Affine groups of flat surfaces

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Introduction

According to Thurston's classification, a diffeomorphism of a closed oriented surface is either elliptic, reducible or a pseudo-Anosov diffeomorphism. The structure of pseudo-Anosov diffeomorphisms, e.g. their dilatation coefficients, and the corresponding measured foliations have been intensely investigated since. Rather than focusing on properties of a single diffeomorphism, the purpose of this chapter is to study the flat surfaces the pseudo-Anosov diffeomorphisms live on together with their whole group of affine diffeomorphisms.

A *flat surface* is a pair (X, ω) consisting of a Riemann surface X together with a holomorphic one-form $\omega \in \Gamma(X, \Omega^1_X)$. Equivalently, flat surfaces arise from gluing rational-angled planar polygons by parallel translations along their edges. Furthermore, flat surfaces naturally arise when studying the trajectories of a ball on a rational-angled billiard table.

One of the basic invariants of a flat surface (X, ω) is the *affine group* $\mathrm{SL}(X, \omega)$ (also called *Veech group*) defined as follows. Let $\mathrm{Aff}^+(X, \omega)$ be the group of orientation-preserving diffeomorphisms that are affine on the complement of the zero set of ω with respect to the charts defined by integrating ω. The linear part of the affine map is independent of the charts and provides a map

$$D : \mathrm{Aff}^+(X, \omega) \to \mathrm{SL}_2(\mathbb{R}).$$

The image of D is called the affine group $\mathrm{SL}(X, \omega)$. The interest in these groups stems from Veech's paper [37], where flat surfaces are constructed whose affine groups are non-arithmetic lattices in $\mathrm{SL}_2(\mathbb{R})$. We will discuss them and recent developments in Section 4.

This chapter touches the following aspects of affine groups. The affine group is often said to be trivial for a generic surface. If we define generic to be meant in the strata of a natural stratification of the space of flat surfaces, this is a little imprecise. Our first goal is to give a complete description of the generic affine group. Next we recall Thurston's construction of pseudo-Anosov diffeomorphisms using a pair of multicurves. This construction has a lot of flexibility and produces rather large affine groups. Results of McMullen resp. of Hubert-Lanneau show that in genus two all pseudo-Anosov diffeomorphisms arise in this way but that this holds no longer for $g \geq 3$.

In Sections 4 and 5 we review the known constructions of very large affine groups: lattices and infinitely generated affine groups. In the last section we discuss some relations between the size of the affine group and the closure of $\mathrm{SL}_2(\mathbb{R})$-orbit of the corresponding flat surface in the moduli space of flat surfaces.

We remark that the affine group is similarly defined for pairs (X, q) of a Riemann surface X and a quadratic differential q. But such a surface admits a canonical double covering which is a flat surface. Hence up to passing to finite index subgroups all the information is contained in affine groups of flat surfaces.

The whole topic is not nearly completely understood at the time of writing. Consequently, the content of this chapter reflects simply the present state of knowledge and almost all sections are concluded by an open problem.

The author thanks Erwan Lanneau for a helpful discussion on the proof of Theorem 1.1

1 Basic properties of affine groups

Our first aim is to realize that for a general flat surface nothing exciting happens. In order to define what 'general' means, we define the parameterizing space of flat surfaces. Let M_g denote the moduli space of curves of genus g. Over M_g there is a vector bundle of rank g whose fiber over a point corresponding to the surface X is the vector space of holomorphic one-forms (or *abelian differentials*) on X. Let ΩM_g be the total space of this vector bundle minus the zero section. By construction, flat surfaces correspond to points of ΩM_g. The space ΩM_g is stratified into subspaces

$$\Omega M_g = \bigcup_{\sum_{i=1}^n k_i = 2g-2} \Omega M_g(k_1, \ldots, k_n)$$

according to the number and multiplicities of the zeros of the holomorphic one-form ω. Some of the strata are not connected, see [17]. A component of a stratum is called *hyperelliptic component*, if it consists exclusively of hyperelliptic curves, i.e. curves with a degree two map to the projective line.

The strata are complex orbifolds that carry a natural complex coordinate system, called period coordinates, whose definition will be recalled below. We say that (X, ω) is generic in its stratum, if it lies outside a countable union of real codimension one submanifolds in its stratum.

Theorem 1.1. *For $g(X) \geq 2$, the affine group of a generic surface (X, ω) is $\mathbb{Z}/2$ or trivial, depending on whether (X, ω) belongs to a hyperelliptic component or not.*

Before we can give the proof we need to recall some facts on flat surfaces and to classify affine diffeomorphisms in order to explain the notions in Thurston's theorem stated at the beginning of the introduction.

As stated in the introduction, a *flat surface* is a pair (X, ω) of a Riemann surface X together with a holomorphic one-form ω. A flat surface has a finite number of zeros of ω, called *singularities*. These correspond to points where the total angle with respect to $|\omega|$ exceeds 2π. On a flat surface we may talk of geodesics with respect to the metric $|\omega|$. Such a geodesic has a well-defined *direction* in $\mathbb{RP}^1$. A geodesic joining two singularities or a singularity to itself is called *saddle connection*.

Definition 1.2. A diffeomorphism φ of X is called *elliptic* if it is isotopic to a diffeomorphism of finite order. A diffeomorphism φ is called *reducible* if it is isotopic to a diffeomorphism fixing a (real) simple closed curve on X. If φ is neither reducible nor elliptic, then φ is called *pseudo-Anosov* .

We alert the reader, that we follow the common abuse of the notion diffeomorphism for homeomorphisms that are C^1 outside a finite set of points ([7], Exposé V).

It is easy to see that an affine diffeomorphism φ of (X, ω) is elliptic, if it is of finite order. In particular $D(\varphi)$ is of finite order. Conversely, if $D(\varphi)$ is of finite order, then φ is of finite order, since $\mathrm{Ker}(D)$ consists of holomorphic diffeomorphisms of X and consequently $\mathrm{Ker}(D)$ is finite by Hurwitz' theorem.

If φ is a pseudo-Anosov diffeomorphism, there exists a pair (X, q) such that φ is an affine diffeomorphism of (X, q). As stated above, we will restrict to the case that $q = \omega^2$. Moreover, (X, ω) can be chosen such that φ stretches the horizontal lines by some factor $\lambda > 1$, called *dilatation coefficient*, and contracts the vertical lines by the same factor λ. Thus, $|\mathrm{tr} D(\varphi)| > 2$ for an affine pseudo-Anosov diffeomorphism.

