ON SOME ASPECTS OF CURVATURE

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Abstract. We survey recent results on generalizations of scalar, Ricci and sectional curvature of a Riemannian manifold. Through the different generalizations, several aspects of these curvatures will be revealed. These aspects are of differential geometric, metric, integral-geometric, measure theoretic and combinatorial nature. Several open problems and some speculations about the possible developments of the theory are formulated.

1. Introduction

One of the pleasant tasks of a mathematician is that of generalization. Most often, there is no straightforward way of generalizing a mathematical concept or object. One phenomenon which might occur is that one object admits different generalizations, each of which shares some properties with the original object, while other properties are not preserved. As a consequence, one can better recognize and distinguish the various aspects of the original object or concept. Even in the known, classical situation this can lead to new insights. As an example, in distribution theory one generalizes the notion of smooth function in order to obtain solutions of linear partial differential equations and shows afterwards that these solutions are actually smooth functions.

In this survey article, we will see how this was successfully worked out in the last twenty years for the concept of *curvature*.

Most mathematicians would agree that curvature is a very difficult object. However, it is not the definition which makes problems, but to get a good feeling of what a curvature condition really *means*. On the other hand, curvature is an important invariant in geometry and topology, with a wide range of applications, for instance in Topology, General Relativity or Computational Geometry. It is thus a promising task to shed some light on curvature by looking at its various generalizations and by separating the different aspects united in this single object. This separation will not be very restrictive and the numerous relations between the different curvature notions will provide some further information about *curvature*.

Before turning our attention to generalizations, we should say what we understand here by curvature, i.e. describe more precisely what is the *classical situation*.

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Classically, curvature is thought of as a *differential geometric*, or better a *Riemannian* invariant. To a Riemannian manifold, one associates numbers, tensors or differential forms which are defined locally and which measure the non-flatness of the space. The best known curvature expressions are the Riemannian curvature tensor, sectional curvature, Ricci tensor and scalar curvature. Besides these, there are curvature expressions which can be computed from the Riemannian curvature tensor, for instance the integrand in the Chern-Gauss-Bonnet theorem or the coefficients in the asymptotic development of the heat kernel on a Riemannian manifold.

As was stated in [59], the generalization R^c of a curvature expression R to a singular space X should satisfy the following two requirements:

- It should be an invariantly defined local measurement of the intrinsic geometry of X which vanishes if X is flat.
- The significance of R^c should be analogous to that of R. More precisely, consider some formula which expresses a certain analytic, geometric, or topological measurement of X in terms of R. If this measurement still makes sense in the singular case, then the formula should continue to hold with R replaced by R^c.

We will see that there are different ways of obtaining generalizations which satisfy the above requirements. Let us now see which aspects of curvature are unraveled by generalizations. By doing so, we also give the plan of the paper.

The first section is a very brief introduction to different curvature conditions. We will recall the definition of the Riemannian tensor, Ricci tensor and scalar curvature and some of the main theorems in the smooth situation.

The sectional curvature, although a difficult expression in the metric and its derivatives, has a clear geometric meaning. The sectional curvature $K(P)$ of a two-dimensional subspace P in the tangent space T_pM of a Riemannian manifold (M,g) measures how fast two geodesics whose initial velocities form an orthogonal base of P spread at small times. This brings us to the metric aspect of sectional curvature. As was said before, a generalization will in general lose some of the initial properties. In our situation, we will see that not the sectional curvature itself (as a numerical value associated to two-dimensional planes) can be generalized to a metric space, but only the notion of a lower or upper bound on the sectional curvature. The corresponding metric spaces are called Alexandrov spaces with curvature bounded from below or above.

In the second section, we will survey some of the development leading to these spaces and several applications. As this is a very active field of research, we will restrict ourselves to some of the main results, without any completeness.

We now turn our attention to the scalar curvature. In the classical situation, it arises as average over all sectional curvatures in a point of a Riemannian manifold. Thus, one gets a function on the manifold. Since the process of averaging depends not only on the metric, but also on a measure, there is no notion of scalar curvature on metric spaces. One should at least consider metric measure spaces in the sense of [120], but even then there seems to be no satisfactory theory. However, the idea of considering certain averages leads to a generalization of scalar curvature which shows that it has an integral geometric flavor.

First, we consider the case of a compact embedded Riemannian manifold (M, g) of dimension n, such that the ambient space is $\mathbb{R}^N$. Consider the space of $N - n + 2$-dimensional affine spaces P in $\mathbb{R}^N$. It can be endowed in a natural way with a measure which is invariant under the induced action of the group of Euclidean motions of $\mathbb{R}^N$. For such a plane P in general position, the intersection $M \cap P$ is a two-dimensional manifold. We take the Euler-characteristic of this intersection and average over all planes P. The result will be, up to a constant, the total scalar curvature of (M, g):

$$\int_M s_g \mu_g = C_{N,n} \int_{\mathcal{G}_{N,N-n+2}} \chi(M \cap P) dP \tag{1}$$

We thus see that the total scalar curvature of M can be computed by averaging Euler-characteristics of intersections of M with planes. Also local versions of this *linear kinematic formula* exist and show that the scalar curvature measure of M can be computed in a similar way.

It turns out that the right hand side of (1) is defined not only for compact manifolds, but for a very large class of singular spaces, which contains, among other spaces, subanalytic sets and convex sets.

The resulting scalar curvature is in general no longer a function, but a signed measure, which shares many of the properties of the classical scalar curvature of Riemannian manifolds. It also turns out that this measure is in some sense independent of the way the set is embedded in an Euclidean space.

We will develop these ideas in the third section, where we will explain how the linear kinematic formula is generalized to subanalytic sets by using stratified Morse theory and to convex and more general sets by Steiner type formulas.

In the fourth section, we will look at scalar curvature from the viewpoint of Geometric Measure Theory. The main idea is to associate to certain singular subspaces of Euclidean space (or of a Riemannian manifold) a current in the unit tangent bundle of the ambient space. This current carries metric and topological information of the singular space. It opens the door for other generalizations of curvature expressions. We will explain the different steps which led to the construction of this so-called normal cycle of a singular space.

The normal cycle is shown in the fifth section to be useful for generalizing certain combinations of scalar curvature, Ricci tensor and Riemannian curvature tensor to singular spaces. We will meet variational formulas related to these curvature expressions which imply such seemingly different things as Chern-Gauss-Bonnet theorem, Hilbert's variational formula for the total scalar curvature of a Riemannian manifold and Schläfli's variational formula for polyhedra.

In the sixth section, we make some notes about further generalizations of curvature. In the non-commutative setting, there is a construction which formally looks a bit like the integral geometric interpretation of scalar curvature. This is related to the nature of the total scalar curvature as a *spectral invariant*.

We consider combinatorial analogues of curvature due to Budach and Forman; the approaches of Cheeger-Colding and Sturm-von Renesse to Ricci curvature bounds and Delladios generalized Gauss graphs.

Finally, the last section is devoted to some questions concerning generalized curvatures.

At this point, the reader might wonder what relations exist between the different generalizations of curvatures to singular spaces. There are indeed a lot of them. For instance, in the subanalytic setting, the curvature bounds from the metric theory imply curvature bounds for the (integral-geometric generalization of the) scalar curvature. We will explicitly remark such links at the place where they occur.

Resuming this introduction, we see that there are lots of generalizations of the classical curvature from Riemannian geometry to a large class of singular spaces. They reveal metric, integral geometric, measure theoretic, non-commutative and combinatorial aspects.

2. Differential geometric aspect of curvature

Let us briefly recall the definition of the curvature tensor of a Riemannian manifold (M, g). Let D denote the Levi-Cevita connection. Set $R(X, Y)Z = -D_X D_Y Z + D_Y D_X Z + D_{[X,Y]} Z$, where X, Y, Z are vector fields. Then R is called $(1, 3)$-Riemannian curvature tensor. If X and Y are orthogonal vectors spanning the plane $P \subset T_p M$, $K(P) := g(R(X, Y)X, Y)$ is called the sectional curvature of P.

The trace of R is called Ricci curvature and is a $(0, 2)$-tensor. Taking again the trace, we obtain the scalar curvature, a smooth function on the manifold.

It is clear from the definition that a curvature condition like positive sectional curvature is stronger than positive scalar curvature.

We will in this section just scetch some of the most important relations between curvature conditions and topology. A detailed survey with a large bibliography on this subject is [21].

It is not known which manifolds carry a metric of positive sectional curvature. Myer's theorem states that an n-dimensional manifold of Ricci curvature at least $(n - 1)\kappa > 0$ has diameter not larger than $\frac{\pi}{\sqrt{\kappa}}$. Therefore, it has finite fundamental group. This applies of course to manifolds with a strictly positive lower bound on the sectional curvature. However, there are only few examples of manifolds admitting a metric of positive sectional curvature. Besides some exceptional cases, they are homogeneous manifolds. One can completely classify the positively curved homogeneous spaces ([17], [20], [160]). If there are many other examples, nobody knows.

Concerning non-negative curvature, the list is a bit longer. For instance, one can take products of positively curved spaces and a longer list of homogeneous spaces.

For positive Ricci curvature, besides the finiteness of the fundamental group and the conditions for positive scalar curvature, no other restriction is known. In the three-dimensional case, only quotients of spheres admit Ricci-positive metrics by a theorem of Hamilton ([127]).

On the other hand, positive scalar curvature is better understood. One can say precisely which simply-connected manifolds (of dimension ≥ 5) admit a metric of positive scalar curvature. It depends on whether or not there is a spinor structure and on the α-genus.

By the Cartan-Hadamard theorem, a simply connected manifold of non-positive curvature is diffeomorphic to $\mathbb{R}^n$. The non simply-connected case (in particularly the compact case) is a very rich and difficult subject with relations to number theory.

