PILLOWCASE COVERS: COUNTING FEYNMAN-LIKE GRAPHS ASSOCIATED WITH QUADRATIC DIFFERENTIALS

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Abstract. We prove the quasimodularity of generating functions for counting pillowcase covers, with and without Siegel-Veech weight. Similar to prior work on torus covers, the proof is based on analyzing decompositions of half-translation surfaces into horizontal cylinders. It provides an alternative proof of the quasimodularity results of Eskin-Okounkov and a practical method to compute area Siegel-Veech constants.

A main new technical tool is a quasi-polynomiality result for 2-orbifold Hurwitz numbers with completed cycles.

Contents

1. Introduction

Mirror symmetry for elliptic curves can be phrased in its tropical version by stating that the Hurwitz number counting covers of elliptic curves can be computed as Feynman integrals and that the corresponding generating functions are quasimodular forms ([BBBM17], [GM18], [Dij95], [KZ95], [EO01]). A Feynman integral is physics inspired terminology for an integral over a product of derivatives of propagators (i.e. Weierstrass $\wp$-functions), the form of the product being encoded by a (Feynman) graph.

The goal of this paper is to show that the mirror symmetry story has a complete analog in the scope of pillowcase covers, covers of the projective line with profile

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$(2,\dots,2)$ over 3 points, with profile $(\nu,2,\dots,2)$ over a special point and possibly a fixed finite number of even order branch points elsewhere (see Section 2). These pillowcase covers arise naturally in the volume computation for strata of quadratic differentials.

Our generalized Feynman graphs have a special vertex 0 corresponding to the branch point with profile $(\nu,2,\dots,2)$ and, most important, come with an orientation of the half-edges, so that the edge contribution to the Feynman integrand is $\wp(z_i \pm z_j)$ according to whether the half-edges are inconsistently or consistently oriented along the edge. Those generalized Feynman integrals are quasimodular forms, now for the subgroup $\Gamma_0(2)$, rather than for $\mathrm{SL}(2,\mathbb{Z})$ in the case of torus covers. The argument is a rather straightforward generalization of the torus cover case, see Theorem 5.6 and Theorem 6.1, and compare to [GM18, Section 5 and 6]. The two papers are intentionally parallel whenever possible, to facilitate comparison. In particular, both papers start with a correspondence theorem (Proposition 2.1) that can certainly be rephrased in terms of covers of tropical curves ([GM18, Section 8], [BBBM17]).

To arrive from there at our main goals, we need moreover a structure theorem for the algebra of shifted quasi-polynomials and a polynomiality theorem for orbifold double Hurwitz numbers, see the end of the introduction. Altogether, we can first give another proof of the following theorem of Eskin-Okounkov ([EO06]). We let $N^\circ(\Pi) = \sum N_d^\circ(\Pi) q^d$ be the generating series of torus covers with branching profile Π.

Theorem 1.1. *(= Corollary 7.4) For any ramification profile* Π *the counting function* $N^\circ(\Pi)$ *for connected pillowcase covers of profile* Π *is a quasimodular form for the group* $\Gamma_0(2)$ *of mixed weight less or equal to* $|\Pi| + \ell(\Pi)$*.*

The starting point of this paper was to obtain the following version for a weighted count, motivated by the computation of area-Siegel-Veech constants (see Section 6 for a brief introduction, see [EMZ03] and [EKZ14] for more background.).

Theorem 1.2. *[= Corollary 8.2] For any ramification profile* Π *and any odd integer* $p \geq -1$ *the generating series* $c_p^\circ(\Pi)$ *for counting connected pillowcase covers with* p*-Siegel-Veech weight is a quasimodular form for the group* $\Gamma_0(2)$ *of mixed weight at most* $|\Pi| + \ell(\Pi) + p + 1$*.*

To explain the use of this result, we compare the knowledge about strata of the moduli space of abelian differentials $\Omega\mathcal{M}_g(\mu)$ and quadratic differentials $\mathcal{Q}(\mu)$ with respect to Masur-Veech volumes and Siegel-Veech constants at the time of writing. In the abelian case our understanding is nearly complete. Siegel-Veech constants can be computed recursively by computing ratios of Masur-Veech volumes of boundary strata ([EMZ03]). These volumes can be computed efficiently by counting torus covers and closed formulas derived from this ([EO01], [CMZ18]). The volumes have an interpretation as intersection numbers of tautological classes ([Sau18], [CMS18]) and the formulas are well-understood, so as to give large genus asymptotics in all detail ([CMZ18], [Agg18], [CMS18]).

For the moduli space of quadratic differentials much less is known, except for strata of genus zero surfaces whose volumes are explicitly computable ([AEZ16]). Siegel-Veech constants are also related to Masur-Veech volumes by a recursive procedure ([MZ08], [Gou15]). But these volumes are much harder to evaluate for higher genus, despite the work of [EO06], and some hints being given in [Eng17]. The behavior of the large genus asymptotics is conjectured in [DGZZ18] for the sequence of principal strata. Only for the principal strata an interpretation as intersection number is known ([DGZZ18]).

In the current status of knowledge, Theorem 1.1 and Theorem 1.2 provide (besides structural insight) a somewhat reasonable practical way to compute volumes and Siegel-Veech constants for strata of quadratic differentials by computing the coefficients for sufficiently many small d in order to determine the quasimodular form uniquely and then using the growth rate of the coefficients. This procedure is explained, along with the technical steps of the proof, in an example in Section 9. Algorithms that compute volumes and Siegel-Veech constants for quadratic differentials as efficiently as in the abelian case still have to be found.

Finally, we explain the 'local' polynomiality results that are the intrinsic reason for quasimodularity. In the case of torus covers, double Hurwitz numbers arise naturally by slicing the torus. These numbers are polynomials if one uses completed cycles at each slice, see e.g. [SSZ12]. Together with a theorem that shows that certain graph sums with polynomial local contributions are quasimodular and a graph combination argument to pass to completed cycles p_ℓ we obtained in [GM18] quasimodularity for torus covers.

In the case of pillowcase covers, slicing the pillowcase, some 2-orbifold Hurwitz numbers arise naturally at special slices. We show in Theorem 7.2 that these 2-orbifold Hurwitz numbers are quasi-polynomials (rather than just piece-wise quasi-polynomials) if the 2-orbifold carries only products of the completed cycles $\overline{p}_k$ in the algebra of shifted symmetric quasi-polynomials (see Section 3). Note that quasi-polynomiality of 2-orbifold Hurwitz numbers fails even for the completed cycles p_ℓ. The *quasi*-polynomiality is the cause of quasimodularity of the associated generating series for the subgroup $\Gamma_0(2)$ rather than the full group $\mathrm{SL}(2,\mathbb{Z})$.

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2. Counting covers of the pillow by global graphs

The goal of this section is the basic correspondence theorem Proposition 2.1 and its variants. It allows to count covers of the pillow by counting graphs with various additional decorations. As in the abelian case, the correspondence theorem works only if we count coverings without unramified components. We thus start with standard remarks on the passage between the various ways of imposing connectivity in the counting problems.

2.1. **Covers of the pillow and their Hurwitz tuples.** We give a short introduction to Hurwitz spaces of covers of the pillow $B \cong \mathbb{C}P^1$ and recall some basic notions needed in the sequel. We provide the pillow with the flat metric that identifies B with two squares of side length $1/2$ glued back to back. We will denote the corners of the pillow by $P_1, \dots, P_4$.

A *pillowcase cover* is a cover of degree $2d$ of B fully branched with d transpositions over three corners of the pillow, with all odd order branching stocked together with transpositions over the remaining corner of the pillow, and with all other even order branch points at arbitrary points different from the corners.

Let $\Pi = (\mu^{(1)}, \cdots, \mu^{(n+4)})$ consist of the following types of partitions. We impose that $\mu^{(1)} = (\nu, 2^{d-|\nu|/2})$ where ν is a partition of an even number into odd parts, we require that $\mu^{(2)} = \mu^{(3)} = \mu^{(4)} = (2^d)$ and finally that $\mu^{(i+4)} = (\mu_i, 1^{2d-\mu_i})$ with μ_i a cycle. We call Π a *ramification profile* and we define g by the relation

$$\ell(\mu) + \ell(\nu) - |\mu| - |\nu|/2 \;=\; 2 - 2g\,,$$

where $\ell(\cdot)$ denotes the length of a partition and $|\cdot|$ the size of a partition. We write $\Pi_\emptyset$ for the profile with $n = 4$ and $\mu^{(1)} = (2^d)$.

Let $H_d(\Pi)$ (or just H if the parameters are fixed) denote the n-dimensional *Hurwitz space* of degree $2d$, genus g, coverings $p : X \to \mathbb{P}^1$ of a curve of genus zero with $n+4$ branch points and ramification profile Π, i.e. we require that over the i-th branch point P_i there are $\ell(\mu^{(i)})$ ramification points, of ramification orders respectively $\mu_j^{(i)}$.

Let $\rho : \pi_1(\mathbb{P}^1 \setminus \{P_1, \ldots, P_{n+4}\}, \mathbb{Z}) \to S_{2d}$ be the monodromy representation in the symmetric group of $2d$ elements associated with a covering in H. We use the convention that loops (and elements of the symmetric group) are composed from right to left. The elements $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \gamma_1, \cdots, \gamma_n)$ as in the left picture of Figure 1 generate the fundamental group $\pi_1(\mathbb{P}^1 \setminus \{P_1, \ldots, P_{n+4}\}, \mathbb{Z})$ with the relation

$$\alpha_1\alpha_4\gamma_1 \ldots \gamma_n = \alpha_2^{-1}\alpha_3^{-1} \tag{1}$$

Given such a homomorphism ρ, we let $\boldsymbol{\alpha}_i = \rho(\alpha_i)$, and $\boldsymbol{\gamma}_i = \rho(\gamma_i)$ and call the

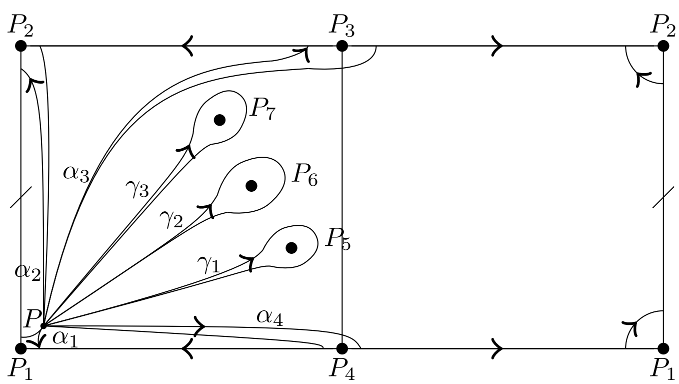


FIGURE 1. Standard presentation of $\pi_1(\mathbb{P}^1 \setminus \{P_1, \ldots, P_{n+4}\})$

tuple

$$h = (\boldsymbol{\alpha}_1, \cdots, \boldsymbol{\alpha}_4, \boldsymbol{\gamma}_1, \cdots, \boldsymbol{\gamma}_n) \in (S_{2d})^{n+4}. \tag{2}$$

the *Hurwitz tuple* corresponding to ρ and the choice of generators. Conversely, a Hurwitz tuple as in (2) satisfying (1) and generating a transitive subgroup of S_{2d} defines a homomorphism ρ and thus a covering p. We denote by $\mathrm{Hur}_d^0(\Pi)$ the set of all such Hurwitz tuples, the upper zero reflecting that we count connected coverings only. The set of all Hurwitz tuples (i.e. without the requirement of a transitive subgroup) is denoted by $\mathrm{Hur}_d(\Pi)$. As important technical intermediate notion we need covers *without unramified components*, i.e. covers $p : X \to \mathbb{P}^1$ that do not have a connected component X' such that $p|_{X'} = \pi_T \circ p'$ factors into an unramified covering p' and the torus double covering $\pi_T : E \to \mathbb{P}^1$ branched at $P_1, \dots, P_4$. We let $\boldsymbol{\alpha}_1^0 \in S_{2d}$ be the permutation with all transpositions of $\boldsymbol{\alpha}_1$ replaced by the the identity. In terms of Hurwitz tuples, we define the tuples without unramified components equivalently as

$$\mathrm{Hur}'_d(\Pi) \;=\; \{h \in \mathrm{Hur}_d(\Pi) \,:\, \langle \boldsymbol{\alpha}_1^0, \boldsymbol{\gamma}_1, \cdots, \boldsymbol{\gamma}_n\rangle \text{ acts non-trivially on every } \langle h\rangle\text{-orbit}\}\,.$$

The corresponding countings of covers (as usual with weight $1/\mathrm{Aut}(p)$) differ from the cardinalities of these sets of Hurwitz tuples by the simultaneous conjugation of the Hurwitz tuple, hence by a factor of $d!$. Consequently, we let

$$N_d(\Pi) \;=\; \frac{|\mathrm{Hur}_d(\Pi)|}{d!}, \quad N'_d(\Pi) \;=\; \frac{|\mathrm{Hur}'_d(\Pi)|}{d!}, \quad N^0_d(\Pi) \;=\; \frac{|\mathrm{Hur}^0_d(\Pi)|}{d!}\,, \tag{3}$$

and package these data into the generating series

$$N(\Pi) \;=\; \sum_{d=0}^{\infty} N_d(\Pi) q^d, \quad N'(\Pi) \;=\; \sum_{d=0}^{\infty} N'_d(\Pi) q^d, \quad N^0(\Pi) \;=\; \sum_{d=0}^{\infty} N^0_d(\Pi) q^d\,. \tag{4}$$

The connected components of a covering induce a partition of the branch points of $\boldsymbol{\alpha}_1^0$ and $\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_n$. This implies that

$$N'(\Pi) \;=\; N(\Pi)/N(\Pi_\emptyset)\,. \tag{5}$$

Similarly, the inclusion-exclusion expression for counting unramified covers in terms of covers without unramified components carries over from the case of torus covers (e.g. [GM18, Proposition 2.1]).

2.2. **Covers of the projective line with three marked points.** We need coverings of the projective line branched over three points with two types of parametrizations. As in the case of torus coverings (see [GM18, Section 2.2] for more details and remarks on numbered vs. unnumbered enumeration) we define

$$\begin{aligned}\mathrm{Cov}(\mathbf{w}^-, \mathbf{w}^+, \mu) \;=\; \big\{ &(\pi : S \to \mathbb{P}^1, \sigma_0, \sigma_\infty) \,:\, \deg(\pi) = \sum w_i^+ = \sum w_i^-\,, \\ &\pi^{-1}(1) = [\mu],\; \pi^{-1}(0) = \mathbf{w}^-, \;\; \pi^{-1}(\infty) = \mathbf{w}^+ \big\}\end{aligned}$$

to be the set of coverings of $\mathbb{P}^1$ with fixed profile μ over 1 with profile $\mathbf{w}^- = (w_1^-, \dots, w_{n^-}^-)$ and $\mathbf{w}^+ = (w_1^+, \dots, w_{n^+}^+)$ over 0 and ∞ respectively, and where σ_0 and σ_∞ are labelings of the branch points over 0 and ∞. We usually consider $\mathbf{w}^-$ and $\mathbf{w}^+$ as 'input' and 'output' tuples of variables. We denote by

$$A(\mathbf{w}^-, \mathbf{w}^+, \mu) \;=\; \sum_{\pi \in \mathrm{Cov}(\mathbf{w}^-, \mathbf{w}^+, \mu)} \frac{1}{\mathrm{Aut}(\pi)} \tag{6}$$

the automorphism-weighted count of these numbers and refer to this quantity as *triple Hurwitz numbers* (although some authors e.g. [SSZ12] call them double Hurwitz numbers referring to two sets $\mathbf{w}^\pm$ of free variables).

The second type of covering has only one set of variables and a product of transpositions of the point at ∞. That is, we define

$$\begin{aligned}\mathrm{Cov}_2(\mathbf{w}, \nu) \;=\; \big\{ &(\pi : S \to \mathbb{P}^1, \sigma_0) \,:\, \deg(\pi) = \sum w_i\,, \pi^{-1}(1) = [\nu, 2^{(\deg(\pi) - |\nu|)/2}]\,, \\ &\pi^{-1}(0) = \mathbf{w}, \;\; \pi^{-1}(\infty) = [2^{\deg(\pi)/2}] \big\}\end{aligned}$$

be the set of coverings with fixed profile over 1 and ∞ (but stabilized by transpositions rather than by adding ones as usual!) and with variable profile $\mathbf{w} = (w_1, \dots, w_k)$ over 0. Finally, we let

$$A_2(\mathbf{w}, \nu) \;=\; \sum_{\pi \in \mathrm{Cov}_2(\mathbf{w},\nu)} \frac{1}{\mathrm{Aut}(\pi)}\,. \tag{7}$$

and we refer to them as *simple Hurwitz numbers with* 2*-stabilization.*

As usual, all these notions have their respective variants for coverings without unramified components (decorated by a prime) and for connected coverings (decorated by an upper zero).

2.3. **Global graphs and cylinder decompositions.** We normalize the pillow to be the quotient orbifold $B = E_i/\pm$ where $E_i = \mathbb{C}/\mathbb{Z}[i]$ is the rectangular torus provided with the unique up to scale quadratic differential q_B such that $\pi_T^* q_B$ is holomorphic on E. The pillow comes with the distinguished points $P_1, \dots, P_4$ that are the images of $0, \frac{i}{2}, \frac{i+1}{2}, \frac{1}{2} \in E_i$ respectively. For the remaining branch points we usually use in the sequel the *branch point normalization* that the i-th branch point P_i is the π_T-image of a point with coordinates $z_i = x_i + \sqrt{-1}\varepsilon_i$ with $0 \le \varepsilon_5 < \varepsilon_6 < \cdots < \varepsilon_{n+4} < 1/2$ and any $x_i \in [0, 1)$.

The horizontal foliation with respect to q_B on B and thus on every pillowcase cover $p : X \to B$ with respect to $p^* q_B$ is periodic. There are two possible variants to encode the covering by a graph and local data: First, we might use that the complement of the leaves through all the preimages of the P_i consists of cylinders only. Second, one can use that the cylinders can be continued across the leaves 'at height $\frac{1}{2}$', i.e. the keeping the leaves through P_2 and P_3 still gives a cylinder decomposition. This can be pushed further by realizing that the leaves 'at height 0' joining two simple transposition preimages of P_1 and P_4 can also be added to the cylinders. In this paper we use the second viewpoint throughout, i.e. extending cylinders as much as possible over fake saddle connections. Said differently, we mark X only at the points where q has a zero or pole, not at the preimages of the P_i where q is regular and only remove saddle connections between those marked points to get a horizontal cylinder decomposition.

The *global graph* Γ associated with the pillowcase covering surface $(X, q = p^* q_B)$ of ramification profile Π is the graph Γ with $n + 1 = |\Pi| - 3$ vertices. The vertex with the special label 0 corresponds to the union of leaves through a preimage of P_1 or P_4 and the remaining vertices, labeled by $j \in \{1, \dots, n\}$ correspond to the leaves through P_{j+4}. below. If the partition ν is empty, then there is no vertex 0. The edges $E(\Gamma)$ of Γ are in bijection with the core curves of the horizontal cylinder decomposition described in the second viewpoint above.

We illustrate this using Figure 2 that gives a covering in the stratum $\mathcal{Q}(2, 1, -1^3)$, in other terms it has a ramification profile given by $n = 1$, $\nu = (3, 1, 1, 1)$ and $\mu_5 = 2$ in the notations of Section 2.1. (For more background on strata of quadratic differentials, including the notation $\mathcal{Q}(2, 1, -1^3)$, see for example [Zor06].) The small triangles (with different orientations) are the three simple poles, the diamond indicates the simple zero. These points map to the black square on the pillow. The white circle indicates the double zero, mapping the white circle on the pillow. This corresponds to the point P_5 while $P_1, \dots, P_4$ are the corners of the pillow.

We provide Γ with an *orientation of its half-edges* as follows. We provide the pillowcase without the special layers with one of the two choices of an orientation

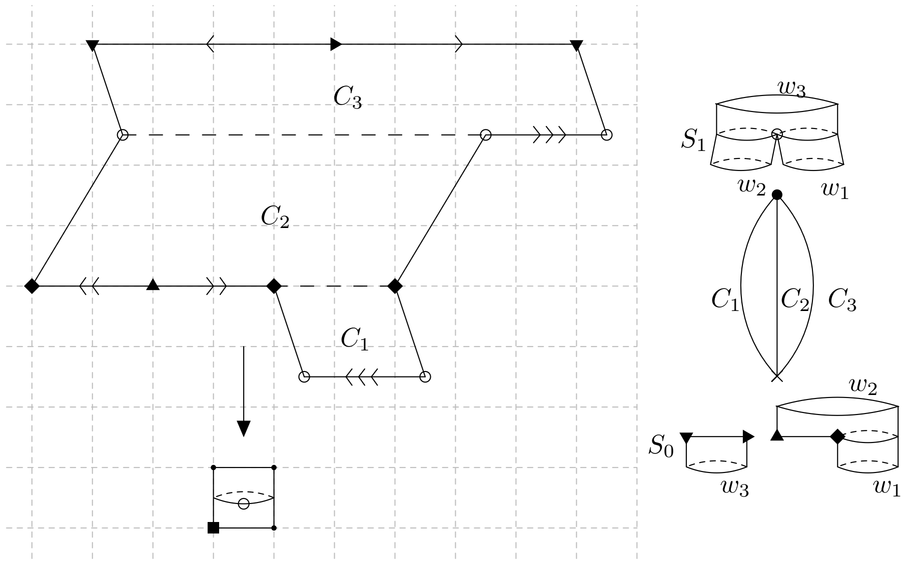


FIGURE 2. A pillowcase cover, the global graph and the local surfaces

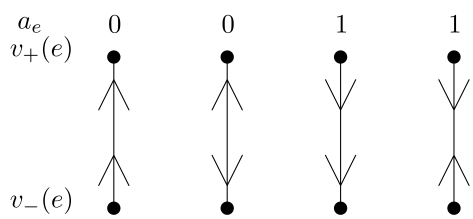


FIGURE 3. Orientation of half-edges and height minimum a_e

of the vertical direction, say the upward pointing. We orient a half-edge at a vertex v outward-pointing, if the orientation of the cylinder pointing towards the boundary representing the half-edge is consistent with the chosen global ("vertical") orientation, and we orient the half-edge inward-pointing otherwise. In particular, all the half-edges starting at the vertex 0 (if it exists) are oriented outward-pointing. Recall that cylinders may cross the special layers at height 0 or $1/2$ any number of times. The two half-edges corresponding to a cylinder are oriented consistently (see Figure 3, leftmost and third arrow) if and only if the cylinder crosses the special layers an even number of times. We refer to this extra datum as an *orientation* $G \in \Gamma$.

