The Grothendieck conjecture for affine curves

Oberseminar Arithmetische Homotopietheorie

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Introduction

Grothendieck’s “anabelian geometry”, as first laid out in his letter [Gro83] to Faltings, is concerned with the question to which extent arithmetic-geometric properties of a variety $X$ over a field $k$ are determined by the exact sequence of profinite groups

$$1 \to \overline{\pi} \to \pi \to G \to 1,$$

where $\pi = \pi_{\text{ét}}(X, \overline{\mathbf{x}})$ and $\overline{\pi} = \pi_{\text{ét}}(\overline{X}, \overline{\mathbf{x}})$ denote the étale fundamental group of $X$ and the base change $\overline{X} = X \otimes_k \overline{k}$, respectively, and where $G = \text{Gal}(\overline{k}/k)$ denotes the absolute Galois group of $k$.

Among other things, he conjectured the following:

**Conjecture.** Let $X$ be a hyperbolic curve over a field $k$ that is finitely generated over $\mathbb{Q}$. Then the isomorphism class of $\pi \to G$ uniquely determines the isomorphism class of $X/k$.

In his doctoral thesis [Tam97], Tamagawa managed to prove an even stronger form of the above conjecture for affine hyperbolic curves:

**Theorem** (Tamagawa ’97, [Tam97, Theorem 0.3]). Let $k$ be a field finitely generated over $\mathbb{Q}$. Let $X$ and $X'$ be affine hyperbolic curves over $k$ with étale fundamental groups $\pi$ and $\pi'$ respectively. Then the canonical map of sets

$$\text{Isom}_{k}(X, X') \sim \to \text{Isom}_{G}^{\text{out}}(\pi, \pi'),$$

where $\text{Isom}_{G}^{\text{out}}(\pi, \pi')$ denotes the set of outer $G$-isomorphisms, is bijective.

Astonishingly, Tamagawa proves his theorem by first proving a finite field version [Tam97, Theorem (0.5)] (something that Grothendieck didn’t even conjecture to be true!) and then reducing the statement over fields finitely generated over $\mathbb{Q}$ to this case.

The goal of this seminar is to work through Tamagawa’s thesis [Tam97] and the above two theorems [Tam97, Theorem (0.5)] and [Tam97, Theorem (0.3)] in detail.
Literature

The main source is Tamagawa’s original paper [Tam97] but it might be handy to have [BLR00, §12, §13] close by. For some of the topics, there are some further helpful references. These are listed in the relevant paragraph.

Time and Place

• We’ll meet in person every two weeks on Wednesday in either Heidelberg (SR tba) or Frankfurt (Hörsaal 16).
• After the session at 08.06., there will be a one-time shift of a week, which means the next session will be at 29.06..
• Usually, a session will consist of two talks:
  ○ Heidelberg
    The first talk will be from 11:30 - 13:00 and the second one from 14:15 - 15:45.
  ○ Frankfurt
    The first talk will be from 12:15 - 13:45 and the second one from 14:30 - 16:00.
Ideally, each talk is prepared with about 10-15 minutes for discussion in mind.
• Talks 7–9 will be held in one longer session, details tba.
• After the last talk, we’ll hike along the philosophers’ path and afterwards have dinner together at tba.

Schedule

Overview  
(Online, 20.04.)

0. Provide an overview of the seminar and distribute the topics.

Generalities on the fundamental groups of curves  
(Frankfurt, 27.04.)

[BLR00, §12, §13], [AM69, §3]

1. Introduce the setup of [Tam97, §1] (up to [Tam97, (1.1)]). Define the notion of a “full” class $\mathcal{C}$ of finite groups and define the maximal pro-$\mathcal{C}$ quotient $G^\mathcal{C}$ of a profinite group $G$. Explain [Tam97, (1.1)] and its corollaries [Tam97, (1.2), (1.4)].

2. Discuss torsion-, and centrefreeness of fundamental groups of curves:
   Prove [Tam97, (1.5)] and deduce that the pro-$\mathcal{C}$ quotient $\tilde{\pi}^\mathcal{C}$ of the geometric fundamental group $\tilde{\pi}$ of a curve is torsionfree [Tam97, (1.6)]. Prove that $\tilde{\pi}^\mathcal{C}$ is centrefree [Tam97, (1.11)] (a more accessible proof can be found in [Col98, Lemma 1]).
Characterization of Decomposition Groups

(Heidelberg, 11.05.)

\[ \text{[Col98, Theorem 3], [Sti13, §4], [Sti13, Proposition 54]} \]

