Orbifold Points on Prym-Teichmüller Curves

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Flat Surfaces & Teichmüller Curves

A **flat surface** is a pair (X, ω) where X is a compact Riemann surface of genus g and ω is a non-zero holomorphic differential. Integrating ω endows X (outside of the zeros of ω) with an atlas where all chart changes are (locally) translations. We may therefore picture (X, ω) as a polygon in the plane, whose sides are identified by translations.

A flat surface admits a natural $\mathrm{SL}_2(\mathbb{R})$-action by affine shearing of the flat structure. Consider now $\Omega\mathcal{M}_g$, the **moduli space of flat surfaces**, which admits a natural projection $\pi\colon \Omega\mathcal{M}_g \to \mathcal{M}_g$ to the moduli space of genus g curves. In the rare case that the projection $\pi(\mathrm{SL}_2(\mathbb{R})(X,\omega))$ is a curve in $\mathcal{M}_g$, we call this image a **Teichmüller curve (generated by** (X,ω)**)**.

The situation can be summarised by the following commutative diagram (note that $\mathrm{SO}(2)$ acts holomorphically):

$$\begin{array}{ccc} \mathrm{SL}_2(\mathbb{R}) & \xrightarrow{\;F\;} & \Omega\mathcal{M}_g \\ \downarrow & & \downarrow \\ \mathbb{H} \cong \mathrm{SO}(2)\backslash\mathrm{SL}_2(\mathbb{R}) & \xrightarrow{\;f\;} & \mathbb{P}\Omega\mathcal{M}_g \\ \downarrow & & \downarrow \pi \\ \mathcal{C} = \mathbb{H}/\Gamma & \longrightarrow & \mathcal{M}_g \end{array}$$

where the map F is given by the action $A \mapsto A \cdot (X,\omega)$.

Note that a Teichmüller curve is never compact, but always admits a finite number of **cusps**.

Orbifold Points

An **orbifold point** of an orbifold $\mathbb{H}/\Gamma$ is the projection of a fixed point of the action of Γ, i.e. a point $s \in \mathbb{H}$ such that $\mathrm{PStab}_\Gamma(s) \leq \mathrm{PSL}_2(\mathbb{R})$ is non-trivial. We call the cardinality of $\mathrm{PStab}_\Gamma(s)$ the **(orbifold) order of** s.

For a Teichmüller curve, this can be expressed in terms of the flat structure:

Lemma. *Let $\mathcal{C} = \mathbb{H}/\Gamma$ be a Teichmüller curve. Then (X,ω) corresponds to an orbifold point on $\mathcal{C}$ if and only if X admits a holomorphic automorphism σ such that*

$$\sigma^*\omega = \lambda\omega \text{ with } \lambda \in \mathbb{C}^* \setminus \{\pm 1\}.$$

For a curve $\mathcal{C}$, denote by χ the **orbifold Euler characteristic**, by h_0 the number of connected components, by C the number of cusps and by e_d the number of points of order d. Then this determines the **genus** g:

$$2h_0 - 2g = \chi + C + \sum_d e_d \left(1 - \frac{1}{d}\right).$$

McMullen's Prym Construction

Not many infinite families of (primitive) Teichmüller curves are known. For low genus, the following construction by McMullen gives a rich set of examples.

Let D be a **(real) discriminant**, i.e. $D > 0$ is not a square and $D \equiv 0$ or $1 \bmod 4$, and denote by $\mathcal{O}_D$ the unique **quadratic order** of discriminant D in $\mathbb{Q}(\sqrt{D})$.

Let $A = \mathbb{C}^2/\Lambda$ be a (polarised) **abelian surface**. Then we say that A admits **real multiplication by** $\mathcal{O}_D$ if there is an embedding $\iota\colon \mathcal{O}_D \hookrightarrow \mathrm{End}(A)$ that is self-adjoint with respect to the polarisation of A. Moreover, we say that real multiplication is **proper** if it cannot be extended to any larger order in $\mathbb{Q}(\sqrt{D})$.

