Mini course: Curvature tensors and measures of singular spaces

Geometry beyond Riemann: Curvature and Rigidity
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Abstract. The following is a slightly extended version of the mini course I gave at the Erwin Schrödinger institute in the summer school of the thematic program "Geometry beyond Riemann: Curvature and Rigidity", organized by Ivan Izmestiev, Athanase Papadopoulos, Marc Troyanov and Sumio Yamada.

We show how integral geometric methods, as introduced by Chern, Weyl, Alesker and others, can be used to define very natural curvature quantities associated to certain singular, but tame sets.

In the first lecture, we review the background from integral geometry in flat spaces. The main point is the notion of valuation on convex bodies. The second lecture generalizes this notion to smooth manifolds. We introduce the normal cycle of a sufficiently tame set and introduce the notion of smooth valuations and curvature measures. In the third lecture we introduce the intrinsic volumes (also called Lipschitz-Killing curvatures) on Riemannian manifolds. We give two different approaches: the one due to Allendoerfer-Weil, Weyl and Alesker, based on isometric embeddings, and the one due to Chern and Fu-Wannerer based on the Cartan formalism.

In the last lecture, we use these results in order to associate certain curvature measures to singular sets, in particular a scalar curvature measure. The classical variational formula for the Einstein-Hilbert action can be generalized to the singular setting and allows to define a distributional Einstein tensor of singular spaces.

Contents

CHAPTER 1

Lecture I: Translation invariant valuations

In this lecture, we introduce the notion of valuation, which will be of crucial importance in the remaining lectures. We state the most important theorems concerning translation invariant valuations. Then we extend this theory to certain classes of singular, but tame, sets.

1. Valuations

Let V be an n-dimensional real vector space. Let $\mathcal{K}(V)$ denote the set of non-empty compact convex bodies. If we fix a euclidean scalar product on V with unit ball B, then the Hausdorff distance is defined by

$$d(K,L)=\inf\{\epsilon : K \subset L+\epsilon B, L \subset K+\epsilon B\}, K,L \in \mathcal{K}(V).$$

The distance depends on the choice of B, but the corresponding topology on $\mathcal{K}(V)$ does not.

Definition 1.1. *A valuation is a functional $\phi : \mathcal{K}(V) \to \mathbb{R}$ such that*

$$\phi(K \cup L)+\phi(K \cap L)=\phi(K)+\phi(L)$$

whenever $K, L, K \cup L \in \mathcal{K}(V)$.

Remarks: Many variations are possible.

- Sometimes $\mathbb{C}$-valued valuations are considered. This is not a big difference. Sometimes valuations with values in some abelian semi-group are considered, for instance the space of measures on V (this will be important for curvature measures), the space of measures on the unit sphere (important for area measures), or the space of compact convex bodies with the Minkowski addition (Minkowski valuations).
- Instead of taking $\mathcal{K}(V)$, one can consider valuations on a smaller set like the set of all polytopes, or the set of integer polytopes. The first case was considered in Max Dehn's solution to Hilbert's 3rd problem. The second case is relevant in combinatorics (Ehrhart theory).
- Valuations on function spaces, where union and intersection are replaced by minimum and maximum, are a very active topic (in particular in Vienna).

Some properties that valuations may have:

- Continuity: this is with respect to the Hausdorff metric defined above.
- Translation invariance: $\phi(K+v)=\phi(K)$ for all $v \in V$.
- Rotation invariance: $\phi(gK)=\phi(K)$ for all $g \in \mathrm{SO}(n)$ (here we assume that V is euclidean).
- Homogeneity: $\phi(tK)=t^\lambda \phi(K), t \geq 0$ for some λ.
- Parity: $\mu(-K)=\epsilon\mu(K)$. $\epsilon=1$ means even, $\epsilon=-1$ means odd valuation.

Let Val denote the vector space of continuous translation invariant valuations on $\mathcal{K}(V)$. This is an infinite-dimensional space. Let Val_k^ϵ denote the subspace of k-homogeneous valuations of parity ϵ.

Theorem 1.2 (McMullen). *There is a decomposition*

$$\mathrm{Val}=\bigoplus_{k=0,\ldots,n,\epsilon=\pm} \mathrm{Val}_k^\epsilon .$$

The spaces $\mathrm{Val}_0 \cong \mathbb{R}\cdot\chi$ *and* $\mathrm{Val}_n \cong \mathbb{R}\cdot \mathrm{vol}$ *are* 1*-dimensional.*

An example for an element $\phi \in \mathrm{Val}$ is given by the valuation

$$\phi(K):=\mathrm{vol}(K+A),$$

where A is a fixed compact convex body. Using mixed volumes, the decomposition of ϕ into homogeneous elements is given by

$$\phi(K)=\mathrm{vol}(K+A)=V(K+A,\ldots,K+A)=\sum_k \binom{n}{k} V(K[k],A[n-k]).$$

As a corollary, one can prove that

$$\|\phi\|:=\sup\{|\phi(K)|: K \subset B\}$$

defines a norm on Val, where B is any compact convex set with non-empty interior. The normed space $(\mathrm{Val},\|\bullet\|)$ is a Banach space, and another choice of B results in an equivalent Banach space.

2. The intrinsic volumes

Let ω_i denote the volume of the i-dimensional unit ball.