3. Define (geometric) (quasi-)sections \[ \text{[Tam97, (2.3)]} \] and the property “A” of a field \( k \) \[ \text{[Tam97, p. 150]. Explain [Tam97, (2.6)] (see [Sti12, Lemma 7] for a more elaborate version) and use it to prove the first item of the main result [Tam97, (2.8)] regarding the characterization of decomposition groups.} \]

4. Prove the second item of [Tam97, (2.8)]. Define the properties “B” and “C” of a field \( k \) \[ \text{[Tam97, p. 150]. Discuss and, time permitting, prove the remaining items of [Tam97, (2.8)] (also see [Col98, Theorem 3]). Focus on accessibility rather than completeness.} \]

Characterization of various invariants

(Frankfurt, 25.05.)

\[ \text{[BLR00, §12.3]} \]

5. Explain how to group-theoretically characterize the characteristic \( p \geq 0 \) of the ground field \[ \text{[Tam97, (3.1)]} \] as well as the geometric fundamental group \( \bar{\pi} \) over finite fields \[ \text{[Tam97, (3.3)]. Prove the third item of [Tam97, (3.4)]. State and, time permitting, sketch how to characterize the geometric fundamental group \( \bar{\pi} \) over fields finitely generated over \( \mathbb{Q} \) [Tam97, (3.2)].} \]

6. Explain how to group-theoretically recover the cardinality \( q = \# k \) of the ground field and the Frobenius element \( \varphi_k \) (for \( k \) finite) (the first two items of [Tam97, (3.4)]). Prove how to recover the genus \( g \) of the curve as well as the number \( n \) of points of the boundary [Tam97, (3.5)]. State and, time permitting, prove how to recover the number of rational points [Tam97, (3.8)] of the curve as well as the kernel “I(\( \pi \))” [Tam97, (3.7)].

The Grothendieck conjecture for curves over finite fields

(Heidelberg, 08.06.)

The goal of this section is to prove the tame case of [Tam97, (4.3)].

7. Introduce the setup considered in [Tam97, §4] and elucidate the general strategy of the proof of (the tame case of) [Tam97, (4.3)]. Prove [Tam97, (4.1)]. Quickly introduce class field theory of function fields and explain how, together with the characterizations of the preceding talks, it can be applied to obtain a multiplicative isomorphism \( \varphi^\times : K_1^{\times} \sim K_2^{\times} \), where \( K'_j \) denotes the maximal pro-\( \mathfrak{c} \) Galois extension of the composite \( K_j \cdot k_i^{\text{sep}} \) of the function field of the curve and the separable closure of the ground field under consideration, respectively ([Tam97, (4.3)] up to, including, [Tam97, (4.5)]).

8. Assuming the critical claim [Tam97, (4.6)] that the isomorphism \( \varphi^\times \) is additive, finish the proof of [Tam97, (4.3)]. State [Tam97, (4.7)] and explain why it is applicable in the situation at hand. Prove [Tam97, (4.16)] and mention that we’ll eventually reduce [Tam97, (4.7)] to it.
9. Finish the proof of the main result [Tam97, (4.3)] over finite fields by proving [Tam97, (4.7)]:

Define minimal elements in function fields [Tam97, (4.8)], state and, time permitting, sketch [Tam97, (4.11)] and use it to prove that “(a)(b)(c) \implies (a)(b)(c)” [Tam97, p. 171]. Prove [Tam97, (4.13), (4.15)] and explain how this reduces the missing claim “(a)(b)(c’) \implies additivity of \(\phi^x\)” [Tam97, p. 172] to the already proven [Tam97, (4.16)].

"Anabelian" criterion for good reduction

10. Introduce the setup of [Tam97, §5]. Define the notion of having good reduction for tuples \((X, D)\) consisting of a curve \(X\) and an effective étale divisor \(D\) at a point \(s\) of the base scheme \(S\) [Tam97, (5.1)]. Prove Tamagawa’s “anabelian” criterion [Tam97, (5.3)] for detecting whether such a tuple \((X, D)\) has good reduction in terms of \(\pi_1^{\text{ét}}(X \setminus D)\).

11. Prove the key lemma [Tam97, (5.5)] and use it and Tamagawa’s anabelian criterion for good reduction to group-theoretically recover the fundamental group of the special fibre of a curve over a DVR from the fundamental group of the curve itself [Tam97, (5.7)].

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12. Reduce the main result in characteristic 0 [Tam97, (6.3)] to [Tam97, (6.6)].

13. Complete the proof of [Tam97, (6.3)] by proving the remaining claim [Tam97, (6.6)].

14. Hike along the philosophers’ path with dinner at tba.
References


