Seminar
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Seminar

P=W Conjecture

Sommersemester 2021

Non-Abelian Hodge theory gives a real analytic isomorphism between two algebraically quite different varieties associated to a Riemann surface X: The Betti moduli space, the moduli space of representation of the fundamental group of X into a complex reductive algebraic group G and the Dolbeault moduli space, the moduli space of G-Higgs bundles on X. It is a generalization of the Narasimhan-Seshadri correspondence between unitary representations of the fundamental group and stable vector bundles.

Analyzing the mixed Hodge structure of the character variety for $G = \mathsf{GL}(2, \mathbb{C})$ Hausel and Rodriguez-Villegas ([HR08]) found an unexpected symmetry, the curious hard Lefschetz theorem. (We will study parts of this paper, but note that it has been generalized meanwhile to the $\mathsf{GL}(n, \mathbb{C})$-character variety by Mellit [Mel19]).

The $P = W$-conjecture asks, if this special symmetry reflects a certain structure on the Higgs bundle moduli space - the so-called Hitchin system. More precisely, it asks, whether the weight filtration on the cohomology of the Betti moduli space is the same as the perverse filtration on the cohomology of the (diffeomorphic but not biholomorphic) Dolbeault moduli space defined by the Hitchin map. This conjecture is proven for the case $G = \mathsf{GL}(2, \mathbb{C})$ on smooth curves of any genus in [CHM12], as well as for the case $G = \mathsf{GL}(n, \mathbb{C})$ for any n but $g(X) = 2$ in [CMS19].

There is an even deeper conjecture, the geometric $P = W$-conjecture by Katzarkov and Noll and Pandit and Simpson [KNPS15], that aims to explain the $P = W$-conjecture by considering the dual boundary complex of a nice compactification of the Dolbeault moduli space. This will be mentioned in the overview talk, but not studied in detail.

A good (but brief) overview of the whole topic is given in the survey [Mig17].

Schedule We will meet on a two-weekly bases on Thursday afternoon 3.15 having two one-hour talks with a break in-between.

What we learn. On the one hand, we will survey the Betti, de Rham, and Dolbeault moduli space as well as the Non-Abelian Hodge correspondence. On the other hand, we will learn about the following useful tools that can be applied to study the cohomology of varieties over $\mathbb{C}$: Mixed Hodge structures, computing the E-polynomial by counting points over finite fields, intersection cohomology, perverse sheaves, perverse filtrations associate to algebraic maps. In particular, the talks 4 and 9 are independent of the remainder of the seminar.

Talks for Ph.D. Students. The program includes introductory talks to several topics of independent general interest that should be suitable to (first year) Ph.D. students, in particular

- Geometric invariant theory (Talk 2). Taking representations up to conjugation by GL_n requires to understand what GIT is about, and which notions of quotients are provided. No details on the construction, we assume its existence and apply it to explore the geometry of the Betti moduli space.
- Mixed Hodge structures (Talk 3). The talk assumes some familiarity with the Hodge decomposition and Hodge numbers as basic invariants of a smooth projective variety. If the variety is not smooth or only quasi-projective, the extra datum of the weight filtration encodes the required correction.
- Riemann-Hilbert correspondence (Talk 5). Here one can learn about flat bundles and local systems, which appear in many places in complex/algebraic geometry. There are different points of view according to the mathematical taste of the speaker. The algebro-geometric reference [Sza09] develops everything starting from covering theory and needs no prerequisites. The complex geometric reader, finds an easy accessible reference in [Kob87]. This correspondence is a baby version of Hilbert's 21'th problem solved by Deligne in much greater generality.
- Dolbeault moduli space (Talk 6). Here one can learn about holomorphic vector bundles and Higgs bundles on Riemann surfaces. One considers easy examples and basics on holomorphic vector bundles on Riemann surfaces. The existence of the moduli spaces is taken as black box. Finally one considers the very concrete example of $\mathsf{GL}(1,\mathbb{C})$-Higgs bundles and discusses the Hodge correspondence in this case as a leading example. Prerequisites: Riemann surfaces.
- Hitchin map and spectral data (Talk 7). Spectral data is the theory of eigenvalues and eigenspace of Higgs fields. So in some sense it is linear algebra. However, as the Higgs bundle is defined on a Riemann surface the local linear algebra glues to a interesting global picture. The eigenvalues define a complex curve covering the original Riemann surface and the eigen spaces define line bundles on this cover. This type of consideration has its origin in the theory of integrable systems. Prerequisites: Covering theory of Riemann surfaces and complex algebraic curves.
- Intersection cohomology (Talk 9). The speaker should not be afraid of dealing with complexes and derived categories. (Almost) no proofs, just comparison to (more) familiar statements, constructions, and examples. We need an overview what the device intersection cohomology is good for. Similar things can be said about Talk 10. We recommend [dM10] for a quick overview.

