Buildings, valuated matroids and tropical linear spaces

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The compactified affine building

Let K be a complete field with respect to a nonarchimedean valuation $\mathrm{val} : K \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ and V a finite-dimensional vector space over K. A **norm** on V is a map $||.||: V \to \mathbb{R}$ that fulfills the following axioms:

(i) For all $v \in V$ we have $||v|| \geq 0$ and $||v|| = 0$ if and only if $v = 0$.

(ii) For all $v \in V$ and $\lambda \in K$ we have
$$||\lambda \cdot v|| = |\lambda| \cdot ||v|| \,.$$

(iii) For all $v, w \in V$ the strong triangle inequality
$$||v + w|| \leq \max\{||v||, ||w||\}$$
holds.

If in (i) we only require $||v|| \geq 0$ and allow vectors $v \in V - \{0\}$ with $||v|| = 0$ we say that $||.||$ is a **seminorm**.

For any basis $B = (b_1, \ldots, b_n)$ of V and $\vec{a} = (a_1, \ldots, a_n) \in \overline{\mathbb{R}}^n$ we obtain a **diagonalizable** seminorm
$$||\sum_{i=1}^{n} \lambda_i b_i||_{B,\vec{a}} = \max_{i=1,\ldots,n} \{|\lambda_i| e^{-a_i}\}.$$

Two seminorms $||.||_1, ||.||_2$ are said to be **homothetic**, if there is a constant $c > 0$ such $||.||_1 = c \cdot ||.||_2$.

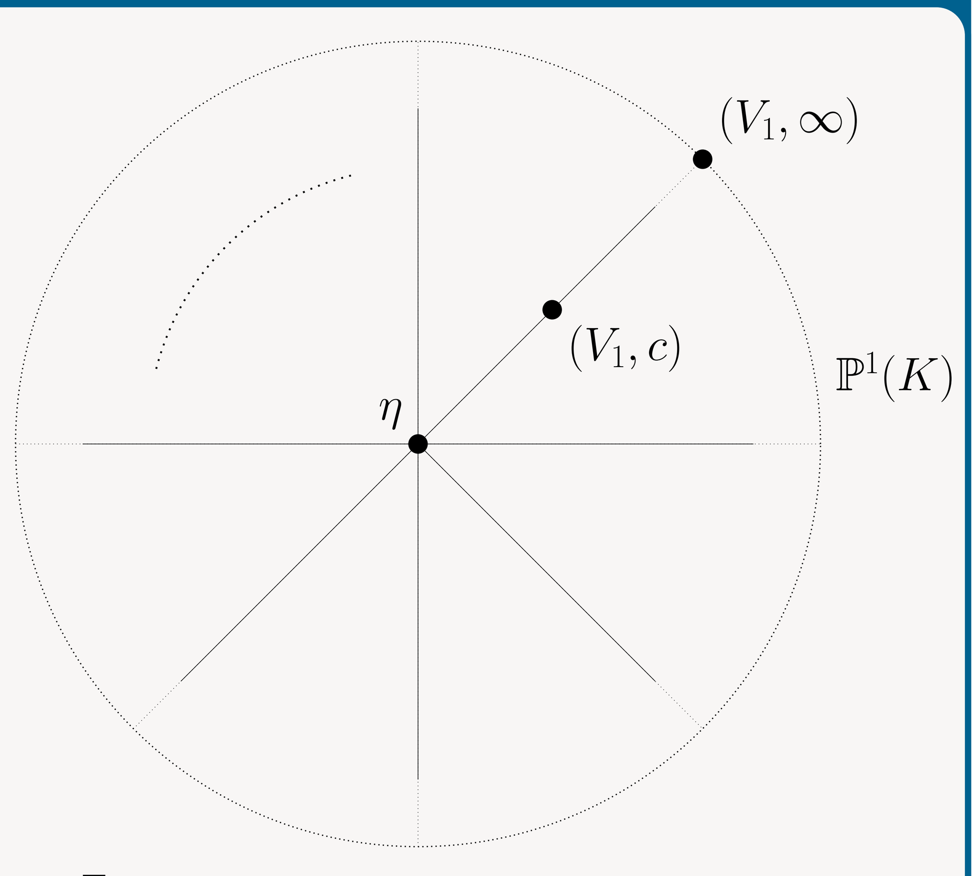
Definition

The (bordified) **Bruhat–Tits building** $\overline{\mathcal{B}}_r(K)$ of PGL_r is defined to be the quotient space of nontrivial diagonalizable seminorms on $(K^{r+1})^*$ by homothety. We equip $\overline{\mathcal{B}}_r(K)$ with the topology of pointwise convergence.

