Non-hypergeomtric E-functions

GAUS AG-Seminar - WiSe 2022/2023

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An *E-function* is a formal power series with algebraic coefficients that is annihilated by some non-zero differential operator, and whose coefficients satisfy some growth condition. E-functions were introduced by Siegel [12] in 1929 with the goal of generalizing the theorems of Hermite, Lindemann and Weierstrass about transcendence of values of the exponential function. Examples of E-functions include the exponential function, the Bessel function, and *hypergeometric E-functions*

$$F\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| \lambda z^{q-p} \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \lambda^n z^{n(q-p)}$$

for integers $0 \leq p < q$, rational parameters $a_1, \dots, a_p, b_1, \dots b_q \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$, and an algebraic scalar λ. Here $(x)_n := x(x+1)\cdots(x+n-1)$. After noticing that any polynomial expression in E-functions with algebraic coefficients is again an E-function, Siegel asked the following question:

does the $\bar{\mathbb{Q}}[z]$-algebra generated by hypergeometric E-functions contain all E-functions?

While this is true for E-functions satisfying a differential equation of order at most two [10], a negative answer has been provided in 2021 by J. Fresán and P. Jossen [4] who found a way to construct non-hypergeometric E-functions annihilated by a third-order linear operator. An explicit example is given by the power series

$$\sum_{n=0}^{\infty} \left(\sum_{m=0}^{\lfloor 2n/3 \rfloor} \frac{\left(\frac{1}{4}\right)_{n-m}}{(2n-3m)!(2m)!} \right) z^n .$$

The goal of the seminar is to understand the Fresán-Jossen construction of non-hypergeometric E-functions.

Contents and structure of the seminar

The study of Fresán-Jossen's paper gives us the opportunity to learn about several topics in algebraic geometry and number theory, including: E-functions and G-functions, differential Galois theory, Tannakian categories, differential modules ($\mathcal{D}$-modules), and p-adic differential equations. More precisely, the seminar is structured as follows:

• The first two talks concern the classical theory of E-functions and of the closely related family of G-functions. In Talk 1 the basic definitions and fundamental examples are given; next the relation between E-functions and G-functions in terms of Laplace transform and Fourier transform of the associated differential operators is discussed. In Talk 2 we introduce a special class of ODEs, called Picard-Fuchs differential equations, whose solutions are G-functions. This permits to establish a relation between special values of G-functions and periods of algebraic varieties. Next we show that special values of G-functions appear in the asymptotic expansion of E-functions. This gives extra motivation to study Siegel's question: a positive answer would imply a severe restriction on the set of numbers that can appear as periods of algebraic varieties, in a way that is not compatible with the current standard conjectures.

• Talks 3 to 6 present several aspect of the theory of linear differential operators that will be needed in the construction of the non-hypergeometric E-functions. This series of talks can be intended as a crash course on the development of some aspects of the theory of linear differential equations in the algebraic setting. Differential Galois theory (Talk 3) associates to a linear operator an algebraic group, the *differential Galois group*. Similarly to classical Galois theory, a correspondence between differential Galois groups and finite extensions of differential fields is described. In several circumstances, it is more convenient to think of the differential Galois group as of the Tannaka group of an object in a category of differential modules (Talk 4). The basic theory of differential modules (*$\mathcal{D}$-modules*) on the affine line and of their structure is the content of Talk 5. Finally in Talk 6 we study the singularities of $\mathcal{D}$-modules, and in particular we introduce the concept of *slope* and the Levelt-Turrittin decomposition.

• In Talks 7-8 we put together what we learned from Talks 1-2 and Talks 3-6. Inspired by E-functions and G-functions, we define *$\mathcal{D}$-modules of type E and type G*. They are related by Fourier transform and live into suitable categories equipped with a special fiber functor (Talk 7). Among $\mathcal{D}$-modules of type E there are $\mathcal{D}$-modules arising from hypergeometric E-functions; in the light of Siegel's question, the goal is to prove that not every $\mathcal{D}$-module of type E is generated by the hypergeometric $\mathcal{D}$-modules. The strategy of the proof partially relies on the fact that the categories of $\mathcal{D}$-modules of type E and G are Tannakian, as we prove in Talk 8. The proof of this fact will give us the opportunity to explore some p-adic aspects of G-functions and differential equations.