Consequently, an affine diffeomorphism φ with $|\mathrm{tr} D(\varphi)| = 2$, i.e. such that $D(\varphi)$ is parabolic, is a reducible affine diffeomorphism. We briefly recall the structure of such a *parabolic diffeomorphism* . Say the horizontal direction is the eigendirection of $D(\varphi)$. Then some power of φ fixes all the finitely many horizontal saddle connections and the complement of these saddle connections has to consist of metric cylinders.

In order to define coordinates on a stratum of ΩM_g, fix locally on some open set U a basis of the integral homology $H_1(X, Z(\omega), \mathbb{Z})$ relative to $Z(\omega)$, the zeros of ω. The cardinality of the basis is $N = 2g - 1 + n$, where n is the number of zeros of ω. The map $U \to \mathbb{C}^N$, that maps (X, ω) to the integrals of ω along the fixed basis, is a local diffeomorphism ([36], see [28] for an algebraic proof). The system of coordinates is called index *period coordinates*.

There is a natural action of $\mathrm{GL}_2^+(\mathbb{R})$ on ΩM_g. In terms of period coordinates, consider

$$\mathbb{C}^N \cong \mathbb{R}^N \otimes_{\mathbb{R}} \mathbb{R}^2$$

and let $\mathrm{GL}_2^+(\mathbb{R})$ act naturally on $\mathbb{R}^2$. This is equivalent to letting an element of $\mathrm{GL}_2^+(\mathbb{R})$ act on the local complex charts of X given by integration of ω as real linear map. This action leaves the affine group essentially unchanged, we have for $A \in \mathrm{GL}_2^+(\mathbb{R})$

$$\mathrm{SL}(A \cdot (X, \omega)) = A \cdot (\mathrm{SL}(X, \omega)) \cdot A^{-1}.$$

Proof of Theorem 1.1. For each of the countably many pseudo-Anosov diffeomorphisms φ in the mapping class group there is a unique flat surface (X, q) or (X, ω) up to the action of $\mathrm{GL}_2^+(\mathbb{R})$, such that φ is an affine diffeomorphism on (X, q) or (X, ω) respectively. Consequently, the set of flat surfaces whose affine group contains a pseudo-Anosov element is a countable union of real 4-dimensional subspaces. Since we assume $g(X) \geq 2$, the generic flat surface does not carry any affine pseudo-Anosov diffeomorphism.

Suppose that $\mathrm{SL}(X, \omega)$ contains a parabolic element. By the classification above, (X, ω) decomposes into metric cylinders in some direction. The boundaries of these cylinders consist of saddle connections and, since $g \geq 2$, at least two of them, say γ_1 and γ_2, are not homologous, i.e. they are linearly independent elements of $H_1(X, Z(\omega), \mathbb{Z})$. Since the saddle connections are parallel, the periods of γ_1 and γ_2 are $\mathbb{R}$-linearly dependent. The locus of surfaces where γ_1 and γ_2 are linearly dependent is of $\mathbb{R}$-codimension at least one in period coordinates. Since the γ_i in question are two elements in the countable group $H_1(X, Z(\omega), \mathbb{Z})$, the generic flat surface does not contain an affine parabolic element.

The remaining discussion serves to prove that the number of elliptic affine diffeomorphisms is as small as claimed. Suppose the generic flat surface (X, ω) in a stratum contains such a diffeomorphism φ of finite order. Each stratum contains square-tiled surfaces and their affine group is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ (see Section 4). In ΩT_g, the pullback bundle to Teichmüller space, the presence of an affine diffeomorphism of finite order is a closed condition. Consequently, a generic affine diffeomorphism must be reflected in the affine group of all square-tiled surfaces, hence $\mathrm{ord}(D(\varphi)) \in \{1, 2, 3, 4, 6\}$.

First suppose that $D(\varphi)$ is the identity or minus the identity, in particular $D(\varphi) \in \mathrm{SO}_2(\mathbb{R})$. Then φ is in fact an automorphism of X and fixes $q = \omega^{\otimes 2}$. We are faced with the problem of classifying strata of half-translation surfaces that consist entirely of pullbacks of half-translation surfaces of lower genus. This classification was solved in [18], although precisely the case of squares of abelian differential was excluded from the discussion in loc. cit. The difference between the cases is apparent whenever the dimension count is involved, since

$$\dim Q_g(k_1, \ldots, k_n) = 2g - 2 + n \quad \text{while} \quad \dim \Omega M_g(k_1, \ldots, k_n) = 2g - 1 + n.$$

We let g_0 denote the genus of the quotient surface $X/\langle\varphi\rangle$ and we let d be the degree of the covering, i.e. the order of φ. Moreover, let p be the number of poles of q, let r be the number of zeros of q over which the covering $\pi : X \to X/\langle\varphi\rangle$ is unramified, and let m be the number of regular points of q, over which π is ramified. Finally, let n be the number of zeros of q over which π is ramified.

The first case is $D(\varphi) = \mathrm{id}$. Then ω is the pullback of an abelian differential and we obtain, as in loc. cit. using the Riemann-Hurwitz formula, that

$$(d - 1)(2g_0 - 2 + n + r) \leq -m.$$

We deduce $g_0 = 0$, which is absurd since the projective line carries no abelian differentials.

The second case is $D(\varphi) = -\mathrm{id}$. Then ω is the pullback of a strictly quadratic differential. An analysis of the covering results this time in

$$(d - 1)(2g_0 - 2 + m + n + r) \leq \begin{cases} m(d - 2) + pd/2 - 1 & \text{if } d \text{ is even} \\ m(d - 2) + p(d - 1)/2 - 1 & \text{if } d \text{ is odd.} \end{cases}$$

This implies $g_0 = 0$ and for $d = 2$ one obtains the hyperelliptic components. For $d \geq 3$ we deduce $n + r \leq 1$ and since $p \geq 4$ the case $n + r = 1$ is absurd. If $n + r = 0$, we conclude that $p = 4$, that d is even and that $m \in \{1, 2\}$. This case is excluded in the same way as the corresponding case in the proof of Theorem 1 in [18].

