Buildings, polytopes and tropical convexity

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Tropical convexity

Tropical semiring:

We endow $\mathbb{R}$ with the tropical addition $a \oplus b = \max\{a, b\}$ and the tropical multiplication $a \odot b = a + b$.
Component-wise tropical addition and tropical scalar multiplication

$$\lambda \odot (x_0, \ldots, x_n) = (\lambda \odot x_0, \ldots, \lambda \odot x_n) = (\lambda + x_0, \ldots, \lambda + x_n)$$

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Tropical line segments

For $x, y \in \mathbb{R}^{n+1}$ we define the tropical line segment between x and y by

$$\{(\lambda \odot x) \oplus (\mu \odot y) \text{ for all } \lambda, \mu \in \mathbb{R}\}.$$

Develin/Sturmfels 2004

A subset of $\mathbb{R}^{n+1}$ is called tropically convex if it contains the tropical line segment between any two of its points.

Every tropically convex subset of $\mathbb{R}^{n+1}$ is closed under tropical scalar multiplication. Therefore we look at it in the quotient space $A = \mathbb{R}^{n+1}/\{x \sim \lambda \odot x\} = \mathbb{R}^{n+1}/\mathbb{R} \cdot (1, \dots, 1)$.

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Tropical Polytopes

The tropical convex hull of finitely many points in A, i.e. the smallest tropically convex subset of A containing these points, is called a tropical polytope.

Tropical polytopes

Develin/Sturmfels

Every tropical polytope has the structure of a polyhedral complex. The cells are tropical polytopes and polytopes at the same time, i.e. they are simultaneously convex and tropically convex.

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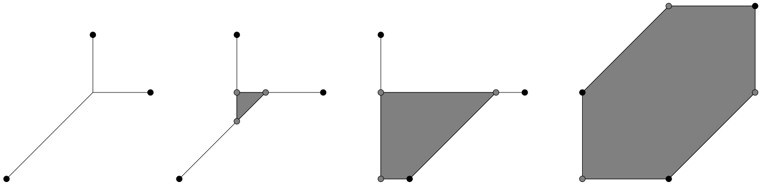
Joswig/Kulas 2010

A full-dimensional bounded subset of A which is simultaneously convex and tropically convex („polytrope“) is the tropical convex hull of $n+1$ vertices, i.e a tropical simplex.

Examples of tropical polytopes



Connection to buildings

Setting:

- Let K be a discretely valued field, for example $K = \mathbb{Q}_p$ or a finite extension of $\mathbb{Q}_p$ or $K = k((X))$ for an arbitrary field k
- and choose π, an element of minimal positive valuation in K
- $\mathcal{O}_K = \{x \in K : |x| \leq 1\}$ is the ring of integers
- A lattice in the vector space K^{n+1} is a free $\mathcal{O}_K$-submodule of full rank in K^{n+1}.
- Two lattices M and N are equivalent: $M \sim N$, if there exists a constant $c \in K^\times$ with $M = cN$.

Connection to buildings

Ad-hoc-definition of the Bruhat-Tits building for $SL_{n+1,K}$

The building $\mathfrak{B}(SL_{n+1})$ is the geometric realization of the following simplicial (flag) complex:

- The vertices are the equivalence classes $\{M\}$ of lattices in K^{n+1}.
- Two vertices $\{M\}$ and $\{N\}$ are adjacent, if there are representatives $M' = cM$ and $N' = dN$ such that

$$\pi M' \subset N' \subset M'.$$

Note that there is a natural continuous action of $SL_{n+1}(K)$ on its building.

Apartments

For every basis $e_0, \ldots, e_n$ of K^{n+1} the subcomplex of all lattice classes

$$M = \mathcal{O}_K a_0 e_0 + \ldots + \mathcal{O}_K a_n e_n$$

generated by multiples of the basis vectors is called an apartment.