• In Talks 9-11 we finally construct the non-hypergeometric E-functions following Fresán-Jossen. We prove that a three-dimensional $\mathcal{D}$-module of type E that is also hypergeometric gives rise to a $\mathcal{D}$-module of type G whose singularities have a special geometric configuration: they are collinear of form an equilateral triangle (Talk 10). It is possible to prove such a result thanks to Katz's classification of the differential Galois groups of hypergeometric $\mathcal{D}$-modules (Talk 9). Finally, we construct $\mathcal{D}$-modules of type G from polynomial maps. Their singular points are given by the critical values of the polynomial. By choosing a suitable fourth-degree polynomial, i.e. such that its critical values are not collinear nor form an equilateral triangle, and using Fourier transform, we finaly construct a $\mathcal{D}$-module of type E that is not generated by hypergeometric modules.

Organization and description of the talks

Format hybrid – in person but will stream simultaneously via Zoom.

Dates We will meet on a two-weekly bases on Thursday afternoon 3.15 having two one-hour talks with a break in-between. We plan to start on November 3. The last talk will take place on January 26.

Places alternating between Darmstadt and Frankfurt (starting from Frankfurt)
• Frankfurt: Room 711 Groß, Robert-Mayer-Straße 10.
• Darmstadt: Room 244 - S2—15, Schlossgartenstraße 7.

Classification of the talks What does $\star$, $\star\star$, $\star\star\star$ mean?
$\star$: Suitable for Masters- or Ph.D. students without much background in algebraic geometry;

talks should be straightforward to prepare.
$\star\star$: Suitable for Ph.D. students and postdocs; usually requires background in algebraic geometry or knowledge of almost all previous talks.
$\star\star\star$: Suitable for ambitious Ph.D. students and postdocs, as well as for Professors. Requires solid background in algebraic geometry (and algebraic groups for Talk 9) and/or the willingness to engage with the material in significant depth.

Getting started It is highly recommended to read the nicely written introduction of the Fresán-Jossen paper [4] in order to have an overview of the problem and of the strategy of the solution.

Final remark The goal of the Gaus-AG is not only to learn advances in current research, but also to get to know each other, discuss and participate actively. We strongly encourage you to contact the organizers in the preparation of the talks, and to discuss with them or with your colleagues about any problem that may arise.

Talk 1: E-functions and G-functions $\star$ (03.11 Frankfurt) [G. Bogo]

Define *E-series* and *G-series* [4, Definition 1.2], and present the basic theory of E-functions follwing [5, Sec 2.1]; in particular, prove that hypergeometric functions are E-functions (example (4) in [5, Sec 2.1]) and discuss the Siegel-Shidlovsky theorem. State Siegel's question [4, pag 904]. Introduce the differential algebras $\mathcal{E}$ and $\mathcal{G}$ [4, Section 1.4], *G-operators* and *E-operators* [4, Def 1.6]. Present their basic properties as given in [4, Theorem 1.8] and [4, Theorem 1.10], but do not discuss singularities of the operators. Introduce the *Fourier transform* [4, Section 1.1], the *Laplace transform* [4, Sections 1.11,1.12,1.13] and give a complete description of their action on the algebras $\mathcal{E}$ and $\mathcal{G}$ [4, Prop 1.14]. Conclude by presenting the example in [4, Section 1.15] in order to shed light on the definition of E-operators.

Talk 2: G functions and geometry $\star\star$ (03.11 Frankfurt) [C. Röhrig]

Introduce *Picard-Fuchs differential equations* on curves by following [6, pag 71-72]. State (without proof) that the solution of a Picard-Fuchs differential equation is a G-function [1, pag 110] and the conjectural converse statement (Bombieri-Dwork conjecture [1, pag 11]). As an example, present the *Legendre family* and its relation with hypergeometric functions as discussed in [9, Sections 2.1-2.2] (the reading of the introduction and of Sections 1,2 of [9] can be helpful in the preparation of the talk.) Introduce and describe the sets G [3, Def 2] and H [3, Section 2.2], and state the main theorem [3, Theorem 1], which relates Siegel's question to periods of algebraic varieties. Prove the theorem (see [3, Section 6]) by discussing the main ingredients: the appearance of arbitrary elements of G in the asymptotic expansion of E-functions [3, Theorem 3] and the asymptotic expansion of hypergeometric E-functions [3, Theorem 4].