The simplest case is that of negative Ricci curvature. By [135], every manifold of dimension at least 3 admits such a metric. Thus, there are no topological instructions or implications. This of course also solves the case of negative scalar curvature.

3. Curvature bounds on metric spaces

Historically, curvature was first considered as a metric concept. Gauss tried to produce exact maps of the earth, i.e. he tried to find *isometries* between parts of the sphere and portions of the plane. Using differential geometry, he showed that this is impossible. Then the study of curvature turned more and more to differential geometric aspects, with Riemann's curvature tensor and all the modern Riemannian geometry. Although there were studies of Alexandrov on curvature bounds on metric spaces, the metric aspects of curvature regained a larger attention only in the last 10-20 years, thanks to the pioneering work of Gromov and others.

It is probably impossible to give a complete survey of this very active research area. In any case, it would go far beyond the size, intention and form of the present paper. Instead, I will try to focus on some main results and present a short introduction into three different metric generalizations of the notion of curvature to metric spaces.

We invite the reader to have a look in the books [32] and [45] for most of the material in this section. The first one concentrates on spaces with upper curvature bound, whereas the second one treats also the case of lower curvature bounds. In the classical book by Alexandrov ([3]), many of the notions discussed below are introduced and studied. Other references are [3], [46] and [141].

3.1. **Basic notions.** A metric space is called *inner* or *intrinsic metric space* or *length space* if the distance between two points is given as the infimum over the length of rectifiable curves between them. A geodesic in an inner metric space is an isometric embedding of an interval. If any two points can be joined by a geodesic, the space is called *geodesic*. Hopf-Rinow Theorem states that a complete, locally compact length space is geodesic.

The most important notion in comparison geometry is that of a comparison triangle. Given three points a, b, c in a metric space (X, d), the triangle inequality implies that there are three points $\tilde{a}, \tilde{b}, \tilde{c}$ in Euclidean plane realizing the same pairwise distances. Such a triangle is well-defined up to a rigid motion of the plane. It is called comparison triangle. If (X, d) is geodesic and if we are given a point q on a geodesic between two of the three points, say between a and b, then there is a point on the line between $\tilde{a}$ and $\tilde{b}$ realizing the same distances. Such a point is called comparison point.

The angle at a of a triangle a, b, c is defined as the angle at $\tilde{a}$ of the comparison triangle and denoted by $\angle_a(b, c)$.

Instead of comparing with Euclidean plane, one can also compare with another model space of constant curvature κ, i.e. with a hyperbolic plane (if $\kappa < 0$) or a sphere (if $\kappa > 0$). In the last case, one has to bound the diameter of the triangle in order to assure the existence of a comparison triangle. The corresponding angles will be denoted by $\angle^\kappa$.

Given two geodesics γ_1, γ_2 with $\gamma_1(0) = \gamma_2(0) = a \in X$, we define the upper angle between γ_1, γ_2 at a by

$$\overline{\angle}_a(\gamma_1, \gamma_2) := \limsup_{t_1, t_2 \to 0} \angle_a(\gamma_1(t), \gamma_2(t))$$

The upper angle defines a pseudo-distance on the space of geodesic germs at a, i.e. it satisfies all axioms of a metric except the fact that two germs can have distance 0 while being different. This happens for instance on a very tiny cone. For instance, rotate the curve $y = x^2, x \geq 0$ around the x-axis, then any two geodesics on the rotation surface have upper angle 0 at the origin.

3.2. **Alexandrov surfaces.** The Gauss-Bonnet theorem for two-dimensional Riemannian manifolds implies that the total (Gaussian) curvature K in a geodesic triangle is given by the angle defect:

$$\int_\Delta K = \alpha + \beta + \gamma - \pi$$

Alexandrov uses this to define a much wider class of spaces, in which curvature bounds exists, but only in a measure-theoretic sense. For this, let (X, d) be a geodesic metric space. Suppose that X is a two-dimensional topological manifold. Consider an open neighborhood U in X which is homeomorphic to an open disc in the plane. A geodesic triangle Δ in U consists of three points a, b, c and three geodesics $[ab], [bc], [ac]$ between them. The triangle separates U in an inner and an outer part, the union of the geodesics being the boundary of both parts.

The triangle is said to be simple if the shortest curve joining two points on the boundary and staying outside the triangle is given by a path on the boundary. The upper angle excess $\overline{Def}(\Delta)$ is defined as the sum of the three upper angles at the vertices minus π.

The space (X, d) is said to be an Alexandrov surface if for each point $p \in X$ there exist an open neighborhood U, homeomorphic to an open disc in the plane, and a real number $K(p)$ such that for any finite collection of simple geodesic triangles $\Delta_1, \ldots, \Delta_k$ in U with disjoint interiors, the sum of the angle excesses is bounded by $K(p)$:

$$\sum_{i=1}^k \left|\overline{Def}(\Delta_i)\right| \leq K(p)$$

Intuitively, this inequality says that the total amount of curvature in U is bounded from below and above.

By counting only those triangles with positive (resp. negative) angle excess, one gets two measures whose difference is a signed measure, called *curvature measure* of the Alexandrov surface.

In Section 4, another definition of curvature measures is given for the class of semialgebraic (and more general) sets. By a result of Bröcker-Kuppe-Scheufler ([34]), a two-dimensional semialgebraic topological manifold is an Alexandrov surface in the above sense and both definitions of curvature measure agree. The proof of this nice result includes a study of geodesics and a version of the Gauss-Bonnet theorem on such spaces. Since also for Alexandrov's measure there is a Gauss-Bonnet theorem, comparison yields (modulo some technical difficulties) that both measures are equal.

3.3. **Gromov-Hausdorff distance.** The space of compact metric spaces carries a very natural metric, first defined by Gromov. Let X, Y be closed subspaces of a metric space Z. Their *Hausdorff-distance* is defined by

$$d_H(X,Y) = \inf\{\epsilon > 0 : X \subset V_\epsilon(Y), Y \subset V_\epsilon(X)\}$$

where V_ϵ stands for the ϵ-neighborhood.

The space of closed subsets of a compact metric space, endowed with the above distance, is again a compact metric space.

In order to define the *Gromov-Hausdorff distance* of two compact metric spaces X and Y, consider all metrics on their disjoint union which induce the given metrics on X and Y. Then the infimum over the Hausdorff distances of X and Y is the Gromov-Hausdorff distance.

The Gromov-Hausdorff distance between two compact metric spaces vanishes if and only if the spaces are isometric. The set of (isometry classes of) compact metric spaces is therefore itself a metric space (but fortunately not a compact one, this avoids a paradox à la Russell). Still it is true by a theorem of Gromov that a sequence of uniformly compact metric spaces has a convergent subsequence. Here *uniformly compact* means that for each $\epsilon > 0$, each space in the sequence can be covered by at most $N(\epsilon)$ balls of radius ϵ.

Using Bishop-Gromov volume comparison, one obtains the following compactness theorem.

Theorem 3.1. (Compactness theorem of Cheeger-Gromov) *(*[120]*) The set of isometry classes of compact Riemannian manifolds of dimension n, with diameter bounded from above by $D > 0$ and with Ricci curvature bounded from below by $r \in \mathbb{R}$ is precompact with respect to Gromov-Hausdorff metric.*

The most spectacular applications of this theorem (like Cheeger's finiteness theorem [49]) use the stronger condition that the sectional curvature is bounded from below. In the next subsection, we will see a compactification of the corresponding space of Riemannian manifolds.

For a lower bound on the Ricci curvature only, the theory is only at its beginning, see 7.3 for some information on this subject.

3.4. **Curvature bounded from below.** Consider the space of all Riemannian manifolds of a fixed dimension with a lower bound on the sectional curvature $K \geq \kappa$ (and upper bound on the diameter). As a corollary of Cheeger-Gromov's compactness result, this space is precompact for the Gromov-Hausdorff distance. In order to find a compactification for this space, it is natural to try to express the (differential geometric) condition $K \geq \kappa$ in terms of the metric. This is indeed possible and is known under the name of Toponogov's comparison theorem, we will formulate this below for metric spaces.

An equivalent formulation is the following:

Proposition 3.2. (4 point condition)
Let M be a Riemannian manifold with sectional curvature $K \geq \kappa$. Then for any point $x \in M$, there is an open neighborhood U of x such that for any 4 points P, A, B, C in U, the following inequality is satisfied:

$$\angle_P^\kappa(A,B) + \angle_P^\kappa(B,C) + \angle_P^\kappa(C,A) \leq 2\pi \tag{2}$$

We say that an inner metric space has *curvature bounded from below* (or is an *Alexandrov space with lower curvature bound κ*) if either the above 4-point condition (2) is satisfied or if the space is isometric to a one-dimensional manifold of diameter at most $\frac{\pi}{\sqrt{\kappa}}$. The latter case is needed for several induction arguments. The definition of Alexandrov spaces is of local nature, but we will see that there are strong global conclusions.

Proposition 3.3. *A locally compact geodesic space is a space of curvature $\geq \kappa$ if and only if each point admits an open neighborhood U such that for each geodesic triangle a, b, c in U and for each point p on the side ab, one has*

$$d(p,c) \geq d(\tilde{p}, \tilde{c})$$

Here $\tilde{p}$ is a comparison point for p on a comparison triangle $\tilde{a}\tilde{b}\tilde{c}$ in the space of constant curvature κ.

This condition is satisfied for manifolds with sectional curvature bounded from below by κ (Toponogov's theorem).

In the manifold case, a lower bound κ on the sectional curvatures implies the lower bound $(n-1)\kappa$ on the Ricci curvature. If $\kappa > 0$, Myer's theorem then implies that the diameter is bounded from above by $\frac{\pi}{\sqrt{\kappa}}$. The same conclusion is true for Alexandrov spaces with a positive lower curvature bound. Let (X, d) be an Alexandrov space with curvature $\geq \kappa > 0$. Then the diameter of X is at most $\frac{\pi}{\sqrt{\kappa}}$.

