Functorial Tropicalization of Logarithmic Schemes over Valuation Rings

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Abstract Polyhedra

Let $\Gamma \subseteq \mathbb{R}$ be a subgroup, Δ be a topological space and Aff a group of real-valued continuous functions on Δ such that

- For all $\gamma \in \Gamma$ the constant function $\Delta \to \mathbb{R}, x \mapsto \gamma$ lies in Aff.
- The quotient Aff/Γ is finitely generated.

The setting one should think about is that Δ is a subset of a vector space and Aff is the group of Γ-affine linear functions on the vector space restricted to Δ.

We define the lattice $N = \mathrm{Hom}(\mathrm{Aff}/\Gamma, \mathbb{Z})$ and $N_{\mathbb{R}} = N \otimes \mathbb{R} = \mathrm{Hom}(\mathrm{Aff}/\Gamma, \mathbb{R})$. Then

$$\mathrm{Hom}_\Gamma(\mathrm{Aff}, \mathbb{R}) = \{x : \mathrm{Aff} \to \mathbb{R} \mid x|_\Gamma = \mathrm{id}\}$$

is an $N_{\mathbb{R}}$-torsor. Each $\varphi \in \mathrm{Aff}$ induces a function $\mathrm{Hom}_\Gamma(\mathrm{Aff}, \mathbb{R}) \to \mathbb{R}$ and cuts out the half plane $\varphi^+ = \{x \in \mathrm{Hom}_\Gamma(\mathrm{Aff}, \mathbb{R}) \mid \varphi(x) \geq 0\}$.

Definition

We call the pair (Δ, Aff) a **Γ-rational (abstract) polyhedron**, if the product map $\Delta \to \mathrm{Hom}(\mathrm{Aff}, \mathbb{R})$ maps Δ homeomorphically onto a Γ-rational polyhedron in the $N_{\mathbb{R}}$-torsor $\mathrm{Hom}_\Gamma(\mathrm{Aff}, \mathbb{R})$, which means that it is cut out by finitely many half-planes $\Delta = \bigcap_i \varphi_i^+$ with $\varphi_i \in \mathrm{Aff}$.

A **morphism** of Γ-rational polyhedra is a continuous map of the underlying topological spaces such that the pull-back on the respective function groups is a well-defined group homomorphism.
Zero sets of the φ_i cutting out Δ are called **faces**. Faces themselves are Γ-rational polyhedra. We call Δ **pointed**, if it has a zero-dimensional face.

Example: Let $\Gamma = \mathbb{Z}$. Consider the Γ-rational polyhedron $\Delta = [0, 1]$. Then

$$\mathrm{Aff} = \{mx + b \mid m \in \mathbb{Z}, b \in \Gamma\}.$$

Let $\Delta' = [\frac{1}{2}, \frac{3}{2}]$, then their groups of affine functions are isomorphic, but the polyhedra are not.

Monoids: a quick reminder

A (commutative) **monoid** is a set P with an associative and commutative binary operation $+$ and a neutral element 0, e.g., $\Gamma^+ = \Gamma \cap \mathbb{R}_{\geq 0}$. To every monoid we can associate its **Grothendieck group** $P^{\mathrm{gp}} := \{p - q \mid p, q \in P\}$. We call a monoid P

- **integral**, if the inclusion morphism $P \to P^{\mathrm{gp}}$ is injective.
- **saturated**, if it is integral and for any natural number $n \in \mathbb{N}$ and $p \in P^{\mathrm{gp}}$ we have that $np \in P$ implies $p \in P$.
- **torsion-free**, if $np = 0$ for some $n \in \mathbb{N}_{>0}, p \in P$ already implies $p = 0$.
- **sharp**, if 0 is the only invertible element in P.
- **divisible**, if for all $p \in P$ and $n \in \mathbb{N}$ there exists an element $q \in P$ with $nq = p$.