Now, let (X,ω) be a flat surface admitting a holomorphic **Prym involution** $\rho\colon X \to X$ such that the quotient X/ρ has genus 2. Then $\Omega(X)$, the space of differentials on X, splits into ρ-eigenspaces $\Omega(X)^\pm$. We call

$$\mathcal{P}(X,\rho) = \ker(\mathrm{Jac}(X) \to \mathrm{Jac}(X/\rho)) = \frac{(\Omega(X)^*)^-}{\mathrm{H}_1(X,\mathbb{Z})^-}$$

the associated **Prym variety**.

Theorem ([McM03; McM06])**.** *Let X be of genus 2, 3 or 4 admitting a Prym involution ρ and differential ω such that*

- *ω has only one zero,*
- *$\rho^*\omega = -\omega$, and*
- *$\mathcal{P}(X,\rho)$ admits real multiplication by some $\mathcal{O}_D$ with ω as an eigenform.*

Then (X,ω) generates a Teichmüller curve $W_D(2g-2)$.

The curves $W_D(2g-2)$ are known as **Prym-Teichmüller** or **Prym-Weierstraß curves** and are non-empty for every discriminant D unless $g = 3$ and $D \equiv 5 \bmod 8$.

Remark. *For the curves $W_D(2)$ in $\mathcal{M}_2$, the cusps were described by McMullen [McM05], the orbifold points by Mukamel [Muk14], and the Euler characteristic was computed by Bainbridge [Bai07].*

Main Result

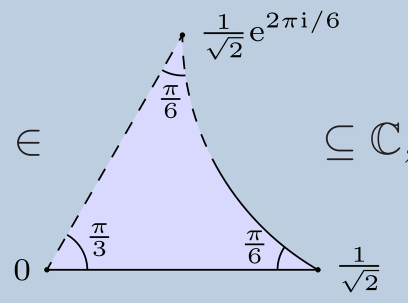
Theorem ([TTZ16a])**.** *For any non-square discriminant $D > 12$, the Prym-Teichmüller curves $W_D(4)$ in $\mathcal{M}_3$ have orbifold points only of order 2 and 3. More precisely:*

- *if D is odd, there are no points of order 2; otherwise*

$$e_2(D) = \#\{a,b,c \in \mathbb{Z} : a^2 + b^2 + c^2 = D\}/24;$$

- *the number of orbifold points of order 3 is*

$$e_3(D) = \#\{a,b,c \in \mathbb{Z} : 2a^2 - 3b^2 - c^2 = 2D,\ (*)\},$$

where condition $(*)$ restricts the set to those $a,b,c \in \mathbb{Z}$ that satisfy $f = \dfrac{c + b\sqrt{-3}}{2(a - \sqrt{D})} \in$  *$\subseteq \mathbb{C}$;*

- *$W_8(4)$ has one point of order 3 and one point of order 4;*
- *$W_{12}(4)$ has a single orbifold point of order 12.*

Moreover, let $D = f_0^2 D_0$ where D_0 is a fundamental discriminant. Then the above sets are all subject to the condition $\gcd(a,b,c,f_0) = 1$.

Remark. *By [LN14] and [Möl14], $W_D(4)$ is empty for $D \equiv 5 \bmod 8$. Moreover, by [LN14], $W_D(4)$ has two components iff $D \equiv 1 \bmod 8$.*

Theorem ([Zac16])**.** *If $D \equiv 1 \bmod 8$ and D is not a square, the two components of $W_D(4)$ are homeomorphic.*

Theorem ([TTZ16b])**.** *For any non-square discriminant $D > 12$, the Prym-Teichmüller curves $W_D(6)$ in $\mathcal{M}_4$ have orbifold points only of order 2 and 3. More precisely:*

- *if D is odd, there are no points of order 2; otherwise*

$$e_2(D) = \begin{cases} h(-D) + h(-\frac{D}{4}), & \text{if } D \equiv 12 \bmod 16, \\ h(-D), & \text{if } D \equiv 0, 4, 8 \bmod 16, \end{cases}$$

where $h(-D)$ is the class number of $\mathcal{O}_{-D}$;

- *the number of orbifold points of order 3 is*

$$e_3(D) = \#\{a,b,c \in \mathbb{Z} : a^2 + 3b^2 + (2c-b)^2 = D\}/12,$$

again subject to the condition that $\gcd(a,b,c) = 1$;