The proof of this version of Myer's theorem is easy if one uses the important globalization theorem.

Theorem 3.4. (Globalization theorem)
Let (X, d) be a complete Alexandrov space with curvature bounded from below by κ. Then inequality (2) is satisfied for any 4 points of (X, d) (such that the angles are defined in case $\kappa > 0$).

The proof is "by induction" on the size of 4-points configurations.

The direct product of two Alexandrov spaces of non-negative curvature is again an Alexandrov space of non-negative curvature.

For Alexandrov spaces with curvature bounded below, there is a dimension theory based on the notion of *burst points*.

Definition 3.5. *Let (X,d) be an Alexandrov space of curvature $\geq \kappa$. An* (n,δ)**-burst point** *is a point $p \in X$ such that there exist n pairs of points a_i, b_i, distinct from p such that*

$$\angle_p^\kappa(a_i,b_i) > \pi - \delta, \angle_p^\kappa(a_i,a_j) > \frac{\pi}{2} - \delta, \angle_p^\kappa(a_i,b_j) > \frac{\pi}{2} - \delta, \angle_p^\kappa(b_i,b_j) > \frac{\pi}{2} - \delta$$

An (n,δ)-burst point looks, up to the error δ, as the origin of $\mathbb{R}^n$, together with one point on each coordinate (half-) axis.

If in the neighborhood of a point $p \in X$ there exist (n,δ)-burst points for every $\delta > 0$, and if n can not be replaced by $n+1$, then n is called the *burst index* near p. It plays the role of a dimension. The next theorem shows that a space with lower curvature bound has everywhere the same local dimension.

Theorem 3.6. *The burst indices near different points of X are equal and coincide with the Hausdorff dimension of X.*

For finite dimensional Alexandrov spaces with curvature bounded below, the completion of the space of germs of geodesics emanating from a fixed point, equipped with the (upper) angle metric, is called *space of directions* and the cone over it the *tangent space* of the point. The space of directions has curvature at least 1, whereas the tangent space has non-negative curvature. More generally, the cone over a space of curvature ≥ 1 has curvature ≥ 0.

The next theorem shows that the space of Alexandrov spaces is indeed a compactification of the space of Riemannian manifolds with corresponding curvature bounds.

Theorem 3.7. (Compactness theorem)
The space of all complete Alexandrov spaces with curvature bounded below by κ, dimension not greater than n and diameter not greater than D is compact with respect to Gromov-Hausdorff distance.

The idea and difficult part of the proof is to show that the space is uniformly compact in the sense described above and to apply the general precompactness theorem for Gromov-Hausdorff distance. For this, one studies the rough volume and the rough dimension of the space, which are defined in a way similar to Hausdorff measure and Hausdorff dimension, but instead of coverings one takes packings by balls of a given radius. The precise value of the volume seems to play no role, but, by proving an upper bound for the rough volume of a finite dimensional complete space with curvature bounded below, one can establish an upper bound for the number of disjoint balls. This is done by an easy inductive argument by passing to spaces of directions (which are of lower dimension). Since this bound depends only on κ, n, D, we get the precompactness. On the other hand, the globalization theorem implies immediately that limit spaces of sequences of spaces with curvature bounded from below by κ have again curvature bounded from below by κ. The analogous statement is true for the dimension and the diameter. □

The boundary of a finite dimensional Alexandrov space with curvature bounded below is defined inductively over the dimension. A one-dimensional Alexandrov space is a manifold and the boundary is to be understood in the usual sense. In a higher-dimensional Alexandrov space, a point is a boundary point if the space of directions has a non-empty boundary.

A difficult theorem by Petrunin ([143]) shows that one can glue two Alexandrov spaces along isometric boundaries:

Theorem 3.8. (Gluing theorem) *Let X, Y be Alexandrov spaces of the same dimension and curvature $\geq \kappa$ with isometric boundaries. Then the gluing of X and Y along their boundary is again a space with curvature $\geq \kappa$.*

This theorem is an application of the concept of a *quasi-geodesic* as defined in the unfortunately unpublished preprint [142]. Intuitively, geodesics are locally shortest curves, whereas quasi-geodesics are locally straightest curves. For manifolds, both concepts are the same, but not for Alexandrov spaces. The advantage in working with quasi-geodesics is that there are quasi-geodesics through every point and every direction and they can be extended through their endpoints. These properties are in general false for geodesics.

Another type of gluing theorem is proved by Kosovskii ([134]). Here one considers two manifolds with boundary such that the sectional curvature of both of them is bounded from below by κ and such that the boundaries are isometric. Kosovskii gives a necessary and sufficient condition such that the gluing along the boundary is an Alexandrov space with curvature bounded from below by κ. This condition can be expressed in terms of the second fundamental forms of the boundaries and appeared also (in the subanalytic setting) in [22].

3.5. **Curvature bounded from above.** The theory for spaces with upper curvature bounds is similar as long as basic properties (products, tangent cones) are concerned. However, although the definitions are almost the same (up to a sign), the deeper parts of both theories are quite different.

Definition 3.9. *A* **space of curvature** $\geq \kappa$ *is a metric space such that for each point p there exists an open ball $B(p,r)$ such that $B(p,r)$ with the induced metric is geodesic and such that for each geodesic triangle a, b, c in $B(p,r)$ and for each point p on the side ab, one has*

$$d(p,c) \leq d(\tilde{p}, \tilde{c})$$

Here $\tilde{p}$ is a comparison point for p on a comparison triangle $\tilde{a}\tilde{b}\tilde{c}$ in the space of constant curvature κ.

A space where this comparison condition holds globally is called *CAT(κ)-space*. Here we see a first difference: there is no theorem analogous to the Globalization theorem (3.4). However, if a complete metric space is simply connected, then the local condition for $\kappa = 0$ implies the global one. This is known as Cartan-Hadamard theorem and was proved (with different degrees of generality) by Cartan-Hadamard ([47], [126]), Gromov ([115]), Ballmann ([12]) and Alexander-Bishop ([2]). It can be used to show that the universal covering of a compact Riemannian manifold X is contractible: just put a metric of non-positive curvature on X (if it exists).

Tangent spaces and spaces of directions are defined in an analogous way to the case of spaces with lower curvature bound. The space of directions of a space with upper curvature bound has non-positive curvature.

Different notions of dimension are studied in [132]. It is in general not possible to triangulate a space with upper curvature bounds, see [138] for a two-dimensional example.

Reshetnyak's gluing theorem ([150]) states that the gluing of two (complete locally compact) metric spaces with the same upper curvature bound along isometric convex subsets has the same upper curvature bound.

This theorem was used in the study of *semi-dispersing billards* by D. Burago, S. Ferleger and A. Kononenko in a long series of papers ([38], [39], [40], [41], [42], [43], [44]). The beautiful ideas involved in these studies are explained in the survey ([37]).

A semi-dispersing billard consists of a complete Riemannian manifold M of non-positive curvature and with positive lower bound on the injectivity radius together with a locally finite collection of smooth convex subsets. A point in such a billard moves in the complement of the convex subsets until it hits some of them. It then gets reflected according to the law of reflection. The study of systems of particles can be reduced to such a situation by considering billards in higher dimensions. For instance, the hard ball model consisting of a finite number of round balls moving freely and colliding elastically in a box or in empty space is such a semi-dispersing billard (cf. [37]).

Under a certain non-degeneracy condition for a semi-dispersing billard, it is shown that there exists an upper bound for the number of collisions near every point of the billard. For a global result, one assumes furthermore that the intersection of the convex subsets is non-empty. Then there is an upper bound (depending on the geometry of the billard) for the number of collisions of a trajectory.

To obtain such bounds, one glues several copies of M along some of the convex subsets. The resulting space has non-positive curvature by Reshetnyak's theorem. Trajectories are transformed to local geodesics in this space. The technical difficulty lies in the fact that one can not work with only one space obtained from finitely many gluings of M, since this would require to glue the "ends" of several gluings, which is not covered by Reshetnyak's theorem. To overcome this difficulty, one *develops* a given trajectory and glues the space only along the convex sets which are hit by it. Then one applies several theorems for non-positively curved spaces to obtain the bounds mentioned above.

Other exciting applications of CAT-spaces and spaces with curvature bounded above are related to group theory and to the large-scale behavior. For the latter, one studies the *boundary* at infinity of a CAT-space, it is given by certain equivalence classes of geodesic rays. This can be done in an even more general setting, that of δ-hyperbolic spaces. Each CAT(κ)-space with $\kappa < 0$ is such a δ-hyperbolic space. Then one can put a quasi-conformal structure on the boundary etc.

Concerning the relation to group theory, we mention that one can put a metric (so called *word metric*) on each finitely generated group. This metric depends on the choice of generators, but it is unique up to a bi-Lipschitz map. If the group, endowed with this metric, is a δ-hyperbolic space, then there are several striking consequences, for instance for the *word problem*.

Since this does not match our definition of curvature as a local measurement of non-flatness, we do not go into details and refer the reader to [32], [30], [120].

4. Integral geometric generalizations of curvature

4.1. **Steiner's formula.** A curios historical fact is that Steiner found a "generalization" of the notion of curvature in 1840, twenty years *before* Riemann introduced his famous curvature tensor. (Of course, Gauss was even earlier, but only in the two-dimensional situation).

Steiner's result concerned compact convex subsets of Euclidean space. Let $K \subset \mathbb{R}^N$ be a compact convex set. For any $r \geq 0$, the parallel body K_r is defined as the set of points at distance at most r. This is again a convex set. Then, as Steiner proved, the N-dimensional volume of K_r is a polynomial in r:

$$\mathrm{vol}(K_r) = \sum_{i=0}^{N} \Lambda_i(K) b_{N-i} r^{N-i} \tag{3}$$

Here the b_{N-i} are convenient normalization factors, given by the volume of the $N-i$ dimensional unit ball. The $\Lambda_i(K)$ are invariants of K, which are nowadays called Lipschitz-Killing curvatures.