From now on we will assume all monoids to be integral and saturated. For any torsion-free monoid P we can assign a divisible monoid $D(P) := \{\frac{p}{n} \mid p \in P, n \in \mathbb{N}_{\geq 1}\}$ containing P. Denote by $Q \oplus_P Q'$ the **fibered sum** of two morphisms $P \to Q$ and $P \to Q'$ in the category of integral and saturated monoids.

Remark: Let Δ be a Γ-rational polyhedron. Then the monoid of non-negative affine functions, denoted by Aff^+, is integral, saturated, and sharp. We have $(\mathrm{Aff}^+)^{\mathrm{gp}} = \mathrm{Aff}$ if and only if Δ is pointed.

Divisible Finiteness

Goal: Find sufficient and necessary conditions such that for an inclusion of monoids $\Gamma^+ \hookrightarrow P$ there is a polyhedron (Δ, Aff) with $P \simeq \mathrm{Aff}^+$ under Γ^+. We need a "correct" finiteness condition on P. A first idea could be the following

Definition

Let $f : P \to Q$ be a morphism of monoids. We call f **finite**, if for some $l \in \mathbb{N}$ there exists a surjective morphism of monoids $P \oplus \mathbb{N}^l \to Q$ restricting to f on P.

Problem: Let $\Gamma = \mathbb{Z} + \pi\mathbb{Z}$ and $\Delta = \{\frac{1}{2}\} \subset \mathbb{R}$. Then Δ is a Γ-rational polyhedron, but the morphism

$$\Gamma^+ \hookrightarrow \mathrm{Aff}^+ = \left\{\frac{a}{2} + b\pi \mid a, b \in \mathbb{Z}, \frac{a}{2} + b\pi \geq 0\right\}$$

is not finite. This can be fixed in the following way:

Definition

We call f **divisibly finite**, if the induced morphism $D(P) \to Q \oplus_P D(P)$ is finite and f^{gp} is finite.

Equivalence of Polyhedra and Monoids

Let Γ^+-*DFMon* denote the category whose objects are injective, divisibly finite monoid morphisms $\Gamma^+ \hookrightarrow P$, where P is integral, saturated and sharp.

Theorem

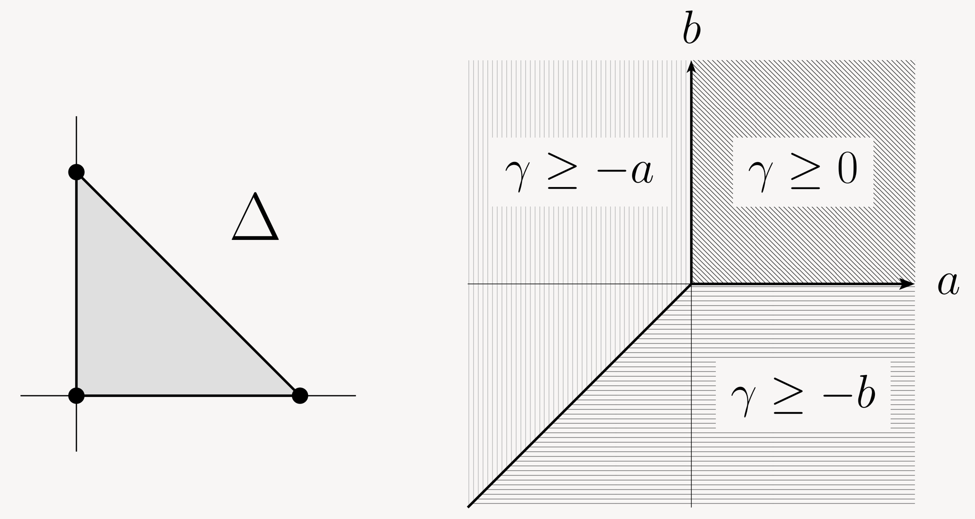
There is a **categorical equivalence** between Γ^+-*DFMon* and the category of Γ-rational pointed polyhedra:

$$[\Gamma^+ \hookrightarrow P] \longmapsto (\mathrm{Hom}_{\Gamma^+}(P, \mathbb{R}_{\geq 0}), P^{\mathrm{gp}})$$
$$[\Gamma^+ \hookrightarrow \mathrm{Aff}^+] \longleftarrow (\Delta, \mathrm{Aff}).$$