We finally have to treat the cases where $D(\varphi)$ has order $3, 4$ or 6. In this cases $D(\varphi)$ is conjugate to an element in $\mathrm{SO}_2(\mathbb{R})$ and φ is actually an automorphism if $D(\varphi) \in \mathrm{SO}_2(\mathbb{R})$. Consequently, for each (X, ω) the $\mathrm{SL}_2(\mathbb{R})$-orbit contains a flat surface where the conjugate of φ is actually an automorphism.

It thus suffices to prove that in each stratum the locus of flat surfaces with an automorphism of order $3, 4$ and 6 is of codimension more than one.

We start with the case $\mathrm{ord}(D(\varphi)) = 3$. Consider the surface $X/\langle\varphi\rangle$ marked with s images of the ramification points. We give the details in the case $\mathrm{ord}(\varphi) = 3$, in all other cases even cruder dimension estimates suffice. The quotient surface has

$$3g_0 - 3 + s = g - 1$$

moduli by Riemann-Hurwitz. If (X, ω) lies in the generic stratum, the locus of flat surfaces with such an automorphism has dimension

$$g - 1 + g < (4g - 3) - 1 = \dim \Omega M_g(1, \dots, 1) - 1,$$

since $g > 1$. If (X, ω) lies in a non-generic stratum S, then the fiber of $S \to M_g$ has at most dimension $g - 1$ and again the locus of flat surfaces with such an automorphism has dimension less that $\dim(S) - 1$.

The case $\mathrm{ord}(D(\varphi)) = 6$ is contained in the previous one by considering φ^2. The same trick allows to reduce the case $\mathrm{ord}(D(\varphi)) = 4$ to the hyperelliptic loci. Again the Riemann-Hurwitz formula yields that the quotient surface has not enough moduli. $\square$

Proposition 1.3. *([37] Proposition 2.7) The group* $\mathrm{SL}(X, \omega)$ *is a discrete subgroup of* $\mathrm{SL}_2(\mathbb{R})$*.*

Proof. Let φ_n be a sequence of affine diffeomorphisms such that $D(\varphi_n)$ converges to the identity. By Arzela-Ascoli and after passing to a subsequence, we may suppose that φ_n converges to some affine diffeomorphism φ uniformly on X. Hence for large enough n, the composition $\varphi_n\varphi_{n+1}^{-1}$ is isotopic to the identity. Using Thurston's classification of diffeomorphisms this is not possible unless $D(\varphi_n) = \mathrm{id}$ for large enough n. $\square$

Concerning the existence of cyclic affine groups, the parabolic case is easy, while the hyperbolic case seems wide open at present. The proof of Proposition 1.4 will be given in the next section.

Proposition 1.4. *In every stratum there exist flat surfaces whose affine group is cyclic generated by a parabolic element.*

Question 1.5. Does there exist a flat surface (X, ω) whose affine group $\mathrm{SL}(X, \omega)$ is cyclic generated by a hyperbolic element?

2 Thurston's construction and implications for the trace field

The following construction first appears in Thurston's famous 1976 preprint ([33]), see also [30], [37], [19], [10] and [24] and in the chapter by Harvey in volume I of this handbook ([9]) for versions and different presentations.

A *multicurve* A on a surface Σ_g of genus g is a union of disjoint essential simple closed curves, no two of which bound an annulus. A pair (A, B) of multicurves *fills* (or *binds*) the surface if for each curve in A and each curve in B the geometric intersection number is minimal in their homotopy classes and if the complement $\Sigma_g \setminus (A \cup B)$ is a simply connected polygonal region with at least 4 sides.

We index the components of A and B such that $A = \cup_{i=1}^{a} \gamma_i$ and $B = \cup_{i=a+1}^{a+b} \gamma_i$ and let C be the (unsigned) intersection matrix of A and B, i.e. for $i \neq j$ we have $C_{ij} = |\gamma_i \cap \gamma_j|$ and $C_{jj} = 0$ for all j.

As additional input datum for the construction we fix a set of multiplicities $m_i \in \mathbb{N}$ for $i = 1, \ldots, a+b$. Since (A, B) fills Σ_g, the intersection graph is connected and the matrix $(m_i C_{ij})$ is a Perron-Frobenius matrix. Hence there is a unique positive eigenvector (h_i) up to scale such that

$$\mu h_i = \sum_{j=1}^{j=a+b} m_i C_{ij} h_j \tag{2.1}$$

for some positive eigenvalue μ.

We now glue a surface X from rectangles $R_p = [0, h_i] \times [0, h_j] \subset \mathbb{C}$ for each intersection point $p \in \gamma_i \cap \gamma_j$. Namely, glue R_p to R_q along the vertical (resp. horizontal) sides whenever p and q are joined by an edge in A (resp. B) of the graph $A \cup B$. The differentials dz^2 on each rectangle glue to a global quadratic differential q on X.

Let τ_i be the Dehn twist around γ_i and define

$$\begin{aligned} \tau_A &= \textstyle\prod_{i=1}^{a} \tau_i^{m_i} \\ \tau_B &= \textstyle\prod_{i=a+1}^{a+b} \tau_i^{m_i}. \end{aligned}$$

Theorem 2.1. *([33]) The flat surface (X, q) constructed above contains affine diffeomorphisms τ_A and τ_B with derivatives*

$$D\tau_A = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \quad \textit{and} \quad D\tau_B = \begin{pmatrix} 1 & 0 \\ -\mu & 1 \end{pmatrix}$$

In particular the elements $\tau_A^n \tau_B$ are pseudo-Anosov diffeomorphisms for n large enough.

Proof: By construction the modulus of the cylinder with core curve γ_i is m_i/μ. Hence the powers of the Dehn twists occurring in the definition of τ_A

and τ_B have linear part as claimed. They fix the boundary of the horizontal resp. vertical cylinders and together define affine diffeomorphisms.

In order to check the last claim, one has to recall that an affine diffeomorphism is pseudo-Anosov if and only if the absolute value of its trace is greater than two. □

Since we are dealing exclusively with flat surfaces in the sequel, we remark that the quadratic differential has a squre root, i.e. $q = \omega^2$ if and only if for a suitable orientation of the γ_i their geometric and algebraic intersection numbers coincide.