To reconstruct a pillowcase covering from a global graph, we need as in the case of torus coverings, two types of extra data that encode the geometry of the cylinders and the geometry of the local surfaces, respectively. The first extra datum is the cylinder geometry.

Each cylinder (corresponding to an edge e) has an integral positive width w_e and a real positive height h_e. The heights h_e are not arbitrary, but related to the position of the branch points. To define the space parametrizing possible heights, we need a finer classification of the set $E(\Gamma)$.

We use an upper index 0 to denote edges having the vertex 0 as an extremity and we use a lower index ℓ to denote loop edges, i.e. edges linking a vertex to itself. For all non-loop edges we denote by $v^+(e)$ (resp. $v^-(e)$) the label of the vertex whose height according to the branch point normalization is higher (resp. lower). We refer to them as the 'upper' (resp. 'lower') vertex of the edge e. For all inconsistently oriented loop edges we extend this notation into $v^+(e) = v^-(e) = v(e)$. Once we provided Γ with an orientation G we can distinguish those edges with consistent orientation. We denote this subset by an upper index $+$. We can now define the *height space* to be

$$\widetilde{\mathbb{N}}^{E(G)} \;=\; \left\{(h_e)_{e\in E(\Gamma)} \,:\, \begin{cases} h_e \in \mathbb{N}_{>0} & \text{if} \quad e \in E^0_\ell(G) \cup E^+_\ell(G) \\ h_e - \Delta(e) \in \mathbb{N}_{\geq a_e} & \text{if} \quad e \in E(G) \setminus (E^0_\ell(G) \cup E^+_\ell(G)) \end{cases} \right\}, \tag{8}$$

where $\Delta(e) = \pm\varepsilon_{v^+(e)} \pm \varepsilon_{v^-(e)}$, where the sign in front of each ε is positive if and only if the edge at the corresponding vertex is incoming and with a_e depending on the orientation as indicated in Figure 3

We claim that the collection of heights the cylinders in a pillowcase covering belongs to the height space and that conversely each element in the height space can be realized by such a covering. The integrality of the heights corrected by $\Delta(e)$ follows directly from the branch point normalization and the conventions of half-edge markings. It remains to justify the lower bounds one for the corrected heights. This happens if and only if the cylinder has to go all the way up to the preimage of the height 1/2-line and down again. Loops based at 0 and consistently oriented loops have this property. For the remaining loops, it depends on the orientation of the half-edges. Note that (second and fourth case in Figure 3) the lower bound a_e is independent of the choice. For non-loop edges the integer a_e encodes whether the cylinder has to go around the pillowcase. This completes the proof of the claim.

The last piece of local information for a cylinder is the *twist* $t_e \in \mathbb{Z} \cap [0, w_e - 1]$. The twist depends on the choice of a ramification point $P^-(e)$ and $P^+(e)$ in each of the two components adjacent to the cylinders and it is defined as the integer part of the real part $\lfloor \Re(\int_s \omega) \rfloor$ of the integral along the unique straight line joining $P^-(e)$ to $P^+(e)$ such that $t_e \in [0, w_e - 1]$. The exact values of the twist will hardly matter in the sequel. It is important to retain simply that there are w_e possibilities for the twist in a given cylinder.

2.4. **The basic correspondence theorem.** The second extra datum needed for the correspondence theorem is the local geometry at the vertices. We let X^0 be the complement of the core curves of the cylinders. We call the union of connected components of X^0 that carry the same label the *local surfaces* of (X, ω). We label these local surfaces also by an integer in $\{0, 1, \ldots, n\}$ according to the ramification point they carry. This labeling is well-defined, since p is a cover without unramified components. The restriction of the cover p to any local surface besides the one corresponding to the special vertex is metrically the pullback of an infinite cylinder branched over one point, as in the case of torus coverings. We thus encode these local surfaces by elements in $\mathrm{Cov}'(\mathbf{w}^-_v, \mathbf{w}^+_v, \mu_v)$ where $\mathbf{w}^-_v$ and $\mathbf{w}^+_v$ are the widths of the incoming and outgoing edges. The restriction of the cover p to a neighborhood of the line at height 0 is precisely the type of cover parameterized by an element in $\mathrm{Cov}_2(\mathbf{w},\nu)$, with $\mathbf{w}$ the tuple of widths of the outgoing edges.

For the following proposition we fix a ramification profile Π and let ν resp. μ_v be the component of the tuple Π that corresponds to the vertex v under the vertex marking conventions explained in Section 2.3.

Proposition 2.1. *There is a bijective correspondence between*

i) *flat surfaces (X,q) with a covering $p: X \to B$ of degree $2d$ and profile Π of the pillow B without unramified components and with $q = \pi^* q_B$, and*

ii) *isomorphism classes of tuples $(G,(w_e,h_e,t_e)_{e\in E(G)},(\pi_v)_{v\in V(G)})$ consisting of*

- *a global graph Γ with labeled vertices including a special vertex 0 if ν is non-empty, without isolated vertices, together with an orientation $G \in \Gamma$ of the half edges such that all half edges emerging from the special vertex are outgoing,*
- *a collection of real numbers $(w_e,h_e,t_e)_{e\in E(G)}$ representing the width, height and twist of the cylinder corresponding to e. The widths w_e are integers, the tuple of heights $(h_e)_{e\in E(G)} \in \widetilde{\mathbb{N}}^{E(G)}$ is in the height space, $t_e \in \mathbb{Z}\cap[0,w_e-1]$ and these numbers satisfy*

$$2\sum_{e\in E(G)} w_e h_e \;=\; 2d\,, \tag{9}$$

- *a collection of $\mathbb{P}^1$-coverings $(\pi_v)_{v\in V(G)\setminus\{0\}} \in \mathrm{Cov}'(\mathbf{w}_v^-,\mathbf{w}_v^+,\mu_v)$ without unramified components where $\mathbf{w}_v^-$ is the tuple of widths at the incoming edges at v, $\mathbf{w}_v^+$ is the tuple of widths at the outgoing edges at v, and μ_v is the ramification profile given by the labels at the vertex v.*
- *and a $\mathbb{P}^1$-covering $\pi_0 \in \mathrm{Cov}_2'(\mathbf{w}_0,\nu)$ where $\mathbf{w}_0$ is the tuple of widths at the outgoing edges at $v=0$ and ν is the ramification profile given by the labels at the vertex $v=0$.*

up to the action of the group $\mathrm{Aut}(\Gamma)$ of automorphisms of the labeled graph Γ.

Proof. With the setup and the orientation of half-edges adapted to pillowcase coverings, the proof proceeds now exactly as in the case of torus covers, see [GM18, Proposition 2.4]. □

2.5. **Variants of the correspondence theorem.** For the proof of the main theorem we will also need variants of the correspondence theorem that arise from counting covers by graphs while declaring a subset of points P_i for $i \in S \subset \{5,\dots n+4\}$ to be part of the layer of the special vertex. For extreme cases $S=\emptyset$ we are back in the situation of the previous situation while for $S=\{5,\dots n+4\}$ the global graph is tautologically just a single vertex with no edges, decorated by a local Hurwitz number which is just the global Hurwitz number we are interested in.

For the concrete statement, we start with the *branch point normalization.* We place the points P_i at $z_i = x_i + \sqrt{-1}\varepsilon_i$ where now $0<\varepsilon_i<\kappa$ for all $i\in S$ and $\kappa<\varepsilon_i<1/2$ for all $i \in S^c = \{5,\dots n+4\}\setminus S$, and moreover within these constraints strictly increasing with i.

The *global graph* associated with $(X, q = p^*q_B)$ is now the following graph Γ_S with $n+1-|S|$ vertices. The special vertex 0 corresponds to the region $R=\{0\le \Im(z)\le\kappa\}$ and the remaining vertices are indexed by S^c. Edges correspond the cylinders that are not entirely contained in a connected component of $p^{-1}(R)$. The notion of an *orientation* $G_S \in \Gamma_S$ carries over verbatim from the above discussion, and the same holds for the height space $\widetilde{\mathbb{N}}^{E_S(G)}$, declaring $\varepsilon_i = 0$ for $i \in S$.

The simplification in the graph is accounted for by a more complex Hurwitz number at the special vertex. We extend the definition of Hurwitz numbers with 2-stabilization by

$$\mathrm{Cov}_2(\mathbf{w}, \{\mu_i\}_{i\in S}, \nu) \;=\; \Big\{(\pi : S \to \mathbb{P}^1, \sigma_0) \,:\, \deg(\pi) = \sum w_i\,, \pi^{-1}(0) = \mathbf{w},$$
$$\pi^{-1}(1) = [\nu, 2^{(\deg(\pi)-|\nu|)/2}]\,, \pi^{-1}(a_i) = [\mu_i]\,(i \in S), \quad \pi^{-1}(\infty) = [2^{\deg(\pi)/2}]\Big\}$$

for some points $a_i \notin \{0, 1, \infty\}$, and set

$$A_2(\mathbf{w}, \{\mu_i\}_{i\in S}, \nu) \;=\; \sum_{\pi\in\mathrm{Cov}_2(\mathbf{w},\{\mu_i\}_{i\in S},\nu)} \frac{1}{\mathrm{Aut}(\pi)}\,. \tag{10}$$

The same proof as above now yields the following proposition.

Proposition 2.2. *There is a bijective correspondence between*

i) *flat surfaces* (X, q) *with a covering* $p : X \to B$ *of degree* $2d$ *and profile* Π *of the pillow* B *without unramified components and with* $q = \pi^* q_B$*, and*

ii) *isomorphism classes of tuples* $(G_S, (w_e, h_e, t_e)_{e\in E(G_S)}, (\pi_v)_{v\in V(G_S)})$ *as in Proposition 2.1, with* $\pi_0 \in \mathrm{Cov}_2'(\mathbf{w}_0, \nu)$ *replaced by* $\pi_0 \in \mathrm{Cov}_2'(\mathbf{w}_0, \{\mu_i\}_{i\in S}, \nu)$*, up to the action of the group* $\mathrm{Aut}(\Gamma)$ *of automorphisms of the labeled graph* Γ.

3. Fermionic Fock space and its balanced subspace

In this section we briefly recall the necessary background material about fermionic Fock space and the balanced subspace for the evaluation of the w-brackets that compute the generating functions of pillowcase covers. (See also [RZ16], [OP06], [EO06], for this formalism) The new result here is Theorem 3.3 stating that the generalized shifted symmetric functions f_ℓ and g_ν are rich enough to generate the algebra $\overline{\Lambda}$.

Recall the definition of the normalized characters $f_\mu(\lambda) \;=\; \mathfrak{z}_\mu \chi^\lambda(\mu)/\dim\chi^\lambda$, where $\mathfrak{z}(\mu) = \prod_{m=1}^\infty m^{r_m(\mu)} \prod_{m=1}^\infty r_m(\mu)! \;=\; \prod_{i=1}^{\ell(\mu)} \mu_i \prod_{m=1}^\infty r_m(\mu)!$ denotes the order of the centralizer of the partition $\mu = 1^{r_1}2^{r_2}3^{r_3}\cdots$. We also write f_ℓ for the special case that σ is a ℓ-cycle. The *algebra of shifted symmetric polynomials* is defined as $\Lambda^* = \varprojlim \Lambda^*(n)$, where $\Lambda^*(n)$ is the algebra of symmetric polynomials in the n variables $\lambda_1 - 1, \ldots, \lambda_n - n$. The functions

$$\widetilde{P}_\ell(\lambda) \;=\; \sum_{i=1}^\infty \big((\lambda_i - i + \tfrac12)^\ell - (-i + \tfrac12)^\ell\big) \quad \text{and} \quad \widetilde{P}_\mu \;=\; \prod_i \widetilde{P}_{\mu_i} \tag{11}$$

belong to Λ^*. We add constant terms corresponding to regularizations to these functions to obtain

$$p_\ell(\lambda) \;=\; P_\ell(\lambda) \,+\, (1 - 2^{-\ell})\,\zeta(-\ell). \tag{12}$$

Here $\ell!\beta_{\ell+1} = (1 - 2^{-\ell})\zeta(-\ell)$ with β_k defined by $B(z) := \frac{z/2}{\sinh(z/2)} = \sum_{k=0}^\infty \beta_k\, z^k$. Recall the first basic structure result.

Theorem 3.1 ([KO94])**.** *The algebra* Λ^* *is freely generated by all the* p_ℓ *(or equivalently, by the* P_ℓ*) with* $\ell \geq 1$*. The functions* f_μ *belong to* Λ^**. More precisely, as* μ *ranges over all partitions, these functions* f_μ *form a basis of* Λ^**.*

Let $f : \mathbb{P} \to \mathbb{Q}$ be an arbitrary function on the set $\mathbb{P}$ of all partitions. That is, we define (following [EO06]) the *w-brackets*

$$\langle f \rangle_w \;=\; \frac{\sum_{\lambda\in\mathbb{P}} w(\lambda) f(\lambda)\, q^{|\lambda|}}{\sum_{\lambda\in\mathbb{P}} w(\lambda) q^{|\lambda|}} \;\in\; \mathbb{Q}[[q]]\,, \tag{13}$$

where the difference to the q-brackets used to discuss torus coverings is the weight function

$$\sqrt{w(\lambda)} \;=\; \frac{\dim(\lambda)}{|\lambda|!} f_{2,\dots,2}(\lambda)^2, \quad w(\lambda) \;=\; \sqrt{w(\lambda)}^{\,2}\,. \tag{14}$$

The main reason for introducing w-brackets is the expression

$$N'(\Pi) = \langle g_\nu f_{\mu_5} \cdots f_{\mu_{n+4}} \rangle_w \tag{15}$$

for the connected Hurwitz numbers. This follows directly from the classical Burnside formula (see [EO01, Section 2]).

The algebra Λ^* is enlarged to the algebra

$$\overline{\Lambda} \;=\; \mathbb{Q}[p_\ell, \overline{p}_k(k,\ell \geq 0)] \tag{16}$$

of *shifted symmetric quasi-polynomials*, where

$$\overline{p}_k(\lambda) \;=\; \sum_{i\geq 0} \Big((-1)^{\lambda_i - i + 1} (\lambda_i - i + \tfrac{1}{2})^\ell - (-1)^{-i+1}(-i+\tfrac{1}{2})^\ell \Big) \;+\; \gamma_k\,, \tag{17}$$

and where the constants γ_i are zero for i odd, $\gamma_0 = 1/2$, $\gamma_2 = -1/8$, $\gamma_4 = 5/32$ and in general defined by the expansion $C(z) = 1/(e^{z/2} + e^{-z/2}) = \sum_{k\geq 0} \gamma_k z^k/k!$. We provide the algebra $\overline{\Lambda}$ with a grading by defining the generators to have

$$\mathrm{wt}(p_\ell) \;=\; \ell + 1 \quad \text{and} \quad \mathrm{wt}(\overline{p}_k) \;=\; k\,.$$

The main reason for introducing Λ^* is the following result. The reason for introducing g_ν will become clear by 15 in Section 4. In this section ν is always a partition consisting of an even number of odd parts.

Theorem 3.2 ([EO06, Theorem 2])**.** *There is a function $g_\nu \in \overline{\Lambda}$ of (mixed) weight less or equal to $|\nu|/2$ such that*

$$g_\nu(\lambda) = \frac{f_{(\nu,2,2,\dots)}(\lambda)}{f_{(2,2,\dots)}(\lambda)} \quad \textit{for } \lambda \textit{ balanced.} \tag{18}$$

Our goal is the following converse, for which we define $\mathrm{wt}(g_\nu) = |\nu|/2$.

Theorem 3.3. *The elements g_ν generate $\overline{\Lambda}$ as a graded Λ^*-module i.e. the subspace of $\overline{\Lambda}$ of weight less or equal to n is generated by expressions hg_ν for $h \in \Lambda^*$ with $\mathrm{wt}(h) + \mathrm{wt}(g_\nu) \leq n$.*

Of course, the elements g_ν do not form a basis as there are many more g_ν than products of $\overline{p}_k$ for a given weight. We recall the main steps of the proof of Theorem 3.2, since we need them for Theorem 3.3.

3.1. **Fermionic Fock space.** Let $\Lambda_0^{\frac{\infty}{2}}V$ be the *charge zero subspace of the half-infinite wedge* or *Fermionic Fock space* over the countably-infinite-dimensional vector space V. We denote the basis elements of V by underlined half-integers. A orthonormal basis of $\Lambda^{\frac{\infty}{2}}V$ is given by the elements

$$v_\lambda \;=\; \underline{\xi_1} \wedge \underline{\xi_2} \wedge \underline{\xi_3} \cdots, \qquad \xi_i = \lambda_i - i + \tfrac{1}{2}\,.$$

indexed by partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$. The basic operators on the half-infinite wedge is for any $k \in \mathbb{Z} + \frac{1}{2}$ the *creation operator* $\psi_k(v) = \underline{k} \wedge v$ and its adjoint ψ^*, the *annihilation operator*. For any function f on the real line we define the (unregularized) operators

$$\widetilde{\mathcal{E}}_k[f] \;=\; \sum_{m \in \mathbb{Z}+\frac{1}{2}} f(m) : \psi_{m-k}\psi_m^* : \,,$$

where the colons denote the normally ordered product. We use the convention that x is the default variable on the real line. Consequently, if T is a term in x (typically a polynomial, or a character like $(-1)^x$ times a polynomial) we write $\widetilde{\mathcal{E}}_k(T)$ as shorthand for $\widetilde{\mathcal{E}}_k[x \mapsto T(x)]$. The regularized operators for exponential arguments are defined by

$$\mathcal{E}_k[e^{zx}] \;:=\; \mathcal{E}_k[x \mapsto e^{zx}] \;=\; \widetilde{\mathcal{E}}_k[e^{zx}] + \delta_{0,k}\frac{B(z)}{z}$$

and in the presence of a character $(-1)^x$ by

$$\mathcal{E}_k[e^{(z+\pi i)x}] \;:=\; \mathcal{E}_k[x \mapsto e^{(z+\pi i)x}] \;=\; \widetilde{\mathcal{E}}_k[x \mapsto e^{(z+\pi i)x}] - i\delta_{k,0}C(z)\,,$$

where

$$C(z) \;:=\; \frac{1}{e^{z/2}+e^{-z/2}} \;=\; \sum_{k\geq 0}\gamma_k\frac{z^k}{k!}\,.$$

We frequently need three special cases of these operators. First

$$\alpha_{-n} \;=\; \mathcal{E}_{-n}[1] \;=\; \sum_{m\in\mathbb{Z}+\frac{1}{2}} : \psi_{m+n}\psi_m^* : \,,$$

whose adjoint is denoted by $\alpha_n = \alpha_{-n}^*$. The *Murnaghan-Nakayama rule* says that

$$\prod_i \alpha_{-\mu_i} v_\emptyset \;=\; \sum_\lambda \chi^\lambda(\mu) v_\lambda\,. \tag{19}$$

Second, the expansion of the (regularized) formal power series

$$\mathcal{E}_0(z) := \mathcal{E}_o[x \mapsto e^{zx}] \;=\; \frac{1}{z} + \sum_{\ell\geq 1}\mathcal{P}_\ell\,\frac{z^\ell}{\ell!}$$

and the expansion of

$$i\mathcal{E}_o[x \mapsto e^{(z+\pi i)x}] \;=\; \sum_{\ell\geq 0}\frac{z^k}{k!}\overline{\mathcal{P}}_k$$

gives operators $\mathcal{P}_\ell$ and $\overline{\mathcal{P}}_k$ with the property

$$\mathcal{P}_\ell v_\lambda \;=\; p_\ell(\lambda)v_\lambda, \quad \overline{\mathcal{P}}_k v_\lambda \;=\; \overline{p}_k(\lambda)v_\lambda\,.$$

Finally, note that the unregularized $\mathcal{E}_0(z)$-operator admits the useful formula

$$\widetilde{\mathcal{E}}_0(z) \;=\; [y^0]\psi(e^z y)\psi^*(y)\,, \tag{20}$$

where the interior expression can be checked by the commutator lemma for vertex operators ([RZ16, Lemma 7.1] or [Kac90, Section 14]) to be

$$\psi(xy)\psi^*(y) \;=\; \frac{1}{x^{1/2}-x^{-1/2}}\exp\Bigl(\sum_{n>0}\frac{(xy)^n-y^n}{n}\alpha_{-n}\Bigr)\exp\Bigl(\sum_{n>0}\frac{y^{-n}-(xy)^{-n}}{n}\alpha_n\Bigr). \tag{21}$$

3.2. **The balanced subspace.** Note that the definition of the algebra Λ excludes the operator p_0, the charge operator, since it is equal to zero on $\Lambda_0^{\frac{\infty}{2}}V$. Similarly the definition of the algebra $\overline{\Lambda}$ excludes the operator

$$\overline{p}_0(\lambda) \;= \frac{1}{2} + \sum_{i\ge 0}\Bigl((-1)^{\lambda_i-i+1}-(-1)^{-i+1}\Bigr).$$

A partition λ is called *balanced* if among the $\lambda_i - i+1$ for $\lambda_i \ge 0$ there are as many odd as even numbers, i.e. if and only if $\overline{p}_0(\lambda) \;= \frac{1}{2}$. Every partition λ determines (by sorting the $\lambda_i - i+1$ into even and odd) two partitions α and β, called the 2-*quotients*, such that

$$\{\lambda_i - i + \tfrac12\} \;=\; \{2(\alpha_i - i+\tfrac12)+\overline{p}_0(\lambda)\}\cup\{2(\beta_i-i+\tfrac12)-\overline{p}_0(\lambda)\}\,.$$

We let $\Lambda^{\rm bal}V$ denote the *balanced subspace* of $\Lambda_0^{\frac{\infty}{2}}V$, i.e. the subspace spanned by the v_λ for λ balanced. It inherits from $\Lambda_0^{\frac{\infty}{2}}V$ the grading by eigenspace of the energy operator, i.e. $\Lambda^{\rm bal}V = \oplus_{d\ge 0,{\rm even}}\Lambda^{\rm bal}V_d$. We use the shorthand notation

$$|[\rho;\overline{\rho}]\rangle \;=\; \frac{1}{\mathfrak{z}(\rho)\mathfrak{z}(\overline{\rho})}\prod_i \alpha_{-\rho_i}\prod_j \overline{\alpha}_{-\overline{\rho_j}}\, v_\emptyset$$

for the following reason ([EO01]).

