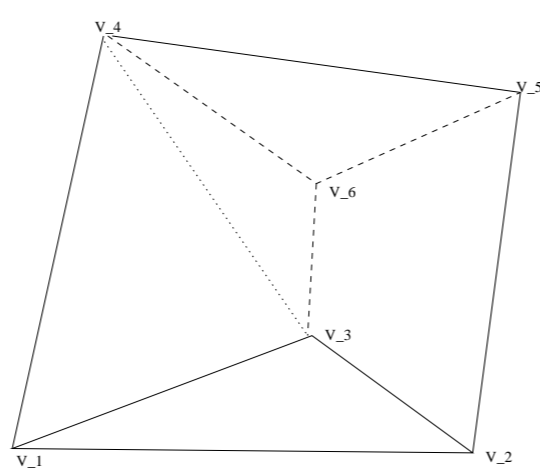
Definition (Laman Graph). Let G be a graph with n vertices and $m = 2n - 3$ edges. If each subset of k vertices spans at most $2k - 3$ edges, we say that G has the *Laman property* and call it a *Laman graph*.

- Laman graphs can be constructed via a sequence of Henneberg steps starting with a simple bar, and all graphs constructed in this way have the Laman property.
- Laman graphs are minimally rigid.

Henneberg I step:

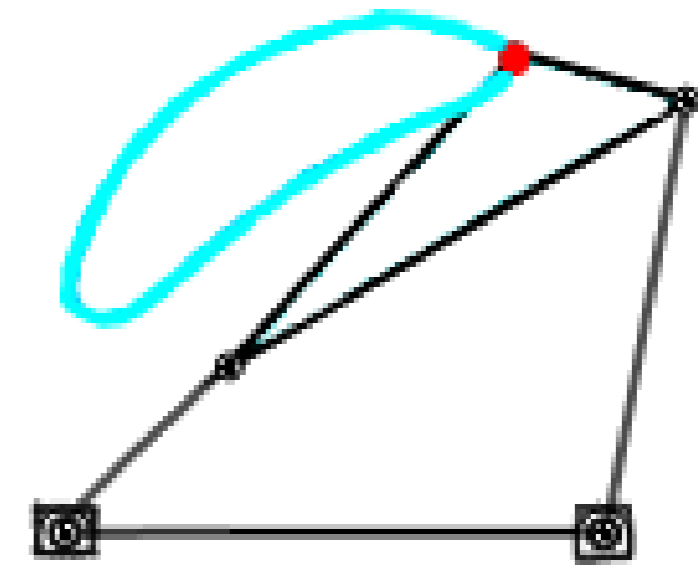


Henneberg II step:



Definition (Rigidity). A graph is *generically rigid* if it is rigid for all embeddings on generic sets of points. An embedding is *rigid* if it cannot be deformed continuously into another (non-congruent) embedding of the same framework.

Definition (1DOF Mechanism). A one-degree-of-freedom (1DOF) mechanism is a framework whose configuration space is a one-dimensional curve.

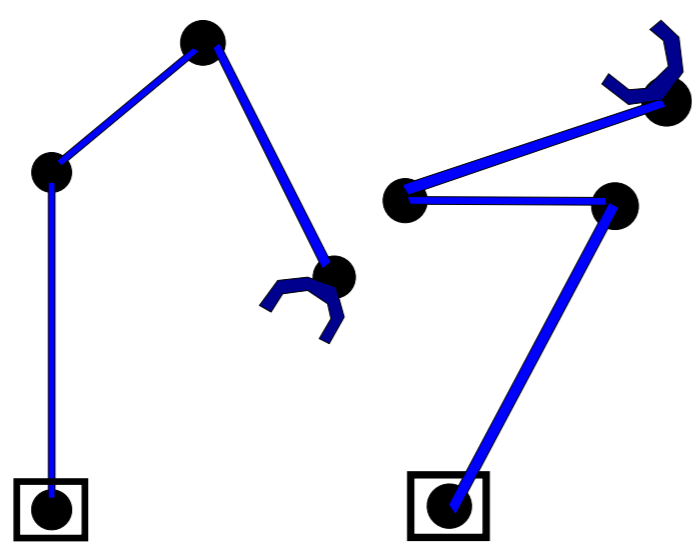


Applications

These concepts appear naturally in the following scenarios.