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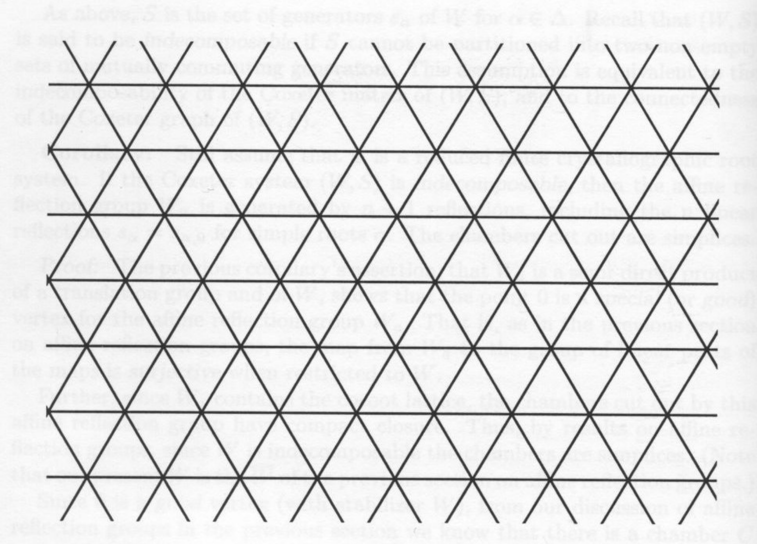
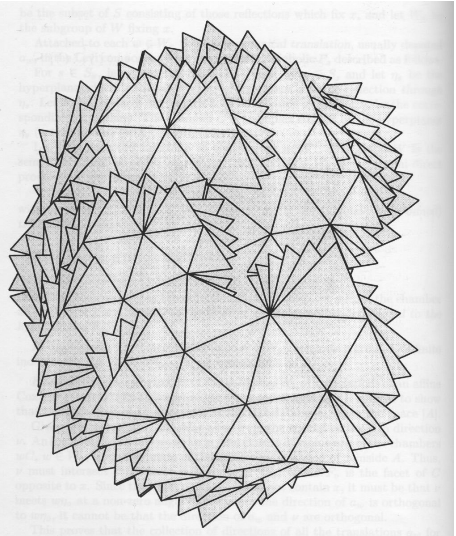
- The building is the union of all its apartments.
- Any two points in the building are contained in a common apartment.
- Apartments are in bijective correspondence with the maximal split tori in SL_{n+1}. Such a maximal split torus in SL_{n+1} is simply an algebraic subgroup which is isomorphic to $\mathbb{G}_{m,K}^n$.

Apartment für SL_3



A part of $\mathcal{B}(SL_3)$



P. Garrett: Buildings and classical groups

Classical group

Let G be a classical group which is the subgroup of all elements in $SL_{n+1,K}$ fixed by an involution i.

Examples: $SO_{n+1,K}$, $Sp_{2n,K}$

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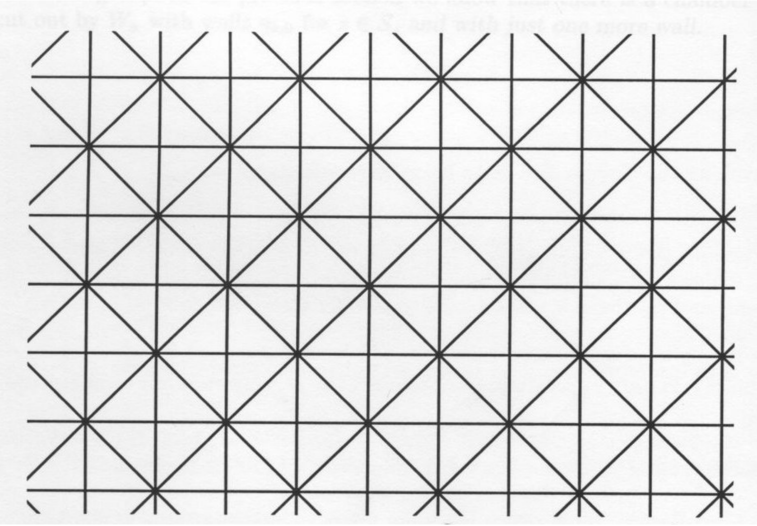
Fact

Then the Bruhat-Tits building $\mathfrak{B}(G)$ associated to G is the fixed point set in $\mathfrak{B}(SL_{n+1})$ of the involution i.

Apartment for Sp_4



Back to the SL_{n+1}-case

Every apartment in $\mathfrak{B}(SL_{n+1})$ can be identified with the cocharacter space of the associated torus, which is isomorphic to $\mathbb{R}^{n+1}/\mathbb{R}(1,\ldots,1)$ (hence to the ambient space A of our tropical polytopes). Denote by a_{ij} the map $\mathbb{R}^{n+1}/\mathbb{R}(1,\ldots,1) \to \mathbb{R}$ (the character) given by $x \mapsto x_i - x_j$.

Simplicial structure of an apartment

The simplical structure on the apartment is induced by the affine hyperplane arrangement

$$\{x \in \mathbb{R}^{n+1}/\mathbb{R}(1,\ldots,1) : a_{ij}(x) = c\} \text{ for all } i \neq j \text{ and } c \in \mathbb{Z}$$

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Root system

$\Phi = \{a_{ij} : i \neq j\}$ is the root system of type A_n induced by the torus in the background.

Tropical convexity in the building

Intrinsic description of tropical polytopes

Let $v_1 = \{M_1\}, \ldots, v_r = \{M_r\}$ be a collection of vertices in one apartment of the building $\mathfrak{B}(SL_{n+1})$. Then the tropical convex hull of $v_1, \ldots, v_r$ is the subcomplex of the apartment generated by all lattice classes of the form

$$\{a_1 M_1 + \ldots + a_r M_r\} \text{ for } a_1, \ldots, a_r \in K.$$

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Note that this gives a definition of the tropical convex hull for an arbitrary finite set of vertices in the building (not necessarily lying in one apartment)!

Joswig, Sturmfels, Yu 2007

Algorithm for computing such tropical convex hulls of finitely many vertices in the building

Tropical convexity in the building

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Open question: Investigate the structure of such tropical polytopes in the building, number of generators, decomposition into faces...