Proposition 3.4. *For $\rho = (\rho_i)$ and $\overline{\rho} = (\overline{\rho_i})$ running over all partitions with entries in $2\mathbb{Z}$, the elements $|[\rho;\overline{\rho}]\rangle$ form an orthogonal basis of $\Lambda^{\rm bal}V$.*

Proof of Theorem 3.2. By (19) the content of the theorem is that the orthogonal projection of $|[\nu, 2^{d-|\nu|/2};\emptyset]\rangle$ to the balanced subspace $\Lambda^{\rm bal}V$ is a linear combination of the projections of $|\prod_i\mathcal{P}_{\mu_i}\prod\overline{\mathcal{P}}_{\overline{\mu_i}}\,2^d\rangle$ with $\mu = (\mu_i)_{i\ge1}$ and $\overline{\mu} = (\overline{\mu_i})_{i\ge1}$ partitions with ${\rm wt}(\mathcal{P}_\mu)+{\rm wt}(\overline{\mathcal{P}}_{\overline{\mu}}) \le |\nu|/2$ with coefficients independent of d. For this purpose one calculates using the commutation laws of the vertex operators that on the one hand

$$\bigl\langle[\rho;\overline{\rho}]\,|\,[\nu,2^{d-|\nu|/2};\emptyset]\bigr\rangle \;=\; \frac{2^{\ell(\nu)-\ell(\overline{\rho})}}{2^{d-|\nu|/2}(d-|\nu|/2)!\mathfrak{z}(\nu)\mathfrak{z}(\overline{\rho})}C(\nu,\overline{\rho}),\quad \text{if}\quad \rho = 2^{d-|\nu|/2}\,, \tag{22}$$

where $C(\nu,\overline{\rho})$ is the number of ways to assemble the parts of $\overline{\rho}$ from the parts of ν, and zero otherwise. In particular $|\nu| = |\overline{\rho}|$ for (22) to be non-zero. The squared norms of the element $|[\rho;\overline{\rho}]\rangle$ for ρ and $\overline{\rho}$ having even parts only is equal to $1/\mathfrak{z}(\rho)\mathfrak{z}(\overline{\rho})$. In particular the scalar product (22) divided by $|||[\rho;\overline{\rho}]\rangle||^2$ is independent of d.

On the other hand one computes using the commutation laws of the vertex operators that the brackets

$$D_{[\rho;\overline{\rho}],(\mu,\overline{\mu})}(d) \;=\; \langle[(\rho,2^{(d-|\rho|)/2};\overline{\rho}]\,|\prod_i\mathcal{P}_{\mu_i}\prod\overline{\mathcal{P}}_{\overline{\mu_i}}\,|\,2^d\bigr\rangle/||\,[(\rho,2^{(d-|\rho|)/2});\overline{\rho}]\rangle||^2 \tag{23}$$

for ρ an partition with only even parts of length different from two are non-zero only if

$$\Delta(\rho,\overline{\rho},\mu,\overline{\mu}) := \mathrm{wt}(\rho) + \mathrm{wt}(\overline{\rho}) - \mathrm{wt}(\mu) + \mathrm{wt}(\overline{\mu}) \geq 0\,.$$

Since an additional factor $\mathcal{P}_1$ in (23) gives an additional factor $(d - \frac{1}{24})$, we first consider $D_{[\rho;\overline{\rho}],(\mu,\overline{\mu})}(d)$ with μ without a part equal to one. Then, if (3.2) is attained, the scalar product $D_{[\rho;\overline{\rho}],(\mu,\overline{\mu})}(d)$ is non-zero if and only if $\mu = \rho/2$ and $\overline{\mu} = \overline{\rho}/2$. In general, $D_{[\rho;\overline{\rho}],(\mu,\overline{\mu})}(d)$ is a polynomial of degree $\Delta(\rho,\overline{\rho},\mu,\overline{\mu})/2$. This also implies that (for fixed weight of $[\rho;\overline{\rho}]$) the matrix $D_{[\rho;\overline{\rho}],(\mu,\overline{\mu})}(d)$ (with entries in $\mathbb{Q}[d]$) is an invertible matrix D, in fact block triangular.

Using these facts we can write

$$g_\nu = \sum_{|\overline{\rho}|=|\nu|} \frac{2^{\ell(\nu)-\ell(\overline{\rho})}}{\mathfrak{z}(\nu)} C(\nu,\overline{\rho}) \sum_{\mu,\overline{\mu}} (D^{-1})_{[\emptyset;\overline{\rho}],(\mu,\overline{\mu})}(p_1 + \tfrac{1}{24}) p_\mu \overline{p}_{\overline{\mu}} \tag{24}$$

where the sum all $(\mu,\overline{\mu})$ with $|\mu| + \ell(\mu) + |\overline{\mu}| \leq \nu/2$. □

Lemma 3.5. *For every fixed d the matrix $C(\nu,\overline{\rho})$, where ν is a partition of $2d$ consisting only of odd parts and $\overline{\rho}$ is a partition of $2d$ consisting only of even parts, has full rank equal to $\mathbb{P}(d)$.*

Proof. We order the rows $\overline{\rho}$ lexicographically and consider the submatrix with columns ν consisting of the partitions

$$\nu(\rho) \;=\; (\overline{\rho}_1 - 1, 1, \overline{\rho}_2 - 1, 1, \ldots, \overline{\rho}_n - 1, 1)$$

formed from $\overline{\rho} = (\overline{\rho}_1 \geq \cdots \geq \overline{\rho}_n)$. Since $C(\nu(\overline{\rho}),\overline{\rho}') \neq 0$ if and only if $\overline{\rho}' \leq \overline{\rho}$ lexicographically, the claim follows. □

Proof of Theorem 3.3. Since the matrix $D_{[\rho;\overline{\rho}],(\mu,\overline{\mu})}(d)$ is invertible, it suffices to prove by induction on the weight that the operators of contraction against

$$b_{\rho,\overline{\rho}} \;=\; \frac{1}{||\langle\, [(\rho, 2^{(d-|\rho|)/2}); \overline{\rho}]||^2} \langle [(\rho, 2^{(d-|\rho|)/2}; \overline{\rho}]\,|$$

are in the Λ^*-module generated by the g_ν. For $\rho = \emptyset$ this follows from Lemma 3.5, in fact those $b_{\emptyset,\overline{\rho}}$ can be spanned by g_ν with constant coefficients. For $\rho \neq \emptyset$ we use the expression of $b_{\rho,\overline{\rho}}$ as linear combination of $p_\mu \overline{p}_{\overline{\mu}}$. The terms with $\mathrm{wt}(p_\mu \overline{p}_{\overline{\mu}}) = \mathrm{wt}(b_{\rho,\overline{\rho}})$ are either $p_{\rho/2}\overline{p}_{\overline{\rho}/2}$ or involve a factor of p_1^j for some $j > 0$ by the properties of the matrix $D_{[\rho;\overline{\rho}],(\mu,\overline{\mu})}(d)$. Consequently, these terms are generated by g_ν as Λ^*-module by induction hypothesis and the extra factor p_1^j does not alter this fact. For the terms with smaller weight the induction hypothesis applies directly. □

4. Hurwitz numbers and graph sums

The goal of this section is to use the correspondence theorems to express any w-bracket in terms of auxiliary brackets that directly reflect the graph sums of the correspondence theorems. The precise form of the goal, Theorem 4.2, will involve in the auxiliary brackets only arguments for which the $A'(\cdot)$-functions will later be proven to be polynomial.

We first define for any function F on partitions

$$A(\mathbf{w}^-, \mathbf{w}^+, F) \;=\; \frac{1}{\prod_i w_i^- \prod_i w_i^+} \sum_{|\lambda|=d} \chi^\lambda_{\mathbf{w}^-} \chi^\lambda_{\mathbf{w}^+} F(\lambda) \tag{25}$$

and we define the connected variant, denoted by $A'(\mathbf{w}^-,\mathbf{w}^+,F)$ by the usual inclusion-exclusion formula (e.g. [GM18, Equation (17) or (25)]). The reason for this definition is that on one hand the triple Hurwitz number introduced in (6) can be written using the Burnside Lemma (see e.g. [GM18, Section 2]) as

$$A'(\mathbf{w}^-,\mathbf{w}^+,\mu) \;=\; A'(\mathbf{w}^-,\mathbf{w}^+,f_\mu)\,.$$

On the other hand, we will use that the function with completed cycles argument

$$\overline{A}'(\mathbf{w}^-,\mathbf{w}^+,\mu) \;:=\; A'\Big(\mathbf{w}^-,\mathbf{w}^+,\frac{P_\mu}{\prod \mu_i}\Big)\,.$$

is a polynomial of even degree for $\mu=(\mu_1)$ being a partition consisting of a single part and for $\mu_1+1-\ell(\mathbf{w}^-)-\ell(\mathbf{w}^+)$ even ([SSZ12], rephrased as [GM18, Theorem 4.1]).

A new feature of pillowcase covers is the use of the one-variable analog

$$A_2(\mathbf{w},F) \;=\; \frac{1}{\prod_i w_i}\sum_{|\lambda|=d}\sqrt{w(\lambda)}F(\lambda)\,, \tag{26}$$

where the second variable has been replaced by the character for the fixed partition $(2,\dots,2)$. We define the connected version $A_2'(\mathbf{w},F)$ by the usual inclusion-exclusion formula. Again, the reason for this definition is two-fold. By the Burnside formula the simple Hurwitz numbers with 2-stabilization introduced in (7) and generalized in (10) can be written as

$$A_2'(\mathbf{w},\{\mu_i\}_{i\in S},\nu) \;=\; A_2'(\mathbf{w},g_\nu\prod_{i\in S}f_{\mu_i})\,. \tag{27}$$

We study polynomiality properties of A_2' for suitable $F=\prod \overline{p}_{k_i}\in\overline{\Lambda}$ in detail in Section 7.

Let Π be a profile as specified in Section 2.1. We decompose the Hurwitz number $N'(\Pi)$ according to the contribution of the global graphs, i.e. we write

$$N'(\Pi) \;=\; \frac{1}{|\mathrm{Aut}(\Gamma)|}\sum_{\Gamma}N'(\Gamma,\Pi)\,,$$

where the sum is over all (not necessarily connected) labeled graphs Γ with $n=|\Pi|$ non-special vertices and possibly a special vertex and where $\mathrm{Aut}(\Gamma)$ are the automorphisms of the graph Γ that respect the vertex labeling. (Note that Γ has neither a labeling nor an orientation on the edges.) Following the results in the correspondence theorem we define an *admissible orientation G of* Γ (symbolically written as $G\in\Gamma$) to be an orientation of the half-edges of Γ such that all the half-edges at the special vertex 0 (if it exists) are outward-pointing. Now the following proposition is an immediate consequence of the correspondence theorem Proposition 2.1.

Proposition 4.1. *The contributions of individual labeled graphs to $N'(\Pi)$ can be expressed in terms of triple Hurwitz numbers as*

$$N'(\Gamma,\Pi) \;=\; \sum_{G\in\Gamma}N'(G,\Pi)\,, \tag{28}$$

where

$$N'(G,\Pi) \;=\; \sum_{\substack{h\in\widetilde{\mathbb{N}}^{E(G)},\\ w\in\mathbb{Z}_+^{E(G)}}} \prod_{e\in E(G)} w_e q^{h_e w_e} \cdot A_2'(\mathbf{w}_0,\nu) \prod_{v\in V(G)\setminus\{0\}} A'(\mathbf{w}_v^-,\mathbf{w}_v^+,\mu_v)\ \delta(v) \tag{29}$$

where $V(G)^* = V(G)\setminus\{0\}$ *and where*

$$\delta(v) = \delta\big(\sum_{i\in e_+(v)} w_i^+ - \sum_{i\in e_-(v)} w_i^-\big). \tag{30}$$

We formalize the type of expression appearing in the previous proposition by defining *auxiliary brackets*

$$[F_1,\dots,F_n;F_0] = \sum_{\Gamma} [F_1,\dots,F_n;F_0]_\Gamma\,, \quad [F_1,\dots,F_n;F_0]_\Gamma = \sum_{G\in\Gamma} [F_1,\dots,F_n;F_0]_G, \tag{31}$$

where the sum is over all labeled graphs Γ with n vertices and over all admissible orientations, respectively, and where

$$[F_1,\dots,F_n;F_0]_G = \sum_{\substack{h\in\widetilde{\mathbb{N}}^{E(G)},\\ w\in\mathbb{Z}_+^{E(G)}}} \prod_{i\in E(G)} w_i q^{h_i w_i} \cdot A_2'(\mathbf{w}_0,F_0) \prod_{v\in V(G)^*} A'(w_v^-,w_v^+,F_{\#v})\ \delta(v)\,.$$

Here $\#v$ denotes the label of the vertex v. This notation is designed so that Proposition 4.1 can be restated as

$$\big\langle f_{\mu_1}\cdots f_{\mu_n} g_\nu\big\rangle_w \;=\; [f_{\mu_1},\dots,f_{\mu_n};g_\nu]\,. \tag{32}$$

More generally, by verbatim the same proof, Proposition 2.2 can be restated as the generalization

$$\big\langle f_{\mu_1}\cdots f_{\mu_n} g_\nu\big\rangle_w \;=\; [\prod_{i\notin S} f_{\mu_i};\prod_{i\in S} f_{\mu_i} g_\nu] \tag{33}$$

for any subset $S\subseteq\{1,\dots,n\}$.

We are now ready to formulate the goal of this section in detail.

Theorem 4.2. *The w-bracket of any element in $\overline{\Lambda}$ can be expressed as a finite linear combination of the auxiliary brackets, i.e. for every $\boldsymbol{\ell} = (\ell_1,\dots,\ell_n)$ and every $\mathbf{k} = (\ell_1,\dots,k_m)$ there exist $c_{(\mathbf{t},\mathbf{s})}\in\mathbb{Q}$ (depending on $(\boldsymbol{\ell},\mathbf{k})$) such that*

$$\big\langle\prod_{j=1}^n p_{\ell_j}\prod_{i=1}^m \overline{p}_{k_i}\big\rangle_w \;=\; \sum_{(\mathbf{t},\mathbf{s})} c_{(\mathbf{t},\mathbf{s})}\Big[p_{t_1},\dots,p_{t_{\ell(\mathbf{t})}};\prod_{i=1}^{\ell(\mathbf{s})}\overline{p}_{s_i}\Big]\,, \tag{34}$$

where the sum is over all $(\mathbf{t},\mathbf{s})$ with $\sum_j(t_j+1)+\sum_i s_i \le \sum_j(\ell_j+1)+\sum_i k_i$.

Proof. The proof is by induction on the weight $w=\sum_{i=1}^m k_i$ of the $\overline{p}_{k_i}$-part of the bracket, the case of weight zero being trivial (no special vertex, i.e. as in the abelian case.)

By Theorem 3.3 we can write the left hand side of (34) as a linear combination of $\big\langle f_{\mu_1}\cdots f_{\mu_n} g_\nu\big\rangle_w$ with $\mathrm{wt}(g_\nu)\le w$. By (32) each such summand is equal to

$$[f_{\mu_1},\dots,f_{\mu_n};g_\nu] \;=\; \sum_{\mathbf{b},\mathbf{a}}[f_{\mu_1},\dots,f_{\mu_n};\prod_{j\ge1} p_{b_j}\prod_{i\ge1}\overline{p}_{a_i}]\,, \tag{35}$$

where the sum is over all partitions **a** and **b**, by Theorem 3.2. In the summands where **b** is the empty partition, we replace f_{μ_i} by a linear combination of products of p_ℓ thanks to Theorem 3.1 and these contributions are of the required form of the right hand side of (34). In all the summands with **b** non-empty we use the converse base change of Theorem 3.1 to write the product of p_{b_j} as a linear combination of a product of f_{μ_j}. We can now use (33) from right to left to express all the terms as a sum w-brackets with $\overline{p}_{k_i}$-part of weight $w - \sum_j b_j + 1$. Since **b** is non-empty, we conclude thanks to the induction hypothesis. □

5. Constant coefficients of quasi-elliptic functions

In this section we consider the constant coefficient (in $z_1, \ldots, z_n$) of a function that is quasi-elliptic in these variables, has a globally a quasimodular transformation behavior and poles at most at two-torsion translates of the coordinate axes and diagonals. We show in Theorem 5.6 that this constant coefficient is indeed a quasimodular form for the subgroup $\Gamma(2)$ of $\mathrm{SL}(2,\mathbb{Z})$.

5.1. **Quasimodular forms.** A *quasimodular form* for the cofinite Fuchsian group $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$ of weight k is a function $f : \mathbb{H} \to \mathbb{C}$ that is holomorphic on $\mathbb{H}$ and the cusps of Γ and such that there exists and integer p and holomorphic functions $f_i : \mathbb{H} \to \mathbb{C}$ such that

$$(c\tau+d)^{-k} f\Big(\frac{a\tau+b}{c\tau+d}\Big) \;=\; \sum_{i=0}^{p} f_i(\tau)\Big(\frac{c}{c\tau+d}\Big)^i \quad \text{for all} \quad \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma\,.$$

The smallest integer p with the above property is called the *depth* of the quasimodular form. By definition, quasimodular forms of depth zero are simply modular forms. The basic examples of quasimodular forms are the Eisenstein series defined by

$$G_{2k}(\tau) \;=\; \frac{(2k-1)!}{2(2\pi i)^{2k}} \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0\}} \frac{1}{(m+n\tau)^{2k}} \;=\; -\frac{B_{2k}}{4k} + \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n\,.$$

Here B_l is the Bernoulli number, σ_l is the divisor sum function and $q = e^{2\pi i\tau}$. For $k \geq 2$ these are modular forms, while for $k = 2$ the Eisenstein series

$$G_2(\tau) = -\frac{1}{24} + \sum_{n=1}^{\infty} \sigma_1(n) q^n$$

is a quasimodular form of weight 2 and depth 1 for $\mathrm{SL}(2,\mathbb{Z})$. By [KZ95] we can write the ring of quasimodular forms for any group Γ with $\mathrm{Stab}_\infty(\Gamma) = \pm\langle\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\rangle$ as $\mathrm{QM}(\Gamma) = \mathbb{C}[G_2] \otimes M(\Gamma)$ in terms of the ring of modular forms $M(\Gamma)$. We will be mainly interested in the congruence groups $\Gamma_0(2)$ and

$$\Gamma(2) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \Gamma_0(4) \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \subset \Gamma_0(2).$$

Since $M(\Gamma_0(2))$ is freely generated by $G_2^{\mathrm{odd}}(\tau) = G_2(\tau) - 2G_2(2\tau)$ and $G_4(2\tau)$ by the transformation formula (e.g. [DS05, Proposition 4.2.1]) and the usual dimension formula for modular forms, we deduce that

$$\mathrm{QM}(\Gamma_0(2)) \;\cong\; \mathbb{C}[G_2(\tau), G_2(2\tau), G_4(2\tau)] \tag{36}$$

is a polynomial ring. For $\Gamma_0(4)$ we restrict our attention to the subring of even weight quasimodular forms. Since $M_{2*}(\Gamma_0(4))$ is freely generated by $G_2^{\rm odd}(\tau)$ and $G_2^{\rm odd}(2\tau)$ (again using the transformation formula together with the isomorphism $\Gamma_0(4) \cong \Gamma(2)$ given by conjugation with $\operatorname{diag}(2,1)$), we deduce that

$$\mathrm{QM}(\Gamma(2)) \;\cong\; \mathbb{C}[G_2(\tau/2), G_2(\tau), G_2(2\tau)]\,. \tag{37}$$

We use the notation $q = e^{2\pi i\tau}$, hence $q^{1/2} = e^{\pi i \tau}$. Note that a typical element $\mathrm{QM}(\Gamma(2))$ has a Fourier expansion in $q^{1/2}$. The following observation allows us to prove quasimodularity by the larger group $\Gamma_0(2)$.