4.2. **Terminology.** Before we continue, a small digression on terminology. Since the time of Steiner, quite a few situations were considered where some invariant appears as coefficient of some (maybe modified) tube volume polynomial. As a consequence of different backgrounds (convex, differential geometric etc.) and of different normalizations, different names were attributed: Minkowski's *Quermasse* (transversal measures), Steiner functionals, intrinsic volumes,... Since we do not want to confuse the reader with this, we will speak of Lipschitz-Killing curvatures (or invariants) in all situations (it will become clear why we speak of *curvatures*). The other invariants can be obtained by renormalization.

We also remark that a very detailed exposition of most of the material described in this and the next section is [137].

4.3. **Principal kinematic formula.** The Lipschitz-Killing curvatures behave like volumes, more precisely Λ_i behaves like an i-dimensional volume even for sets of different dimensions. For instance, it has the same scaling property $\Lambda_i(tK) = t^i \Lambda(K)$. If K, L and $K \cup L$ are compact convex, then

$$\Lambda_i(K) + \Lambda_i(L) = \Lambda_i(K \cap L) + \Lambda_i(K \cup L) \tag{4}$$

The most important formula for Lipschitz-Killing curvatures of compact convex sets is the *principal kinematic formula* of Chern, Santaló and Blaschke. Consider two compact convex sets K and L in the same Euclidean space $\mathbb{R}^N$. Leave K fixed and move L by rigid motions. Then average (with respect to a natural invariant measure of the group of rigid motions) over a

Lipschitz-Killing curvature of the intersection $K \cap gL$. Then the result can be expressed only in terms of the Lipschitz-Killing curvatures of K and L:

$$\int \Lambda_k(K \cap gL)dg = \sum_{i+j=N+k} c(N,i,j,k)\Lambda_i(K)\Lambda_j(L) \tag{5}$$

The constants $c(N,i,j,k)$ only depend on N,i,j,k, but not on K,L.

For a survey on recent results concerning this formula, see [130].

Taking for L a closed ball of radius $r > 0$, one easily obtains that $\Lambda_i(L) = c(N,i)r^i$. Replacing this into the principal kinematic formula, we obtain the tube volume polynomial (3). The principal kinematic formula is therefore a generalization of Steiner's formula.

4.4. **Self-intersections of the tube.** The natural question is for which other sets such formulas hold true. If you think of the union of two disjoint compact convex sets, the r-tube of the union will be the union of the r-tubes. As long as there is no intersection of the tubes, i.e. for r smaller than half the distance between the sets, the volume of the tube is still a polynomial, by Steiner's formula. But as soon as such an intersection occurs, the tube volume stops being polynomial. We should therefore consider compact sets where such a self-intersection of the tube does not occur. Stated otherwise, we need that each point in $\mathbb{R}^N$ has a unique foot-point on the set. However, these sets are precisely the compact convex sets. The hypotheses of Steiner's theorem can therefore not be relaxed.

It seems that this is already the end of the story. But it is only the beginning! Even if the tube volume is not polynomial for all r, it may be polynomial for sufficiently small radius. This will indeed be the case if self-intersections of the tube are only at a positive distance of the set. The example of two disjoint compact convex sets considered above was an example for this.

4.5. **The smooth case.** Another example for a set having the property that self-intersections occur only at positive distance is that of a compact submanifold (with or without boundary). Upper bounds on the eigenvalues of the second fundamental form imply that there are indeed no self-intersections for small distances. The proof that the corresponding tube volume is really polynomial for small radii r was obtained by Hermann Weyl in 1939. This was an easy computation using transformation of variables, or with his words: *So far we have hardly done more than what could have been accomplished by any student in a course of calculus.* What is more important and difficult is that the coefficients of the polynomial (using suitable normalizations) can be expressed as integrals over some polynomial in the curvature tensor. This implies that they are independent of the embedding. Weyl's proof of this fact uses the theory of invariants. We call these coefficients again *Lipschitz-Killing invariants.* Since we can integrate the polynomial in the curvature tensor over any Borel set of the manifold, we obtain a signed Borel measure, called *Lipschitz-Killing measure.*

Several of the Lipschitz-Killing curvatures are well-known invariants of a compact n-dimensional Riemannian manifold (M,g). Let us suppose for simplicity that there is no boundary. Then, as H. Weyl noticed, $\Lambda_{n-i}(M)$

vanishes for odd i. Also $\Lambda_i(M) = 0$ for $i > n$. From the tube formula one trivially obtains $\Lambda_n(M) = \mathrm{vol}(M)$. Furthermore, and this will play an important role later on, $\Lambda_{n-2}(M) = 4\pi \int_M s\mu_g$, where s is the scalar curvature of M and μ_g the volume density. The integral on the right hand side is known as total scalar curvature of M and plays a fundamental role in General Relativity (Einstein-Hilbert functional). Another Lipschitz-Killing invariant which yields a known quantity is $\Lambda_0(M) = \chi(M)$. Hermann Weyl was unaware of this fact, but Allendoerfer ([5]) immediately showed this equality. Expressing $\Lambda_0(M)$ as integral over a polynomial in the curvature tensor yields the Chern-Gauss-Bonnet theorem, generalizing the classic Gauss-Bonnet theorem in dimension 2 to any dimension. A different proof, not using an isometric embedding of the Riemannian manifold in Euclidean space, was obtained by Chern ([63]).

The principal kinematic formula (5) remains true for compact Riemannian (sub-) manifolds. Remark that the intersection of two submanifolds M, N is not necessarily a submanifold. But for almost each rigid motion g, $M \cap gN$ is again a submanifold, so that the principal kinematic formula makes sense.

4.6. **Sets with positive reach.** As we have pointed out, a submanifold has the property that at small distances, there are no self-intersections of the tube. Such sets are called *sets with positive reach*. The precise definition, due to Federer ([87]), is the following:

Definition 4.1. *The* **reach** *of a subset K of $\mathbb{R}^N$ is the largest $r \geq 0$ such that if $x \in \mathbb{R}^N$ and the distance from x to K is smaller than r, then K contains a unique point nearest to x.*

By using geodesic distance instead of Euclidean distance, one can define sets of positive reach on Riemannian manifolds in the same way. Bangert gave a different characterization from which it follows that this property does not depend on the Riemannian metric ([16]).

As Federer showed, the volume of an r-tube around a set with positive reach is a polynomial *as long as r is not bigger than the the reach of the set*. This result generalizes the formulas of Steiner and Weyl, since compact convex sets have reach ∞ and compact submanifolds have positive reach.

The idea of Federer's proof is the following. First he notices that for a set K of positive reach, the boundary ∂K_{r_0} of the parallel body K_{r_0} with r_0 positive and smaller than the reach of K is a C^1-manifold with Lipschitz normals. By an argument as in Weyl's proof, the volume of the parallel body of K_{r_0} is a polynomial for sufficiently small radii. Thus, for $r_0 < r < reach(M)$, $\mathrm{vol}\, K_r = \mathrm{vol}(K_{r_0})_{r-r_0}$ is a polynomial in r. Since this is true for each $0 < r_0 < reach(K)$, we get Federer's tube formula.

Again, the principal kinematic formula (5) holds true in this setting. Here one needs that the intersection $K \cap gL$ is of positive reach for almost all g, provided K and L are of positive reach.

Federer did not only consider real valued invariants but also *measures*. For this, instead of looking at the whole r-tube around a compact set K of positive reach, one looks only at those points in the tube whose foot-points lie in a given Borel subset of $\mathbb{R}^N$. This yields (signed) Radon measures

$\Lambda_i(K, -)$ supported on K. Federer called them *Curvature measures* of K, here we will speak of *Lipschitz-Killing measures*.

4.7. **Convex ring and PL-case.** Although the class of sets of positive reach is natural and big, there are many sets not contained in it. For instance, it is not closed under finite unions (take the union of two touching balls). Also PL-spaces are in general not of positive reach. Real algebraic or semialgebraic sets have different kinds of singularities and are not of positive reach. To treat these classes, a new idea was needed, that of *counting multiplicities*.

Remember the example of the union of two disjoint compact sets. The problem for large tubes was the self-intersection of the tube. Taking the volume of the union, it is smaller than the sum of the volumes of the two tubes. But counting the intersection with multiplicity 2, we will get the right formula even for big tubes. This idea of counting multiplicities indeed works well for very large classes of singular spaces, such as the classes considered above.

Let us first consider the convex ring. It consists of all finite unions of convex sets. This class is clearly closed under finite unions and intersections. Let $K = \cup_i K_i$ be such a finite union. If we want the tube volume to be Euler-additive, i.e. to satisfy the analogue of formula (4), then we have to count the volumes of the tubes of the different K_i, subtract the volumes of the intersections, add the volumes of intersections of three of the sets and so on (inclusion-exclusion principle). Since the tube of an intersection is the intersection of the tubes, and since each tube volume is polynomial, we will get a polynomial expression for the whole tube, which is counted with multiplicities.

The problem with this approach is that there might be several ways to write a given set as union of compact convex sets and it is not clear why the tube volumes should agree. But from the additivity of the Euler characteristic it follows by simple counting that the multiplicity with which a point $x \in \mathbb{R}^N$ is counted in the r-tube is precisely $\chi(B(x,r) \cap K)$.

Starting with this idea, Schneider constructed Lipschitz-Killing invariants on the convex ring ([156]). Here again, the principal kinematic formula is satisfied. In the PL-case, i.e. the case of finite simplicial complexes of $\mathbb{R}^N$, these invariants were also introduced by Banchoff, Cheeger, Wintgen and studied further by the team Cheeger-Müller-Schrader. One of these invariants, the *scalar curvature*, appeared even earlier in the work of Regge ([149]). In all cases, these invariants can be localized to Lipschitz-Killing measures.