Talk 3: Some aspects of differential Galois theory $\star\star$ (24.11 Darmstadt) [R. Çiloğlu]

The goal of the talk is to give an introduction to differential Galois groups and Galois correspondence. The main reference is the book [13]. Introduce differential modules over differential fields (with algebraically closed fields of constants), [13, Definition 1.6.]. Pick your favorite running example to illustrate all the concepts. Introduce Picard-Vessiot extensions [13, §1.3.] and define the differential Galois group [13, Definition 1.25.]. Explain that it is an algebraic group [13, Theorem 1.27.] and prove that the maximal spectrum of the Picard-Vessiot ring is a torsor for the differential Galois group [13, Theorem 1.28.]. State the Galois correspondence [13, Theorem 1.34.] and sketch its proof.

Talk 4: Tannakian categories $\star$ (24.11 Darmstadt) [A. Güthge]

The goal of the talk is to introduce Tannakian categories and relate the Tannaka group of a differential module to its differential Galois group. We mostly follow the article [2], specifically Chapter 2. Define affine group schemes and explain how to recover them from their category of finite-dimensional representations, [2, Chapter 2, Proposition 2.8.]. Define neutral Tannakian categories [2, Chapter 2, Definition 2.19.] and give some examples (such as the category of differential modules or algebraic vector bundles with integrable connection). Sketch the proof of the main theorem [2, Chapter 2, Theorem 2.11.]. Explain how to recover the differential Galois group of a differential module through Tannakian formalism, [13, Theorem 2.33.]. Prove [2, Chapter 2, Proposition 2.20.]. This gives an alternate way to see that the differential Galois group is algebraic

Talk 5: Holonomic $\mathcal{D}$-modules on $\mathbb{A}^1$ $\star$ (01.12 Frankfurt) [M. Müller]

In this talk and the next we follow Claude Sabbah's notes [11] to introduce the basic notions of $\mathcal{D}$-module theory on the affine line. Define the Weyl algebra $A_1(\mathbb{C})$ and the algebras $\mathcal{D}$, $\widehat{\mathcal{D}}$ of local differential operators. Talk about their basic properties [11, I, §1.3]. Introduce the notion of good filtration of a $\mathcal{D}$-module [11, I, Definition 3.2.1.]. Define the characteristic variety [11, I, Definition 3.2.5.] and give an example. Define holonomic $\mathcal{D}$-modules and explain that they are precisely the $\mathcal{D}$-modules of the form $\mathcal{D}/I$ for some non-zero left ideal $I \subset \mathcal{D}$ [11, I, Corollary 3.3.5]. Finally globalize the story following [11, III, §1.1.] to introduce holonomic modules over the Weyl algebra $A_1(\mathbb{C})$. In particular, we need [11, III, Proposition 1.1.5.] whose key input is [11, I, Lemma 2.3.3].

Talk 6: Formal meromorphic connections $\star$ (01.12 Frankfurt) [F. Pennig]

Introduce the notion of formal meromorphic connection [11, Definition 4.3.1.] (replacing K by $\widehat{K}$). This is the same as a differential module over $\widehat{K}$. Prove that any holonomic $A_1(\mathbb{C})$- or $\mathcal{D}$-module determines a formal meromorphic connection by extension of scalars (this is one direction of [11, Theorem 4.3.2.]). Prove [11, Proposition 4.3.3], also known as the cyclic vector lemma. Introduce the Newton polygon of a formal meromorphic connection, define the slopes and regular meromorphic connections. Spend the rest of the talk explaining [11, Theorem 5.4.7.], also known as the Levelt-Turrittin decomposition. One of the main ingredients is [11, Theorem 5.3.1.] which gives a splitting of a formal meromorphic connection according to its slopes.