Lemma 5.1. *For all $k \in \mathbb{N}$ the Eisenstein series $G_{2k}(\tau/2)$, $G_{2k}(\tau)$ and $G_{2k}(2\tau)$ are quasimodular forms for $\Gamma(2)$. Moreover, any even weight quasimodular form for $\Gamma(2)$ whose Fourier expansion is a series in q is in fact a quasimodular form for $\Gamma_0(2)$.*

Proof. The second statement follows immediately from (36) and (37). □

5.2. **Quasimodular forms as constant coefficients of quasi-elliptic functions.** We are now ready to state the first main criterion for quasimodularity, involving the constant coefficients of some quasi-elliptic functions introduced below. We start with a general remark on the domains where the expansions are valid. Suppose that the meromorphic function $f(z_1, z_2, \ldots, z_n; \tau)$ is periodic under $z_j \mapsto z_j + 1$ for each j and under $\tau \mapsto \tau + 1$. We can then write $f(z_1, z_2, \ldots, z_o; \tau) = \overline{f}(\zeta_1, \ldots, \zeta_n, q)$ where $\zeta_j = e^{2\pi i z_j}$ as above. For any permutation $\pi \in S_n$ on we fix the domain

$$\Omega_\pi \;=\; |q^{1/2}| < |\zeta_{\pi(i)}| < |\zeta_{\pi(i+1)}| < 1 \quad \text{for all } i = 1, \ldots, n-1\,.$$

On such a domain the *constant term with respect to all the* ζ_i is well-defined. It can be expressed as integral

$$[\zeta_n^0, \ldots, \zeta_1^0]_\pi\, \overline{f} \;=\; \frac{1}{(2\pi i)^n} \oint_{\gamma_n} \cdots \oint_{\gamma_1} f(z_1, \ldots, z_n; \tau) dz_1 \ldots dz_n$$

along the integration paths

$$\gamma_j : [0,1] \to \mathbb{C}, \qquad t \mapsto i y_j + t\,,$$

where $0 \le y_{\pi(1)} < y_{\pi(2)} < \ldots y_{\pi(n)} < 1/2$. We call these our *standard integration paths* for the permutation π. If the domain Ω_π is clear from the context we also write $[\zeta^0]$ or $[\zeta_n^0, \ldots, \zeta_1^0]$ as shorthand for the coefficient extraction $[\zeta_n^0, \ldots, \zeta_1^0]_\pi$.

Let $\Delta = \Delta_\tau$ be the operator on meromorphic functions defined by

$$\Delta(f)(z) \;=\; f(z+\tau) - f(z)\,.$$

A meromorphic function f is called *quasi-elliptic* (for the lattice $\mathbb{Z}+\tau\mathbb{Z}$) if $f(z+1) = f(z)$ and if there exists some integer e such that $\Delta^e(f)$ is elliptic. The minimal such e is called the *order (of quasi-ellipticity)* of f.

We say that a meromorphic function $f : \mathbb{C}^n \times \mathbb{H} \to \mathbb{C}$ is quasi-elliptic, if it is quasi-elliptic in each of the first n variables. For such a function we write $\mathbf{e} = (e_1, \ldots, e_n)$ for the tuple of orders of quasi-ellipticity in the n variables. Consequently, a quasi-elliptic function of order $(0, \ldots, 0)$ is simply an elliptic function.

We write Δ_i for the operator Δ acting on the i-th variable. Note that these operators Δ_i commute. Let $T = \{0, 1/2, \tau/2, (1+\tau)/2\}$ be the set of 2-torsion points.

The functions we want to take constant coefficients of belong to the space in the following definition. It is similar to the quasi-elliptic quasimodular forms used in [GM18, Definition 5.5]. The difference consists of allowing 2-torsion translates for the poles and requiring a modular transformation law for a smaller group. We do not decorate our new definition of $\mathcal{Q}^{(k)}_{n,\mathbf{e}}$ by an extra symbol 2 to avoid overloading notation.

Definition 5.2. *We define for $n \geq 0$, $k \geq 0$ and $\mathbf{e} \geq 0$ the vector space of $\mathcal{Q}^{(k)}_{n,\mathbf{e}}$ of* quasi-elliptic quasimodular forms *for $\Gamma(2)$ to be the space of meromorphic functions f on $\mathbb{C}^n \times \mathbb{H}$ in the variables $(z_1, \ldots, z_n; \tau)$ that*

i) *have poles on $\mathbb{C}^n$ at most at the $\mathbb{Z}+\tau\mathbb{Z}$- translates of the diagonals $z_i = z_j$, $z_i = -z_j$ and the 2-torsion points $z_i \in T$,*
ii) *that are quasi-elliptic of order $\mathbf{e}$, and*
iii) *that are quasimodular of weight k for $\Gamma(2)$, i.e. f is holomorphic in τ on $\mathbb{H} \cup \infty$ and there exists some $p \geq 0$ and functions $f_i(z_1, \ldots, z_n; \tau)$ that are holomorphic in τ and meromorphic in the z_i such that*

$$(c\tau+d)^{-k} f\Big(\frac{z_1}{c\tau+d}, \ldots, \frac{z_n}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\Big) \;=\; \sum_{i=0}^{p} f_i(z_1, \ldots, z_n; \tau)\Big(\frac{c}{c\tau+d}\Big)^i$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(2)$.

Examples of such quasi-elliptic quasimodular forms will be constructed from the *propagator*, the shift $P(z;\tau) = \frac{1}{(2\pi i)^2}\wp(z;\tau) + 2G_2(\tau)$ of the Weierstrass $\wp$-function, and from the shift $Z(z;\tau) = -\zeta(z;\tau)/2\pi i + 2G_2(\tau)2\pi i z$ of the Weierstraß ζ-function. The reason for this shift, as well as the Fourier and Laurent series expansion of these functions is summarized in [GM18, Section 5.2]. In particular we will need

$$P_{even}(z;\tau) \;=\; 2P(2z;2\tau) \quad \text{and}$$
$$P_{odd}(z;\tau) \;=\; P(z;\tau) - P_{even}(z;\tau)\,.$$

Proposition 5.3. *The functions $P^{(k)}(z_i - a;\tau)$ where $a \in T$, and each of the functions $P^{(k)}(2z_i;\tau)$, $P^{(k)}(2z_i;2\tau)$, $P^{(k)}_{even}(z_i;\tau)$, $P^{(k)}_{odd}(z_i;\tau)$, $P^{(k)}(z_i - z_j;\tau)$ and $P^{(k)}(z_i + z_j;\tau)$ belong to $\mathcal{Q}^{(k+2)}_{n,\mathbf{0}}$.*

The functions $Z(z_i - a;\tau)$ for $a \in T$ belong to $\mathcal{Q}^{(1)}_{n,\mathbf{e}_i}$, where $\mathbf{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, and the functions $Z(z_i - z_j;\tau)$ and $Z(z_i + z_j;\tau)$ belong to $\mathcal{Q}^{(1)}_{n,\mathbf{e}_i+\mathbf{e}_j}$.

Proof. Since P is an elliptic meromorphic function with poles at $\mathbb{Z}+\tau\mathbb{Z}$, and quasimodular in the sense of iii) of weight 2 for $\mathrm{SL}(2,\mathbb{Z})$, more precisely

$$(c\tau+d)^{-2} P\Big(\frac{z}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\Big) \;=\; P(z;\tau) + \frac{c}{c\tau+d}\,,$$

we deduce easily the result for the functions derived from P. In fact, the functions $P^{(k)}(z_i - 1/2;\tau)$ (hence also $P^{(k)}(2z_i;2\tau)$, $P^{(k)}_{even}(z_i;\tau)$ and $P^{(k)}_{odd}(z_i;\tau)$) are quasimodular for the bigger group $\Gamma_0(2)$. Moreover, the functions $P^{(k)}(2z_i;\tau)$ are quasimodular for the full group $\mathrm{SL}(2,\mathbb{Z})$. We proceed similarly for the functions derived from Z, which is quasi-elliptic of order 1, quasimodular of weight one and depth one with $Z_1(z;\tau) = z$. ∎

Proposition 5.4. *The direct sum*

$$\mathcal{Q}_n \;=\; \bigoplus_{k\geq 0}\mathcal{Q}_n^{(k)}, \quad \textit{where} \quad \mathcal{Q}_n^{(k)} \;=\; \bigoplus_{\mathbf{e}\geq 0}\mathcal{Q}_{n,\mathbf{e}}^{(k)}\,,$$

is a graded ring. The derivatives $\partial/\partial z_i$ map $\mathcal{Q}_n^{(k)}$ to $\mathcal{Q}_n^{(k+1)}$ for all $i=1,\dots,n$ and the derivative $D_q = q\frac{\partial}{\partial q}$ maps $\mathcal{Q}_n^{(k)}$ to $\mathcal{Q}_n^{(k+2)}$.

For all $i,j\in\{1,\dots,n\}$, $i\neq j$, the functions

$$L(z_i;\tau) \;=\; -\frac{1}{2}Z^2(z_i;\tau)+\frac{1}{2}P(z_i;\tau)-G_2(\tau)+\frac{1}{12} \tag{38}$$

as well as $L(2z_i;\tau), L(2z_i;2\tau), L(z_i-z_j;\tau)$ and $L(z_i+z_j;\tau)$ belong to $\mathcal{Q}_n^{(0)}\oplus\mathcal{Q}_n^{(2)}$.

Proof. The proof is similar to the $\mathrm{SL}(2,\mathbb{Z})$-case (cf. [GM18, Proposition 5.6]). □

From now on we omit the variable τ in the notation, if not necessary.

Proposition 5.5. *The vector space $\mathcal{Q}_n$ is (additively) generated as $\mathcal{Q}_{n-1}$-module by the functions $Z^e(z_n-a)$ and $Z^e(z_n-a)P^{(m)}(z_n-a)$ for $a\in T$ together with $Z^e(z_n+z_j)$ and $Z^e(z_n+z_j)P^{(m)}(z_n+z_j)$,$Z^e(z_n-z_j)$, $Z^e(z_n-z_j)P^{(m)}(z_n-z_j)$ for $j=1,\dots,n-1$ and for all $e\geq 0$ and $m\geq 0$.*

More precisely, if $f\in\mathcal{Q}_n^{(k)}$ then we can write

$$\begin{aligned} f(z_1,\dots,z_n) \;=\; &\sum_{a\in T}\left(\sum_{e,m}A_{e,m,j}Z^e(z_n-a)P^{(m)}(z_n-a)+\sum_e B_{a,e}Z^e(z_n-a)\right)\\ &+\sum_{e,m,i}C_{e,m,j}Z^e(z_n+z_i)P^{(m)}(z_n+z_i)+\sum_{e,i}D_{e,i}Z^e(z_n+z_i)\\ &+\sum_{e,m,i}E_{e,m,i}Z^e(z_n-z_i)P^{(m)}(z_n-z_i)+\sum_{e,i}F_{e,i}Z^e(z_n-z_i)+G \end{aligned}$$

with $A_{a,e,m}, C_{e,m,i}, E_{e,m,i}\in\mathcal{Q}_{n-1}^{(k-e-m+2)}$, $B_{a,e}, D_{e,i}, F_{e,i}\in\mathcal{Q}_{n-1}^{(k-e)}$ and $G\in\mathcal{Q}_{n-1}^{(k)}$.

Proof. For every n we argue inductively on the order $e=\min_{j\geq 0}\{\Delta_n^j(f)\text{ elliptic}\}$ of quasi-ellipticity with respect to the last variable. Suppose, without loss of generality, that $f\in\mathcal{Q}_n^{(k)}$ is homogeneous of weight k.

We first treat the case $e=0$. We show that we can write

$$f=\sum_{a\in T}\left(\sum A_{m,a}P_a^{(m)}+B_aZ_a\right)+\sum C_{m,i}P_{i,+}^{(m)}+D_iZ_{i,+}+\sum E_{m,i}P_{i,-}^{(m)}+F_{i,j}Z_{i,j}+G$$

with $A_{m,a}, C_{m,i}, E_{m_i}\in\mathcal{Q}_{n-1}^{(k-m-2)}$, $B_a, D_i, F_{i,j}\in\mathcal{Q}_{n-1}^{(k-1)}$ and $G\in\mathcal{Q}_{n-1}^{(k)}$, where $P_{j,a}^{(m)}=P^{(m)}(z_n+a)$ for $a\in T$, $P_{i,-}^{(m)}=P^{(m)}(z_n-z_i)$,$P_{i,+}^{(m)}=P^{(m)}(z_n+z_i)$, for all $m\geq 0$ and all $i=1,\dots,n-1$ $Z_a=Z(z_n+a)-Z(z_n+z_{n-1})+Z(z_{n-1})$ for $a\in T$, $Z_{i,+}=Z(z_n+z_i)-Z(z_n-z_i)-2Z(z_i)$, for all $i=1,\dots,n-1$, $Z_{i,j}=Z(z_n-z_i)-Z(z_n-z_j)+Z(z_i-z_j)$ for all $1\leq i<j\leq n-1$. By Proposition 5.3, these functions are clearly in $\mathcal{Q}_{n,\mathbf{0}}$.

We proceed by induction on the pole orders, first along the divisors z_n-a, then along the divisors z_n+z_i, then along the divisors z_n-z_i. The rest of the proof is then totally similar to [GM18, Proposition 5.4]. The residue theorem ensures that we can eliminate the last poles with the functions $Z_{i,j}$ to end the procedure.

For the case $e>0$, the proof is a straightforward adaptation of the proof of [GM18, Proposition 5.7]. □

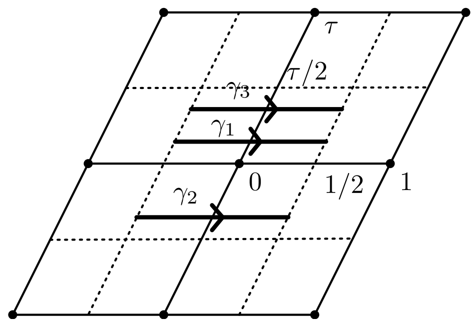


FIGURE 4. Integration paths to evaluate $[\zeta^0]Z^e$

Using this additive basis we can now prove the main result.

Theorem 5.6. *For any permutation π the constant term with respect to the domain Ω_π of a function in $\mathcal{Q}_n^{(k)}$ is a quasimodular form for $\Gamma(2)$ of mixed weight $\leq k$.*

Proof. Again, the proof is exactly the same as in [GM18, Theorem 5.8]: we reduce the problem to the computation of $[\zeta^0]Z^e(z-a)$ by some transformations preserving the weight of the quasimodular form. The last step of the proof is isolated in the statement of the next proposition. □

Proposition 5.7. *The constant coefficient $[\zeta^0]Z^e(z-a)$ for $a \in T$ is a quasimodular form for $\mathrm{SL}(2,\mathbb{Z})$ of mixed weight less or equal to e.*

Proof. Since $Z(z)$ is 1-periodic, we clearly have $[\zeta^0]Z^e(z-1/2) = [\zeta^0]Z^e(z)$ and these coefficients are quasimodular forms for $\mathrm{SL}(2,\mathbb{Z})$ of mixed weight less or equal to e, by [GM18, Proposition 5.9]. Similarly, $[\zeta^0]Z^e(z-\tau/2-1/2) = [\zeta^0]Z^e(z-\tau/2)$ so we just have to compute $[\zeta^0]Z^e(z-\tau/2)$.

Using again the 1-periodicity of Z, we obtain for all ℓ

$$\begin{aligned}[\zeta^0]Z^\ell(z-\tau/2) &= \int_{\gamma_1} Z^\ell(z-\tau/2)dz = \int_{\gamma_2} Z^\ell(z)dz = \int_{\gamma_3} Z^\ell(-z)dz \\ &= [\zeta^0]Z^\ell(-z),\end{aligned} \tag{39}$$

where the integration paths γ_1, γ_2, γ_3 are described in Figure 4. Since Z is odd, the constant coefficients $[\zeta^0]Z^\ell(z-\tau/2)$ are then given by $(-1)^\ell[\zeta^0]Z^\ell$ (see [GM18] for some explicit values). □

Note that the proof of Theorem 5.6 provides an effective algorithm to compute constant coefficients of quasi-elliptic functions. Applications of this algorithm and explicit computations of quasi-modular forms are detailed in Section 9.

6. QUASIMODULARITY OF GRAPH SUMS

The goal of this section is to show the quasimodularity of graph sums of the form (40) below. The motivation for considering these sums will become apparent in comparison with the quasi-polynomiality theorem in Section 7. We encourage the reader to look at Section 9 simultaneously with this one, we hope that all notations will become transparent on the example that we treat in detail.

We will show quasimodularity for graphs that arise as global graphs as in Section 2.3 with the following extra decoration.

A *global graph with distinguished edges* is a graph with vertices with the labels $1, \ldots, n$ and possibly a special vertex 0 and a subset $E^+(\Gamma)$ of $E(\Gamma)$ of distinguished edges such that no extremity of an edge in $E^+(\Gamma)$ is the vertex 0. We let $V^*(\Gamma) = V(\Gamma) \setminus \{0\}$, we let $E^0(\Gamma)$ be the edges adjacent to 0. Finally we define $E^*(\Gamma) = E(\Gamma) \setminus E^0(\Gamma)$ and $E^-(\Gamma) = E^*(\Gamma) \setminus E^+(\Gamma)$. An *admissible orientation* G *of* (Γ, E^+) is an orientation of the half-edges of Γ, such that

- all half-edges adjacent to the vertex 0 are outgoing, and
- the orientations of the two half-edges are consistent on marked edges, and inconsistent on the other edges that are not adjacent to v_0.

We write $G \in (\Gamma, E^+)$ for the specification of an admissible orientation. Obviously every admissible orientation G of Γ (in the sense of Section 4) is admissible for (Γ, E^+) for a uniquely determined subset $E^+ \subset E(\Gamma)$. We define the *set of parity conditions* to be $\mathrm{PC}(\Gamma) = \{0,1\}^{E^0(\Gamma)}$. It specifies a congruence class mod 2 for the width of each edge adjacent to the vertex 0.

We consider here for fixed $\mathbf{m} = (m_1, \ldots, m_{E(\Gamma)})$ and fixed $E^+(\Gamma)$ the graph sums

$$S(\Gamma, E^+, \mathbf{m}, \mathrm{par}) \;=\; \sum_{G \in (\Gamma, E^+)} S(G, \mathbf{m}, \mathrm{par}) \tag{40}$$

over all admissible orientations G of the half edges of Γ, where for $\mathrm{par} \in \mathrm{PC}(\Gamma)$

$$S(G, \mathbf{m}, \mathrm{par}) \;=\; \sum_{\substack{h \in \widetilde{\mathbb{N}}^{E(G)} \\ \mathbf{w}_* \in \mathbb{N}_{>0}^{E(G)^*}}} \; \sum_{\substack{\mathbf{w}_0 \in \mathbb{N}_{>0}^{E^0(\Gamma)} \\ \mathbf{w}_0 \cong \mathrm{par} \bmod 2}} \; \prod_{i \in E(G)} w_i^{m_i+1} q^{h_i w_i} \prod_{v \in V(G)^*} \delta(v)\,. \tag{41}$$

Here $\widetilde{\mathbb{N}}^{E(G)}$ is the height space introduced in (8) and $\delta(v)$ is as in (30). The goal of this section is to show the quasimodularity of these graph sums.

Theorem 6.1. *For a fixed tuple of non-negative even integers* $\mathbf{m} = (m_1, \ldots, m_{|E(\Gamma)|})$ *and for any* $\mathrm{par} \in \mathrm{PC}(\Gamma)$ *the graph sums* $S(\Gamma, E^+, \mathbf{m}, \mathrm{par})$ *are quasimodular forms for the group* $\Gamma_0(2)$ *of mixed weight at most* $k(\mathbf{m}) := \sum_i (m_i + 2)$.

The following is a first simplification step for the computation. Recall the definition of the offsets a_e from Figure 3.

Lemma 6.2. *Replacing the height space* $\widetilde{\mathbb{N}}^{E(G)}$ *by*

$$\widehat{\mathbb{N}}^{E(G)} \;=\; \left\{ (h'_e)_{e \in E(\Gamma)} \,:\, \begin{cases} h'_e \in \mathbb{N}_{>0} & \text{if} \quad e \in E^0_\ell(G) \cup E^+_\ell(G) \\ h'_e \in \mathbb{N}_{\geq a_e} & \text{otherwise} \end{cases} \right\}, \tag{42}$$

does not change the total sum $S(G, \mathbf{m}, \mathrm{par})$.

Proof. We apply the linear change of variables $h'_e = h_e - \Delta(e)$ with $\Delta(e)$ as in the line below (8). This maps $\widetilde{\mathbb{N}}^{E(G)}$ bijectively onto $\widehat{\mathbb{N}}^{E(G)}$. For notational convenience we set $\Delta(e) = 0$ for the remaining edges. Each summand of (41) for fixed $(w_1, \ldots, w_{|E(G)|})$ is multiplied under the variable change by

$$q^{\sum_e \Delta(e) w_e} \prod_{v \in V(G)^*} \delta(v) \;=\; q^{\sum_v \varepsilon_v \left(\sum_{i \in e^-(v)} w_i - \sum_{i \in e^+(v)} w_i\right)} \prod_{v \in V(G)^*} \delta(v) \;=\; 1\,.$$

This implies the claim. □

6.1. **The reduced graph.** We will simplify the graph sums by isolating the contribution from the loop edges $E^0_\ell(G) \cup E^+_\ell(G)$. The *reduced graph* $\overline{\Gamma}$ is obtained from Γ by deleting those loops, i.e. the loops adjacent to the vertex v_0 and the loops among the distinguished edges.