This equivalence of categories above allows us to **extend** polyhedra: if $\Delta = \mathrm{Hom}_{\Gamma^+}(P, \mathbb{R}_{\geq 0})$ define $\overline{\Delta} = \mathrm{Hom}_{\Gamma^+}(P, \overline{\mathbb{R}}_{\geq 0})$.

Example: Let $\Gamma \subset \mathbb{R}$ be arbitrary with $\mathbb{Z} \subset \Gamma$ and let P be the monoid under Γ^+ generated by a, b, c with relations $a + b + c = 1$. Then $P^{\mathrm{gp}} \simeq \Gamma \oplus \langle a, b \rangle_{\mathbb{Z}}$ and one can check that P is both integral and saturated. We can identify $\mathrm{Hom}_\Gamma(P^{\mathrm{gp}}, \mathbb{R})$ with $\mathbb{R}^2$, as we have no relations in P^{gp} on a, b. For any point $x \in \Delta = \mathrm{Hom}_{\Gamma^+}(P, \mathbb{R}_{\geq 0})$ we thus have $x(a) \geq 0, x(b) \geq 0$, and $x(c) = x(1) - x(a) - x(b) \geq 0$. We thus can identify Δ with the triangle

$$\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq 1\}.$$



The right hand side sketches the monoid P, which we identify with the subset of $\Gamma \oplus \langle a, b \rangle_{\mathbb{Z}} \subset \mathbb{R}^3$.

Tropicalization

Let K be a non-archimedean field with value group $\Gamma \subset \mathbb{R}$ and R its valuation ring. Let $\mathcal{X}$ be a scheme of locally finite type over R with generic fiber $X = \mathcal{X} \times_R K$.

The Raynaud Generic Fiber

First, let $\mathcal{X} = \mathrm{Spec}\, A$ for a finite type algebra A. Then we define its **Raynaud generic fiber** $\mathcal{X}^\circ$ as the space of all seminorms $|.| : A \to \mathbb{R}_{\geq 0}$ extending the norm on R with $|.| \leq 1$. There is a canonical morphism $\mathcal{X}^\circ \to X^{\mathrm{an}}$ into the Berkovich analytification after basechanging to K.
In the general case, we can obtain a Raynaud generic fiber $\mathcal{X}^\circ$ by gluing.

Equip $\mathrm{Spec}\, R$ with the divisorial log structure of its closed point, assume that $\mathcal{X} = \mathrm{Spec}\, A$ is affine and that we have a log structure $M_{\mathcal{X}}$ such that