- *$W_5(6)$ has one point of order 3 and one point of order 5;*
- *$W_8(6)$ has one point of order 2 and one point of order 3;*
- *$W_{12}(6)$ has one point of order 2 and one point of order 6.*

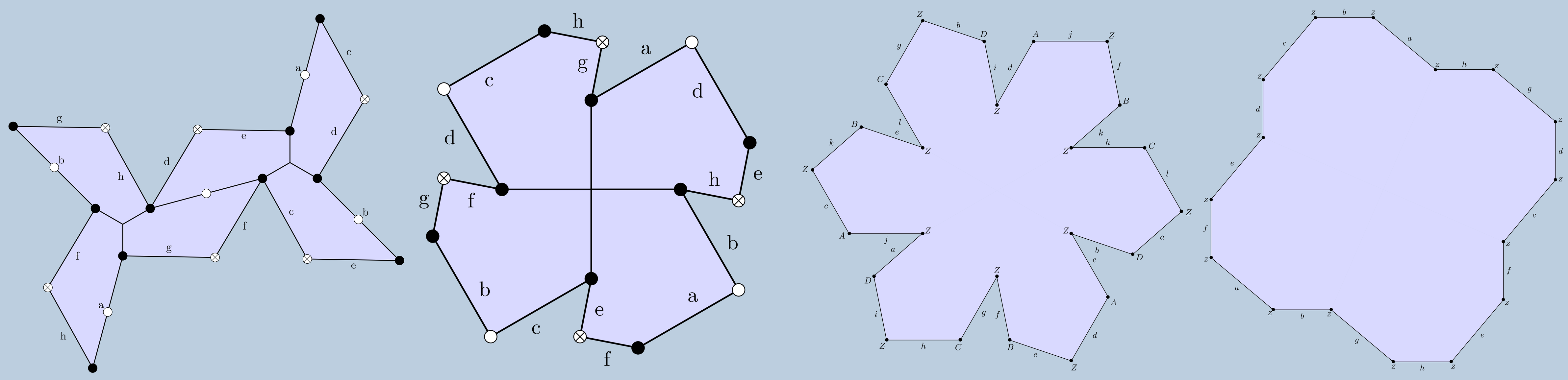
Remark. *The cusps and connected components of $W_D(4)$ and $W_D(6)$ are described in [LN14], while the Euler characteristics are computed in [Möl14]. Thus, this completes the topological classification of the Prym-Teichmüller curves.*

Theorem ([TTZ16b])**.** *There exist constants $C_1, C_2 > 0$, independent of D, such that the genus satisfies*

$$C_1 \cdot D^{3/2} < g(W_D(6)) < C_2 \cdot D^{3/2}.$$

Remark. *By Mukamel [Muk14], the genus of the curves $W_D(2)$ in $\mathcal{M}_2$ is also asymptotically $D^{3/2}$.*

Flat Pictures



$g = 3$: Choosing the side a as a complex parameter yields differentials with a $\mathbb{Z}/6$- and $\mathbb{Z}/4$-action, respectively, and a single 4-fold zero.

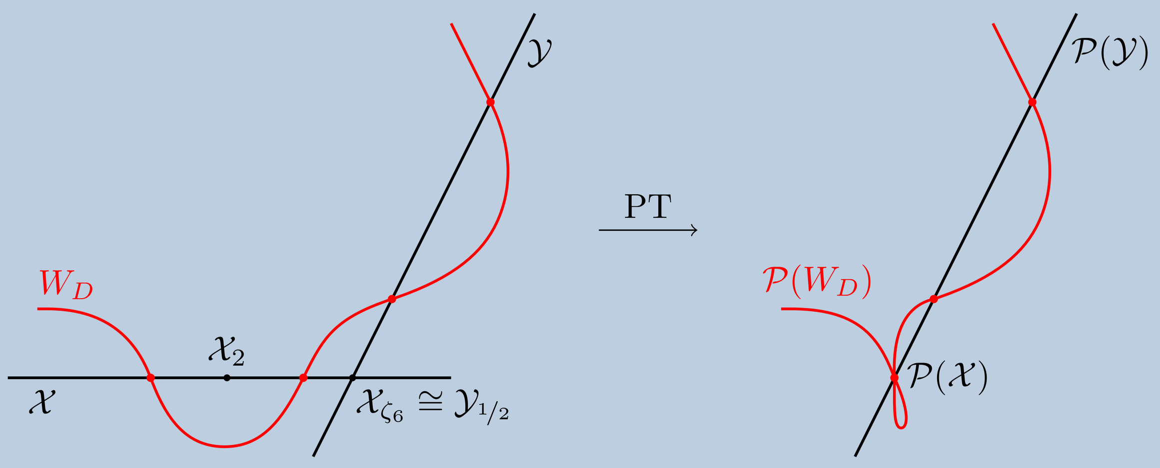
$g = 4$: Choosing the side d as a complex parameter yields differentials with a $\mathbb{Z}/6$- and $\mathbb{Z}/4$-action, respectively, and a single 6-fold zero.