In the PL-case, the Lipschitz-Killing invariant $\Lambda_i(K)$ is obtained by summing the volume of all i-faces multiplied by some *exterior angle*. Lipschitz-Killing measures can be defined in a similar way. One of the results of [59] was to show that there is also an expression in terms of *inner angles*. This shows that $\Lambda_i(K)$ only depends on the inner geometry of K and not on the particular embedding in an Euclidean space. We can say that this is the PL-version of Weyl's result concerning the intrinsic character of Lipschitz-Killing curvatures of smooth manifolds.

The main result of [59] is an approximation result relating the PL and the smooth case. For this, start with a compact Riemannian manifold and a sufficiently fine triangulation. Endow the triangulation with a PL-structure such that the lengths of edges are given by geodesic distances on the manifold. Cheeger-Müller-Schrader show that by taking sufficiently fine and fat triangulations, such a PL-structure exists. Now consider a sequence of such PL-approximations which gets finer and finer but whose fatness remains bounded from below. This means that the simplexes should not become too flat. Then the Lipschitz-Killing measures converge to the corresponding measures of the Riemannian manifold.

The proof of this beautiful result has three steps. First, it is shown that the limit measure is independent of the chosen sequence of approximations. Here a generalized Schläfli formula (they call it Regge formula) plays an important role. We will see in later sections that this formula can be obtained and generalized by using a very general variational formula for Lipschitz-Killing curvatures. In the second step it is shown that the resulting measure can be written as $f\mu_g$, where f is a polynomial expression in the curvature tensor. Finally, sufficiently many properties of f are established to ensure that f is the Lipschitz-Killing curvature of M. For this last step, one uses a characterization of Lipschitz-Killing curvatures due to Gilkey.

In another paper ([60]), Cheeger-Müller-Schrader showed that kinematic and tube formulas hold for PL-spaces. We will come back to this point in the next section.

4.8. **Finite unions of sets with positive reach.** Instead of taking finite unions of convex sets, one also can take finite unions of sets with positive reach. This was elaborated by M. Zähle in [163]. Here one needs the condition that each intersection itself has positive reach. The set of such unions is denoted by $\mathcal{U}_{PR}$.

As in the case of the convex ring, there is at most one way to generalize Lipschitz-Killing invariants to this class. The problem is to show that the result is well-defined, i.e. does not depend on the representation of the set as finite union of sets with positive reach.

This problem can be solved by means of a certain index function, which was introduced for the convex ring by Schneider. This function can be computed by limits of certain Euler-characteristics and is therefore independent of the representation of the set. Then Zähle shows that the Lipschitz-Killing invariants (and measures) can be obtained by integration over the *normal cycle* of the set. We will say much more about this construction in the next section. Zähle could show the principal kinematic formula for finite unions of sets with positive reach, including of course the tube volume formula. The multiplicity of a point is given in terms of the index function.

4.9. **Subanalytic and Whitney-stratified sets.** In all the situations we considered above (convex, positive reach, PL, convex ring, finite unions of positive reach), we counted multiplicities in the tube by the Euler characteristic of the intersection with a closed ball. The resulting tube volume was a polynomial. For this we need that this Euler characteristic is well-defined and in some sense uniformly bounded, at least integrable. The classes of semialgebraic or subanalytic sets have this property.

Recall that a set which can be described by a finite Boolean combination of polynomial equalities and inequalities is called semialgebraic ([75]). Replacing polynomials by analytic functions, we get the notion of *semianalytic set*. Subanalytic sets are defined as projections of semianalytic sets under proper analytic maps ([29]).

A generalization of these classes is that of an *o-minimal structure*. An o-minimal structure is given by a class of sets in $\mathbb{R}^N$, for each $N = 1, 2, \ldots$, which are called *definable*. The class has to be closed under unions, intersections, taking complements and under projections on lower-dimensional spaces. Furthermore, it is required that the graphs of addition and multiplication belong to the structure. The o-minimality is the hypothesis that the definable sets of $\mathbb{R}$ are precisely the finite unions of points and intervals.

Examples for o-minimal structures are the class of semialgebraic sets and the class of those subanalytic sets which remain subanalytic in the projective closure of Euclidean space. Another example is the class of sets defined by polynomials and the exponential function.

Definable sets, i.e. sets belonging to some o-minimal structure, have several nice properties. We refer to [74] for an introduction to this theory, as well as to [84],[85]. For our purposes, we need the following facts.

Definable sets can be stratified, i.e. written as locally finite union of submanifolds, called strata. The frontier condition is that the closure of each stratum is the union of the stratum and lower-dimensional strata. It is possible to stratify these sets such that Whitney's condition ([84]) and Verdier's condition ([159]) are satisfied.

Also there are finiteness properties. For instance, in a definable family, the Euler-characteristic is uniformly bounded. It makes thus sense to consider the modified tube volume $\int \chi(X \cap B(x, r))dx$ for a definable set X.

It was first shown by J. Fu (in the subanalytic case) that this tube volume is indeed a polynomial in r. We will come back to his investigations in the next section.

Two-dimensional semialgebraic sets were considered by Bröcker-Kuppe-Scheufler ([34]). Based on the wing-lemma, they first showed a Gauss-Bonnet-formula for such spaces. The main theorem is that two-dimensional semialgebraic sets are Alexandrov-surfaces in the sense of Section 3.

Using a different approach (and being unaware of Fu's results) Kuppe and Bröcker also studied integral geometric properties of higher-dimensional subanalytic (and more generally definable) sets. The basic notion in their work is that of a *tame set*. This is a Whitney stratified set such that the tangent bundle itself is Whitney stratified and such that the projection map is submersive on each stratum. Compact definable sets are shown to be tame sets.

The new ingredient in the study of tame sets is *stratified Morse theory*. See [108] for stratified Morse theory and [128] for a short proof of the main theorem. The normal Morse index of this theory is used in order to define a modified tube volume. Afterwards, it is shown by counting critical points of the distance function, that formula (5) is satisfied if one of the sets is a closed ball (this is called kinematic ball formula). With approximation of the tame set by manifolds with boundaries it follows that the principal kinematic formula (5) still holds and that the Lipschitz-Killing curvatures are invariant under definable isometries.

These studies are generalized to ambient spaces of constant sectional curvature in [28].

For further information concerning isometry classes of definable sets, Lipschitz-Killing measures and characterizations thereof, see [27].

4.10. **Scalar curvature measure.** In the case of Riemannian manifolds, one of the Lipschitz-Killing measures equals (up to a constant 4π) the integral over the scalar curvature, see [22]. It is thus natural to consider the same quantity in the case of subanalytic sets and to seek analogies with the smooth case.

This program was carried out by Bernig ([22], [23],[24]). The main results relate two different generalizations of curvatures, namely the metric theory of Section 3 and the integral geometric approach of Bröcker-Kuppe.

Recall that in the smooth setting, a bound on the sectional curvature implies a bound on the scalar curvature:

$$K \geq \kappa \implies s \geq n(n-1)\kappa$$
$$K \leq \kappa \implies s \leq n(n-1)\kappa$$

Here n is the dimension of the manifold.

Bernig generalizes these implications to the setting of compact connected subanalytic (or more generally definable) sets. The hypothesis $K \geq \kappa$ is replaced by the assumption that the space is an Alexandrov space with curvature bounded below by κ. The used metric is the induced length metric on the space. This is indeed a geodesic metric, which follows from the fact that the space admits a Whitney-stratification. See [91] and [146] for further information on this metric.

Since the scalar curvature measure of a subanalytic set is a signed measure, the inequality scal $\geq n(n-1)\kappa$ only makes sense for $\kappa = 0$. However, one can replace this inequality by one for measures, namely $\operatorname{scal}(X,-) \geq n(n-1)\kappa \operatorname{vol}(X,-)$ (n denotes the dimension of the set). The main theorem of [22] is that if X is a connected compact definable set of dimension n which is an Alexandrov space with curvature bounded from below by κ, then $\operatorname{scal}(X,-) \geq n(n-1)\kappa \operatorname{vol}(X,-)$. The proof uses a local study of geodesics near the different strata. Although geodesics on definable sets are very difficult objects, it is still possible to get information on angles by using comparison with the outer distance (that of the ambient space).

The analogous result for upper curvature bounds is the subject of [23]. Here one needs a topological assumptions, namely that the space is a topological manifold. Counter-examples are given showing that this assumption is necessary. Then the main theorem states that an upper bound on the sectional curvature in Alexandrov's sense (Section 3) for a connected compact definable topological manifold X implies that $\operatorname{scal}(X,-) \leq n(n-1)\kappa \operatorname{vol}(X,-)$. The proof relies on the possibility of finding a Verdier stratification for X. The other ingredient is a law of reflection at strata of codimension 1.

The generalization of Hilbert's variational formula is contained in [24] and will be presented in Section 6.

5. Generalizations using Geometric Measure Theory, the normal cycle of singular spaces

5.1. **Flat distance.** We saw in the section about metric aspects how metric spaces arise by completing the space of compact Riemannian manifolds under Gromov-Hausdorff distance. Taking another, more complicated distance will yield an embedding of the space of submanifolds in a certain space of currents.

The framework for these investigations is Geometric Measure Theory. The reader is referred to [89] for an overview of this theory and to [88] for a detailed treatment.

We consider the unit normal bundle of an n-dimensional compact submanifold $X \subset \mathbb{R}^N, n < N$. It is an $N-1$-dimensional manifold in the sphere bundle $S\mathbb{R}^N = \mathbb{R}^N \times S^{N-1}$. It has a canonical orientation induced from $\mathbb{R}^N$ and may thus be considered as an $N-1$-current in $S\mathbb{R}^N$. The action on an $N-1$-differential form is given by integration. Since the unit normal bundle has no boundary, we get in fact a cycle, which is the normal cycle of the submanifold.