2.1 Trace fields of affine groups

Given a pair (X, ω) resp. (X, q) we define the *trace field of the affine group* $\mathrm{SL}(X, \omega)$ to be $K = \mathbb{Q}(\mathrm{tr}(A), A \in \mathrm{SL}(X, \omega))$ The notion of trace field is a useful invariant since it turns out to be stable under passing to a finite index subgroup.

Theorem 2.2 ([16], Appendix; [22]). *Let $A = D\varphi \in \mathrm{SL}(X, \omega)$ be any hyperbolic element. Then the trace field of the affine group equals the trace field of φ. More precisely, if $\mathrm{SL}(X, \omega)$ contains a hyperbolic element A, then the $\mathbb{Q}$-vector space generated by the periods of ω is a 2-dimensional K-vector space, where $K = \mathbb{Q}(\mathrm{tr}(A))$.*

With this result we can obviously determine the trace fields of affine groups arising from Thurston's construction.

Corollary 2.3. *If φ is constructed using a pair of multicurves then $K = \mathbb{Q}(\mu^2)$, where μ is as in equation* (2.1).

Hubert and Lanneau have shown that Thurston's construction imposes a restriction on the trace field. We will see below (Corollary 3.1) that this property does not hold for all pseudo-Anosov diffeomorphisms.

Theorem 2.4 ([10]). *If (X, ω) is given by Thurston's construction, then the trace field K of $\mathrm{SL}(X, \omega)$ is totally real, i.e. all embeddings $K \to \mathbb{C}$ factor through $\mathbb{R}$. In particular, if $\mathrm{SL}(X, \omega)$ contains two non-commuting parabolic elements then K is totally real.*

Proof of Theorem 2.4. Let D_m be the diagonal matrix with entries m_i. The square of the largest eigenvalue of the matrix C (as in Thurston's construction) is the largest eigenvalue of the matrix C^2. Hence we have to show that all the eigenvalues of $(D_m C)^2$ are real.

Suppose first for simplicity $m_i = 1$ for all i. Since for some matrix C_0 we have

$$D_m C = C = \begin{pmatrix} 0 & C_0 \\ C_0^T & 0 \end{pmatrix}, \quad \text{hence} \quad (D_m C)^2 = C^2 = \begin{pmatrix} C_0 C_0^T & 0 \\ 0 & C_0^T C_0 \end{pmatrix}.$$

Since C^2 is symmetric, all its eigenvalues are real. Thus $\mathbb{Q}(\mu^2)$ is totally real.

If the m_i are no longer identically one, $(D_m C)^2$ is still similar to a symmetric matrix: Split D_m into two pieces D'_m and D''_m of size a resp. b and let $D'_{\sqrt{m}}$ resp. $D''_{\sqrt{m}}$ denote the diagonal matrix with entries $\sqrt{m_i}$. Then

$$(D_m C)^2 = \begin{pmatrix} D'_m C_0 D''_m C_0^T & 0 \\ 0 & D''_m C_0^T D'_m C_0 \end{pmatrix}.$$

The upper block decomposes as

$$D'_m C_0 D''_m C_0^T = D'_{\sqrt{m}} (D'_{\sqrt{m}} C_0 D''_{\sqrt{m}})(D'_{\sqrt{m}} C_0 D''_{\sqrt{m}})^T (D'_{\sqrt{m}})^{-1}$$

and for the lower block the same trick works. The above conclusion about the eigenvalues thus still holds. □

We can now easily give a proof of a statement from the previous section.

Proof of Proposition 1.4. It is easy to construct in each stratum a flat surface (X, ω) that consists of only one cylinder horizontally. Consequently, $\mathrm{SL}(X, \omega)$ contains a parabolic element φ irrespectively of the lengths of the horizontal saddle connections. Since $g(X) > 1$ by hypothesis, we may arrange that the periods of all horizontal saddle connections generate a K-vector space of dimension two or more, where K is real, but not totally real. By Theorem 2.2 and Theorem 2.4, the affine group $\mathrm{SL}(X, \omega)$ does not contain two non-commuting parabolic elements. Suppose $\mathrm{SL}(X, \omega)$ contains a hyperbolic or an elliptic element ψ. Then φ and $\psi\varphi\psi^{-1}$ are non-commuting parabolic elements. This contradiction completes the claim. □

Remark 2.5. Recent results on Thurston's construction can be found in [19]. E.g., the smallest dilatation coefficients of the pseudo-Anosov diffeomorphisms arising from Thurston's construction are determined there.

3 The Arnoux-Yoccoz surface as a multi-purpose counter-example

The following construction is a special case of what has become known as the construction of zippered rectangles ([35], [20]). We sketch the version in [2] where the construction is given for hyperelliptic curves of genus g. The construction appeared originally in [3]. It will be used below to refute many naive conjectures one could derive seeing only the construction from the preceding sections.

We first construct an *interval exchange transformation* $f : [0,1) \to [0,1)$, i.e. a map that consists of translations on a subdivision of intervals. Here we take α to be the real root of

$$\alpha^g + \alpha^{g-1} + \cdots + \alpha = 1$$

and subdivide $[0,1)$ into g pairs of subintervals I_k of lengths $\alpha^k/2$, k=1,…,g. We let f be the exchange of the pairs of same length composed with a half turn, where $[0,1)$ is identified with a circle. Explicitly, $f = f_r \circ f_s$, where for $k = 0, \dots, g-1$ we let

$$f_s(x) := \begin{cases} x + \alpha^{k+1}/2 & \text{if} \quad x \in [\sum_{i=1}^k \alpha^i, \alpha^{k+1}/2 + \sum_{i=1}^k \alpha^i) \\ x - \alpha^{k+1}/2 & \text{if} \quad x \in [\alpha^{k+1}/2 + \sum_{i=1}^k \alpha^i, \sum_{i=1}^{k+1} \alpha^i) \end{cases}$$

and define

$$f_r(x) = \begin{cases} x + 1/2 & \text{if} \quad x \in [0, 1/2) \\ x - 1/2 & \text{if} \quad x \in [1/2, 1). \end{cases}$$