Likewise, we define the (compactified) **Goldman-Iwahori space** $\overline{\mathcal{X}}_r(K)$ to be the quotient space of all nontrivial seminorms on $(K^{r+1})^*$ by homothety.

While $\overline{\mathcal{X}}_r(K)$ turns out to be compact, $\overline{\mathcal{B}}_r(K)$ need not be in general. If K is spherically complete, e.g. discretely valued, we have $\overline{\mathcal{B}}_r(K) = \overline{\mathcal{X}}_r(K)$.

Example: $\overline{\mathcal{B}}_1(K)$



The building $\overline{\mathcal{B}}_1(K)$ of a trivially valued field K. Homothety classes of seminorms correspond to flags of subspaces of $(K^2)^*$ together with a single coordinate $c \in \overline{\mathbb{R}}_{\geq 0}$. A norm in the homothety class corresponding to (V_1, c) has generic value 1, and value e^{-c} on $V_1 \setminus \{0\}$. In the case of $c = \infty$, we have a proper seminorm with kernel V_1. The central point η corresponds to the class of the seminorm that is 1 everywhere except at 0.

Tropicalization

To any quasiprojective K-variety $X \overset{\iota}{\hookrightarrow} \mathbb{P}^n$ one can associate an (embedded) **tropicalization**
$$\mathrm{Trop}(X, \iota) \subset \mathbb{TP}^n,$$
where $\mathbb{TP}^n = \overline{\mathbb{R}}^{n+1} \setminus \{(\infty, \ldots, \infty)\} / \mathbb{R} \cdot (1, \ldots, 1)$ denotes the tropical projective space. The tropicalization is essentially given by taking coordinate-wise valuations.

Limits of linear tropicalizations

For any linear embedding $\iota = [f_0 : \cdots : f_n] : \mathbb{P}^r \hookrightarrow \mathbb{P}^n$ with $f_0, \ldots, f_n \in (K^{r+1})^*$, we have a continuous and surjective map
$$\pi_\iota : \overline{\mathcal{X}}_r(K) \to \mathrm{Trop}(\mathbb{P}^r, \iota)$$
$$[||.||] \mapsto \big[-\log ||f_0|| : \cdots : -\log ||f_n|| \big].$$

Let I be the **category of linear embeddings** $\mathbb{P}^r \hookrightarrow U \subseteq \mathbb{P}^n$, where U is a torus-invariant open subset of $\mathbb{P}^n$, with toric morphisms commuting with the respective embeddings. Toric morphisms functorially yield continuous maps on the respective tropicalizations.

Theorem A [BKKUV23]

The tropicalization maps induce a natural homeomorphism
$$\overline{\mathcal{X}}_r(K) \xrightarrow{\sim} \varprojlim_{\iota \in I} \mathrm{Trop}\left(\mathbb{P}^r, \iota\right).$$

Hence, for K spherically complete we have:

Slogan: The (compactified) building $\overline{\mathcal{B}}_r(K)$ is the limit of all **linear** tropicalizations.