More generally, the same works if the ambient space is replaced by some Riemannian manifold (M, g). Let us denote by $\mathcal{E}^{N-1}(SM)$ (respectively $\mathcal{E}_{N-1}(SM)$) the space of $N-1$-forms (respectively compactly supported $N-1$-currents) on the sphere bundle SM. Then we obtain a map which associates to a compact submanifold $X \subset M$ a cycle $\tilde{X} \in \mathcal{E}_{N-1}(SM)$.

One of the basic tools in Geometric Measure Theory is the flat distance. We do not want to go into details, but for our purpose it is enough to know that convergence in the flat topology implies weak convergence. The space of compact submanifolds is thus a subspace of $\mathcal{E}_{N-1}(SM)$ and we endow it with the induced topology and distance. The natural question now is: what is a good completion for this space? In order to get useful results, we want the completion to be as small as possible, but on the other hand we want to have a simple description of it.

Let us first state some easy properties of the normal cycle of a Riemannian manifold. As we have seen, it is a cycle. Moreover, it is a compactly supported integer-multiplicity rectifiable current (with multiplicities 1) in the sense of [88]. If α denotes the canonical 1-form on SM, the normal cycle vanishes on α: $\tilde{X} \llcorner \alpha = 0$. Such currents are called *Legendrian*. The space of integer-multiplicity rectifiable compactly supported Legendrian cycles is complete with respect to the flat distance by the famous Federer-Fleming compactness theorem ([88]). This is the completion we looked for and we call it the *space of normal cycles* (not to be confused with normal currents, although they are trivially normal currents).

Besides compact submanifolds, many other classes of spaces can be embedded in the space of normal cycles. This is the case for (compact) PL-sets, sets with positive reach, the convex ring, unions of sets of positive reach in the sense of Zähle, subanalytic sets or definable sets. Briefly: all the classes considered so far.

5.2. **PL and sets with positive reach.** The normal cycle was first introduced by Wintgen in the PL-case and by Zähle for sets with positive reach. A related, although only partial result was proven by Baddeley in [11]. For sets with positive reach, the normal cycles consist of all normal vectors, counted with multiplicity 1. For PL-spaces, one has to count multiplicities in the same (Euler-additive) way we did in the previous section. The main observation of Wintgen and Zähle is that in both cases the Lipschitz-Killing invariants can be obtained by integrating universal differential forms $\Phi_i \in \mathcal{E}^{N-1}(S\mathbb{R}^N)$ over the normal cycle of the set:

$$\Lambda_i(X,B) = \tilde{X} \llcorner B(\Phi_i) \quad B \text{ Borel subset of } \mathbb{R}^N \tag{6}$$

From this description, it follows at once that the (global) Lipschitz-Killing invariants are continuous under the flat distance.

Several authors ([165], [99], [67]) have shown (with different degrees of generality) that if a sequence of PL-spaces approximates a submanifold in a suitable geometric sense, then the normal cycles converge in the flat topology. The flat distance between the normal cycle of a body with smooth boundary and the normal cycle of a *closely inscribed geometric set* is bounded from above in terms of geometric quantities. In particular, using a sequence of restricted Delaunay triangulations of the boundary, one gets a very good convergence of the normal cycles ([67]).

The next case is that of a finite union of sets with positive reach (with the condition explicitly stated in 4.8). Here again, one has at most one way to define normal cycles if one imposes Euler-additivity, which means here

$$\tilde{X} + \tilde{Y} = \widetilde{X \cap Y} + \widetilde{X \cup Y} \tag{7}$$

Zähle was able to show that this extension really exists and that it has the expected properties.

5.3. **Subanalytic sets.** Much more complicated was the construction of the normal cycle for compact subanalytic sets. It was carried out by J. Fu in [102], based on [95], [96] and [97]. He addresses the more general problem which sets do admit normal cycles and under which conditions they are unique. The first question can, up to now, only answered partially, whereas there is a satisfactory uniqueness result concerning the second question.

The basic notion in Fu's work is that of a *Monge-Ampère function*. This is a function f on a manifold M such that df exists in a weak, measure-theoretic sense as current in the cotangent bundle S^*M, denoted by $[df]$. An *aura* for a set $X \subset M$ is a non-negative Monge-Ampère function f such that $f^{-1}(0) = X$. In order to develop his theory, Fu needs an additional assumption on the generalized differentials of such an aura, called non-degeneracy. In general, compact subanalytic sets do not have non-degenerate auras, but here the corresponding assumption is that the aura be subanalytic.

As an example, sets with positive reach admit non-degenerate auras, more precisely the distance function is such an aura. This follows from Bangert's alternative description of sets with positive reach mentioned above. Compact subanalytic sets admit subanalytic auras.

Let $X \subset M$ admit a non-degenerate aura f. Then one can approximate X by the sublevels $f^{-1}([0,\epsilon))$. These sets admit a normal cycle constructed from the differential current $[df]$. The normal cycle of X is defined to be the limit of these normal cycles as ϵ tends to 0. The non-degeneracy condition implies that the limit exists. In the subanalytic case, the limit exists by properties of subanalytic currents.

The obvious problem is that X may have different non-degenerate auras and potentially different normal cycles. Here is where Fu's uniqueness result applies. Stating this result would require much notation from Geometric Measure Theory and from Fu's papers, so we will just give the idea. First, a Chern-Gauss-Bonnet theorem is proven (for non-degenerate or subanalytic auras) by a deformation argument. Working locally, one can suppose that $M = \mathbb{R}^N$. If T is the normal cycle for X, then one can compute from T the Euler-characteristic of the set

$$\{x \in X : \langle x, v\rangle \leq t\}$$

where $v \in S^{N-1}, t \in \mathbb{R}$.

Fu's uniqueness result states that there is at most one Legendrian compactly supported integer-multiplicity rectifiable cycle T which yields these Euler-characteristics for almost all v, t. Based on Chern-Gauss-Bonnet theorem, he then shows that any limit cycle for an approximation with the help of non-degenerate auras has these properties. Therefore, the construction is indeed independent of the choice of a non-degenerate (resp. subanalytic) aura and we get a unique (integer-multiplicity, rectifiable Legendrian) cycle $\tilde{X}$ associated to X.

The Euler-additivity (7) is still satisfied and Lipschitz-Killing measures can be introduced by Equation (6). Fu shows furthermore that these measures only depend on the (subanalytic) isometry class of the set X and not on the embedding (compare with the results of Bröcker-Kuppe).

5.4. **Kinematic formula.** The principal kinematic formula (5) remains true for subanalytic sets. Fu derives this formula from an abstract kinematic formula for cycles. Replacing the normal cycles for subanalytic sets then yields an expression of the form (5), with unknown real constants $c_{N,i,j,k}$. By replacing simple examples for X and Y, one can compute these constants. Of course, they are the same as in all the cases considered above.

The abstract kinematic formula is proved by bundle-theoretic and measure theoretic methods. The main technical tool is that of *current multiplication in bundles*, a natural operation dual to that of fiber-integration.

5.5. **More general singular sets.** It follows from Fu's uniqueness result that a compact subset $X \subset M$ can have at most one normal cycle. A difficult challenge is to determine which sets have this property.

As we said above, sets with positive reach and compact subanalytic sets admit normal cycles. Following the same lines of proof as in Fu's work, one can show that all compact definable sets admit normal cycles.

By formula (6), one can associate curvature measures to any set admitting a normal cycle. If the dimension of the set is n (if some notion of dimension is defined), then $\Lambda_{n-2}(X, -)$ can be considered as a generalized scalar curvature measure. Besides the results in the cases of definable sets mentioned in 4.10, I do not know of generalizations of classical results about scalar curvature to these more singular sets.

6. Ricci and Riemannian tensor, variational formulas

Resuming the last two sections, we see that on large classes of singular sets, we can define curvature measures. There are several ways to characterize them: by using some modified tube volume, by inspecting the terms in the principal kinematic formula or by integrating universal differential forms over the normal cycle of the set. Lipschitz-Killing measures are *real-valued measures.*

When seeking analogous generalizations of the Ricci and Riemannian tensors, it is natural to look for *tensor-valued measures.* There is a technical problem to overcome: namely on general ambient spaces it is not clear what a tensor-valued measure is. Bernig uses the notion of distributional tensor field to overcome this difficulty. The definition mimics the definition of currents, but replaces smooth differential forms by smooth tensor fields. The dual objects are called distributional tensor fields. Then one can define a notion of mass and sees that in Euclidean space, distributional tensors of finite mass are ordinary tensor-valued measures.

6.1. **Cohen-Steiner and Morvan's tensor-valued measure.** In a series of papers ([66], [67], [68]), Cohen-Steiner and Morvan studied tensor-valued invariants of polyhedral and smooth surfaces. The idea is to integrate tensor-valued forms over the normal cycle. If the ambient space is Euclidean, then the measure associated to a full-dimensional compact set with smooth boundary is obtained by integrating over the boundary the second fundamental form. If the ambient space is a Riemannian manifold, then one still gets the second fundamental form but has to interprete the result as a distributional tensor of type $(0, 2)$ in the sense of [25].

In dimension 3, they also construct another tensor-valued measure which yields as result not the second fundamental form, but the form which has the same eigenvectors but flipped eigenvalues. We will see later that this tensor-valued measure could be called *Einstein measure*, since it corresponds in a natural way to the Einstein tensor.

Using Delaunay triangulations, one can approximate a body in $\mathbb{R}^3$ with smooth boundary by a sequence of polyhedra. The general approximation theorem for normal cycles (see 5.2) implies fast convergence of the corresponding measures.

In [66] and [65] the Einstein measure is applied in computational geometry. The idea is that the geometry of a surface changes quickly in those directions where the second fundamental form is big. Since such directions have to be drawn at a small scale, one is very naturally led to the second fundamental form with flipped eigen-values.