There are other connections between buildings and tropical geometry working for arbitrary reductive groups:

Theorem (W. 11)

Stabilizers of points in Bruhat-Tits buildings can be described with tropical linear algebra.

Tropical convexity in other buildings

Question

Define generalizations of tropical convexity for other buildings.
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Define generalizations of tropical convexity for other buildings. The following is joint work with Josephine Yu.

First idea: Look at a classical group $G \subset SL_{n+1}$ which is the fixed point set of an involution. Take finitely many vertices in the building $\mathfrak{B}(G)$, embed them into $\mathfrak{B}(SL_{n+1})$ and take the tropical convex hull there. Look at the intersection of this tropical convex hull with the smaller building $\mathfrak{B}(G)$.

Bad luck:

Often you get nothing new.

For $G = Sp_4 \subset SL_4$ and two vertices v, w in the building of Sp_4, the intersection of the tropical convex hull of v, w in the building of SL_4 with the building of G is in many cases just $\{v, w\}$. Hence all points on the tropical line connecting v and w lie outside the smaller building $\mathfrak{B}(G)$.

Look at polytropes

Recall:

A polytrope is a subset of $\mathbb{R}^{n+1}/\mathbb{R}(1,\ldots,1)$ which is tropically convex and classically convex at the same time. Every tropical polytope has a cell decomposition into polytropes. Polytropes can be generated by $n+1$ vertices.

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Develin/Sturmfels

Every polytrope P can be written as an intersection of hyperplanes parallel to the root hyperplanes:

$$P = \{x \in \mathbb{R}^{n+1}/\mathbb{R}(1,\ldots,1) : a_{ij}(x) \leq c_{ij}\}$$

for real constants c_{ij}.

This can be generalized!

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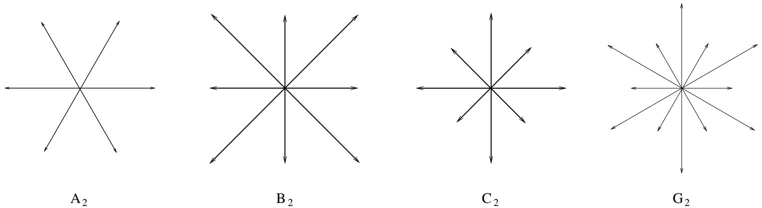
Root systems

A root system is a finite set subset Φ of a Euclidean vector space $(V, (\ ,\ ))$ which generates V and does not contain zero such that the following two conditions hold:
i) For every $a \in \Phi$ the reflection at the hyperplane orthogonal to a leaves Φ invariant.
ii) For all $a, b \in \Phi$ the number $2(a, b)/(b, b)$ is an integer.

The subgroup of the orthogonal group of V generated by all reflections at hyperplanes orthogonal to the roots is called the Weyl group $W(\Phi)$.

Root systems

Examples in dimension two:



There is a classification of root systems in arbitrary dimension by Dynkin diagrams.

Definition

Consider an irreducible root system Φ in the dual space V^* of a finite-dimensional vector space V. An alcoved polytope of type Φ is a bounded subset of V defined as an intersection of affine hyperplanes parallel to the root hyperplanes:

$$\bigcap_{a \in \Phi} \{x \in V : a(x) \leq c_a\}.$$

Alcoved Polytopes

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Alcoved polytopes have already been studied, e.g. in

- Lam/Postnikov 2004: Triangulations, volumes (mostly for type A)
- Payne 2009: Koszul Property

General case

Let P be an alcoved polytope.

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Definition

We call an alcoved polytope symmetric, if it is invariant under the action of the Weyl group.

Coxeter number

Let Φ be a root system in a vector space of dimension n . The root hyperplanes form a finite hyperplane arrangement. The complete fan whose walls are given by these hyperplanes is called the Weyl fan.

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Definition

Select a chamber in the Weyl fan and make a list of its walls $H_1, \ldots, H_n$. Then the product

$$w = s_1 \circ \ldots \circ s_n$$

of the reflections at $H_1, \ldots, H_n$ is called a Coxeter element. The order of w is called the Coxeter number h of Φ.

Coxeter element

Note:

The element w depends on the chamber and on the ordering of the walls, but all those Coxeter elements are conjugate. Hence the Coxeter number h is independent of all choices. The number of roots is equal to nh.

The Coxeter number of the root system of type A_n is $h = n + 1$.

Number of generators

Theorem (W., Yu)

Any symmetric alcoved polytope can be generated by h vertices, where h is the Coxeter number of the root system.

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Note: Since the Coxeter number of the root system of type A_n is equal to $n+1$, this sheds new light on the fact that every polytrope is generated by $n+1$ vertices.

Proof

In most cases the following strategy works: We use results on the orbit decomposition of Φ under the cyclic $\Gamma = \langle w \rangle$ to find a vertex whose orbit under Γ generates the symmetric alcoved polytope.
The remaining cases (F_4 and E_8) are treated individually.
Symmetric E_8-alcoved polytope are interesting objects: They have 19440 vertices. Note that the Coxeter number of E_8 is 30.

To do

Open question

Find an upper bound for the number of generators for general (possibly non-symmetric) alcoved polytopes.