Faithful tropicalization

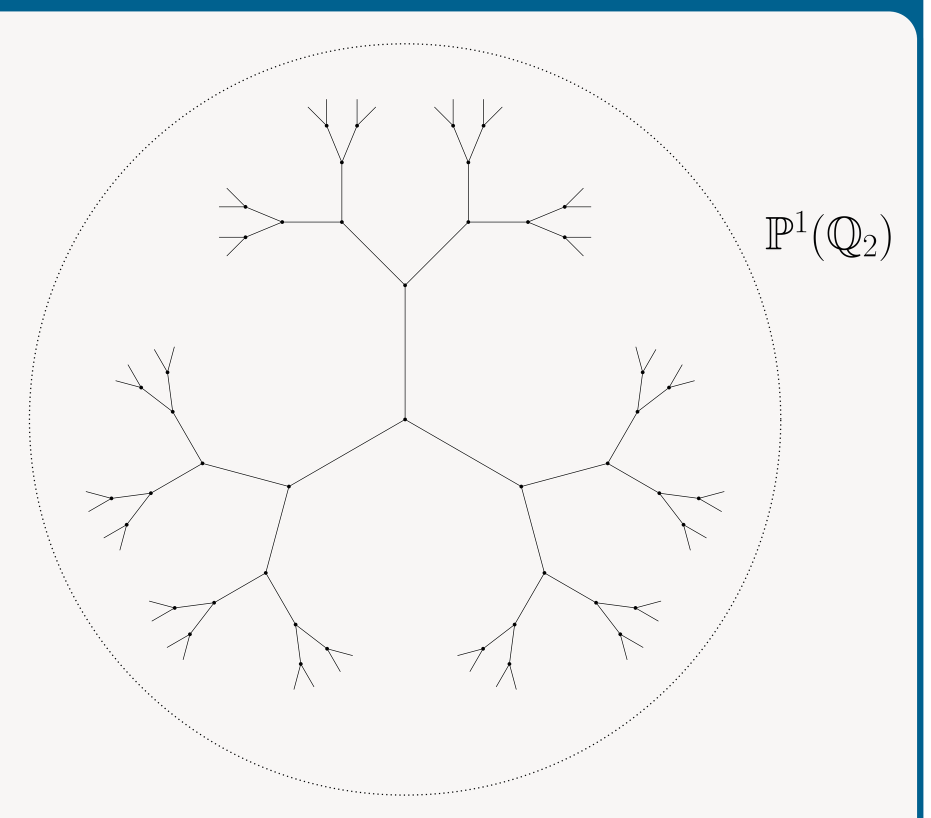
Theorem B [BKKUV23]

Let $\iota : \mathbb{P}^r \hookrightarrow \mathbb{P}^n$ be a linear embedding. Then there is a natural piecewise linear embedding $J : \mathrm{Trop}(\mathbb{P}^r, \iota) \to \overline{\mathcal{B}}_r(K)$ that makes the following diagram commute

$$\begin{array}{ccc} \overline{\mathcal{B}}_r(K) & \xleftarrow{\;J\;} & \mathrm{Trop}\left(\mathbb{P}^r, \iota\right) \\ & {\scriptstyle \pi_\iota} \searrow \quad \swarrow {\scriptstyle \supseteq} & \\ & \mathbb{TP}^n. & \end{array}$$

An **apartment** $\overline{\mathcal{A}}(B)$ of $\overline{\mathcal{B}}_r(K)$ associated to a basis B of $(K^{r+1})^*$ is given by the homothety classes of seminorms diagonalized by B. The map π_ι induces a piecewise linear homeomorphism between the union of apartments $\bigcup \overline{\mathcal{A}}(B)$, where B ranges over the bases of the matroid associated to ι, and the tropicalized linear subspace $\mathrm{Trop}(\mathbb{P}^r, \iota)$.

Example: The Bruhat-Tits tree $\overline{\mathcal{B}}_1(\mathbb{Q}_2)$



The affine building $\mathcal{B}_r(\mathbb{Q}_p)$ has the description as a flag simplicial complex whose vertices correspond to (equivalence classes of) lattices, i.e. free $\mathbb{Z}_p$-submodules of $\mathbb{Q}_p^{r+1}$ of rank $r+1$. Concretely, $\mathcal{B}_1(\mathbb{Q}_p)$ is an infinite trivalent tree. An apartment $\mathcal{A}(B)$ associated to a basis $B = (b_1, b_2)$ of $\mathbb{Q}_p^2$ is an infinite path in the tree which uses all $\mathbb{Z}_p$-lattices with basis $(p^{u_1} b_1, p^{u_2} b_2)$ where $(u_1, u_2) \in \mathbb{Z}^2$. The boundary of $\overline{\mathcal{B}}_1(\mathbb{Q}_2)$ can be identified with $\mathbb{P}^1(\mathbb{Q}_2)$.