Abbreviations and Symbols

*	Very accessible topic
**	Medium level topic
***	Advanced topic
ITA	If time allows.

1. **What is $P = W$-conjecture about?** (60 min) 22.04.21 (Martin Möller)

   Overview of the framework of the conjecture. Give the context for the following detailed talks.

   References: [Mig17].

2. ***The Betti moduli space** (60 min) 22.04.21 (Felix Göbler)

   Introduce (twisted) $\mathsf{GL}(n, \mathbb{C})$-representation varieties [Mig17, Section 2.1], [HR08, Section 2.2]. Recall the construction of a categorical quotient by the spectrum of the invariants. GIT as a black box! Defined the (twisted) representation variety. Discuss smoothness, freeness of the action by conjugation and smoothness of the quotient in the twisted case [HR08, Section 2.2].
   ITA: Indicate that GIT quotient in untwisted is more complicated (non-trivial stabilizers, non-closed orbits). In this case, the GIT quotient is the moduli space of semi-simple representations (poly-stable points).

   References: [Mig17], [HR08], Background on GIT: [Bri].

3. ***Hodge structures and mixed Hodge structures** (60 min) 06.05.21 (Paul Kiefer)

   Briefly recall the Hodge structure of compact Kähler manifolds [Dur83, Section 1]. Introduce mixed Hodge structures and their properties for smooth/projective varieties [Dur83, Section 2]. State Deligne's theorem [Dur83, Section 3]. Give the examples of [Dur83, Section 4, 5, 6], most importantly for our seminar Section 6. Define mixed Hodge polynomial and E-polynomial [HR08, Definition 2.1.4].
   ITA: State the existence theorem for mixed Hodge structures of smooth varieties [PS08, Theorem II.4.2.]

   References: [HR08], [Dur83], Advanced reference: [PS08, Chapter I, Section II.4].

4. *****The mixed Hodge structure of the Betti moduli space** (60 min) 06.05.21 (Matti Würthen)

   State the curious Hard Lefschetz Theorem [HR08, Theorem 1.1.5], see also the summary in [Mig17, Section 4.1]. Give an outline how Hausel and Rodriguez-Villegas computed the E-polynomial by point counts over finite fields (Theorem 2.18 by Katz). The solutions of equations in finite groups (here: $\mathrm{GL}_n(\mathbb{F}_q)$) is given by a generalization of the Frobenius formula in Proposition 2.3.2. Indicate why the number of solutions is polynomial in q. Deduce the main formulas in Theorem 3.5.1 and Theorem 3.5.2. Use this for curious Poincaré duality Corollary 3.5.3 and ITA connectivity. (The proof of the curious Hard Lefschetz Theorem depends on the computation of the full Hodge polynomial, in this in turn on the structure of the cohomology ring. Time will not allow to enter this discussion here. We return to this in Talk 11.)

   References: [Mig17], [HR08].