6.2. **Intrinsic distributional invariants.** Closely related is ([25]), where Bernig gives the higher-dimensional analogues of the Einstein measure of Cohen-Steiner and Morvan. The basic idea is that of an *intrinsic* distributional invariant. One is interested in curvature properties of a singular set and not in the properties of the embedding. If for instance a subanalytic set is embedded in two different, isometric ways into Riemannian manifolds, then the corresponding Lipschitz-Killing measures are the same. Intrinsic invariants have the same kind of behavior, there is a compatibility condition for different embeddings which can be interpreted as independence of the embedding.

Bernig then studies the space of all intrinsic distributional invariants. He defines a sequence of tensor-valued intrinsic invariants starting with the Lipschitz-Killing measures. The next term is the generalization of Cohen-Steiner and Morvan's Einstein measure to higher-dimensional ambient spaces. This invariant can be defined on any space admitting a normal cycle, since as the Lipschitz-Killing curvatures it is defined via integration over a universal differential form. When evaluated for a compact submanifold, one obtains the (dual of the) Einstein tensor $\mathrm{ric} - \frac{s}{2}g$. In general, this invariant can therefore be thought of as analogue of the Einstein tensor. Its trace is (a multiple of) the scalar curvature. Using this invariant, Bernig shows how to compute the Einstein tensor of a Riemannian manifold from its PL-approximations. He also studies the Einstein invariant on subanalytic and convex sets.

The next term in the series yields an intrinsic distributional invariant which on submanifolds is given by integration over the (dual of the) curvature tensor $\tilde{R} := R - \mathrm{ric} \cdot g + \frac{s}{4} g \cdot g$. Here R denotes the Riemannian curvature tensor, ric the Ricci tensor, s the scalar curvature and the dot is the Kulkarni-Nomizu product transforming two symmetric $(0,2)$-tensors into a $(0,4)$-tensor having the same symmetry properties as R. The tensor $\tilde{R}$ is divergence-free (as the Einstein tensor) and its trace is a multiple of the Einstein tensor. It seems that $\tilde{R}$ does not appear naturally in Riemannian geometry.

6.3. **Ricci curvature bounds.** It is an interesting fact that the Ricci tensor and the Riemannian curvature tensor do not appear as intrinsic distributional invariants. This should be related to the fact that they are not divergence-free.

However, it is possible to define Ricci curvature bounds for subanalytic spaces. This is analogous to the situation for sectional curvature, where the bounds could be generalized to metric spaces, although the precise values were not defined.

Bernig ([26]) gives the definition of compact subanalytic sets with upper or lower bounds on the Ricci curvature. The main result generalizes the following implications, which are all well-known and easy in the case of Riemannian manifolds.

Theorem 6.1. *Let $X \subset (M,g)$ be an n-dimensional compact subanalytic subset of the real-analytic Riemannian manifold (M,g). Suppose that X is a connected topological manifold.*

- *If X is an Alexandrov space with curvature bounded below by κ, then the Ricci curvature is bounded below by $(n-1)\kappa$.*
- *If X has Ricci curvature bounded below by $(n-1)\kappa$, then its scalar curvature is bounded below by $n(n-1)\kappa$.*
- *If X is an Alexandrov space with curvature bounded from above by κ, then its Ricci curvature is bounded from above by $(n-1)\kappa$.*
- *If X has Ricci curvature bounded from above by $(n-1)\kappa$, then its scalar curvature is bounded from above by $n(n-1)\kappa$.*

The basic idea of proof is similar to that in [22] and [23]. On the other hand, the use of *quasi-geodesics* introduced by Petrunin and Perelman ([142]) considerably simplifies the proof.

Although this theorem seems to be complete, the study of Ricci curvature bounds is far from being finished. Such simple questions as the generalization of Myer's theorem remain open.

6.4. **Variational formulas.** Hilbert's variational formula for compact smooth Riemannian manifolds (M, g) concerns the total scalar curvature of the manifold, i.e. $\operatorname{scal}(M, g) := \int_M s\mu_g$ with μ_g the volume density. This is considered as a functional which associates to a Riemannian metric a real number. Hilbert computed the directional derivatives in terms of the Einstein tensor.

More precisely, let h be a symmetric bilinear form and set $g_t := g + th$, where t is a sufficiently small real parameter. Then g_t is again a Riemannian metric and we have

$$\left.\frac{d}{dt}\right|_{t=0} \operatorname{scal}(M, g_t) = \int_M \left\langle \frac{s}{2} g - \operatorname{ric}, h \right\rangle \mu_g \tag{8}$$

Since we know what the generalization of the total scalar curvature to compact subanalytic sets is (namely one of the Lipschitz-Killing invariants multiplied by 4π) and since we have defined the Einstein tensor of such spaces, it is natural to ask if formula (8) still holds in this setting. This is the main result of [24]. In fact, this is the way the Einstein tensor was found. Instead of changing the metric of the space, one takes a variation of the metric of the ambient space.

The proof relies heavily on the normal cycle construction. It permits a reduction of the difficult geometry of such spaces to calculations with differential forms (these calculations are still non-trivial). Instead of taking subanalytic sets, one can as well consider the class of compact PL-spaces. Then one has two ways of computing the left hand side of equation (8), namely by the main variational formula of [24] and by a direct computation. Therefore, the difference of the results has to vanish. This implies the classical Schläfli formula for PL-spaces, as well as all known generalizations ([59], [151], [155]).

Another application of this variational formula is a proof of the Chern-Gauss-Bonnet theorem by a cutting argument.

When the metric is fixed, but the set moves along a smooth vector field, the directional derivatives of the Lipschitz-Killing invariants were computed by J. Fu ([104]) in the case of an ambient space of constant curvature. The (easy) generalization to general ambient spaces is contained in [25].

7. Further generalizations

In this section, we collect and describe some further generalizations of the classical curvature notions. We do not claim that this description is complete, but we tried to collect as much material as possible.

7.1. **Lipschitz-Killing curvatures on angular partially ordered sets.** Budach ([36]) considers *homogeneous finite partially ordered sets. Homogeneous* means that all maximal chains $x_1 < x_2 < \cdots < x_k$ have the same length k. The basic example for such sets are homogeneous n-dimensional simplicial complexes, i.e. simplicial complexes such that each face is contained in an n-dimensional simplex.

The *incidence algebra* of a homogeneous finite partially ordered set P consists of all real valued functions $f : P \times P \to \mathbb{R}$ such that $f(x, y) \neq 0$ only if $x \leq y$. Besides the obvious sum and product, one has the following associative multiplication:

$$(fg)(x, y) := \sum_{x \leq z \leq y} f(x, z)g(z, y)$$

In the case of a finite cell complex, one has three distinguished elements in this algebra: the interior angle function α, the exterior angle function β and the volume function. The Sommerville and McMullen equations relate α and β, in particular β can be computed from α.

Budach defines an *angular partially ordered set* as a homogeneous finite partially ordered set, endowed with an "interior angle" function and a "volume function". The exterior angle function is computed by the Sommerville and McMullen relations.

The Lipschitz-Killing curvatures of an angular partially ordered set are defined like in the case of PL-spaces: sum over elements of rank i and multiply the volume by the (total) exterior angle.

Budach shows that the Lipschitz-Killing curvatures remain unchanged under subdivision and satisfy a product formula as in the PL-case (subdivisions and products are defined in the setting of angular partially ordered sets).

7.2. **Combinatorial Ricci curvature.** A different approach to combinatorial analogues of curvature was given by Forman ([92],[93]). Based on a Bochner-like decomposition of the combinatorial Laplacian, he defines Ricci curvature bounds for cell complexes. This is a purely combinatorial notion, no metric is involved. It depends only on how many neighbors a given cell has. In contrast to the scalar curvature measure and to the Ricci curvature bounds of [26], Forman's curvature bounds only depend on low-dimensional cells, i.e. cells of dimension at most 2. He then shows the analogue of Myer's theorem. For this, he defines combinatorial analogues of Jacobi fields and mimics the classical proof. He also shows the analogue of Bochner's theorem.

Remember that, by [135], any smooth manifold of dimension at least 3 admits a Riemannian metric with negative Ricci curvature. The combinatorial counter-part is that any simplicial complex, which is a manifold of dimension at least 2, has a subdivision with strictly negative Ricci curvature. Note that this is even true in dimension 2 which implies that there is no Gauss-Bonnet theorem for this Ricci curvature.

7.3. **Cheeger-Colding's approach to Ricci curvature.** In a series of papers, Cheeger and Colding ([52],[54],[56],[55]) study limits of (pointed) sequences of Riemannian manifolds with a lower bound on the Ricci curvature. They show among other things that the n-dimensional Hausdorff measure of a ball in the limit space is the limit of the volumes of the n-dimensional balls in the manifolds of the sequence. If the sequence is non-collapsing, which means that the volumes of the unit balls around the base points remain uniformly bounded from below by a positive constant, then it follows that the volume comparison of Bishop-Gromov still holds for the limit space.

In the collapsing case, one has to renormalize the n-dimensional Hausdorff measure in order to obtain a limit measure. Then the volume comparison still holds, but the limit measure depends on the choice of a subsequence.

One can define tangent cones for the limit set. A *regular point* is one where the tangent cone is unique and isometric to some Euclidean space. The set of singular points is shown to have measure zero (for the limit measure).

The study of Ricci curvature reveals analytic aspects of curvature. We refer the reader to the survey papers [69], [70] for further information.

7.4. **Non-commutative scalar curvature and heat equation approach.** The idea of replacing a space by an algebra of functions acting on it is central in algebraic geometry. Instead of considering an algebraic variety, one studies the algebra of algebraic functions on it. One can then define notions (such as dimension) in purely algebraic terms, which make sense for (commutative) algebras independent of their origin as function algebra over an algebraic variety.