The interval exchange f enjoys a remarkable 'self-similarity' property,, inherited by the form of the minimal polynomial of α. The map

$$\varphi_I : \begin{cases} [0,1) & \to \quad [0,\alpha) \\ x & \mapsto \quad \begin{cases} \alpha x + (\alpha + \alpha^{g+1})/2 & \text{if} \quad x \in [0, (1-\alpha^g)/2) \\ \alpha x - (\alpha + \alpha^{g+1})/2 & \text{if} \quad x \in [(1-\alpha^g)/2, 1), \end{cases} \end{cases}$$

which shrinks the interval linearly by the factor α and then exchanges two pieces (of unequal length), commutes with f resp. its induction on the subinterval $[0, \alpha)$, i.e.

$$\varphi_I \circ f = f|_{[0,\alpha)} \circ \varphi_I,$$

where

$$f|_{[0,\alpha)}(x) = f^n(x), \text{ where } n \in \mathbb{N}_{>0} \text{ is minimal such that } f^n(x) \in [0, \alpha).$$

Suppose we let each of the intervals I_i be the bottom of a rectangle of height h_i and glue the top of these rectangles to $[0,1)$ according to f. Under some conditions (see e.g. [35] or the survey [38] §5 for details) the sides of the rectangles can be glued to yield a translation surface. Here we take

$$h_i = \sum_{j=1}^{g-k} \alpha^j \quad \text{for} \quad i = 2k+1, 2k+2.$$

The resulting flat surface $(X_{\mathrm{AY}}, \omega_{\mathrm{AY}})$ has genus g with two singularities of type $g-1$, i.e. with angle $2g\pi$.

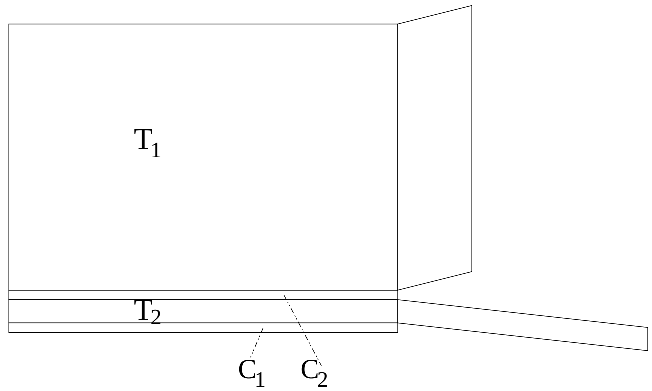

Figure 1. Topology of a special diagonal direction on the Arnoux-Yoccoz surface, displayed horizontally

The main point in this choice of heights is that the self-similarity of the base interval is also reflected in the gluing of the vertical sides. That is, there is a map $\varphi : X_{\mathrm{AY}} \to X_{\mathrm{AY}}$ that restricts to φ_I on the segment $[0,1)$ and which stretches the vertical side by α^{-1}. Obviously φ is a pseudo-Anosov diffeomorphism with dilatation α. Since $\mathbb{Q}(\alpha + 1/\alpha)$ is not totally real, we conclude:

Corollary 3.1. *([10]) There exist flat surfaces with a pseudo-Anosov diffeomorphism whose trace field is not totally real. In particular, there exist flat surfaces with a pseudo-Anosov diffeomorphism that does not arise via Thurston's construction.*

The Arnoux-Yoccoz surfaces were thought to be good candidates to answer Question 1.5 affirmatively. But at least for $g = 3$ this is not the case.

Theorem 3.2. *([15]) For $(X_{\mathrm{AY}}, \omega_{\mathrm{AY}})$ the Arnoux-Yoccoz surface with $g(X) = 3$ the group $\mathrm{SL}(X, \omega)$ is not cyclic.*

We sketch the proof in order to illustrate a phenomenon that yet needs deeper investigation. First, there exist many (diagonal) directions on $(X_{\mathrm{AY}}, \omega_{\mathrm{AY}})$ that topologically look like the horizontal one in Figure 1, called $2T2C$-direction in [15]. The vague 'looks like topologically' can be made precise using numerical invariants of a given direction, like the widths, heights and twists and some finite data, called *combinatorics*. The reader may consult [15] for the definition of these invariants.

Second, many of these $2T2C$-directions have the same combinatorics and the same projectivised tuple of numerical invariants. Consequently, for each such pair of directions p, q, there exists an affine diffeomorphism of (X, ω), that maps p to q. Necessarily, such a diffeomorphism is pseudo-Anosov.

Finally, there exist pairs of $2T2C$-directions on $(X_{\mathrm{AY}}, \omega_{\mathrm{AY}})$ with the same projectivised invariants, such that the corresponding pseudo-Anosov diffeomorphism is not a power of the diffeomorphism φ constructed above. Consequently, the affine group is not cyclic.

Question 3.3. How large is the affine group of $(X_{\mathrm{AY}}, \omega_{\mathrm{AY}})$? Is it finitely generated or infinitely generated?

We finally mention three more questions, the Arnoux-Yoccoz surface provides a negative answer to. First, the dilatation coefficient of a pseudo-Anosov diffeomorphism of a surface of genus g is an algebraic number of degree r at most $2g$ over $\mathbb{Q}$. In Thurston's original examples r turned out to be even, but the Arnoux-Yoccoz surface shows that odd r is possible too ([3]).

Second, the directional flow on a flat surface defines an interval exchange transformation (IET). An IET has an easily computable invariant, the SAF-invariant (compare [1] for the definition), that vanishes if the directional flow has periodic orbits only. For a surface of genus two the converse holds, but the Arnoux-Yoccoz surface shows that the converse does not hold in genus three ([1]).

The third question concerns $\mathrm{SL}_2(\mathbb{R})$-orbit closures and will be dealt with in the last section.

4 Large affine groups: Veech surfaces

A flat surface (X, ω) is called a *Veech surface* if $\mathrm{SL}(X, \omega)$ is a lattice in $\mathrm{SL}_2(\mathbb{R})$. We do not want to address the dynamics of flat surfaces here, but we mention the most striking result, Veech's dichotomy ([37]), for later use. If (X, ω) is a Veech surface then for each direction either

- all geodesics are uniformly distributed, in particular dense, or,
- all geodesics are closed or a saddle connection. Such directions are called *periodic*.