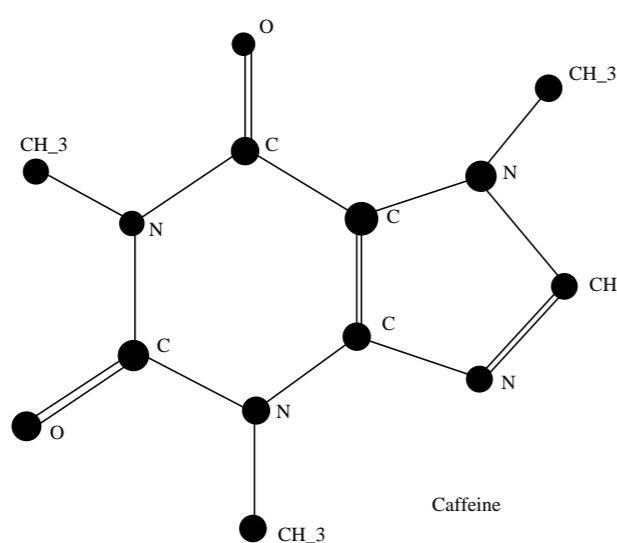
- Robot Kinematics

→ Planar non-colliding robot arm motion planning

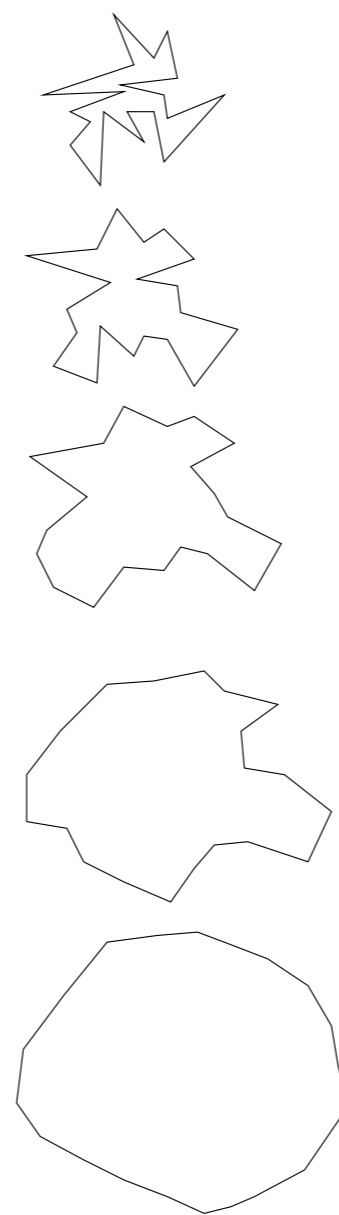


- Molecular biology

→ Efficient computer simulation of molecule conformation.



- The **Carpenter's Rule Problem:** Is it possible to move a closed linkage continuously from an arbitrary initial configuration to any final configuration avoiding collisions between the bars?



→ Always possible (Connelly, Demain and Rote 2000)

→ Algebraic-combinatorial algorithm by Streinu 2000

Combinatorial and algebraic questions

- How many embeddings has a given framework of a minimally rigid graph?
 - Best bound so far, Borcea and Streinu '02: For a generic choice of edge lengths, a Laman Graph with n vertices has at most $\binom{2n-4}{n-2}$ distinct embeddings in \mathbb{R}^2 , up to rigid motions.
 - This is a bound for the complex solutions.
 - Even over \mathbb{C} it is in general not sharp.
- Given a 1-degree-of-freedom framework, how can one characterize and compute the trajectory of the vertices?