Idea of Proof

Let (X,ω) be an orbifold point of order $2d$. Then X admits a holomorphic automorphism σ of order $2d$ that fixes the (single) zero of ω and descends to X/ρ.

Idea: study families of curves with such an automorphism admitting an eigenform with a single zero and check when the Prym part of such a curve admits real multiplication, i.e. **count intersections with the Teichmüller curve**.

For any such family $\mathcal{X}$, consider therefore the **Prym-Torelli image** $\mathrm{PT}(\mathcal{X})$, i.e. the family of abelian surfaces with fibres $\mathcal{P}(\mathcal{X}_t, \sigma^d)$. We must check these fibres for **real multiplication**. A sketch of the situation in $\mathcal{M}_3$:



More precisely, we obtain:

- For $g(X) = 3$, we have $g(X/\rho) = 1$, hence $\overline{\sigma}$ is of order $d = 2, 3, 4$ or 6 and $g(X/\sigma) = 0$. We thus obtain families of **cyclic covers of** $\mathbb{P}^1$ and in these cases the σ-eigenspace decomposition of $\Omega(X)$ is understood. In fact, all orders occur ($d = 4$ and 6 give 0-dimensional families).
- For $g(X) = 4$, we have $g(X/\rho) = 1$, hence $\overline{\sigma}$ is of order $d = 2, 3, 4, 5, 6, 8$ or 10. Using Riemann-Hurwitz, one can show that only $d = 2, 3, 5$ and 6 occur. The case $d = 2$ is special, because in this case the quotient is an elliptic curve. But this family can be constructed as a **fibre-product of the quotient elliptic curves**. Again, $d = 5$ and 6 give 0-dimensional families.

As the curves have many automorphisms, one can use **Bolza's method** and other tricks to calculate the endomorphism rings of the Prym part explicitly.

The following positive-dimensional families occur:

$g(\mathcal{X}_t)$	d	$\dim\mathcal{X}$	$\dim\mathrm{PT}(\mathcal{X})$	$\mathrm{Aut}(\mathcal{X}_t)$	$\mathrm{End}(\mathcal{P}(\mathcal{X}_t, \sigma^d))$
3	2	1	0	G	order in $M_2(\mathbb{Q}[i])$
3	3	1	1	$\mathbb{Z}/6$	order in $\left(\frac{2,-3}{\mathbb{Q}}\right)$
4	2	2 (1)	1	D_8	$M_2(\mathrm{End}(E_t))$
4	3	1	0	$\mathbb{Z}/6 \times \mathbb{Z}/2$	$M_2(\mathbb{Z}[\zeta_6])$

Here, $G = \mathbb{Z}/2 \ltimes (\mathbb{Z}/2 \times \mathbb{Z}/4)$, $E_t\colon y^2 = x(x-1)(x-t)$, and $\left(\frac{2,-3}{\mathbb{Q}}\right)$ denotes the quaternion algebra over $\mathbb{Q}$.

Remark. *Note that $g = 3 = d$ gives the Shimura curve uniformised by $\Delta(2,6,6)$, explaining the hyperbolic triangle in the theorem.*

Remark. *Observe that for $g = 4$, $d = 2$, the D_8-family is 2-dimensional. However, restricting to curves that admit an eigendifferential with a sixfold zero reduces the dimension.*

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