This old idea of algebraic geometry has a relatively recent counter-part in differential geometry. Instead of working with a *space* (Riemannian manifold), one works with an involutive algebra $\mathcal{A}$ of operators on a Hilbert-space $\mathcal{H}$. A fixed self-adjoint operator D plays the role of the inverse of the metric element ds. The triple $(\mathcal{A}, \mathcal{H}, D)$ is called *spectral triple* by A. Connes ([72]). A compact Riemannian manifold endowed with a spinor bundle S induces the spectral triple $(C^\infty(M), L^2(M, S), D)$, where D is the Dirac operator acting on spinors. We set $ds := D^{-1}$.

One can recover basic geometric invariants from the spectral triple only, and there is no need to assume that $\mathcal{A}$ is commutative (as in the case of Riemannian manifolds). For instance, the dimension can be defined using the asymptotic behavior of the eigen-values of D. However, this is not a single number, but a subset of $\mathbf{C}$. Concerning curvature, one can at least define the total scalar curvature of such a spectral triple.

More precisely, one can build an integral $⨍ T$ for operators like $T = ds$. It depends on the *Dixmier trace*, which in turn is related to the asymptotic development of the eigenvalues of $\sqrt{T^*T}$. Then the total scalar curvature of a compact Riemannian manifold (M, g) can be obtained as

$$⨍ ds^{n-2} = c_n \int_M s\mu_g$$

Let us explain why this formula is not so surprising.

First note that ds is the inverse of the Dirac operator D and $\Delta := D^*D$ is the Laplace operator. Therefore, the integral on the left hand side is related to the asymptotic development of the eigenvalues of Δ. Now recall the following formula (based on the trace of the heat kernel of (M, g)) for the eigenvalues λ_i of the Laplacian:

$$\sum_{i=0}^\infty e^{-\lambda_i t} \sim (4\pi t)^{-\frac{n}{2}} \sum_{k=0}^\infty a_k t^k$$

with real numbers a_k ([152]). The first coefficient is given by the volume of (M, g): $a_0 = \mathrm{vol}(M, g)$. The next one is in fact the total scalar curvature: $a_1 = \frac{1}{6}\int_M s\mu_g$. The other coefficients are polynomials in the curvature tensor and its derivatives, but it seems to be hard to extract geometric information from them ([107], [152]).

7.5. **Ricci curvature and entropy.** Unlike sectional curvature, which is a purely metric quantity (see Section 3), Ricci curvature depends not only on the metric, but also on an underlying measure (which is the volume measure in the smooth case). One may try to define Ricci cuvature bounds for metric measure spaces. A *metric measure space* is a metric space (X, d) endowed with a measure μ such that the distance function $d : X \times X \to \mathbb{R}$ is measurable with respect to $\mu \times \mu$.

One possible generalization of lower Ricci curvature bounds is based on the entropy of measures ([158]). For a metric space (X, d), let $\mathcal{P}^2(X, d)$ denote the space of probability measures ν on (X, d) such that $\int_X d^2(x, y)\nu(dy) < \infty$ for all $x \in X$. The L^2-*Wasserstein distance* between $\nu_1, \nu_2 \in \mathcal{P}^2(X, d)$ is defined as

$$d_2^W(\nu_1, \nu_2) := \inf_{\nu} \left\{ \int_{X\times X} d(x_1, x_2)^2 \nu(dx_1 dx_2) \right\}^{\frac{1}{2}}$$

where the infimum runs over all measures ν on $X \times X$ with $\nu(A \times X) = \nu_1(A)$ and $\nu(X \times A) = \nu_2(A)$ for all measurable sets $A \subset X$.

The *entropy* of $\nu \in \mathcal{P}^2(X, d)$ is defined as

$$Ent(\nu) := \int_X \frac{d\nu}{d\mu} \log \frac{d\nu}{d\mu} d\mu$$

if ν is absolutely continuous with respect to μ and $\int_X \frac{d\nu}{d\mu}[\log \frac{d\nu}{d\mu}]_+ d\mu < \infty$ and $Ent(\nu) := \infty$ otherwise.

A possible definition of Ricci curvature bounds for metric measure spaces is given by the following. Recall that a real-valued function f on a geodesic metric space (X, d) is called κ-convex if for each unit speed geodesic $\gamma : I \to X$ (where $I \subset \mathbb{R}$ is an interval) the function $t \mapsto f(\gamma(t)) - \frac{\kappa}{2}t^2$ is convex on I.

Definition 7.1. *A metric measure space* (X, d, μ) *has* **Ricci curvature bounded from below by** κ *if the function* $Ent : \mathcal{P}^2(X, d) \to \mathbb{R} \cup \{\infty\}$ *is* κ*-convex.*

One of the main results of [158] is that a connected Riemannian manifold with its volume measure has Ricci curvature $\geq \kappa$ in the above sense if and only if it has Ricci curvature $\geq \kappa$ in the usual sense. They also give characterizations of lower Ricci curvature bounds on manifolds in terms of heat kernels and in terms of uniform distributions on distance spheres.

7.6. **Generalized Gauss graphs and unit normal bundles.** Closely related to the normal cycles are *generalized Gauss graphs* and *generalized unit normal bundles.* A generalized unit normal bundle is a Legendrian current with a further positivity condition. Also generalized Gauss graphs encode curvature information. One can indeed recover the Riemannian curvature tensor from the Gauss graph of a Riemannian manifold and this idea is used to define a Riemannian curvature tensor for generalized Gauss graphs in [79]. In contrast to the curvature tensors of Subsection 6.2, these objects depend not only on a point in the base space, but also on a direction. The reader can consult the papers by Delladio ([78], [79], [80], [81]) and by Anzelotti et al. ([8], [9], [10]) for definitions of these currents and applications to geometric variational problems.

8. Some questions and speculations

Let us finish this survey with some open questions and some speculations. The choice for these problems, as the choice of the content of the whole paper, is of course subjective and reflects my personal taste.

Let us start with some questions concerning PL-spaces. The study of PL-spaces with curvature bounded from above or below seems to be a promising field of research. There are many results (see [32]), but mainly in the case of upper curvature bounds. One can formulate questions from Riemannian geometry in this setting. For instance, is it true that non-negative sectional curvature in Alexandrov sense implies non-negative Euler-characteristic (Hopf-conjecture)? The corresponding question for non-positively curvature is known under the name of *Chern-Hopf-Thurston conjecture* ([76]) and also open.

Since the Chern-Gauss-Bonnet-integrand of a 4-dimensional non-negatively curved manifold is non-negative, one may expect that the lowest Lipschitz-Killing measure of a non-negatively curved 4-dimensional PL-space is non-negative. In higher dimensions, one would rather expect this to be false (as in the smooth setting). However, the global Lipschitz-Killing invariant could still be positive. Also it might be true that non-negative sectional curvature implies that the $n-4$th Lipschitz-Killing curvature is a positive measure and it would be interesting to show this fact on wide classes of spaces, such as manifolds, PL-spaces, subanalytic spaces.

The advantage in working with PL-spaces is that one can try to prove things by induction on the dimension, the spherical sections are M_1-simplicial complexes in the terminology of [32]. If the space has non-negative (non-positive) curvature, then the spherical sections have curvature at least 1 (at most 1). On the other hand, one can try to approximate manifolds by PL-spaces. As was described above, the Lipschitz-Killing measures converge. But if the manifold has non-negative curvature, can we find an approximation by non-negatively curved PL-spaces? If we use the Lipschitz-distance ([120]), then the answer is no, since only manifolds whose curvature tensor is positive in a strong sense can be approximated in this way ([145]). Maybe with other distances such as the flat distance one gets better results. Also an analogous approximation result in the non-positively curved case is in general false (see [77]), except for hyperbolic manifolds ([48]).

Another big question is to which spaces the theory of normal cycles applies. Fu's uniqueness result shows that there can be at most one normal cycle associated to a compact set, but which sets do admit normal cycles? The study of this class seems to be promising, since it is a good framework for integral-geometric studies. There were two attempts ([104],[82]) to prove that the projection of the carrier of a normal cycle is a C^2-rectifiable set in the sense of [9], but, according to J. Fu, both "proofs" have serious gaps, so the problem seems to remain open.

In the metric theory, one gets a lot of information about how a sequence of manifolds with lower curvature bound can degenerate (collapse) to an Alexandrov space. An analogous theory for the flat distance would be highly interesting. For this, one considers sequences of submanifolds in a fixed ambient space which converge in the flat topology to some normal cycle. What can be said about this convergence? In connection with Federer-Fleming's compactness theorem, this could yield powerful tools for the study of submanifolds as well as singular spaces.

A related problem is to develop a more intrinsic theory, i.e. one that does not (or less) depend on an ambient space. One first try could consist of identifying isometric embeddings of a singular space in ambient spaces. Then one looks for invariants of the equivalence classes, they will be quantities of the singular space and not of the embedding. A similar approach was worked out by Pflaum ([146]) concerning *smooth structures on stratified spaces*.

A different problem is to relate the non-commutative approach of Connes to the theory of Lipschitz-Killing curvatures. It could be true that all these curvature measures (and not only the total scalar curvature) can be recovered from the spectral triple alone (in the case of a compact Riemannian manifold). However, I think that the total scalar curvature only appears because it is a spectral invariant, whereas no formula expressing the other Lipschitz-Killing curvatures as spectral invariants seems to be known (except for Euler characteristic and volume of course). However, for the Laplacian *acting on forms*, Donelly [83] gave such an expression for Lipschitz-Killing curvatures. It is probably a very hard problem to prove an analogue of his formula for convex subsets. Cheeger [51] gave such a formula for piecewise linear spaces.

The same kind of questions can be raised for the entropy approach to Ricci curvature (7.5). What does positive Ricci curvature on a piecewise linear space or on a compact subanalytic set means? Of course, the PL-case should be relatively easy, whereas the subanalytic case is probably too hard. Maybe somewhere between there are more interesting classes to study: manifolds with boundary or manifolds with corners.

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