Our viewpoint

A framework (G, L) can be formulated by the following system of 4 linear and $|E| - 1$ quadratic equations in $2|V|$ unknowns.

$$\begin{aligned} x_1 - c_1 &= 0 \\ y_1 - c_2 &= 0 \\ x_2 - (L_{1,2} + c_1) &= 0 \\ y_2 - c_2 &= 0 \\ (x_i - x_j)^2 + (y_i - y_j)^2 - L_{i,j}^2 &= 0 \quad \forall \{i, j\} \neq \{1, 2\} \in E \end{aligned}$$

The real roots of this system define a real algebraic variety. Questions concerning the framework (G, L) can be answered by studying the components of this variety.

Our Tools

Definition (Mixed Volume). Let P_1, \dots, P_n be n polytopes in \mathbb{R}^n . Then the *mixed volume* of P_1, \dots, P_n is defined as

$$MV_n(P_1, \dots, P_n) = \sum_{Q \text{ mixed cell of a mixed subdivision of } \sum P_j} \text{vol}_n(Q)$$

Definition (Newton Polytope). For $f = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^n} c_{\alpha} x^{\alpha} \in \mathbb{C}[x_1, \dots, x_n]$ the *Newton polytope* of f is defined as

$$\text{conv}(\{\alpha \in \mathbb{Z}_{\geq 0}^n : c_{\alpha} \neq 0\}).$$

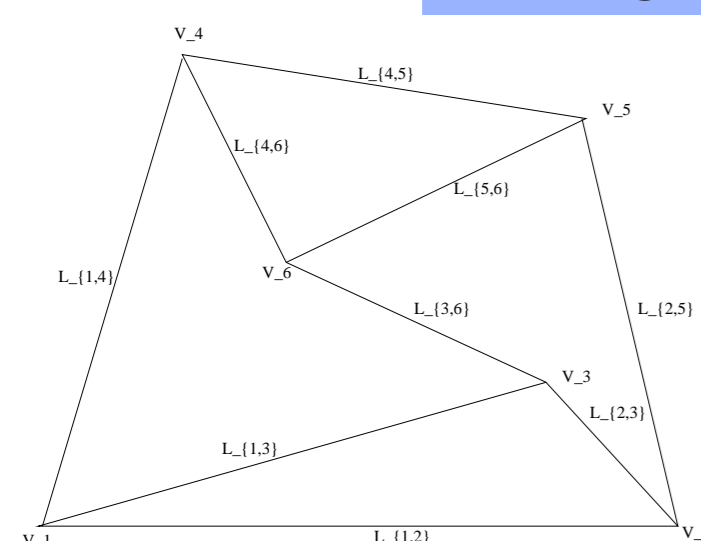
Theorem (Bernstein '75). Let $f_1, \dots, f_n \in \mathbb{C}[x_1, \dots, x_n]$ be polynomials with finitely many common zeros in $(\mathbb{C}^*)^n$, and let P_1, \dots, P_n be the Newton Polytopes of the f_i 's. Then the number of common zeros in $(\mathbb{C}^*)^n$ is less than or equal the mixed volume $MV_n(P_1, \dots, P_n)$. Equality holds if the coefficients of the f_i are generic.

Lemma (Separation Lemma). Let $\Gamma_1, \dots, \Gamma_k$ be polytopes in \mathbb{R}^{m+k} and $\Delta_1, \dots, \Delta_m$ be polytopes in $\mathbb{R}^m \subset \mathbb{R}^{m+k}$. Then

$$MV_{m+k}(\Delta_1, \dots, \Delta_m, \Gamma_1, \dots, \Gamma_k) = MV_m(\Delta_1, \dots, \Delta_m) * MV_k(\pi(\Gamma_1), \dots, \pi(\Gamma_k)),$$

where $\pi : \mathbb{R}^{m+k} \rightarrow \mathbb{R}^k$ denotes the projection on the last k coordinates.