The presence of saddle connections on (X, ω) forces the lattice $\mathrm{SL}(X, \omega)$ to be non-cocompact. Up to coverings, all Veech surfaces known at the time of writing except for one arise from two fundamental constructions which we explain below:

- Quotients of cyclic coverings of the projective line branched at 4 points.
- Eigenforms for real multiplication by a quadratic field in genus two and the Prym variants in genus $g \leq 5$.

From the point of view of affine groups we can state this result as follows:

Theorem 4.1. *([5]) All triangle groups* (m, n, ∞) *for* $1/m + 1/n < 1$ *and* $m, n \leq \infty$ *arise as affine groups of Veech surfaces.*

Theorem 4.2. *([21], [6]) All real quadratic fields arise as trace fields of lattice affine groups.*

Before we give the proofs we need to recall some background. Recall from Section 1 the action of $\mathrm{GL}_2^+(\mathbb{R})$ on flat surfaces. By a theorem of Smillie (see [32] for a recent proof), the orbit of (X, ω) is closed in ΩM_g if and only if $\mathrm{SL}(X, \omega)$ is a lattice in $\mathrm{SL}_2(\mathbb{R})$. In this case the image C of the orbit in M_g is a complex, in fact algebraic curve, called a *Teichmüller curve*. We mention that such an algebraic curve is a totally geodesic subsurface for the Teichmüller metric, whence the name, but we won't need details on Teichmüller theory. Instead of considering a Teichmüller curve as a curve in the moduli space of curves, it is often useful to restrict the universal family over M_g to a family $f : \mathcal{X} \to C$ over the Teichmüller curve and to study f instead.

Proof of Theorem 4.1. We extract from [5] the special case where m, n are both odd, finite and coprime. This case illustrates almost all ideas, except for a fiber product construction needed to cover the general case.

The basic idea is to study a family of cyclic coverings ramified over the projective line at 4 points. There is a criterion ([26], we will apply the version [5] Theorem 1.2 (b)) that detects Teichmüller curves by the existence of an eigenspace of the relative de Rham cohomology, whose monodromy group is the affine group. For appropriate cyclic coverings, there is such an eigenspace whose monodromy group is the desired triangle group. But the family of cyclic coverings does not quite match the cohomological criterion as we shall see, so we need furthermore to find a suitable quotient family.

Consider the family of cyclic degree N covering

$$\mathcal{Y}_t : \qquad y^N = x^{a_1}(x-1)^{a_2}(x-t)^{a_3}$$

of $\mathbb{P}^1_x$ with t varying in $\mathbb{P}^1_t \smallsetminus \{0, 1, \infty\}$, where $N = 2mn$ and

$$a_1 = 2mn - m + n, \quad a_2 = 2mn + m - n, \quad a_3 = 2mn + m + n, \quad a_4 = 2mn - m - n.$$

The coverings is ramified precisely over $x = 0$, $x = 1$, $x = t$ and $x = \infty$. Let $\mathbb{L}(i)$ denote the ζ_N^i-eigenspace of the relative de Rham cohomology for the automorphism $\varphi : (x, y) \mapsto (x, \zeta_N y)$. The local systems $\mathbb{L}(1)$, $\mathbb{L}(-1)$, $\mathbb{L}(mn+1)$, $\mathbb{L}(mn-1)$ are isomorphic and the a_i are chosen such that the monodromy group is $\Delta(m, n, \infty)$.

We claim that we can lift the automorphisms of $\mathbb{P}^1_x$ that interchange the points $\{0, 1, t, \infty\}$ in pairs to an automorphism group H of $\mathcal{Y}_t$ such that the stable model of the fibers $\mathcal{Y}_0/H$ and $\mathcal{Y}_1/H$ are smooth. Let $\mathcal{X} = \mathcal{Y}/H$ be the quotient and denote by $f : \mathcal{X} \to \mathbb{P}^1_t$ the corresponding family of curves. Given the claim, the local system

$$\mathbb{L}(1) \oplus \mathbb{L}(-1) \oplus \mathbb{L}(\alpha) \oplus \mathbb{L}(-\alpha)$$

is made to be H-invariant. Thus, the de Rham cohomology of $\mathcal{X}_t$ has a local subsystem of rank two with discrete monodromy group $\Delta(m, n, \infty)$ and the fibers of the family $\mathcal{X}_t$ are smooth precisely over $\mathbb{H}/\Delta(m, n, \infty)$. We conclude using the characterization of Teichmüller curves given in [5] Theorem 1.2 (b).

To establish the claim, choose elements $t^{1/n}, (t-1)^{1/m} \in \overline{\mathbb{C}(t)}$ and define

$$c = (t-1)^{\sigma_2+\sigma_3}, \qquad d = t^{\sigma_1+\sigma_3}. \tag{4.1}$$

We now define $H = \langle \sigma, \tau \rangle$ by

$$\sigma(z) = cd\frac{x(x-1)}{y(x-t)} = cd\frac{-y}{(x-t)^2}$$

and for $\alpha \equiv 1 \mod m$ and $\alpha \equiv -1 \mod n$

$$\tau(z) = d\frac{y(\alpha)}{x^2}.$$

The fiber over $t = 0$ consists of two smooth components with affine charts $y^N = x^{a_1}(x-1)^{a_3}$ (since branch points of type a_1 and a_3 have come together) and $y^N = x^{a_2}(x-1)^{a_4}$. They meet in $\gcd(a_1 + a_3, N)$ points transversally. One of the elements in H exchanges the two components and, in fact, fixes the intersection points of the components. Consequently, the quotient is smooth as claimed. See [5] for the details. □

Proof of Theorem 4.2. Consider a curve X of genus two, such that its Jacobian $\mathrm{Jac}(X)$ has more endomorphisms than just multiplication by an integer, namely such that $\mathrm{End}(\mathrm{Jac}(X))$ is an order $\mathfrak{o}_D$ in a real quadratic field $\mathbb{Q}(\sqrt{D})$. These endomorphisms act on the space of holomorphic one-forms of $\mathrm{Jac}(X)$, which is in natural bijection with the space of holomorphic one-forms on X. Let $\mathcal{E}_D \subset \Omega M_2$ be the locus of flat surfaces (X, ω), such that $\mathrm{Jac}(X)$ has real multiplication by $\mathfrak{o}_D$ and such that ω is an eigenform for the action of $\mathfrak{o}_D$ on the space of holomorphic one-forms. Obviously, $\mathcal{E}_K \subset \Omega M_2$ is a closed subvariety and the main point is to show that $\mathcal{E}_D$ is invariant under the action of $\mathrm{SL}_2(\mathbb{R})$. Granted this, the intersection $W_D = \mathcal{E}_D \cap \Omega M_2(2)$ is again closed and $\mathrm{SL}_2(\mathbb{R})$-invariant. A local dimension count shows that the image of W_D in M_2 is a curve, by construction a Teichmüller curve.