Lemma 6.3. *The graph sums* $S(\Gamma, \mathbf{m}, \mathrm{par})$ *factor as*

$$S(\Gamma, E^+, \mathbf{m}, \mathrm{par}) \;=\; S_{loops}(\Gamma, \mathbf{m}, \mathrm{par})\, S(\overline{\Gamma}, E^+, \mathbf{m}, \mathrm{par})$$

where

$$S_{loops}(\Gamma, \mathbf{m}, \mathrm{par}) \;=\; \sum_{h \in \mathbb{N}_{>0}^{E^0_\ell(\Gamma)\cup E^+_\ell(\Gamma)}} \;\sum_{\mathbf{w} \in \mathbb{Z}_+^{E^+_\ell(\Gamma)}} \;\sum_{\substack{\mathbf{w}_0 \in \mathbb{N}_{>0}^{E^0_\ell(\Gamma)} \\ \mathbf{w}_0 \cong \mathrm{par} \bmod 2}} \;\prod_{i \in E^0_\ell(\Gamma)\cup E^+_\ell(\Gamma)} w_i^{m_i+1} q^{h_i w_i}\,.$$

Proof. The length constraints $\delta(v)$ for $v \in V(G)^*$ are unchanged under removing a loop edge $e \in E^+(G)$ that contributes equally to both incoming and outgoing weight. The length parameters at the vertex 0 is always unconstrained. This proves the factorization we claim. □

Lemma 6.4. *If* $\mathbf{m}$ *is even,* $S_{loops}(\Gamma, \mathbf{m}, \mathrm{par})$ *is a quasimodular form for* $\Gamma_0(2)$ *of mixed weight* $k(\mathbf{m})$.

Proof. The graph sum $S_{loops}(\Gamma, \mathbf{m}, \mathrm{par})$ is a product of

$$\begin{aligned}
S_m &= \sum_{w,h,=1}^{\infty} w^{m+1} q^{2wh} &&= G_{m+2}(q) - G_{m+2}(0) \\
S_{m,even} &= \sum_{w,h=1}^{\infty} (2w)^{m+1} q^{(2w)h} &&= 2^{m+1}(G_{m+2}(q^2) - G_{m+2}(0)) \\
S_{m,odd} &= \sum_{w,h,=1}^{\infty} (2w-1)^{m+1} q^{(2w-1)h} &&= S_m - S_{m,even}\,.
\end{aligned}$$

All the right hand sides are quasimodular forms for $\Gamma_0(2)$ by (36). □

6.2. **Contour integrals.** We now write the sum of the reduced graph as contour integral of suitable derivatives of the following variants of the propagator. Let $P_{even}(z;\tau) = 2P(2z; 2\tau)$ and $P_{odd}(z;\tau) = P(z;\tau) - P_{even}(z;\tau)$. We also use P_i for $i \in \mathbb{Z}/2$ to refer to these two functions. For a reduced graph $\overline{\Gamma}$, for $\mathbf{m}$ even, and for a given parity condition par $\in \mathrm{PC}(\Gamma)$, define

$$\begin{aligned}
P_{\overline{\Gamma},E^+,\mathbf{m},\mathrm{par}}(\mathbf{z}) \;=\; \prod_{i \in E^0(\overline{\Gamma})} P^{(m_i)}_{\mathrm{par}_i}(z_{v_+(i)}) \cdot \prod_{i \in E^+(\overline{\Gamma})} P^{(m_i)}(z_{v_-(i)} - z_{v_+(i)}) \cdot \\
\cdot \prod_{i \in E^-(\overline{\Gamma})} P^{(m_i)}(z_{v_-(i)} + z_{v_+(i)})\,,
\end{aligned} \tag{43}$$

where $v_+(i)$ and $v_-(i)$ are the two ends of the edge i, with v_- being the one of lower index (so for $i \in E^0(\overline{\Gamma})$ necessarily $v_-(i) = 0$). Note that since the graph is reduced, $v_+(i)$ and $v_-(i)$ can be the same, but only if $i \in E^-(\overline{\Gamma})$. Note that the variable z_0 does not appear in the expression $P_{\overline{\Gamma},\mathbf{m},\mathrm{par}}(\mathbf{z})$ at all.

Proposition 6.5. *For a tuple of non-negative even integers* $\mathbf{m}_1$ *and a parity condition* par *we can express the graph sum as*

$$S(\overline{\Gamma}, E^+, \mathbf{m}, \mathrm{par}) \;=\; [\zeta^0_n, \ldots, \zeta^0_1]\, P_{\overline{\Gamma},E^+,\mathbf{m},\mathrm{par}}(\mathbf{z};\tau)\,, \tag{44}$$

where the coefficient extraction is for the expansion on the domain $|q^{1/2}| < |\zeta_i| < |\zeta_{i+1}| < 1$ *for all* i*.*

Proof. The proof is analogous to the proof of [GM18, Proposition 6.7] and we suggest to read the two proofs in parallel since we will not reproduce the bulky main formulas. We rather indicate the main changes. First note that in the domain specified above the inequalities

$$|q| < |\zeta_i\zeta_j| < 1 \quad \forall i,j, \qquad |q| < |q^{1/2}| < |\zeta_i/\zeta_j| < 1 \quad \forall j > i$$

hold, and hence the following Fourier expansions

$$\begin{aligned} P^{(m)}_{even}(z;\tau) &= \sum_{w\geq 1, even} w^{m+1}\left(\zeta^w \sum_{h\geq 0} q^{wh} + \zeta^{-w}\sum_{h\geq 1} q^{wh}\right) \\ P^{(m)}_{odd}(z;\tau) &= \sum_{w\geq 1, odd} w^{m+1}\left(\zeta^w \sum_{h\geq 0} q^{wh} + \zeta^{-w}\sum_{h\geq 1} q^{wh}\right) \\ P^{(m)}(z_i+z_j;\tau) &= \sum_{w\geq 1} w^{m+1}\left((\zeta_i\zeta_j)^w \sum_{h\geq 0} q^{wh} + (\zeta_i\zeta_j)^{-w}\sum_{h\geq 1} q^{wh}\right) \ \forall i,j \\ P^{(m)}(z_i-z_j;\tau) &= \sum_{w\geq 1} w^{m+1}\left((\zeta_i/\zeta_j)^w \sum_{h\geq 0} q^{wh} + (\zeta_i/\zeta_j)^{-w}\sum_{h\geq 1} q^{wh}\right) \ \forall j>i \end{aligned} \tag{45}$$

are valid. For the proof we first consider the factors in (43) that involve the edges E_1 adjacent to the vertex 1, i.e. those involving the variable z_1. The propagators $P, P_{\rm odd}$ or $P_{\rm even}$ in (43) are chosen so that the parity conditions for w_e specified in (41) hold. Each of the propagators in (45) has (for fixed (w,h)) two summands, that we consider as incoming (ζ-exponent $+w$) or outgoing (ζ-exponent $-w$). Consequently, expanding the product of propagators involving the edges in E_1 is a sum over all partitions $E_1 = J_1 \cup K_1$ of the incoming and outgoing terms. The integration with respect to z_1 forces that all contributions vanish except for those where the incoming w_e are equal to the outgoing w_e. This ensures the appearance of the factor $\delta(v_1)$ in (41). The proof proceeds by similarly considering the vertex 2 and expanding the propagator factors that involve the edges E_2 adjacent this vertex but not already in E_2, which produces a sum over all partitions $E_2 = J_2 \cup K_2$ according to whether the incoming or outgoing summand of the propagator has been taken.

The main difference to the abelian case is the consequence of orienting the half-edges. Suppose that e joins v_1 to v_2. If $e \in E^+(\Gamma)$ then in all admissible orientations e is incoming at v_1 and outgoing at v_2, or vice versa. If $e \in J_1$ is incoming at v_1, we have to make sure that the propagator terms have ζ_2^{-w}, i.e. we have to use $P^{(m)}(z_1 - z_2)$. On the other hand if $e \in E^-(\Gamma)$, then in all admissible orientations e is incoming or outgoing simultaneously at v_1 and v_2. I.e. ζ_1 and ζ_2 have to appear with the same w-exponent, whence the use of $P^{(m)}(z_1+z_2)$. The reader can check that this orientation convention is also consistent for the special vertex 0 and that the range of the sums $h \geq 0$ versus $h \geq 1$ in (43) is consistent with the conditions of the height space that appear in the h-summation in (41). □

Proof of Theorem 6.1. This is now a direct consequence of Proposition 6.4 for the loop contribution, of Proposition 6.5 for the reduced graph and of Theorem 5.6 for quasimodularity (for $\Gamma(2)$) of contour integrals, combined with Lemma 5.1 to get quasimodularity for the bigger group $\Gamma_0(2)$. □

7. Quasipolynomiality of 2-orbifold double Hurwitz numbers

The main result of this section is the quasi-polynomiality of the simple Hurwitz numbers with 2-stabilization $A_2'(\mathbf{w}, F)$ in the case that F is a product of $\overline{p}_k$. The meaning of quasi-polynomiality is that the restriction to a congruence class mod 2 in each variable is a polynomial. The crucial statement for the quasi-modularity is that these polynomials are global, i.e. not piece-wise polynomials depending on a chamber decomposition of the domain of $\mathbf{w}$. As a first application we combine this with the correspondence and quasimodularity theorems of the previous section to give in Corollary 7.4 another proof the Eskin-Okounkov theorem on the quasimodularity of the number of pillowcase covers.

7.1. **The one-sided pillowcase operator.** Our goal here is to write $A_2'(\mathbf{w}, F)$ in terms of vertex operators. For this purpose we define the *one-sided pillowcase operator*

$$\Gamma_{\sqrt{w}} \;=\; \exp\Bigl(\sum_{i>0} \tfrac{\alpha_{-i}^2}{2i}\Bigr).$$

Proposition 7.1. *The simple Hurwitz numbers with* 2*-stabilization can be expressed using the one-sided pillowcase operator as*

$$A_2(\mathbf{w}, F) \;=\; \frac{1}{\prod w_i}\Bigl\langle 0\,|\, \prod_{i=1}^{\ell(\mathbf{w})} \alpha_{w_i} \mathcal{F}\,\Gamma_{\sqrt{w}}\,|\,0\Bigr\rangle \tag{46}$$

where $\mathcal{F} v_\lambda = F(\lambda) v_\lambda$.

Proof. We first observe that

$$\langle \Gamma_{\sqrt{w}} v_\emptyset, v_\lambda\rangle \;=\; \sum_{\nu} \chi^\lambda(2\nu) \prod_{i\geq 1} \frac{(1/2i)^{r_i(\nu)}}{r_i(\nu)!}, \tag{47}$$

where 2ν is the partition obtained by repeating twice each row of the Young diagram of ν, i.e. if $\nu = 1^{r_1(\nu)} 2^{r_2(\nu)} \cdots$ is written in terms of the multiplicities of the parts then $2\nu = 1^{2r_1(\nu)} 2^{2r_2(\nu)} \cdots$. This observation follows from developing the exponential in $\Gamma_{\sqrt{w}}$ and the Murnaghan-Nakayama rule. It thus remains to show that

$$\sum_{\nu} \chi_\lambda(2\nu) \prod_{i=1}^{\nu_1} \frac{(1/2i)^{r_i(\nu)}}{r_i(\nu)!} \;=\; f_{2,2,\ldots,2}(\lambda)^2 \frac{\dim\lambda}{|\lambda|!} \;=\; \sqrt{\mathbf{w}(\lambda)}$$

or equivalently that

$$f_{2,\ldots,2}(\lambda)^2 \;=\; \sum_{\nu} f_{2\nu}(\lambda) \cdot \prod_{i\geq 1} (i^{r_i}(2r_i-1)!!). \tag{48}$$

Let C_2 be the conjugacy class corresponding to the partition $(2, \ldots, 2)$ and $C(2\nu)$ the conjugacy class of 2ν. Define

$$n^C_{C_2^2} \;=\; \#\{(a,b) \in C_2^2 \,|\, ab \in C\}$$

and let $e_C = \sum_{g\in C}[g]$ be the central elements the group ring $\mathbb{Z}[G]$. Then $e_{C_2}e_{C_2} = \sum_C n^C_{C_2^2} e_C$. Note that e_C acts on any irreducible representation λ as multiplication by the scalar $f_C(\lambda)$. Consequently, the claim (48) is equivalent to

$$n^C_{C_2^2} = \begin{cases} \prod_i (i^{r_i}(2r_i-1)!!) & \text{if } C = C(2\nu) \\ 0 \text{ otherwise}. \end{cases}$$

To prove this, note first that the product $\tau = \rho\sigma$ of two permutations ρ, σ in C_2 is in $C(2\nu)$. This follows be recursively proving that $\sigma\tau^{-n}(P) = \tau^n(\sigma(P)$, i.e. the cycles starting at P and $\sigma(P)$ have the same length. To justify the combinatorial factor, assume first that all $r_i = 1$. In order to specify the factorization it is necessary and sufficient to specify for each i and for some point in an i-cycle its σ-image in the other i-cycle. The rest of the factorization is determined by the requirement of profile $(2,\ldots,2)$ and this initial choice. In the general case $r_i > 1$ we moreover have to match the $2r_i$ cycles of length i in pairs (which corresponds to the factor $(2r_i-1)!!$) and make a choice as above for each pair.

The proposition follows from (47) using again the Murnaghan-Nakayama rule. □

We next collect some rules to evaluate the right hand side of (46). The standard commutation law between creation operators is

$$\alpha_n^a \alpha_{-n}^b = \sum_{k=0}^{\min(a,b)} \binom{a}{k} \frac{b!}{(b-k)!} n^k \alpha_{-n}^{b-k}\alpha_n^{a-k}. \tag{49}$$

Together with $[\alpha_m, \alpha_n] = 0$ for $m \neq -n$ this implies that we can deal with the $\alpha_{\pm n}$ for all n separately, i.e.

$$\langle \prod_n f_n(\alpha_n) g_n(\alpha_{-n}) v_\emptyset, v_\emptyset\rangle = \prod_n \langle f_n(\alpha_n) g_n(\alpha_{-n}) v_\emptyset, v_\emptyset\rangle,$$

where $f_n(\alpha_n) = \sum c_{j,n}\alpha_n^j$ and $g_n(\alpha_{-n}) = \sum c'_{j,n}\alpha_{-n}^j$. From the commutation relation

$$e^{c_n\alpha_{-n}} e^{d_n\alpha_n} = e^{-nc_nd_n} e^{d_n\alpha_n} e^{c_n\alpha_{-n}},$$

that follows directly from (49), we deduce successively the following relations involving the factors of $\Gamma_{\sqrt{w}}$. First

$$\Pi := \prod_{i=1}^s \left(e^{c_{i,n}\alpha_{-n}} e^{d_{i,n}\alpha_n}\right) = e^{-nA_n} e^{D_n\alpha_n} e^{C_n\alpha_{-n}}$$

where $C_n = \sum_{i=1}^s c_{i,n}$, $D_n = \sum_{i=1}^s d_{i,n}$ and $A_n = \sum_{j=1}^s d_{j,n}\sum_{i=1}^j c_{i,n}$. Next, from

$$\langle 0| e^{c_n\alpha_{-n}} e^{d_n\alpha_n} e^{\alpha_{-n}^2/2n}|0\rangle = e^{nd_n^2/2}$$
$$\langle 0|\alpha_n e^{c_n\alpha_{-n}} e^{d_n\alpha_n} e^{\alpha_{-n}^2/2n}|0\rangle = n(d_n+c_n)e^{nd_n^2/2}$$

we obtain

$$\begin{aligned} \langle 0|\prod_{i=1}^s \left(e^{c_{i,n}\alpha_{-n}} e^{d_{i,n}\alpha_n}\right) e^{\alpha_{-n}^2/2n}|0\rangle &= e^{n\tilde{A}_n} e^{nD_n^2/2} \\ \langle 0|\alpha_n\prod_{i=1}^s \left(e^{c_{i,n}\alpha_{-n}} e^{d_{i,n}\alpha_n}\right) e^{\alpha_{-n}^2/2n}|0\rangle &= n(D_n+C_n) e^{n\widetilde{A}_n} e^{nD_n^2/2} \end{aligned} \tag{50}$$

where $\widetilde{A}_n = D_n C_n - A_n = \sum_{j=2}^{s} d_{j,n} \sum_{i=1}^{j-1} c_{i,n}$, and most generally

$$\langle 0|\alpha_n^\ell \, \Pi \, e^{\alpha_{-n}^2/2n}|0\rangle \;=\; \sum_{i=0}^{\ell/2} \binom{\ell}{2i} \frac{(2i)!}{2^i (i)!} n^{\ell-i} (C_n + D_n)^{\ell-2i} e^{n\widetilde{A}_n} e^{nD_n^2/2} \,. \tag{51}$$

7.2. **Polynomiality.** We can now evaluate the vertex operator expressions and prove the main result of this section.

Theorem 7.2. *The simple Hurwitz number with 2-stabilization $A_2'(\mathbf{w}, F)$ without unramified components is a quasi-polynomial if F is a product of $\overline{p}_k$, i.e. for each coset $\mathbf{m} = (m_1, \ldots, m_t) \in \{0,1\}^n$ with $\sum m_i$ even there exists a polynomial $R_{F,\mathbf{m}} \in \mathbb{Q}[w_1, \ldots, w_t]$ such that*

$$A_2'(\mathbf{w}, F) \;=\; R_{F,\mathbf{m}}(\mathbf{w}) \quad \text{for all} \quad \mathbf{w} \in 2\mathbb{N}^t + \mathbf{m}\,.$$

The proof relies on matching piece-wise polynomial on sectors like $w_1 > w_2$ to form a global polynomial. The parity constraints to match the piece-wise polynomials do not work out for elements in Λ^*, not even for $F = p_\ell$, if there is more than one boundary variable w_i. In fact for any $w_1, w_2 \in \mathbb{N}$ with $w_1 + w_2$ even

$$\frac{1}{5} A_2'((w_1, w_2), p_5) \;=\; \frac{7}{8} u^3 + \frac{13}{8} u v^2 - u\,,$$

where $u = \min(w_1, w_2)$ and $v = \max(w_1, w_2)$ is only piece-wise polynomial, while e.g. on the coset $\mathbf{m} = (0,0)$

$$\frac{1}{4} A_2'((2w_1, 2w_2), \overline{p}_4) \;=\; 10(2w_1)^2 + 10(2w_2)^2 - 3$$

is globally a polynomial.

The basic source of polynomiality is the following lemma, relevant for the case of $F = \overline{p}_k$.

Lemma 7.3. *For each $k \geq 1$ there is a polynomial Q_k such that for $n \in \mathbb{N}$*

$$Q_k(n) \;=\; [y^{-2n}][z^k] D\,, \quad \text{where} \quad D(y,z) \;=\; \frac{1 + y^{-2} e^{-z}}{\sqrt{(1 - y^{-2})(1 - y^{-2} e^{-2z})}}\,.$$

Moreover, Q_k is even for k even and Q_k is odd for k odd and $Q_k(0) = 0$ in both cases.

Proof. We abbreviate $\partial_z = \partial/\partial z$. Since $[z^k] = k! \partial_z f|_{z=0}$ it suffices to write $\partial_z^k D(y,z)|_{z=0} = R_k(y)/(1 - y^{-2})^{k+1}$ for some polynomial $R_k(y^{-2})$ of degree $\leq k$ without constant coefficient. The relation $D(1/y, -z) = D(y,z)$ implies that R_k is palindromic, i.e. in the span of $y^{-s} + y^{-(2k-s)}$ for $s = 2, 4, \ldots, 2k-2$. Since by the binomial theorem

$$[y^{-2n}] \frac{1}{(1 - y^{-2})^{k+1}} \;=\; \frac{1}{k!} (n + k - 1) \cdots (n+1) n$$

agrees with a polynomial for integers $n \geq 1 - k$, we obtain the polynomiality claim. The parity claim follows from R_k being palindromic. □

Proof of Theorem 7.2. We may shift $\overline{p}_k$ by the regularization constant γ_k in order to use (20) and (21) and assume that $F = \prod_{j=1}^{s} (\overline{p}_{k_j} - \gamma_{k_j})$ for some k_j, not necessarily distinct. Proposition 7.1 now translates into

$$A_2(\mathbf{w}, F) \;=\; [y_1^0 \cdots y_s^0][z_1^{k_1} \cdots z_s^{k_s}] \prod_{n \geq 1} \frac{1}{n^{r_n(\mathbf{w})}} \langle 0|\alpha_n^{r_n(\mathbf{w})} \Psi_F e^{\alpha_{-n}^2/2n}|0\rangle\,, \tag{52}$$

where $r_n(\mathbf{w})$ is the multiplicity of n in $\mathbf{w}$ and where

$$\Psi_F = \prod_{j=1}^{s} \frac{1}{e^{z_i/2}+e^{-z_i/2}} \exp\Big(\frac{y_j^n((-e^{z_j})^n-1)}{n}\alpha_{-n}\Big) \exp\Big(\frac{y_j^{-n}(1-(-e^{-z_j})^n)}{n}\alpha_n\Big).$$

By (50), the first factor common to the evaluation of the brackets (52) for all n is $e^{nD_n^2/2}$, results in a product of

$$D_{[j]} \;:=\; \exp\Big(\sum_{n>0}\frac{y_j^{-2n}(1-(-e^{-z_j})^n)^2}{2n}\Big) \;=\; \frac{1+y_j^{-2}e^{-z_j}}{\sqrt{(1-y_j^{-2})(1-y_j^{-2}e^{-2z_j})}}\,.$$

and

$$D_{[ij]} \;:=\; \frac{(1+y_iy_je^{-z_i})(1+y_iy_je^{-z_j})}{(1-y_iy_j)(1-y_iy_je^{-z_i-z_j})}\,.$$

The second common factor $e^{n\widetilde{A}_n}$ for all n results in a factor of

$$\widetilde{A} \;:=\; \prod_{j=2}^{s}\prod_{i=1}^{j-1}\frac{(1+\frac{y_j}{y_i}e^{-z_i})(1+\frac{y_j}{y_i}e^{z_j})}{(1-\frac{y_j}{y_i})(1-\frac{y_j}{y_i}e^{z_j-z_i})}\,.$$

In order to built an arbitrary covering with boundary lengths $\mathbf{w}$ from a covering without unramified components, we have to choose for each length n of the boundary components among the $\ell_n = r_n(\mathbf{w})$ an even number $2i$ of boundary components that are glued together in pairs to form cylinders, and the number of such gluing is $\frac{(2i)!}{2^i(i)!}$, the number of fixed point free involutions. That is, the combinatorial factor $\binom{\ell_n}{2i}\frac{(2i)!}{2^i(i)!}$ in front of the summand in (51) counts precisely these possibilities. Consequently this formula implies that

$$A_2'(\mathbf{w},F) \;=\; [y_1^0\cdots y_s^0][z_1^{k_1}\cdots z_s^{k_s}]\left(\frac{\widetilde{A}\prod_{i<j}D_{[ij]}\prod_j D_{[j]}}{\prod_j(e^{z_j/2}+e^{-z_j/2})}\prod_{n:r_n(\mathbf{w})\geq 1}K_n^{r_n(\mathbf{w})}\right) \tag{53}$$

where $K_n = C_n + D_n$ as in the vertex operator manipulations above, i.e.

$$K_n \;=\; \sum_{j=1}^{s}\frac{y_j^n((-e^{z_j})^n-1)}{n} \;+\; \frac{y_j^{-n}(1-(-e^{-z_j})^n)}{n}$$

The claim follows if we can show two statements, first that the expression (53) is piece-wise polynomial and second that this expression with each K_n replaced by $\widetilde{K_n} = nK_n$ is globally a polynomial. The factor $\prod_j(e^{z_j/2}+e^{-z_j/2})$ results just in a shift of z_j-degrees and will be ignored in the sequel. Note that we can write the last factor in (53) equivalently as $\prod_{n:r_n(\mathbf{w})\geq 1}K_n^{r_n(\mathbf{w})} = \prod_{i=1}^{t}K_{w_i}$.