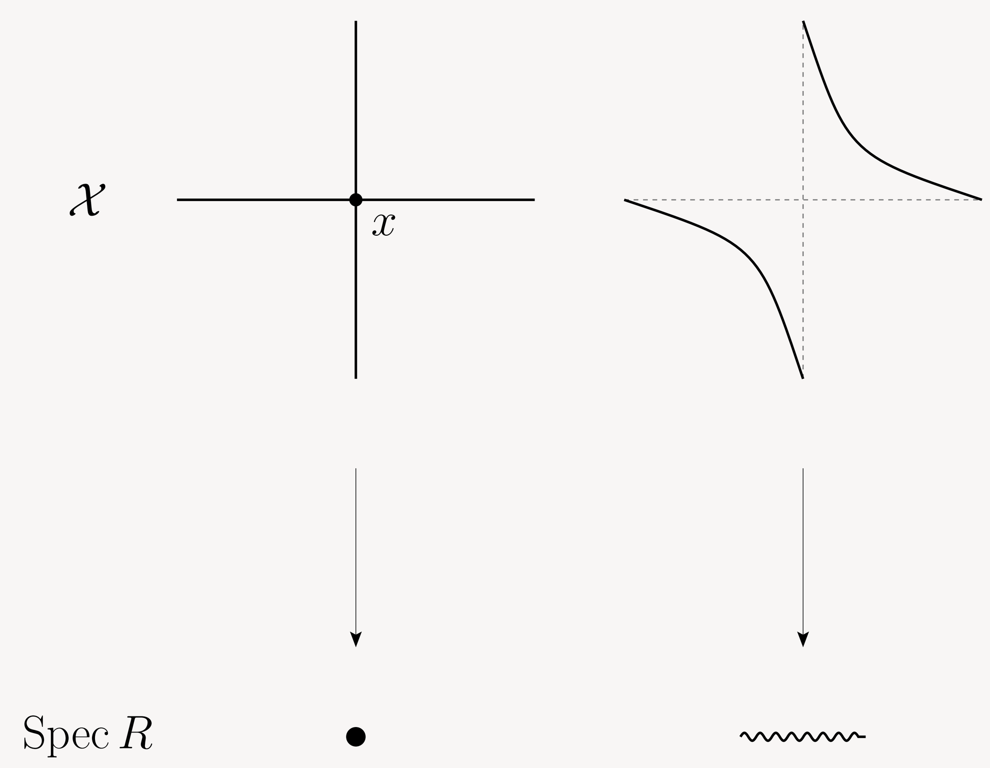
- $M_{\mathcal{X}}$ is associated to a chart $\alpha : P \to \mathcal{O}_{\mathcal{X}}$,
- $P = \overline{M}_{\mathcal{X}}(\mathcal{X})$, and
- $\{x \in \mathcal{X} \mid P = \overline{M}_{\mathcal{X},x}\}$ is non-empty and connected.

Given a morphism of log schemes $\mathcal{X} \to \mathrm{Spec}\, R$ such that the induced morphism $\Gamma^+ \to P$ is in Γ^+-*DFMon*, we define the **tropicalization map**

$$\mathrm{trop} : \mathcal{X}^\circ \to \overline{\Delta} = \mathrm{Hom}_{\Gamma^+}(P, \overline{\mathbb{R}}_{\geq 0})$$
$$|.| \mapsto [p \mapsto -\log |\alpha(p)|].$$

Example: Let $\mathbb{Z} \subset \Gamma$ and assume that there is a section of the valuation $s : \Gamma^+ \to R$. Denote by $t \in R$ the image of 1 under this section. Consider $\mathcal{X} = \mathrm{Spec}\, A$, where $A = R[x, y]/xy - t$ and let $P = \Gamma^+ \oplus_{\mathbb{N}} \mathbb{N}^2$ via the diagonal morphism $\mathbb{N} \to \mathbb{N}^2$. We equip $\mathcal{X}$ with the log structure $M_{\mathcal{X}}$ associated to $P \to R, (\gamma, (a, b)) \mapsto s(\gamma) x^a y^b$. Then there is a unique point $x \in \mathcal{X}$ with $P = \overline{M}_{\mathcal{X},x}$ and the image of the tropicalization map is the polyhedron

$$[0, 1] \simeq \mathrm{Hom}_{\Gamma^+}(P, \overline{\mathbb{R}}_{\geq 0}).$$



Outlook

The following is currently in progress.

- **Glue together** the tropicalization maps to a (generalized) polyhedral complex when the log structure on $\mathcal{X}$ is given by étale local charts of the form above.
- Show **functoriality** and continuity of the the tropicalization map.
- Explore applications to **polystable degenerations of moduli spaces**.
- Explore applications to toroidal **bordifications of reductive algebraic groups** over valuation rings.
- Interpret the tropicalization as a **skeleton** of the Berkovich space.