5. ***The de Rham moduli space and Riemann-Hilbert correspondence** (60 min) 20.05.21 (Felix Röhrle)

Here the de Rham moduli space is introduced. It can be seen either as the moduli space of flat/integrable connections or as the moduli space of local systems on a closed surface. Start by introducing these objects. To a flat connection/local system one can associate a monodromy representation of the fundamental group of the closed surface by parallel transport. This defines one direction of the Riemann-Hilbert correspondence to the Betti moduli space. We obtain a flat bundle from a representation by a quotient construction [Kob87, Equation §I.(2.4)]. The correspondence is described in short in [Wen16, Section 3.1]. An easy accessible complex geometric approach is given in [Kob87, Section I.§1, 2] (here what we referred to as local system is denoted as flat bundle). An algebro-geometric treatment starting from the theory of covering spaces can be found in [Sza09, Section 2] emphasizing the point of view of local systems. After this pointwise Riemann-Hilbert correspondence define the de Rham moduli space as the space of isomorphim classes of semi-simple flat connections. (This is again a GIT quotient.) State the biholomorphism to the (untwisted) Betti moduli space. Mention that there is twisted de Rham moduli space [Mig17, Section 2.6] and that the Riemann-Hilbert correspondence generalizes to this case. ITA: Explain how this is related to Hilbert's 21th problem ([Mig17, Section 2.2],[Sza09, Section 2.7.10]).

References: [Kob87], [Sza09], [Wen16], Advanced reference: [Sim95, Section 6, 7].

6. ****The Dolbeault moduli space and the abelian Hodge correspondence** (60 min) 20.05.21 (Johannes Schwab)

   Recall holomorphic vector bundles on Riemann surfaces: Topological classification by rank and degree, Dolbeault formalism [Sch13, Section 1.2, 1.3]. Introduce $\mathsf{GL}(n, \mathbb{C})$-Higgs bundles on a closed Riemann surface and the notion of (semi-)stability [Mig17, Definition 2.10, 2.11]. Give local description of Higgs field as matrix-valued holomorphic one-forms. Give global example of rank 2 Higgs bundle, where the holomorphic vector bundle is a sum of two line bundles (ask Johannes for details). State the existence of the moduli space and some basic properties [Mig17, Page 473 (1)-(4)]. Explain the Non-Abelian Hodge correspondence for $\mathsf{GL}(1, \mathbb{C})$ (ask Johannes for details). Stress non-complex nature of diffeomorphism between Dolbeault and de Rham moduli and transcendental nature of correspondence between de Rham and Betti moduli space.

   References: [Mig17, Section 2.3] 2.3, Advanced references: [Nit91], [GX08], [Sim90, Introduction].

7. ****The Hitchin map and spectral data** (60 min) 10.06.21 (Riccardo Zuffetti)

   Introduce the Hitchin map for $\mathsf{GL}(n, \mathbb{C})$ and its main properties [Mig17, Section 4]. Define spectral curves and explain how to identify the generic Hitchin fibers as abelian varieties [Mig17, Section 3], [Hit87, Section 5.1] 5.1. Indicate how this generalizes to the case of integral spectral curves using torsion-free sheaves instead of line bundles [BNR89, Proposition 3.6]. ITA: Point out that this defines the structure of an algebraically completely integrable system on a dense subset of the Higgs bundle moduli space [Hit87, § 3], [Fre99, Definition 3.1].

   References: [Mig17, Section 3], [Hit87], [BNR89].

8. *****Non-Abelian Hodge Correspondence** (60 min) 10.06.21 (Jakob Stix)

   Start by putting on the (white)board a diagram of four moduli spaces. Dolbeault moduli space, de Rham moduli space, moduli of solutions to Hitchin's equation, moduli space of harmonic flat bundles. We get from the Dolbeault to the moduli of solutions to Hitchin's equation by the Hitchin-Simpson theorem [San, Theorem 3.3] (Black Box). We get form the De-Rham to the moduli of harmonic flat bundles by finding an harmonic hermitian metric on the flat bundle (Black Box: Coreltte-Donaldson Theorem [San, Theorem 2.9]). The correspondence between solutions to Hitchin's equation and harmonic flat bundles is a computation [San, Pages 6&7]. A more complete reference is [Wen16] and [LeP91]. Some familiarity with complex differential geometry is helpful. ITA: Emphasis that NAH is not complex analytic by pointing to the hyperkähler structure induced in this way [Wen16, Section 3.4.1].

   References: [Mig17, Section 2.4], [San] [Wen16], [LeP91].