Talk 7: Divisor of a module in $\mathrm{Conn}_0(\mathbb{G}_m)$ and hypergeometric modules $\star\star\star$ (22.12 Darmstadt)[K. Jakob]

Define Fourier transform of $\mathcal{D}$-modules and introduce the set of solutions of a $\mathcal{D}$-module in a differential algebra $\mathcal{A}$, [4, §2.1 & 2.2]. Introduce the Tannakian categories $\mathbf{Conn}_0(\mathbb{G}_m)$ and $\mathbf{RS}_0(\mathbb{A}^1)$ and explain their relation via Fourier transform of $\mathcal{D}$-modules [4, Proposition 2.3., §2.4]. Define modules of type E and G [4, Def 2.9] and the associated categories $\mathbf{E}$ and $\mathbf{G}$. Discuss [4, Theorem 2.12 (1)], which relates $\mathcal{D}$-modules to G-functions, and the analogous statement for $\mathbf{E}$ [4, Theorem 2.13]. Construct the monoidal functor

$$\Psi : \mathbf{Conn}_0(\mathbb{G}_m) \to \{\overline{\mathbb{Q}}_\ell - \text{graded vector spaces}\}$$

and conclude that the Galois group of M contains an explicit algebraic torus. Define the divisor $\mathrm{div}(M)$ of a module $M \in \mathbf{Conn_0}(\mathbb{G}_m)$ [4, §2.6., Lemma 2.7., 2.8]. Finally, introduce hypergeometric $\mathcal{D}$-modules [4, Definition 3.6] and prove [4, Theorem 3.7.], computing their divisor.

Talk 8: André's theorem $\star$ (22.12 Darmstadt)[Y. M. Wong]

The goal of the talk is to prove that $\mathbf{E}$ is a Tannakian subcategory of $\mathbf{Conn}_0(\mathbb{G}_m)$ [4, Theorem 2.14]. The main tool is André's theorem [4, Theorem 2.12 (2)] whose proof uses Bombieri's characterization of G-functions in terms of their p-adic radius of convergence. the presentation is based on André's book [1]. Introduce the notion of *global radius of a D-module* [1, IV, 3.3]. This requires some basic concepts in p-adic differential equations tat can be found in [1, IV, 1-2]. State that G-modules are precisely the modules with finite global radius [4, pag 76] (see also [4, Sec 1.7,1.8]). Prove the first two points of Lemma 2 in (André, IV,3). Complete the proof for the category $\mathbf{E}$ by following the proof of [4, Theorem 2.14].

Talk 9: Differential Galois groups of hypergeometric $\mathcal{D}$-modules $\star\star\star$ (12.01 Frankfurt) [J. Chen]

Recall the definition of generalized hypergeometric equations following [8, Chapter 3, §1]. Compute the slope and state [8, Theorem 3.6.]. Define Lie-irreducibility [8, §2.7.] and explain the relation to Kummer induction. The rest of the talk will focus on the computation of differential Galois groups. Following [7, §2.5. & 2.6.] introduce the local differential Galois group, the upper numbering filtration and the unique index N-subgroup. Explain the relation to the slopes. Prove Katz's Main D.E. Theorem [8, 2.8.1.] using Gabber's torus trick [8, Theorem 1.0]. Use the various recognition results for semisimple Lie algebras as a black box.

Talk 10: The symmetry constraint $\star\star\star$ (12.01 Frankfurt) [Y. Li]

The talk follows closely [4, Sec 4]. Define *Lie-generated objects* of a Tannakian category [4, Def 4.3] and motivate this definition as in [4, Sec 4.4]. Prove [4, Theorem 4.7], which shows that an object of $\mathbf{E}$ with Galois group SL_3 which is Lie-generated by objects of $\mathbf{H}$ is Lie-generated by only one specific object. It is not necessary to prove all the special cases in the proof. Finally, prove [4, Theorem 4.9], which describes the symmetry constraint on the Fourier transform of a three-dimensional object of $\mathbf{E}$ with Galois group containing SL_3.