We now single out the role of genus two rather than rigorously proving the main point. Let $A = \mathbb{C}^g/\Lambda$ be a g-dimensional abelian variety. An endomorphism of A consists of an endomorphism of the lattice Λ plus a linear map of $\mathbb{C}^g$ with the obvious compatibility condition. Suppose that (X, ω) is an eigenform for real multiplication and T a generator of $\mathfrak{o}_D$. For any $M \in \mathrm{GL}_2^+(\mathbb{R})$ let $M \cdot (X, \omega) = (Y, \eta)$. By definition of the $\mathrm{GL}_2^+(\mathbb{R})$-action, there is an affine diffeomorphism $\varphi_M : X \to Y$. The map $\varphi_M \circ T \circ \varphi_M^{-1}$ defines a map of Λ, where $\mathrm{Jac}(Y) = \mathbb{C}^g/\Lambda$. Moreover this map preserves the complex line

$$M \cdot \langle \Re\omega, \Im\omega \rangle = \langle \Re\eta, \Im\eta \rangle.$$

Since here $g = 2$, the orthogonal complement of $\langle \Re\omega, \Im\omega \rangle$ with respect to the symplectic form is also a complex line, also preserved by $\varphi_M \circ T \circ \varphi_M^{-1}$. Consequently, (Y, ω) has also real multiplication by $\mathfrak{o}_D$. See [21] for the missing details. □

In fact, the orbifold Euler characteristic of the quotients $\mathbb{H}/\mathrm{SL}(X, \omega)$ for (X, ω) as constructed in Theorem 4.2 has been determined by Bainbridge ([4]). The complete description of the structure (starting with the number of elliptic elements) of these Veech groups is an open question. Only for the 12 smallest examples, when $\mathbb{H}/\mathrm{SL}(X, \omega)$ is a rational curve, the affine groups are known by generators and relations. It would be interesting to have such a description of the Veech groups for the whole series.

Square-tiled surfaces, covering constructions

Let (Y, η) be a Veech surface. A point P on Y is called a *periodic point*, if the orbit $\mathrm{SL}(X, \omega) \cdot P$ is finite. A covering surface $\pi : X \to Y$ provided with the flat structure $\omega = \pi^*\eta$ is again a Veech surface if and only if π is branched at most over periodic points ([8]).

For the rest of this section we suppose that Y is the torus. In this case, periodic points are precisely the torsion points on Y, if we normalize $0 \in Y$ to be one of the branch points. For more on periodic points on flat surfaces of higher genera, see [27]. Composition of $X \to Y$ with the multiplication on Y ensures that the composition map is ramified over the origin only. These flat surfaces are called *square-tiled surfaces*, sometimes also *origamis*. By [8] the affine group of a square-tiled surface is a subgroup of finite index in $\mathrm{SL}_2(\mathbb{Z})$. We mention two results indicating that many types of subgroups of $\mathrm{SL}_2(\mathbb{Z})$ arise as affine group.

Theorem 4.3 ([12], [31]). *With the exception of the covering consisting of three squares, the affine groups of square-tiled surfaces in ΩM_2 are non-congruence subgroups. In any genus $g \geq 2$ there are square-tiled surfaces, whose affine group is a non-congruence subgroup.*

Theorem 4.4 ([31]). *All congruence subgroups of* $\mathrm{SL}_2(\mathbb{Z})$ *with possibly* 5 *exceptions occur as affine groups of square-tiled surfaces.*

Question 4.5. Is there a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ that is *not* the affine group of a square-tiled surface?

5 Also large affine groups: infinitely generated

There exist two constructions for infinitely generated affine groups. McMullen's construction ([22]) gives a complete description in genus two but the techniques apply to genus two only. On the other hand, the construction of Hubert and Schmidt ([13]) is a way to construct a flat surface with infinitely generated affine group starting from a Veech surface with a special point. The resulting surfaces have genus at least four.

We sketch both constructions and conclude with a number of open questions concerning the precise structure of the limit of these infinitely generated affine groups.

Theorem 5.1. *([22]) Suppose that* $(X, \omega) \in \Omega M_2(1,1)$ *has a hyperbolic element in its affine group, but* (X, ω) *is neither in the* $\mathrm{GL}_2^+(\mathbb{R})$*-orbit of the regular decagon nor obtained as a covering of the torus. Then* $\mathrm{SL}(X, \omega)$ *is infinitely generated.*

Sketch of proof. Veech surfaces in $\Omega M_2(1,1)$ are either in the orbit of the decagon or torus coverings ([27], [23]). Hence it suffices to show that once $\mathrm{SL}(X, \omega)$ contains a hyperbolic element the limit set of $\mathrm{SL}(X, \omega)$ is the whole S^1. For that purpose it is enough to show that each direction s joining a zero and a Weierstraß point decomposes the surface into cylinders of commensurable moduli, since then the affine group contains a parabolic element in such a direction and since those directions are dense in S^1.

In order to prove this, one first shows that the presence of the hyperbolic element implies that the SAF-invariant of the induced interval exchange transformation (IET) on a transverse interval to s vanishes. (The Galois flux used in [22] is a quantity equivalent to the SAF-invariant.) In genus two, due to the bad approximation of quadratic irrationals, this implies that the IET is not minimal. Topological considerations using the Weierstraß point imply that the direction s decomposes into cylinders. Using the presence of the hyperbolic element again, one checks that the moduli of the cylinders have to be commensurable. □

Theorem 5.2. *([13]) For* $g \geq 4$ *there exist flat surfaces* (X, ω)*, whose affine group is infinitely generated.*

More precisely, take any of the Veech surfaces in genus two with trace field $K \neq \mathbb{Q}$ *(see Theorem 4.2) and normalize it by* $\mathrm{GL}_2^+(\mathbb{R})$ *to have periods in* $K(i)$*. Then a covering ramified over a Weierstraß point and a non-Weierstraß point with coordinates in* $K[i]$ *has infinitely generated Veech group.*

Proof. Recall the definition of a periodic point from Section 4. In order to ensure that $\mathrm{SL}(X, \omega)$ is infinitely generated, the branch points must not be exclusively periodic points on the one hand and not too general either for

$\mathrm{SL}(X,\omega)$ might become trivial then. A *connection point* P on Y has the property that every straight line emanating from a singularity of Y and passing through P ends in a singularity, i.e. yields a saddle connection.