Realizable Valuated Matroids

Let $\iota = \big[f_0 : \cdots : f_n\big] : \mathbb{P}^r \hookrightarrow \mathbb{P}^n$ with $f_0, \ldots, f_n \in (K^{r+1})^*$ be a linear embedding. Then we can associate to it a **realizable valuated matroid** v of rank $r+1$ on $[n] = \{0, \ldots, n\}$ by
$$v : \binom{[n]}{r+1} \longrightarrow \overline{\mathbb{R}}$$
$$\{a_0, \ldots, a_r\} \longmapsto \mathrm{val}\left(\det\big[f_{a_0} \cdots f_{a_r}\big]\right).$$

To any valuated matroid v on $[n]$ one can associate a **tropical linear space** $\mathcal{L}(v) \subset \mathbb{TP}^n$ which is a polyhedral complex $\mathcal{L}(v)$ of pure dimension r. By [Spe08, Proposition 4.2] we have
$$\mathcal{L}(v) = \mathrm{Trop}(\mathbb{P}^r, \iota).$$

Infinite Tropicalization

Let v be a valuated matroid of rank $r+1$ on a possibly infinite ground set E. We can associate to it the tropical linear space $\mathcal{L}(v) \subset \mathbb{TP}^E$ defined as the set of $(u_e)_{e \in E} \in \mathbb{TP}^E$ such that for any $\tau \in \binom{E}{r+2}$ the minimum $\min_{e \in \tau} v(\tau \setminus \{e\}) + u_e$ is attained at least twice. The tropical linear space can be written as an inverse limit of all full rank restrictions of v to finite subsets of E, hence it comes naturally equipped with a limit topology.

Theorem C [BKKUV23]

For the **universal realizable valuated matroid**
$$w_{\mathrm{univ}} : \binom{E = K^{r+1}}{r+1} \longrightarrow \overline{\mathbb{R}}$$
induced by the permutation-invariant map $\mathrm{val} \circ \det : K^{(r+1) \times (r+1)} \to \overline{\mathbb{R}}$ we have
$$\overline{\mathcal{X}}_r(K) = \mathcal{L}(w_{\mathrm{univ}}).$$

Hence, for K spherically complete we have:

Slogan: The (compactified) building $\overline{\mathcal{B}}_r(K)$ is the tropical linear space associated to the universal realizable valuated matroid w_{univ}.

Analytification

To an algebraic variety X over K one can functorially associate a larger "analytic" space, the **Berkovich analytification** $X^{\mathrm{an}} \supset X(K)$. If X is quasiprojective, X^{an} maps continuously and surjectively onto any tropicalization $\mathrm{Trop}(X, \iota)$, where $\iota : X \to \mathbb{P}^n$ is an embedding of X into a projective space.

[Pay09]: The analytification X^{an} of a quasiprojective algebraic variety X over K is the inverse limit of **all** tropicalizations with respect to all the embeddings in toric varieties.

There is a natural continuous surjective restricition map $\tau : \mathbb{P}^{r,\mathrm{an}} \to \overline{\mathcal{X}}_r(K)$ that factors the tropicalization map. If K is a local field, there is a natural embedding of $\overline{\mathcal{B}}_r(K) = \overline{\mathcal{X}}_r(K)$ into $(\mathbb{P}^r)^{\mathrm{an}}$. Theorem A then tells us that the collection of all linear re-embeddings $\mathbb{P}^r \hookrightarrow \mathbb{P}^n$ recovers exactly $\overline{\mathcal{B}}_r(K) = \overline{\mathcal{X}}_r(K)$. The main result of [Pay09], on the other hand, tells us that, once we also allow non-linear algebraic re-embeddings of $\mathbb{P}^r$ into suitable toric varieties, we recover $(\mathbb{P}^r)^{\mathrm{an}}$.

References:

[BKKUV23] L. Battistella, K. Kuehn, A. Kuhrs, M. Ulirsch, and A. Vargas. *Buildings, valuated matroids, and tropical linear spaces.* 2023. arXiv: 2304.09146 [math.AG].

[Pay09] S. Payne. "Analytification is the limit of all tropicalizations". In: *Math. Res. Lett.* 16.3 (2009), pp. 543–556. ISSN: 1073-2780.

[Spe08] D. E. Speyer. "Tropical linear spaces". In: *SIAM J. Discrete Math.* 22.4 (2008), pp. 1527–1558. ISSN: 0895-4801.