9. ****The constructible derived category and intersection complexes** (60 min) 24.06.21 (Anton Güthge)

   The talks 9 and 10 should be prepared jointly. We recommend [dM10] for a quick overview. At the end of the two talks we should have seen and understood the objects in the decomposition theorem for proper morphisms, see [CM05, Theorem 2.3.3] for a recent source with full proofs, and [dM09, Theorem 1.6.1] and [Cat17, Theorem 1.6.2] for versions in nice survey papers, see also [PS08] IV.13. We propose to split the tasks as follows:

   Departing from the smooth situation, give the decomposition arising from the degeneration of the Leray spectral sequence and state how mixed Hodge structures and intersection cohomology deal with singularities [dM09, Section 1.3]

   Briefly recall derived categories. Introduce constructible complexes and the constructible bounded derived category. Define intersection complexes and give some of the examples of [dM10] and [dM09, Section 2.2].

   No proofs, just comparison to (more) familiar statements, constructions, and maybe examples (which in turn may require proof).

   References: [dM10], [dM09], [CM05], [Cat17], [PS08].

10. ****Perverse sheaves and the topology of algebraic maps** (60 min) 24.06.21 (Can Yaylali)

    The talks 9 and 10 should be prepared jointly. We recommend [dM10] for a quick overview. At the end of the two talks we should have seen and understood the objects in the decomposition theorem for proper morphisms, see [CM05, Theorem 2.3.3] for a recent source with full proofs, and [dM09, Theorem 1.6.1] and [Cat17, Theorem 1.6.2] for versions in nice survey papers, see also [PS08] IV.13.

    Introduce perverse sheaves and the middle perversity t-structure. Give some properties of perverse sheaves, e.g. finite extension of intersection complexes, the category of perverse sheaves is abelian, artinian and noetherian and invariant under Verdier duality.

State the decomposition theorem with its symmetries (duality, hard Lefschetz). All this is covered by [dM09, Section 2.3, 2.4, 5.7] with alternative versions in the sources below.

Apply this in the end to Hitchin map and formulate P=W conjecture, as in the the introduction of [CHM12]. Keep an eye at [CHM12, Section 1.4] for the objects (and notation we need). Define the perverse Leray filtration and give the geometric interpretation, when Y is affine.

References: [Mig17], [CM05], [dM09], [CHM12] [PS08] IV.13.

11. *****P=W for tautological classes** (60 min) 08.07.21 (Martin Ulirsch)

Explain the multiplicative generators for the cohomology of the Dolbeault moduli space (and Betti moduli space). (These are given in the introduction of [CHM12], see also [CMS19, Section 0.3] for an overview, or the original results independently by Hausel-Thaddeus and Markman. Take as black box that they generate.)

Show that their place in the perverse filtration [CHM12, Theorem 3.1.1] agrees with their place in the weight filtration [HR08, Section 4] (or Hausel-Thaddeus).

Then show that the assumption [CHM12, Assumption 1.4.5] on the support of the intersection complex implies the multiplicativity of the perverse filtration [CHM12, Proposition 1.4.11]. Check with Talk 12 and move this to Talk 12, if necessary. [1]

References: [HR08],[CHM12], [CMS19].

12. *****P=W for $G = \mathsf{GL}(2, \mathbb{C})$** (60 min) 08.07.21 (Luca Battistella)

Show that the assumptions for multiplicativity hold by proving [CHM12, Corollary 2.1.5], see the outline around Remark 2.1.6. The upper bound is given using the combinatorics of fibrations with A_k-singularities and compactified Jacobians. The lower bound is technical. Try to explain the Outline of the strategy for the proof of Theorem 2.3.1.

Wrap up the proof of the $P = W$-conjecture in the simplest case, e.g. [CHM12, Theorem 4.2.2].

References: [CHM12].

13. **Discussion of topics for WS 2021/2022** (60 min) 15.07.21 ()

Discussion of topics for next semester's seminar.

References: Bring your own book.

[1] This multiplicativity of the perverse filtration is equivalent to the $P = W$-conjecture, see [CMS19, Conjecture 0.3] proven in this paper as [CMS19, Theorem 0.6] with no hypothesis on the genus of the curve. Mention this! However this proof uses the full toolbox of their Theorems 0.4 and 0.5. We thus propose to present here and in the next talk to present the older and more direct proof, learning more about the cohomology of weak abelian fibrations.

Literatur

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