We start with the case $s=1$, illustrating the main idea. Let t be the number of n with $r_n(\mathbf{w})\geq 1$, say these are $n_1,\ldots,n_t$. First, we want to show that

$$\mathbf{n}=(n_1,\ldots,n_t) \;\mapsto\; [y_1^0]\Big([z_1^{k-j}]\prod_{i=1}^{t}\widetilde{K}_{n_i}\;\cdot\;[z_1^j]D_{[1]}\Big)$$

is polynomial in the n_i for each $j\in[0,k]$ (and zero otherwise). To evaluate this, we can choose in each $\widetilde{K}_{n_i}$-factor the $y_i^{n_i}$-term or the $y_i^{-n_i}$-term and then sum over the contributions of all choices. For each $\boldsymbol{\delta}=(\delta_1,\ldots,\delta_t)\in\{\pm 1\}^t$ we consider the linear form $f_{\boldsymbol{\delta}}(n_1,\dots,n_t)=\sum_{i=1}^t \delta_i n_i$. We claim that already the sum of the contributions of $f_{\boldsymbol{\delta}}$ and $f_{-\boldsymbol{\delta}}$ is polynomial, i.e. that

$$\begin{aligned}\mathbf{n}\mapsto\ &[z_1^{k-j}]\prod_{i=1}^t((-1)^{m_i}e^{\delta_i n_i z_1}-1)[y_1^{f_{\boldsymbol{\delta}}(\mathbf{n})}][z_1^j]D_{[1]}\\ &+[z_1^{k-j}]\prod_{i=1}^t((-1)^{m_i}e^{-\delta_i n_i z_1}-1)[y_1^{f_{-\boldsymbol{\delta}}(\mathbf{n})}][z_1^j]D_{[1]}\end{aligned}$$

is the restriction of a polynomial to any collection of natural numbers $n_i\equiv m_i$ mod (2) is the fixed coset. If we denote $f^+_{\boldsymbol{\delta}}=\max(0,f_{\boldsymbol{\delta}})$, the claim follows from the observation that $Q_j(\frac12 f^+_{\boldsymbol{\delta}}(\mathbf{n}))+(-1)^jQ_j(\frac12 f^+_{-\boldsymbol{\delta}}(\mathbf{n}))=Q_j(\frac12 f_{\boldsymbol{\delta}}(\mathbf{n}))$ is globally a polynomial for $\mathbf{n}$ in a fixed congruence class and for Q_j with the parity as in Lemma 7.3. The polynomiality for $s=1$ and the $\widetilde{K_n}$-version follows by summing up these expressions.

Second, we argue that $A_2'(\mathbf{w},\bar p_k)$ without the additional factors n in $\widetilde{K_n}$ is a piecewise polynomial, i.e. that the polynomial expression obtained previously using $\widetilde{K_n}$ is indeed divisible by n. The divisibility by n_i follows from adding the contribution of f_δ and $f_{\delta'}$ where δ' differs from δ precisely in the i-th digit, since $(n_i+n_j)^k+(n_i-n_j)^k$ is divisible by n_i independently of the parity of k.

For the general case $s\geq 1$ follows along the same lines. We first prove a generalization of Lemma 7.3, stating that for $\mathbf{k}=(k_1,\dots,k_s)$ there is a polynomial $Q_{\mathbf{k}}$ in s variables g_i such that

$$Q_{\mathbf{k}}(g_1,\dots,g_s)\ =\ [y_1^{-2g_1}\cdots y_s^{-2g_s}][z_1^{k_1}\cdots z_s^{k_s}]\Big(\widetilde{A}\prod_{i<j}D_{[ij]}\prod_j D_{[j]}\Big)$$

if all $g_i\geq 0$. Moreover this polynomial has the parity

$$Q_{\mathbf{k}}(g_1,\dots,-g_j,\dots,g_s)\ =\ (-1)^{k_j}Q_{\mathbf{k}}(g_1,\dots,-g_j,\dots,g_s)\,. \tag{54}$$

To see this, we write

$$\begin{aligned}&\partial_{z_1}^{k_1}\cdots\partial_{z_s}^{k_s}\Big(\widetilde{A}\prod_{i<j}D_{[ij]}\prod_j D_{[j]}\Big)|_{z_1=\cdots=z_s=0}\\ =\ &\sum_{\boldsymbol{\ell},\mathbf{m},\mathbf{n}}R_{\mathbf{k},\boldsymbol{\ell},\mathbf{m},\mathbf{n}}(y_1,\dots,y_s)\cdot\prod_i\frac{1}{(1-y_i^2)^{\ell_i}}\cdot\prod_{i<j}\frac{1}{(1-y_iy_j)^{m_{ij}}}\cdot\prod_{i<j}\frac{1}{(1-y_j/y_i)^{n_{ij}}}\end{aligned}$$

for some polynomial $R_{\mathbf{k},\boldsymbol{\ell},\mathbf{m},\mathbf{n}}$ with the components of $\boldsymbol{\ell},\mathbf{m}$ and $\mathbf{n}$ bounded in terms of $\mathbf{k}$. Using the binomial expansion of this expression we see that the coefficient $[y_1^{-2g_1}\cdots y_s^{-2g_s}]$ of this expression is a sum of polynomial expressions in non-negative integers u_i, v_{ij} and w_{ij} over the bounded simplex defined by

$$u_i+\sum_j v_{ij}+\sum_{j<i}n_{ij}-\sum_{j>i}n_{ij}\ =\ g_i,\quad (i=1,\dots,s)\,.$$

This sum is again a polynomial and this implies the polynomiality claim. Moreover, the argument $\widetilde{A}\prod_{i<j}D_{[ij]}\prod_j D_{[j]}$ in the definition of $Q_{\mathbf{k}}$ is unchanged under the transformation $(y_i,z_i)\mapsto(1/y_i,-z_i)$ since each $D_{[i]}$ has this property and since this transformation swaps $D_{[ij]}$ with the (i,j)-factor of $\widetilde{A}$. As in the case $s=1$ this implies palindromic numerators and the parity statement (54).

We show similarly that for each tuple $(j_1,\dots,j_s)$ separately the function

$$\mathbf{n}=(n_1,\dots,n_t)\;\mapsto\;[y_1^0]\Big([z_1^{k_1-j_1}\cdots z_s^{k_s-j_s}]\prod_{i=1}^{t}\widetilde{K}_{n_i}\;\cdot\;[z_1^{j_1}\cdots z_s^{j_s}]\widetilde{A}\prod_{i<j}D_{[ij]}\prod_{j}D_{[j]}\Big)$$

is a polynomial. Evaluation of $\prod_{i=1}^t \widetilde{K}_{n_i}$ now leads to s linear forms $f_1(\mathbf{n}),\dots,f_s(\mathbf{n})$ with each n_i appearing in exactly one of the f_j, and with coefficient ± 1. Given one such tuple, the contributions of $\pm f_1,\dots,\pm f_s$ add up to a global polynomial thanks to (54), as in the case $s=1$. Finally adding the contribution of $f_1,\dots,f_s$ and the linear form with precisely the sign of n_i flipped gives the divisibility by n_i that was still left to prove. □

7.3. **Quasimodularity of the number of pillowcase covers.** Recall that we introduced in Section 2 the generating functions $N^0(\Pi)$ and $N'(\Pi)$ of the number of pillowcase covers that are connected resp. without unramified components.

Corollary 7.4. *For any ramification profile* Π *the counting function* $N^0(\Pi)$ *for connected pillowcase covers of profile* Π *is a quasimodular form for the group* $\Gamma_0(2)$ *of mixed weight less or equal to* $\mathrm{wt}(\Pi)=|\Pi|+\ell(\Pi)$.

Proof. In (15) we recalled that $N'(\Pi)$ is the w-bracket of some element in $\overline{\Lambda}$. By Theorem 4.2 this series is thus a linear combination of auxiliary brackets with entries p_ℓ in the first arguments and a product of $\overline{p}_k$'s as last argument. The classical polynomiality for triple Hurwitz numbers with p_k-arguments (summarized as [GM18, Theorem 4.1]) and polynomiality in Theorem 7.2 imply that both auxiliary functions $A'(\cdot)$ and $A_2'(\cdot)$ appearing in the definition (31) of auxiliary brackets are indeed polynomials. That is, the auxiliary bracket is the sum over all subsets $E^+(\Gamma)$ and all parity conditions par $\in \mathrm{PC}(\Gamma)$ of graph sums of the form defined by (40) and (41). By Theorem 6.1 such a graph sum defines a quasimodular form of the weight as claimed. This gives the result for $N'(\Pi)$ and the claim for $N^0(\Pi)$ follows from inclusion-exclusion, see e.g. [GM18, Proposition 2.1]. □

8. Application to Siegel-Veech constants

In this section we show that counting pillowcase covers with certain weight functions also fall in the scope of the quasimodularity theorems and we prove Theorem 1.2.

Let $\lambda=(\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_k)$ be a partition. For $p\in\mathbb{Z}$ we define the *p-th Siegel-Veech weight* of λ to be $S_p(\lambda)=\sum_{j=1}^k\lambda_j^p$ .With the conventions of Section 2, in particular the definition of Hurwitz tuples in (2), the core curves of the horizontal cylinders have the monodromies

$$\begin{aligned}
\boldsymbol{\sigma}_0 &= \boldsymbol{\alpha}_1\boldsymbol{\alpha}_4=\boldsymbol{\alpha}_2\boldsymbol{\alpha}_3(\boldsymbol{\gamma}_1\dots\boldsymbol{\gamma}_n)^{-1}\\
\boldsymbol{\sigma}_1 &= \boldsymbol{\alpha}_1\boldsymbol{\alpha}_4\boldsymbol{\gamma}_1=\boldsymbol{\alpha}_2\boldsymbol{\alpha}_3(\boldsymbol{\gamma}_2\dots\boldsymbol{\gamma}_n)^{-1}\\
&\dots\\
\boldsymbol{\sigma}_n &= \boldsymbol{\alpha}_1\boldsymbol{\alpha}_4\boldsymbol{\gamma}_1\dots\boldsymbol{\gamma}_n=\boldsymbol{\alpha}_2\boldsymbol{\alpha}_3\,.
\end{aligned}$$

Motivated by the relation to area-Siegel-Veech constants in Proposition 8.3 below, we define the Siegel-Veech weighted Hurwitz numbers of a Hurwitz tuple h to be

$$S_p(h)\;=\;\sum_{i=0}^{n}S_p(\boldsymbol{\sigma}_i(h))\,. \tag{55}$$

Next, for $* \in \{', 0, \emptyset\}$ we package them into the generating series

$$c_p^*(d,\Pi) \;=\; \sum_{j=1}^{|\mathrm{Hur}_d^*(\Pi)|} S_p(\alpha^{(j)})\,, \quad \text{and} \quad c_p^*(\Pi) = \sum_{d\geq 0} c_p^*(d,\Pi) q^d\,. \tag{56}$$

These series admit the following graph sum decomposition. Let $\widetilde{\mathbb{N}}_{\mathrm{reg}}^{E(G)}$ be the special case of the height space defined in (8) with all the horizontal cylinders on the base pillow of the same height, i.e. with $\varepsilon_{4+i} = i/2(n-1)$ for $i = 1, \ldots, n$.

Proposition 8.1. *The generating series $c_p'(\Pi)$ can be expressed in terms of graph sums of triple Hurwitz numbers as*

$$c_p'(\Pi) \;=\; \sum_{\Gamma} \frac{1}{|\mathrm{Aut}(\Gamma)|} c_p'(\Pi,\Gamma) \quad \textit{where} \quad c_p'(\Pi,\Gamma) \;=\; \sum_{G\in\Gamma} c_p'(\Pi,G)$$

and where

$$c_p'(\Pi,G) = \sum_{\substack{h\in\widetilde{\mathbb{N}}_{\mathrm{reg}}^{E(G)},\\ w\in\mathbb{Z}_+^{E(G)}}} \Big(\sum_{e\in E(G)} h_e w_e^p\Big) \prod_{e\in E(G)} w_e q^{h_e w_e} A_2'(\mathbf{w}_0,\nu) \prod_{v\in V(G)} A'(\mathbf{w}_v^-,\mathbf{w}_v^+,\mu_v)\;\delta(v)\,.$$

Proof. The definition (55) together with the definition of the Siegel-Veech weight is made such that a covering defined by a Hurwitz tuples is counted with the weight given by the sum over all horizontal cylinders C of $h(C)w(C)^p$. This results in the extra factor in the formula for $c_p'(\Pi,G)$ in comparison with the formula for $N'(\Pi,G)$ in (29). The summation over $\widetilde{\mathbb{N}}_{\mathrm{reg}}^{E(G)}$ is needed to ensure that each strip of each cylinder is counted with the same weight. The whole formula is a direct consequence of the correspondence theorem, i.e. of Proposition 2.1. □

Corollary 8.2. *For any ramification profile Π and any odd $p \geq -1$ the generating series $c_p'(\Pi)$ for counting pillowcase covers without unramified components and with p-Siegel-Veech weight as well as the generating series $c_p^0(\Pi)$ for connected counting with p-Siegel-Veech weight are quasimodular forms for the group $\Gamma_0(2)$ of mixed weight $\leq \mathrm{wt}(\Pi) + p + 1$.*

Roughly, this follows from the polynomiality of $A'(\cdot)$ and $A_2'(\cdot)$ (i.e. from Theorem 7.2) in a similar way as Corollary (7.4). The extra factor $\sum_{e\in E(G)} h_e w_e^p$ raises the degree of the polynomial by $p+1$ and this results in the shifted weight. We explain the procedure in detail in Section 8.2

8.1. **Relation to area Siegel-Veech constants.** Siegel-Veech constants measure the growth rates of the number of saddle connections or closed geodesics or equivalently embedded cylinders. Among the various possibilities of weighting the count, the area weight is the most important due to its connection to the sum of Lyapunov exponents ([EKZ14]). In detail,

$$c_{\mathrm{area}}(X) \;=\; \lim_{L\to\infty} \frac{N_{\mathrm{area}}(T,L)}{\pi L^2}, \quad \text{where} \quad N_{\mathrm{area}}(T,L) \;=\; \sum_{\substack{Z\subset X \text{ cylinder},\\ w(Z)\geq L}} \frac{\mathrm{Area}(Z)}{\mathrm{Area}(X)}.$$

is called the *(area) Siegel-Veech constant* of the flat surface X. This constants are interesting both for generic flat surfaces of a given singularity type and for pillowcase covers.

Proposition 8.3. *The area Siegel-Veech constant is related to Siegel-Veech weighted Hurwitz numbers by*

$$c_{area}(d,\Pi) \;=\; \frac{3}{\pi^2}\frac{c^0_{-1}(d,\Pi)}{N^0_d(\Pi)}\,.$$

In particular, knowing the numerator and the denominator of the right hand side to be quasimodular forms, and thus knowing the asymptotic behavior of both $c^0_{-1}(d,\Pi)$ and $N^0_d(\Pi)$ as $d\to\infty$ allows to compute the area Siegel-Veech constant of a generic surface with a given singularity type.

Proof. The proof of [EKZ14, Theorem 4] or [CMZ18, Theorem 3.1] is easily adapted from torus covers to pillowcase covers. □

8.2. **Quasimodularity of Siegel-Veech weighted graph sums.** The goal of this section is to prove Corollary 8.2. This will follow from the following proposition. Recall from Section 6 the definition of the distinguished edges $E^+(\Gamma)$ and the parity conditions $\mathrm{par}\in\mathrm{PC}(\Gamma)$.

Proposition 8.4. *If $\mathbf{m}=(m_1,\dots,m_{|E(\Gamma)|})$ is a tuple of even integers, then for each $e_0\in E(G)$ the graph sum $S^{SV}_{e_0}(\Gamma,E^+,\mathbf{m},\mathrm{par})=\sum_{G\in(\Gamma,E^+)}S^{SV}_{e_0}(G,\mathbf{m},\mathrm{par})$, where*

$$S^{SV}_{e_0}(G,\mathbf{m},\mathrm{par}) \;=\; \sum_{\substack{h\in\hat{\mathbb{N}}^{E(G)},\\ w\in\mathbb{N}_+^{E(G)^*}}}\;\sum_{\substack{\mathbf{w}_0\in\mathbb{N}^{E^0(\Gamma)}_{>0}\\ \mathbf{w}_0\cong\mathrm{par}\bmod 2}}\frac{h_{e_0}}{w_{e_0}}\prod_{e\in E(G)}w_i^{m_i+1}q^{h_iw_i}\prod_{v\in V(G)^*}\delta(v)\,,$$

is a quasimodular form of mixed weight at most $k(\mathbf{m})=\sum_i(m_i+2)$.

Proof. We may reduced to the reduced graph $\overline{\Gamma}$ by computing the loop contributions separately, compare Lemma 6.3 or rather [GM18, Lemma 7.5]. If the e_0 is not a loop, then the loop contributions are quasimodular by Lemma 6.4. If e_0 is a loop, we also need to take the extra factor h/w into account and note that $\sum_{w,h=1}^\infty hw^mq^{hw}=D_qS_m$ is quasimodular (for $m\geq 2$ even) and similarly for the odd and even variants appearing the proof of Lemma 6.4.