Suppose that (Y,η) admits a non-periodic connection point P. The subgroup $\mathrm{SL}(P) \subset \mathrm{SL}(X,\omega)$ that fixes P is not of finite index, since P is not periodic. On the other hand, for each direction of a geodesic from a singularity to P there is a parabolic element σ in $\mathrm{SL}(X,\omega)$ by Veech dichotomy and the definition of a connection point. A suitable power of σ fixes all saddle connections, hence lies in $\mathrm{SL}(P)$.

Since the set of directions joining P to a singularity is dense in S^1, there is a dense set of directions in S^1 fixed by some parabolic element in $\mathrm{SL}(P)$. Said differently, the limit set of $\mathrm{SL}(P)$ is S^1. Consequently, $\mathrm{SL}(P)$ is infinitely generated.

It thus suffices to find a Veech surface with a non-periodic connection point. The periodic points of the Veech surfaces from Theorem 4.2 are precisely the Weierstraß points by [27]. A Veech surface normalized as in the second statement of the theorem is said to have *strong holonomy type*, if the set of periodic directions is precisely $\mathbb{P}^1(K)$. It is straightforward to check that strong holonomy type implies that point with coordinates in $K[i]$ are connection points. Finally, [22] Theorem A.1 implies that all the Veech surfaces in question are of strong holonomy type. □

We remark that the abundance of surfaces of strong holonomy type is a particular property of genus $g = 2$, too.

Some more results on the structure of infinitely generated affine groups are known ([14]). For example $\mathbb{H}/\mathrm{SL}(X,\omega)$ has infinitely many cusps and infinitely many infinite ends. Yet, many questions concerning these infinitely generated groups both for the case of [13] and [22] remain open, in particular the convergence behavior of the associated Poincaré series.

6 The size of the affine group compared to the size of the orbit closure

In Section 4 we have encountered Veech surfaces. Their affine group is, by definition, large, and the $\mathrm{GL}_2^+(\mathbb{R})$-orbit is closed in ΩM_g, it projects to a Teichmüller curve. In genus two, actually the motto 'the larger the affine group the smaller the orbit closure' holds.

Theorem 6.1. *([25]) Suppose that $g(X) = 2$ and that $\mathrm{SL}(X,\omega)$ contains a hyperbolic element. Then the closure of the $\mathrm{GL}_2^+(\mathbb{R})$-orbit of $\mathrm{SL}(X,\omega)$ projects to an orbifold of dimension one or two in M_2.*

In fact, this projection is a Teichmüller curve if $(X, \omega) \in \Omega M_2(2)$. If $(X, \omega) \in \Omega M_2(1,1)$, then the projection is a Teichmüller curve or it is the preimage of a Hilbert modular surface in the moduli space of abelian surfaces under the Torelli map.

In genus three this motto no longer holds:

Theorem 6.2. *([15]) The affine group of the Arnoux-Yoccoz surface contains a hyperbolic element and nevertheless the orbit closure is as big as possible, namely the hyperelliptic locus $\mathcal{L}$ in the connected component of $\Omega M_3(2,2)$ which does not consist entirely of hyperelliptic surfaces.*

There is a $\mathrm{GL}_2^+(\mathbb{R})$-equivariant map from $\mathcal{L}$ to the stratum $Q(1,1,1,1)$ of quadratic differentials in genus 2 with 4 simple zeros. Consequently, the above statement can be rephrased as follows: There is a surface in $Q(1,1,1,1)$, whose affine group contains a hyperbolic element and whose $\mathrm{GL}_2^+(\mathbb{R})$-orbit closure is the whole stratum $Q(1,1,1,1)$.

We now explain the idea of proof of both theorems. The starting point is to reduce the orbit closure question for flat surfaces to a question in a homogeneous space, where Ratner's theorem predicts how orbit closures look like. For that purpose, one needs to cut the surface along saddle connections in some fixed direction into tori and cylinders. In order to be able to do so in a neighborhood of the surface, too, the slitting configuration has to stable under small deformation. This means that the saddle connections have to be homologous.

Such sets of homologous saddle connections are rather rare, but in genus two each surface admits such a set ([25]) and in the locus $\mathcal{L}$ the generic surface does. In $\mathcal{L}$, the horizontal saddle connections in Figure 1 split the surface into two tori and two cylinders. Not all surfaces in $\mathcal{L}$ admit such a $2T2C$-direction, but the Arnoux-Yoccoz surface does.

Cut the surface in pieces along the homologous saddle connections. The difference between genus two and genus three becomes apparent in the application of Ratner's theorem to the splitting pieces. In genus two, if $\mathrm{SL}(X, \omega)$ contains a hyperbolic element, then the two splitting pieces are isogenous tori and the orbit closure is a ('small') unipotent subgroup of $(\mathrm{GL}_2^+(\mathbb{R}))^2$. The major remaining step to complete the proof of Theorem 6.1 consists in showing that flat surfaces that split into isogenous tori have Jacobians with real multiplication. Consequently, compare to the proof of Theorem 4.2, the $\mathrm{GL}_2^+(\mathbb{R})$-orbit closure is contained in the preimage of a Hilbert modular surface

In the case of the Arnoux-Yoccoz surface however, the splitting pieces are 'as incommensurable as possible' despite the presence of a hyperbolic element in $\mathrm{GL}_2^+(\mathbb{R})$. Consequently, an application of Ratner's theorem yields a large orbit closure and a second application in a different $2T2C$-direction implies that the orbit closure is the whole locus $\mathcal{L}$.

As a first, and maybe important, step towards extending a Ratner type theorem from genus two to genus three we are thus led to ask:

Question 6.3. How can one describe the locus of flat surfaces in $\mathcal{L}$ that admit a $2T2C$-direction?

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