Talk 11: A non-hypergeometric E-operator $\star\star$ (26.01 Darmstadt) [M. Zhang]

The talk follows [4, Sec 5]. Prove [4, Prop 5.1], which describes a fundamental matrix of solutions of $\mathcal{D}_{\mathbb{G}_m}$-module induced by a polynomial map. Discuss then the case of fourth order polynomials: if the critical values are not collinear, the associated $\mathcal{D}_{\mathbb{G}_m}$-module is simple [4, Lemma 5.3] and its Galois group of contains SL_3 [4, Lemma 5.3]. Finally, prove the main result [4, Theorem 5.7], which provides infinitely may examples of non-hypergeometric E-functions. Conclude by discussing [4, Sec 5.8] and giving a concrete example of such E-function.

Talk 12: E-functions and geometry (26.01 Darmstadt) [J. Fresán]

I will explain why every exponential period function of the form $\int_\sigma e^{-zf}\omega$, where f is a regular function on an algebraic variety X defined over the field of algebraic numbers, ω is an algebraic differential form on X, and σ is a rapid decay cycle on $X(C)$, is a linear combination of E-functions "with monodromy" with coefficients in the field generated by usual periods, special values of the gamma function and Euler's constant. This is how E-functions arise from geometry and gives some intuition of why a positive answer to Siegel's question about hypergeometric E-functions was extremely unlikely. (Joint work with Peter Jossen).

References

[1] Yves André. *G-functions and geometry.* Aspects of Mathematics, E13. Friedr. Vieweg & Sohn, Braunschweig, 1989.

[2] Pierre Deligne and J. S. Milne. Tannakian categories. In *Hodge cycles, motives, and Shimura varieties, Lect. Notes Math. 900, 101-228 (1982).* Springer Berlin, Heidelberg, 1982. available at `https://www.jmilne.org/math/xnotes/tc2018.pdf`.

[3] S. Fischler and T. Rivoal. On Siegel's problem for E-functions. *Rendiconti del Seminario Matematico dell'Università di Padova*, 2019. `https://arxiv.org/abs/1910.06817`.

[4] Javier Fresán and Peter Jossen. A non-hypergeometric E-function. *Ann. of Math. (2)*, 194(3):903–942, 2021.

[5] Javier Fresán. *Motifs exponentiels, fonctions E et sommes exponentielles.* Habilitation thesis.

[6] Nicholas M. Katz. On the differential equations satisfied by period matrices. *Publications Mathématiques de l'IHÉS*, 35:71–106, 1968.

[7] Nicholas M. Katz. On the calculation of some differential Galois groups. *Invent. Math.*, 87(1):13–61, 1987.

[8] Nicholas M. Katz. *Exponential sums and differential equations*, volume 124 of *Annals of Mathematics Studies.* Princeton University Press, Princeton, NJ, 1990.

[9] Maxim Kontsevich and Don Zagier. *Periods*, pages 771–808. Springer Berlin Heidelberg, Berlin, Heidelberg, 2001.

[10] T. Rivoal and J. Roques. Siegel's problem for E-functions of order 2. In *Transcendence in algebra, combinatorics, geometry and number theory*, volume 373 of *Springer Proc. Math. Stat.*, pages 473–488. Springer, Cham, [2021] ©2021.

[11] Claude Sabbah. Introduction to algebraic theory of linear systems of differential equations. In *Éléments de la théorie des systèmes différentiels. $\mathcal{D}$-modules cohérents et holonomes (Nice, 1990)*, volume 45 of *Travaux en Cours*, pages 1–80. Hermann, Paris, 1993. available at `https://perso.pages.math.cnrs.fr/users/claude.sabbah/livres/sabbah_cimpa90.pdf`.

[12] C. L. Siegel. Über einige Anwendungen diophantischer Approximationen. *Abh. Preuß. Akad. Wiss., Phys.-Math. Kl.*, 1929(1):70 s., 1929.

[13] Marius van der Put and Michael F. Singer. *Galois theory of linear differential equations*, volume 328 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, 2003.