To deal with $\overline{\Gamma}$, we combine the construction of Section 6.2 and the Siegel-Veech weight in the proof of [GM18, Theorem 7.3]. More precisely, we define if $e_0\in E^*(\Gamma)$

$$P^{SV}_{\overline{\Gamma},E^+,\mathbf{m},\mathrm{par}}(\mathbf{z}) \;=\; \frac{D_qP^{(m_{e_0}-2)}(z_{v_1(e_0)}\pm z_{v_2(e_0)})}{P^{(m_{e_0})}(z_{v_1(e_0)}\pm z_{v_2(e_0)})}\cdot P_{\overline{\Gamma},E^+,\mathbf{m},\mathrm{par}}(\mathbf{z})$$

if $m_{e_0}\geq 2$ and in the remaining case $m_{e_0}=0$ we let

$$P^{SV}_{\overline{\Gamma},E^+,\mathbf{m},\mathrm{par}}(\mathbf{z}) \;=\; \frac{L(z_{v_1(e_0)}\pm z_{v_2(e_0)})}{P(z_{v_1(e_0)}\pm z_{v_2(e_0)})}\cdot P_{\overline{\Gamma},E^+,\mathbf{m},\mathrm{par}}(\mathbf{z})$$

with L as in (38). In both cases the sign is chosen according to $e_0\in E^\pm(\Gamma)$. This definition replaces the factor in $P_{\overline{\Gamma},E^+,\mathbf{m},\mathrm{par}}(\mathbf{z})$ corresponding to the edge e_0 in $P_{\overline{\Gamma},E^+,\mathbf{m},\mathrm{par}}(\mathbf{z})$ is replaced by one with the extra factor h_{e_0}/w_{e_0}. This follows from the power series expansion of P and L given in [GM18, Equation (41)], compare also the proof of Theorem 7.3 in loc. cit. If $e_0 \in E^0(\Gamma)$, we define similarly

$$\begin{aligned} P^{SV}_{\overline{\Gamma},E^+,\mathbf{m},\mathrm{par}}(\mathbf{z}) &= \frac{D_q P^{(m_{e_0}-2)}_{\mathrm{par}_{e_o}}(2z_{v_2(e_0)})}{P^{(m_{e_0})}_{\mathrm{par}_{e_o}}(2z_{v_2(e_0)})} \cdot P_{\overline{\Gamma},E^+,\mathbf{m},\mathrm{par}}(\mathbf{z}) \quad \text{or} \\ P^{SV}_{\overline{\Gamma},E^+,\mathbf{m},\mathrm{par}}(\mathbf{z}) &= \frac{L_{\mathrm{par}_{e_0}}(2z_{v_2(e_0)})}{P_{\mathrm{par}_{e_0}}(2z_{v_2(e_0)})} \cdot P_{\overline{\Gamma},E^+,\mathbf{m},\mathrm{par}}(\mathbf{z}) \end{aligned}$$

according to $m_{e_0} \geq 2$ or $m_{e_0} = 0$ respectively, where $L_{\mathrm{even}}(z,\tau) = 2L(2z, 2\tau)$ and $L_{\mathrm{odd}} = L - L_{\mathrm{even}}$. We this modified prefactor, the same proof as in Proposition 6.5 shows that

$$S^{SV}(\overline{\Gamma}, E^+, \mathbf{m}, \mathrm{par}) = [\zeta_n^0, \ldots, \zeta_1^0]\, P^{SV}_{\overline{\Gamma},E^+,\mathbf{m},\mathrm{par}}(\mathbf{z}; \tau)\,. \tag{57}$$

Each of the factors in the definition of $P^{SV}_{\overline{\Gamma},E^+,\mathbf{m},\mathrm{par}}(\mathbf{z})$ is a quasi-elliptic quasimodular form by Proposition 5.3 and Proposition 5.4 As in the case without Siegel-Veech weight, the claim follows from Theorem 5.6 and the upgrade Lemma 5.1 to get quasimodularity by the group $\Gamma_0(2)$. □

Proof of Corollary 8.2. We want to apply Proposition 8.4. First, we pretend for the moment that the summation in 8.1 is over the normalized height space $\widehat{\mathbb{N}}^{E(G)}$ rather than over $\widetilde{\mathbb{N}}^{E(G)}_{\mathrm{reg}}$ on prove quasimodularity of the corresponding sum. Second, to reduce the graph sum expression for Siegel-Veech constants in Proposition 8.1 to those with polynomial entries, we have to mimic the argument leading to Theorem 4.2. Let

$$[F_1, \ldots, F_n; F_0]^{p-SV} = \sum_{\Gamma} \sum_{G \in \Gamma} [F_1, \ldots, F_n; F_0]^{p-SV}_G$$

and

$$\begin{aligned} &[F_1, \ldots, F_n; F_0]^{p-SV}_G \\ &= \sum_{\substack{h \in \widetilde{\mathbb{N}}^{E(G)},\\ w \in \mathbb{Z}_+^{E(G)}}} \Big(\sum_{e \in E(G)} h_e w_e^p \Big) \prod_{e \in E(G)} w_e q^{h_e w_e} A_2'(\mathbf{w}_0, F_0) \prod_{v \in V(G)} A'(\mathbf{w}_v^-, \mathbf{w}_v^+, F_{\#_v})\ \delta(v)\,. \end{aligned}$$

Then Proposition 8.1 can be generalized using the correspondence theorem in the form Proposition 2.2 to

$$c_p'(\Pi) = [\prod_{i \notin S} f_{\mu_i}; \prod_{i \in S} f_{\mu_i} g_\nu]^{p-SV} \tag{58}$$

for any subset $S \subseteq \{1, \ldots, n\}$. To prove quasimodularity we start with $S = \emptyset$ and decompose g_ν as linear combination of $P_{I,J} = \prod_{jinJ} p_{b_j} \prod_{i \in I} \overline{p}_{a_i}$ for some a_i and b_j. For the summands where $J = \emptyset$ we write the first argument of the bracket as linear combination of products of p_k and use the polynomiality for triple Hurwitz numbers with p_k-argument and Theorem 7.2 to conclude thanks to Proposition 8.4.

To deal with the summands where $J = \emptyset$, we write $P_{I,J}$ a linear combination with each time a product of f_k's and one g_ν. This is possible by Theorem 3.3. Since $J = \emptyset$, each term involves at least one f_k. For each term we now apply (58) twice to move the f_k-product to the first argument of the auxiliary bracket. By this procedure we have reduced the weight to the g_ν in the second argument and we conclude by induction.

We have to justify the first simplification concerning height spaces. Note that the shift to $\widehat{\mathbb{N}}^{E(G)}$ does not change the q-exponents by Lemma 6.2, so we may focus on the Fourier coefficients. Since the expression for $c'_p(\Pi)$ in Proposition 8.1 involves a summation over all orientations we may combine the contributions of G and the reverse orientation $-G$, where the arrows of all edges except for those emmanating from v_0 have been inverted. We thus obtain a pre-factor of

$$\sum_{h\in\mathbb{N}_{\geq a_e}} \frac{h-\Delta(e)}{w} + \sum_{h\in\mathbb{N}_{\geq(1-a_e)}} \frac{h+\Delta(e)}{w} \quad \text{instead of} \quad \sum_{h\in\mathbb{N}_{\geq a_e}} \frac{h}{w} + \sum_{h\in\mathbb{N}_{\geq(1-a_e)}} \frac{h}{w}$$

when using $\widehat{\mathbb{N}}^{E(G)}$. The difference is only the term $h=0$, i.e. without an h/w-prefactor, and we recursively know these graph sums to be quasimodular (in fact of smaller weight).

Finally, the quasimodularity for $c_p^0(\Pi)$ follows from the usual inclusion-exclusion formulas, see [CMZ18, Proposition 6.2] for the version with Siegel-Veech weight. □

8.3. **Siegel-Veech weight and representation theory.** The reader familiar with [CMZ18] will recall that counting function for Hurwitz tuples, even with Siegel-Veech weight, can be expressed efficiently using the representation theory of the symmetric group. More precisely, for Π the profile of a torus cover

$$N_d(\Pi) \;=\; \sum_{\lambda\in\mathcal{P}(d)} \prod_{i=1}^{n} f_{\mu^{(i)}}(\lambda) \quad \text{and} \quad c_p(d,\Pi) \;=\; \sum_{\lambda\in\mathcal{P}(d)} \prod_{i=1}^{n} f_{\mu_i}(\lambda) T_p(\lambda)\,, \tag{59}$$

where $T_p(\lambda) = \sum_{\xi\in Y_\lambda} h(\xi)^{p-1}$ and where $h(\xi)$ is the hook-length of the cell ξ of the Young diagram Y_λ.

It would be very useful to have a similar formula for the Siegel-Veech weighted counting of pillowcase covers. We are only aware of the following much more complicated formula.

Proposition 8.5. *The number of all covers of degree d with profile Π counted with p-Siegel-Veech weight is*

$$c_p(d,\Pi) \;=\; \frac{1}{l(\mu)+1} \sum_{k=0}^{l(\mu)} \sum_{C} S_p(C) \sum_{\lambda,\lambda'} \sqrt{w(\lambda)w(\lambda')}|C|\chi^{\lambda}(C)\chi^{\lambda'}(C)\cdot$$
$$\cdot\, g_\nu(\lambda') \prod_{i=1}^{k} f_{\mu_i}(\lambda') \prod_{i=k+1}^{l(\mu)} f_{\mu_i}(\lambda)\,.$$

In particular we are not aware of an operator on Fock space whose q-trace computes the generating series with Siegel-Veech weight. Note that the $\mathfrak{W}$-operator of [EO06, Theorem 4] has the property $\langle v_\lambda|\mathfrak{W}|v_\lambda\rangle = w(\lambda)$, but it is not true that $\langle v_\lambda|\mathfrak{W}|v_\nu\rangle = \sqrt{w(\lambda)w(\nu)}$ for $\lambda\neq\nu$. Finding a vertex operator with this property would be a way to use Proposition 8.5 to express Siegel-Veech weighted generating series as q-traces.

Proof. The monodromy of the core curves of the cylinders of a Hurwitz tuple $h \in \mathrm{Hur}_d(\Pi)$ is given by

$$\begin{aligned}\boldsymbol{\sigma}_0 &= \boldsymbol{\alpha}_1\boldsymbol{\alpha}_4 = \boldsymbol{\alpha}_2\boldsymbol{\alpha}_3(\boldsymbol{\gamma}_1\dots\boldsymbol{\gamma}_n)^{-1}\\ \boldsymbol{\sigma}_1 &= \boldsymbol{\alpha}_1\boldsymbol{\alpha}_4\boldsymbol{\gamma}_1 = \boldsymbol{\alpha}_2\boldsymbol{\alpha}_3(\boldsymbol{\gamma}_2\dots\boldsymbol{\gamma}_n)^{-1}\\ &\dots\\ \boldsymbol{\sigma}_n &= \boldsymbol{\alpha}_1\boldsymbol{\alpha}_4\boldsymbol{\gamma}_1\dots\boldsymbol{\gamma}_n = \boldsymbol{\alpha}_2\boldsymbol{\alpha}_3\,.\end{aligned}$$

To count Hurwitz tuples with Siegel-Veech weight, say for the k-th cylinder, we split the defining equation as

$$\boldsymbol{\alpha}_1\boldsymbol{\alpha}_4\boldsymbol{\gamma}_1,\dots,\boldsymbol{\gamma}_k \;=\; c \;=\; \boldsymbol{\alpha}_2\boldsymbol{\alpha}_3\boldsymbol{\gamma}_n^{-1}\dots\boldsymbol{\gamma}_{k+1}^{-1} \tag{60}$$

and count the solutions of each side separately. That is, we denote by $C_1, C_2, C_3, C_4, C, C_1^{\mu}, \dots, C_n^{\mu}$, respectively, the conjugacy classes of permutations of type

$$(\nu, 2^{|\nu|/2-d}), (2^d), (2^d), (2^d), (k^{c_k}), (\mu_1, 1^{d-\mu_1}), \dots, (\mu_n, 1^{d-\mu_n}).$$

We denote by $c_p(S_{2d}; C_1, C_4, C_1^{\mu}, \dots, C_k^{\mu}, C)$ the number of solutions of

$$\boldsymbol{\alpha}_1\boldsymbol{\alpha}_4\boldsymbol{\gamma}_1,\dots,\boldsymbol{\gamma}_k \;=\; c \tag{61}$$

with $\boldsymbol{\alpha}_i$ of conjugacy class C_i and $\boldsymbol{\gamma}_i$ of conjugacy class C_i^{ν}), c of conjugacy class C, counted with weight $S_p(c) = S_p(C)$. We claim that

$$\begin{aligned}c_p(S_{2d}; C_1, C_4, C_1^{\mu}, \dots, C_k^{\mu}, C) = \frac{|C_1||C_4||C_1^{\mu}|\dots|C_k^{\mu}||C|}{|G|}S_p(C)\cdot\\ \cdot\sum_{\chi}\frac{\chi(C_1)\chi(C_4)\chi(C_1^{\mu})\dots\chi(C_k^{\mu})\chi(C)}{\chi(1)^{k+1}}\,.\end{aligned}$$

To see this, we revisit the proof of the orthogonality relations, see [Ser08, Theorem 7.2.1]. We introduce the class function

$$\phi(x) = S_p(C)1_{\{x\in C\}} = \sum_{\chi} c_\chi \chi$$

with

$$c_\chi = \int_G S_p(C)1_{\{x\in C\}}\overline{\chi}(x)dx = \frac{|C|}{|G|}S_p(C)\chi(C).$$

We have

$$\begin{aligned}I(\phi) &= \int_{G^{k+2}} \phi(t_1\boldsymbol{\alpha}_1 t_1^{-1} t_4\boldsymbol{\alpha}_4 t_4^{-1} s_1\boldsymbol{\gamma}_1 s_1^{-1}\dots s_k\boldsymbol{\gamma}_k s_k^{-1} y)dt_1dt_4ds_1\dots ds_k\\ &= \sum_{\chi} c_\chi \frac{\chi(C_1)\chi(C_4)\chi(C_1^{\mu})\dots\chi(C_k^{\mu})\chi(y)}{\chi(1)^{k+2}}\end{aligned}$$

and the left hand side is $I(\phi) = N_p(S_{2d}; C_1, C_4, C_1^{\mu}, \dots, C_k^{\mu}, C)/|G|^{k+2}$. Taking $y = 1$ we get the formula of the claim.

In the second step we count the solution of the right equality of (60) with weight $c_p(S_{2d}; C_1, C_4, C_1^{\mu}, \dots, C_k^{\mu}, C)$. The class function is now

$$\phi(x) \;=\; \frac{1}{|C|}c_p(S_{2d}; C_1, C_4, C_1^{\mu}, \dots, C_k^{\mu}, x)1_{x\in C}\,,$$

and its coefficients are

$$c_\chi \;=\; \frac{1}{|G|}N_p(S_{2d}; C_1, C_4, C_1^{\mu}, \dots, C_k^{\mu}, C)\overline{\chi}(C)\,.$$

With the argument as above we conclude that the number of solutions counted with weight is

$$|C_2||C_3||C_n^\mu|\dots|C_{k+1}^\mu|\sum_\chi c_\chi \frac{\chi(C_2)\chi(C_3)\chi(C_n^\mu)\dots\chi(C_{k+1}^\mu)}{\chi(1)^{n-k+1}}\,.$$

We sum now on all conjugacy classes C and use the definition of f_μ and g_ν to obtain the result. □

9. Example: $\mathcal{Q}(2,1,-1^3)$

In this section, we treat the example of the stratum $\mathcal{Q}(2,1,-1^3)$ from A to Z to illustrate all sections of the paper. This stratum is the lowest dimensional example exhibiting all the relevant aspects.

Integer points in the stratum $\mathcal{Q}(2,1,-1^3)$ correspond in our setting to covers of the pillow ramified over five points: the four corners $P_1,\dots P_4$ and an additional point P_5 (see Figure 1), with the ramification profile $\Pi = (\mu^{(1)},\mu^{(2)},\mu^{(3)},\mu^{(4)},\mu^{(5)})$ where $\mu^{(1)} = (3,1,1,1,2^{d-3})$ over P_1, $\mu^{(2)} = \mu^{(3)} = \mu^{(4)} = (2^d)$ over P_2, P_3, P_4 and $\mu^{(5)} = (2,1^{2d-2})$ over P_5. Here $2d$ is the degree of the cover. In this particular case, we cannot have covers with at least two ramified connected components, so $N'(\Pi) = N^0(\Pi)$.

9.1. **Counting covers.** By [EO06] and Theorem 1.1 the generating series $N^0(\Pi)$ is a quasimodular form for $\Gamma_0(2)$ of mixed weight less or equal to 6. Computing the first coefficients of the series we get that[1]

$$\begin{aligned} N^0(\Pi) \;=\; & 360G_2(q^2)^3 - 360G_2(q)G_2(q^2)^2 + 72G_2(q)^2G_2(q^2) - 30G_4(q^2)G_2(q^2) \\ & - \frac{5}{4}G_4(q^2) + 3G_2(q)^2 + 15G_2(q^2)^2 - 15G_2(q)G_2(q^2). \end{aligned}$$

Our goal is to retrieve this result by considering all graph contributions. Standard Hurwitz theory (see (26) and [EO06]) gives that

$$N^0(\Pi) = \frac{1}{N(\Pi_\emptyset)}\sum_\lambda g_{3,1,1,1}(\lambda)f_2(\lambda)\mathrm{w}^2(\lambda)q^{|\lambda|/2}$$

where $g_{3,1,1,1}(\lambda) = f_{3,1,1,1,2,\dots,2}(\lambda)/f_{2,\dots,2}(\lambda)$ and $w(\lambda)$ as in (14).

Following the strategy in Theorem 4.2 we need to express the graph sums with local contributions $g_{3,1,1,1}$ and f_2 by graphs sums using the functions p_k and $\overline{p}_k$ for which we have nice polynomiality results. We compute that

$$g_{(3,1,1,1)} \;=\; -\frac{1}{4}\overline{p}_1p_1 + \frac{1}{108}\overline{p}_1^3 - \frac{1}{36}\overline{p}_2\overline{p}_1 + \frac{3}{8}\overline{p}_1 + \frac{2}{27}\overline{p}_3, \quad f_2 \;=\; \frac{1}{2}p_2$$

The definition

$$\begin{aligned} \overline{g}_{(3,1,1,1)} &:= \frac{1}{108}\overline{p}_1^3 - \frac{1}{36}\overline{p}_2\overline{p}_1 + \frac{3}{8}\overline{p}_1 + \frac{2}{27}\overline{p}_3 \\ g^{deg}_{(3,1,1,1)} &:= -\frac{1}{4}\overline{p}_1 \end{aligned}$$

and the resulting decomposition

$$g_{(3,1,1,1)} = \overline{g}_{(3,1,1,1)} + g^{deg}_{(3,1,1,1)}p_1.$$

[1] We choose in this paper to consider series in q^d rather than as series in q^{2d} as in [EO06]. The integer d can be seen as the area of the cover while $2d$ is the degree of the cover.

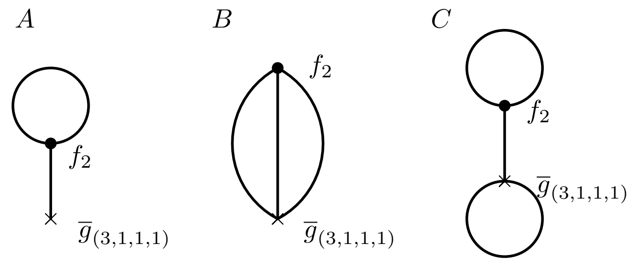


FIGURE 5. Admissible graphs for $\overline{g}_{(3,1,1,1)} \cdot f_2$

isolates the products of $\overline{p}_i$'s so we can use the polynomiality results of Section 7. Writing $\overline{g}_{3111}f_2 = \frac{1}{2}f_2f_1g_{11} + f_2g_{3111}$ and $g^{deg}_{3111}p_1f_2 = -\frac{1}{2}g_{11}f_2f_2 + \frac{1}{48}g_{11}f_2$ we can make geometric sense of the formal decomposition, as counting degenerate covers, e.g. in the stratum $\mathcal{Q}(2,0,-1^2)$. We determine the simple Hurwitz numbers with 2-stabilization A'_2 for products of $\overline{p}_i$'s, proven to be quasi-polynomials in Theorem 7.2 by computing the first few terms. As a result

$$A'_2\big((w),\overline{g}_{(3,1,1,1)}\big) = \frac{1}{24}w^2 + \frac{1}{3}$$
$$A'_2\big((w_1,w_2,w_3),\overline{g}_{(3,1,1,1)}\big) = \begin{cases} \frac{1}{2} & \text{if two } w_i \text{ are odd} \\ \frac{3}{2} & \text{if all } w_i \text{ are even} \end{cases}$$
$$A'_2\big((w),g^{deg}_{(3,1,1,1)}\big) = -\frac{1}{4}$$

We recall that these polynomials are only defined for $\sum w_i$ even. Similarly, the double Hurwitz numbers A' for products of p_i's are polynomials, in fact

$$A'\big((w),(w),p_1\big) \;=\; 1\,, \quad A'\big((w_1),(w_2,w_3),f_2\big) \;=\; 1\,,$$

We have thus collected all local polynomials.

Now we glue the local surfaces together, encoding the gluings by the various possible global graphs. The contribution of $\overline{g}_{(3,1,1,1)} \cdot f_2$ is encoded in graphs with two vertices, one special vertex v_0 of valency one or three, and another trivalent vertex v_1 since all other valencies result in a zero local polynomial and thus in a zero contribution. By convention, we represent v_0 as the bottom of the graph, marked with a cross. Disregarding orientations, we obtain three admissible graphs, shown in Figure 5.

Similarly, the contribution of $g^{deg}_{(3,1,1,1)} \cdot p_1 \cdot f_2$ is encoded in graphs with three vertices, one special vertex v_0 of valency 1, a vertex v_1 of valency 2 and a vertex v_2 of valency 3. The graphs are listed in Figure 6.

Next, we are supposed to sum over all possible orientations of these graphs. We sort the orientations by the subset of coherently oriented edges, i.e. by those distinguished with a + in the notation of Section 6. Note that certain decorations give trivial contributions, as the graph A with the loop decorated with +. In fact, considering all possible orientations of half-edges compatible with this decoration, we see that we get incompatible width conditions for the vertex v_1. This implies

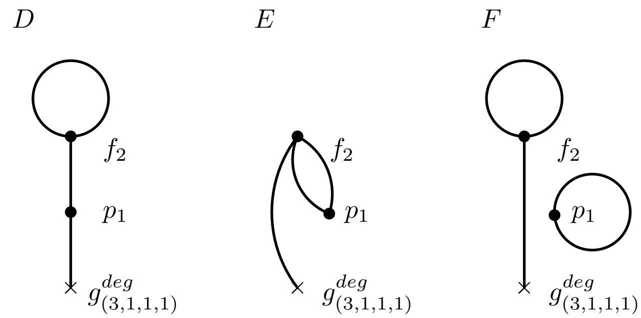


FIGURE 6. Admissible graphs for $\overline{g}_{(3,1,1,1)} \cdot f_2$

that integrating the corresponding propagator we will get terms like $\delta(w_2 = w_1+w_2)$ that are always trivial.

The next step is to associate to each decorated graph its propagator, and then to integrate this propagator (get its ζ^0-coefficient) to obtain the contributions of each individual graph. In the following table we consider only decorations with non trivial contributions. For each vertex, we indicate the corresponding integration variable z_i (on the left of the vertex) and the corresponding local polynomial (on the right). For each decorated graph we give the associated propagator and the contribution to the volume. The contributions were computed using three independent methods: the reduction algorithm used in the proof of Theorem 5.6, the computation of the first terms of the q-series using graph sums (41), and, independently, using extraction of $[\zeta^0]$ coefficients, combined with a numerical test of quasimodularity (test of linear dependency with the basis of quasimodular forms).