Desargues Graph



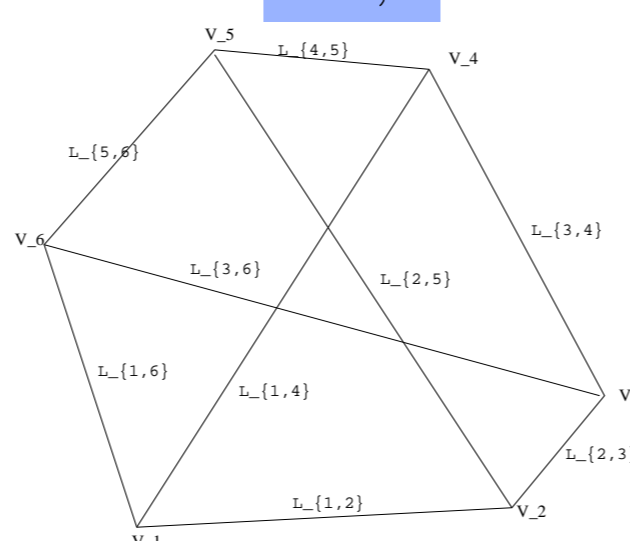
$$\begin{aligned} x_1 - 1 &= 0 \\ y_1 - 1 &= 0 \\ x_2 - (L_{1,2} + 1) &= 0 \\ y_2 - 1 &= 0 \\ (x_1 - x_3)^2 + (y_1 - y_3)^2 - L_{1,3}^2 &= 0 \\ x_2^2 - x_1^2 - 2x_2x_3 + 2x_1x_3 + y_2^2 - y_1^2 - 2y_2y_3 + 2y_1y_3 - L_{2,3}^2 + L_{1,3}^2 &= 0 \\ (x_1 - x_4)^2 + (y_1 - y_4)^2 - L_{1,4}^2 &= 0 \\ 2x_4 + 2x_5 - 2x_4x_5 + 2y_4 + 2y_5 - 2y_4y_5 - L_{4,5}^2 + L_{1,4}^2 + L_{2,5}^2 &= 0 \\ (x_2 - x_5)^2 + (y_2 - y_5)^2 - L_{2,5}^2 &= 0 \\ (x_3 - x_6)^2 + (y_3 - y_6)^2 - L_{3,6}^2 &= 0 \\ 2x_4 + 2x_6 - 2x_4x_6 + 2y_4 + 2y_6 - 2y_4y_6 - L_{4,6}^2 + L_{1,4}^2 + L_{3,6}^2 &= 0 \\ 2x_5 + 2x_6 - 2x_5x_6 + 2y_5 + 2y_6 - 2y_5y_6 - L_{5,6}^2 + L_{2,5}^2 + L_{3,6}^2 &= 0 \end{aligned}$$

There is a choice of edge lengths giving **24** distinct real embeddings of this framework. Borcea and Streinu's theorem tells us, that there are at most $\binom{8}{4} = 70$. With our approach we get a better bound.

The truncated system of equations corresponding to this framework is:

The Mixed Volume of this system is **32**.

$K_{3,3}$



Again, the above theorem gives 70 as a bound on the number of possible embeddings in \mathbb{R}^2 , and as above we can improve that bound using Bernstein's theorem. The Mixed Volume of the truncated system of equations corresponding to this framework is **32**.

Perspective

- Frameworks coming from Laman graphs that involve only Henneberg I steps in their construction are well understood. The (sharp) upper bound for the number of embeddings is $2^{|V|-2}$, which we can obtain with our method or even much simpler considerations. But graphs which involve Henneberg II steps still pose some difficulties.
- On the left we see the only examples of Non-Henneberg I-constructable graphs with 6 vertices. Since the Separation Lemma allows to 'split' larger graphs and compute the mixed volume for each component separately we can already treat a large class of graphs with our method.
- We would like to obtain a bound that respects the structure of the graph. (i.e. That takes into account the number of Henneberg I and Henneberg II steps needed to construct it.)