We provide details of the method of Theorem 5.6 for the graphs A and D, to illustrate the algorithm in the proof. We denote by $T = \{0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}\}$ the set of 2-torsion points and let $T^* = T \setminus \{0\}$. Note that, considering the residues at the poles at T, we have

$$P(2z;\tau) \;=\; \frac{1}{4}\sum_{a\in T} P(z-a;\tau)\,.$$

Consequently, $P(2z)P(z)$ is an elliptic function with a pole of order 4 at 0, and poles of order 2 at T^*, whose residues are easily compensated by $\frac{1}{4}\frac{1}{6}P''(z)$, $\frac{1}{4}P(z-a)P(a)$ for $a \in T^*$ respectively. Then, compensating the remaining pole of order 2 at 0, we get

$$P(2z)P(z) = \frac{1}{24}P''(z) + G_2P(z) + \frac{1}{4}\sum_{a\in T^*} P(a)\,(P(z) + P(z-a)) + \frac{5}{3}G_4 - 4G_2^2.$$

Since P is the derivative of a $1-$periodic function Z, the contour integral is then reduced to

$$[\zeta^0]P(2z)P(z) \;=\; \frac{5}{3}G_4 - 4G_2^2\,.$$

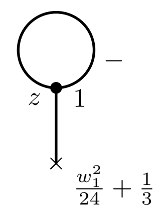
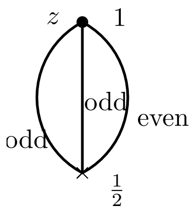
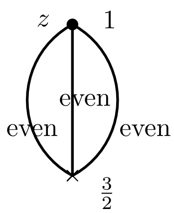
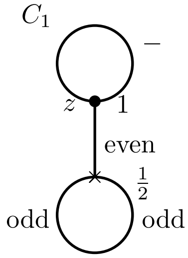
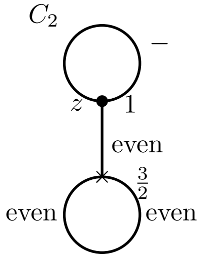
Graph Γ	$\frac{1}{\lvert\mathrm{Aut}(\Gamma)\rvert}$	Propagator	Contribution
A 	$\frac{1}{2}$	$\left(\frac{1}{24}P^{(2)}(z)+\frac{1}{3}P(z)\right)\cdot$ $\cdot P(2z)$	$\frac{7}{30}G_6-\frac{8}{3}G_4G_2+\frac{5}{9}G_4-\frac{4}{3}G_2^2$
B_1 	$\frac{1}{2}$	$\frac{1}{2}P_{\mathrm{odd}}(z)^2P_{\mathrm{even}}(z)$	$-\frac{1}{3}(16G_{22}G_{42}+8G_2G_{42})+$ $\frac{1}{5}(-192G_{22}^3+608G_2G_{22}^2-$ $384G_2^2G_{22}+64G_2^3)$
B_2 	$\frac{1}{6}$	$\frac{3}{2}P_{\mathrm{even}}(z)^3$	$48G_{22}G_{42}-16G_2G_{42}+$ $\frac{1}{5}(-832G_{22}^3+768G_2G_{22}^2-$ $384G_2^2G_{22}+64G_2^3)$
C_1 	$\frac{1}{4}$	$\frac{1}{2}S_{0,\mathrm{odd}}P_{\mathrm{even}}(z)P(2z)$	$\left(G_2-2G_{22}-\frac{1}{24}\right)\cdot$ $\left(\frac{5}{6}G_4-2G_2^2\right)$
C_2 	$\frac{1}{4}$	$\frac{3}{2}S_{0,\mathrm{even}}P_{\mathrm{even}}(z)P(2z)$	$3\left(2G_{22}+\frac{1}{12}\right)\left(\frac{5}{6}G_4-2G_2^2\right)$

FIGURE 7. Graphs for $\mathcal{Q}(2,1,-1^3)$: Contribution of $\overline{g}_{(3,1,1,1)}\cdot f_2$

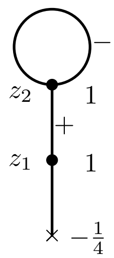
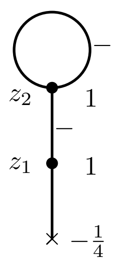
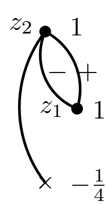
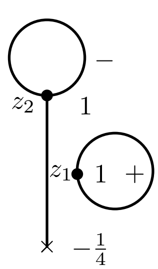
Graph Γ	$\frac{1}{\lvert\mathrm{Aut}(\Gamma)\rvert}$	Propagator	Contribution
D_a 	$\frac{1}{2}$	$-\frac{1}{4}P(z_1)P(z_1-z_2)\cdot$ $\cdot\, P(2z_2)$	$A-3B$
D_b 	$\frac{1}{2}$	$-\frac{1}{4}P(z_1)P(z_1+z_2)\cdot$ $\cdot\, P(2z_2)$	$A+B$
E 	1	$-\frac{1}{4}P(z_2)P(z_1-z_2)\cdot$ $\cdot\, P(z_1+z_2)$	$A+B$
F 	$\frac{1}{2}$	$-\frac{1}{4}S_0P(z_2)P(2z_2)$	$-\frac{1}{4}\left(G_2+\frac{1}{24}\right)\left(-\frac{20}{3}G_{42}+\right.$ $\left.80G_{22}^2-80G_2G_{22}+16G_2^2\right)$

FIGURE 8. Graphs for $\mathcal{Q}(2,1,-1^3)$: Contribution of $g^{deg}_{(3,1,1,1)}\cdot p_1\cdot f_2$

Proceeding similarly with the term $P(2z)P''(z)$ we get

$$\begin{aligned} P(2z)P''(z) &= \frac{1}{80}P^{(4)}(z) + \frac{G_2}{2}P''(z) + \frac{49}{2}G_4P(z) \\ &\quad + \frac{1}{4}\sum_{a\in T^*}\left(P(a)P''(z) + P''(a)P(z-a)\right) + \frac{28}{5}G_6 - 64G_4G_2\,, \end{aligned}$$

hence

$$[\zeta^0]P(2z)P''(z) \;=\; \frac{28}{5}G_6 - 64G_4G_2\,,$$

which gives the contribution of the graph A. The contributions of the graphs B and C are computed similarly using the decomposition

$$P(2z;2\tau) \;=\; \frac{1}{4}\left(P(z;\tau) + P(z-1/2;\tau)\right).$$

Computing the contribution of the graph D_a we will see quasi-elliptic functions appearing. We first decompose $P(z_1-z_2)P(2z_2)$ in the additive basis with respect to z_2. For this purpose decompose as usual $P(2z_2)$ into the sum of the four contributions of the 2-torsion points. The term $P(z_1-z_2)P(z_2)$ has a pole of order 2 at $z_2 = z_1$ (which is compensated by $P(z_1-z_2)P(z_1)$), a pole of order 2 at $z_2 = 0$ (which is compensated by $P(z_1)P(z_2)$), and it also has a pole of order one at $z_2 = z_1$ (which is compensated by $Z(z_1-z_2)P'(z_1)$), and a pole of order 1 at $z_2 = 0$ (finally compensated by $Z(z_2)P'(z_1)$) Proceeding similarly for the three other two-torsion points, we get

$$\begin{aligned} P(z_1-z_2)P(2z_2) = \frac{1}{4}\sum_{a\in T}\Big(&[P(z_2-a)+P(z_2-z_1)]P(z_1-a) \\ &+ [Z(z_2-a)-Z(z_2-z_1)]P'(z_1-a)\Big) + R(z_1) \end{aligned}$$

where

$$R(z_1) \;=\; \frac{4}{3}P''(2z_1) - \frac{1}{4}\sum_{a\in T}\Big(P'(z_1-a)Z(z_1-a)\Big) - 4P(2z_1)G_2 + \frac{1}{6}G_2^2 - \frac{5}{3}G_4.$$

As a result,

$$[\zeta_2^0]P(z_1)P(z_1-z_2)P(2z_2) \;=\; P(z_1)R(z_1)\,.$$

We already showed how to treat terms like $P(z_1)P''(2z_1)$, $P(z_1)P(2z_1)$, so we focus on $P(z_1)P'(z_1-a)Z(z_1-a)$ in the sequel. The product $S_2 = P(z_1)P'(z_1-a)$ is an elliptic function, so examining the pole orders we get

$$S_2 \;=\; \begin{cases} \frac{1}{12}P^{(3)}(z_1) + 2P'(z_1)G_2 & \text{if } a=0 \\ P''(a)[Z(z_1-a)-Z(z_1)] + P(a)P'(z_1-a) & \text{if } a=1/2 \\ P''(a)[Z(z_1-a)-Z(z_1)+\frac{1}{2}] + P(a)P'(z_1-a) & \text{if } a\in\{\tau/2,(1+\tau)/2\}\,. \end{cases}$$

When multiplying the right hand side by $Z(z_1-a)$ the right hand side belongs to the additive basis, except for the case $Z(z_1)Z(z_1-a)$ for $a\in T^*$. But since

$$\Delta\Big(Z(z_1)Z(z_1-a)\Big) \;=\; \frac{1}{2}\Delta\Big(Z(z_1)^2 + Z(z_1-a)^2\Big)\,,$$

the function $Z(z_1)Z(z_1-a) - \frac{1}{2}(Z(z_1)^2 + Z(z_1-a)^2)$ is elliptic with poles of order less or equal to 2 at 0 and a, so we get

$$Z(z_1)Z(z_1-a) = \frac{1}{2}\Big(Z(z_1)^2 + Z(z_1-a)^2 - P(z_1) - P(z_1-a)\Big) + 3G_2 - \frac{1}{2}P(a) + \begin{cases} 0 & \text{if } a = 1/2 \\ \frac{1}{2}[Z(z-a) - Z(a)] + \frac{1}{8} & \text{if } a = \tau/2, (1+\tau)/2 \end{cases}$$

and finally

$$[\zeta_1^0]Z(z_1)Z(z_1-a) \;=\; G_2 - \frac{1}{2}P(a) + \begin{cases} \frac{1}{6} & \text{if} a = 1/2 \\ -\frac{5}{24} & \text{if } a = \tau/2, (1+\tau)/2\,. \end{cases}$$

In total the contribution of graph D_a is a combination of the quasimodular forms

$$A = (-40G_{22} + 10G_2)G_{42} + 352G_{22}^3 - 408G_2G_{22}^2 + 144G_2^2G_{22} - 16G_2^3$$
$$B = -5/4G_{42} + 12G_{22}^2 - 12G_2G_{22} + 3G_2^2$$

as indicated in the table. Note that each contribution is a quasimodular form for $\Gamma(2)$, but also a series in q, so in fact it is a quasimodular form for $\Gamma_0(2)$ (as we remarked already Lemma 5.1). The sum of all contributions in the table is finally the quasimodular form given at the beginning of this subsection.

9.2. **Siegel-Veech weight.** Everything is ready to compute the contributions of these graphs with Siegel-Veech weight. We have the same graphs with the same local polynomials, we just associate a slightly modified propagator to take care of the weight. The recipe to get this propagator from the old one is simple, for example if they are no distinguished loops: for each edge of the graph replace the corresponding factor P by L and P^m by D_qP^{m-2} if $m \geq 2$, and then sum over all edges of the graph (see Section 8.2 for precise statement). The last step of integration is then similar to the previous case. We give the results in the Table 9 (we do not copy the factors $1/|\mathrm{Aut}(\Gamma)|$ which are the same; we group the graphs of same type). In the column contribution, *lwt* stands for lower weight terms. In the compilation of the table we used

$$S_0^{SV} \;=\; \sum_{w,h=1}^{\infty} hq^{wh} \;=\; S_0 \;=\; G_2 + \frac{1}{24}$$
$$S_{0,even}^{SV} \;=\; \sum_{w,h=1}^{\infty} hq^{(2w)h} \;=\; G_{22} + \frac{1}{24}$$

and $S_{0_odd}^{SV} \;=\; S_0^{SV} - S_{0,even}^{SV}$. Summing up, we get the generating series

$$c_{-1}^0(\Pi) \;=\; 245G_{22}^3 - 245G_2G_{22}^2 + 49G_2^2G_{22} - \frac{245}{12}G_{42}G_{22} + lwt.$$

The lower weight terms are weight 2 and 4 terms, as we can expect from the weight of $N^0(\Pi)$ (as L contains lower weight terms). This series is proportional to $N^0(\Pi)$, since this stratum is non-varying (see [CM12] for more explanation). We will see in the next section that the coefficient of proportionality is the Siegel-Veech constant of the stratum.

Graph	Propagator	Contribution
A	$\frac{1}{2}\Big[\Big(\frac{1}{24}D_qP(z)+\frac{1}{3}L(z)\Big)P(2z)$ $+\Big(\frac{1}{24}P^{(2)}(z)+\frac{1}{3}P(z)\Big)L(2z)\Big]$	$\frac{1}{2}\Big[-\frac{880}{3}G_{22}^3+340G_2G_{22}^2-120G_2^2G_{22}+\frac{100}{3}G_{42}G_{22}+\frac{40}{3}G_2^3+-\frac{25}{3}G_{42}G_2\Big]+lwt$
B	$\frac{1}{2}\cdot\frac{1}{2}\Big(2L_{odd}(z)P_{odd}(z)P_{even}(z)$ $+P_{odd}(z)^2L_{even}(z)\Big)$ $+\frac{1}{6}\cdot\frac{3}{2}\Big(3P_{even}(z)^2L_{even}(z)\Big)$	$\frac{1}{2}\Big[-\frac{560}{3}G_{22}^3+280G_2G_{22}^2-160G_2^2G_{22}+\frac{100}{3}G_{42}G_{22}+\frac{80}{3}G_2^3-\frac{50}{3}G_{42}G_2\Big]$
C	$\frac{1}{4}\Big[\Big(\frac{1}{2}S_{0,odd}+\frac{3}{2}S_{0,even}\Big)\cdot$ $\Big(L_{even}(z)P(2z)+P_{even}(z)L(2z)\Big)$ $+\Big(\frac{1}{2}S^{SV}_{0,odd}+\frac{3}{2}S^{SV}_{0,even}\Big)\cdot$ $P_{even}(z)P(2z)\Big]$	$\frac{1}{4}\Big[180G_{22}^3-115G_2G_{22}^2-29G_2^2G_{22}-15G_{42}G_{22}+13G_2^3-\frac{65}{12}G_{42}G_2\Big]+lwt$
D	$\frac{1}{2}\cdot(-\frac{1}{4})\cdot\Big(L(z_1)P(z_1-z_2)P(2z_2)$ $+P(z_1L(z_1-z_2)P(2z_2)$ $+P(z_1)P(z_1-z_2)L(2z_2)$ $+L(z_1)P(z_1+z_2)P(2z_2)$ $+P(z_1L(z_1+z_2)P(2z_2)$ $+P(z_1)P(z_1+z_2)L(2z_2)\Big)$	$\frac{1}{2}\Big[352G_{22}^3-408G_2G_{22}^2+144G_2^2G_{22}-40G_{42}G_{22}-16G_2^3+10G_{42}G_2\Big]+lwt$
E	$-\frac{1}{4}\Big(L(z_1)P(z_1-z_2)P(z_1+z_2)$ $+P(z_1L(z_1-z_2)P(z_1+z_2)$ $+P(z_1)P(z_1-z_2)L(z_1+z_2)\Big)$	$264G_{22}^3-306G_2G_{22}^2+108G_2^2G_{22}-30G_{42}G_{22}-12G_2^3+\frac{15}{2}G_{42}G_2+lwt$
F	$\frac{1}{2}\cdot(-\frac{1}{4})\cdot\Big(P(z_2)P(2z_2)+$ $L(z_2)P(2z_2)+P(z_2)L(2z_2)\Big)$	$\frac{1}{2}\Big[-\frac{65}{2}G_2G_{22}^2+\frac{65}{2}G_2^2G_{22}-\frac{13}{2}G_2^3+\frac{65}{24}G_{42}G_2\Big]+lwt$

FIGURE 9. Graphs for $\mathcal{Q}(2,1,-1^3)$: Siegel-Veech contribution

9.3. **Contributions to volumes of strata and Siegel-Veech constants.** Evaluation of volumes and Siegel-Veech constants of strata are closely related to the asymptotics of the generating series $N^0(\Pi)$ and $c_{-1}(\Pi)$ as q tends to 1 (or equivalently τ tends to 0), as stated in [EO06] (see also [Gou16, Proposition 7]). This asymptotics can be easily obtained thanks to the quasimodularity property for Eisenstein series: the transformation $\tau \to -1/\tau$ relates the asymptotics as $\tau \to 0$ to the asymptotics as $\tau \to i\infty$.

We overview briefly the results of [CMZ18, Section 9] here and define two polynomials describing the growth of a quasimodular form (for $\Gamma_0(2)$) near $\tau = 0$, so at the same time the average growth of its Fourier coefficients.

We recall that $QM(\Gamma_0(2))$ is the space of even weight quasimodular forms for $\Gamma_0(2)$, and it is generated as a polynomial ring by G_2, G_{22}, G_{42}, see (36).

Graph Γ	$\mathrm{ev}[N^0(\Gamma,\Pi)](h)$	$\mathrm{ev}[c_{-1}(\Gamma,\Pi)](h)$
A	$\frac{4}{45}\frac{\pi^4}{h^5}+O\big(\frac{1}{h^4}\big)$	$O\big(\frac{1}{h^4}\big)$
B	$\frac{2}{15}\frac{\pi^4}{h^5}+O\big(\frac{1}{h^4}\big)$	$\frac{5}{36}\frac{\pi^4}{h^5}+O\big(\frac{1}{h^4}\big)$
C	$\frac{1}{9}\frac{\pi^4}{h^5}+O\big(\frac{1}{h^4}\big)$	$\frac{11}{144}\frac{\pi^4}{h^5}+O\big(\frac{1}{h^4}\big)$
D	$O\big(\frac{1}{h^4}\big)$	$O\big(\frac{1}{h^4}\big)$
E	$O\big(\frac{1}{h^4}\big)$	$O\big(\frac{1}{h^4}\big)$
F	$-\frac{1}{36}\frac{\pi^4}{h^5}+O\big(\frac{1}{h^4}\big)$	$-\frac{13}{288}\frac{\pi^4}{h^5}+O\big(\frac{1}{h^4}\big)$
Total	$\mathrm{vol}=\pi^4/3072$	$\frac{\pi^2}{3}c_{area}=\frac{49}{72}$

FIGURE 10. Graphs for $\mathcal{Q}(2,1,-1^3)$: growth polynomials and contribution to the volume and the Siegel-Veech constant of the stratum

Definition 9.1. *We define the map* Ev *as the unique algebra homomorphism from* $QM(\Gamma_0(2))$ *to* $\mathbb{Q}[X]$ *sending* G_2 *to* $-X/24-1/2$, G_{22} *to* $-X/96-1/4$ *and* G_{42} *to* $X^2/3840$.

Setting $h=-2\pi i\tau$, we define

$$\mathrm{ev}[F](h) \;=\; \frac{1}{h^k}\,\mathrm{Ev}[F]\Big(-\frac{4\pi^2}{h}\Big) \;\in \mathbb{Q}[\pi^2][1/h]$$

for $F\in QM_{2k}(\Gamma_0(2))$ (weight $2k$ quasimodular form). This polynomial describes the growth of $F(\tau)$ near $\tau=0$ directly (also for mixed weight forms), as proved in the following Proposition.

Proposition 9.2. *For* $F\in QM(\Gamma_0(2))$ *we have*

$$F(i\varepsilon)=\mathrm{ev}[F](2\pi\varepsilon)+(\textit{small})\quad(\varepsilon\searrow 0)$$

where "small" means terms that tends exponentially quickly to 0.

Proof. This is directly derived from the modularity properties $G_2(-1/\tau)=\tau^2G_2(\tau)-\tau/4\pi i$ and $G_4(-1/\tau)=\tau^4G_4(\tau)$. $\square$

In Figure 10 we give the h-evaluation of all individual graphs. Note that the h-evaluation of lower weight terms is $O\big(\frac{1}{h^4}\big)$. In this table

$$\mathrm{vol}=\frac{2\dim}{2^{\dim}\dim!}\cdot\lim_{h\to 0}\big(\mathrm{ev}[N^0(\Pi)](h)\cdot h^{\dim}\big)$$

is the volume of $\mathcal{Q}(2,1,-1^3)$ (with respect to the Eskin-Okounkov convention ; the volume for the Athreya-Eskin-Zorich convention has an additional factor $4^5/2\cdot 3!=3072$, see [Gou16, Lemma 2] for discussion concerning volume normalizations), and $\dim=\dim_{\mathbb{C}}\mathcal{Q}(2,1,-1^3)=5$.

In the general case, the Siegel-Veech constant c_{area} is computed using

$$\frac{\pi^2}{3}c_{area} \;=\; \lim_{D\to\infty}\frac{\sum_{d=1}^D c_{-1}^0(d,\Pi)}{\sum_{d=1}^D N^0(d,\Pi)} \;=\; \frac{\lim\limits_{h\to 0}\big(\mathrm{ev}[c_{-1}^0(\Pi)](h)\cdot h^{\dim}\big)}{\lim\limits_{h\to 0}\big(\mathrm{ev}[N^0(\Pi)](h)\cdot h^{\dim}\big)},$$

since both limits are finite.

In our example, the stratum is non varying so the series $c^0_{-1}(\Pi)$ and $N^0(\Pi)$ are proportional (not the individual graph contributions though). The Siegel-Veech constant is just the proportionality factor. We get the following constant,

$$\frac{\pi^2}{3} c_{area}(\mathcal{Q}(2,1,-1^3)) = \frac{49}{72}.$$

in agreement with the value computed in [CM12].

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