Theorem 2.1 (Steiner [**21**]). *Let V be a euclidean vector space of dimension n and $K \in \mathcal{K}(V)$. Then there are real numbers $\mu_0(K),\ldots,\mu_n(K)$ such that the volume of the r-tube around K is given by*

$$\mathrm{vol}(K+rB)=\sum_{k=0}^n \mu_k(K)\omega_{n-k}r^{n-k}.$$

The $\mu_i, i=0,\ldots,n$ are continuous valuations, called intrinsic volumes. Moreover, if $V \subset W$ is an isometric embedding, then $\mu_k^W|_V=\mu_k^V$ (with the convention that $\mu_k^V=0$ if $k>\dim V$).

Examples:

- $\mu_0(K) = 1$. For reasons that will become clear afterwards, μ_0 is called Euler characteristic and denoted by χ.
- $\mu_n(K) = \operatorname{vol}(K)$.
- $\mu_{n-1}(K) = \frac{1}{2}\operatorname{vol}_{n-1}(\partial K)$.
- If the boundary of K is smooth, then

$$\mu_k(K) = \frac{1}{(n-k)\omega_{n-k}} \int_{\partial K} \sigma_{n-1-k}(x)dx,$$

where $\sigma_{n-1-k}(x)$ is the $(n-1-k)$th elementary symmetric function of the principal curvatures at x.

Let $\operatorname{Val}^{\mathrm{SO}(n)}$ be the subspace of rotation invariant elements. Clearly $\mu_k \in \operatorname{Val}^{\mathrm{SO}(n)}$.

THEOREM 2.2 (Hadwiger [**18**]). *The vector space* $\operatorname{Val}^{\mathrm{SO}(n)}$ *is spanned by* $\mu_0, \ldots, \mu_n$*, in particular it is of dimension* $(n+1)$*.*

We will also need a local version of Steiner's formula. Let $U \subset V$ be a Borel subset. Inside the r-tube $K + rB$, consider only those points whose foot points on K belong to U. Then

$$\operatorname{vol}((K + rB) \cap \pi^{-1}(U)) = \sum_{k=0}^{n} \Phi_k(K, U)\omega_{n-k}r^{n-k},$$

and the coefficient $\Phi_k(K, U)$ is called Lipschitz-Killing curvature measure. For fixed K, the map $U \mapsto \Phi_k(K, U)$ is a measure. For fixed U, the map $K \mapsto \Phi_k(K, U)$ is a (non-continuous) valuation.

3. Alesker's irreducibilty theorem

The group $\mathrm{GL}(n)$ acts naturally on Val by

$$g\phi(K) = \phi(g^{-1}K).$$

It is obvious that the subspaces $\operatorname{Val}_k^\epsilon$ are invariant under this action. A very deep and important theorem by Alesker shows that these spaces can not be further decomposed:

THEOREM 3.1 (Alesker's irreducibility theorem). *The spaces* $\operatorname{Val}_k^\epsilon, k = 0, \ldots, n, \epsilon = \pm$ *are irreducible, i.e. every non-trivial* $\mathrm{GL}(n)$*-invariant subspace of* $\operatorname{Val}_k^\epsilon$ *is dense.*

DEFINITION 3.2. *A valuation* $\phi \in \operatorname{Val}$ *is called smooth if it is a finite linear combination of valuations of the form* $\phi(K) = \operatorname{vol}(K + A)$ *with* A *a smooth convex body with positive curvature.*

By Alesker's irreducibility theorem, smooth valuations form a dense subspace, denoted by $\operatorname{Val}^\infty$ in Val.

Remark: there are equivalent definitions. The original one, due to Alesker, is the following: ϕ is smooth, if the map $g \mapsto g\phi$ is a smooth map from the Lie group $\mathrm{GL}(n)$ to the Banach space $(\mathrm{Val}, \|\bullet\|)$. The equivalence of the two definitions was independently shown by Knörr and van Handel. Later on, we will see another equivalent definition.

4. Algebraic structures

THEOREM 4.1 (B'-Fu). *There is a (commutative, associative) convolution product on the space* Val^∞ *of smooth valuations that is defined by bilinearity and the following property: if* $\phi_i(K) = \mathrm{vol}(K + A_i), i = 1, 2$ *with* A_i *a smooth convex body with positive curvature, then*

$$\phi_1 * \phi_2(K) = \mathrm{vol}(K + A_1 + A_2).$$

The neutral element is the volume valuation. The convolution is compatible with McMullen's grading in the sense that if $\phi_i \in \mathrm{Val}^\infty_{k_i}$*, then* $\phi_1 * \phi_2 \in \mathrm{Val}^\infty_{k_1+k_2-n}$*. It satisfies a version of Poincaré duality: if* $\phi \neq 0 \in \mathrm{Val}^\infty_k$*, we find* $\psi \in \mathrm{Val}^\infty_{n-k}$ *such that* $\phi * \psi \neq 0$*.*

THEOREM 4.2 (Alesker). *There is a product on the space* Val^∞ *of smooth valuations that is defined by bilinearity and the following property: if* $\phi_i(K) = \mathrm{vol}(K + A_i), i = 1, 2$ *with* A_i *a smooth convex body with positive curvature, then*

$$\phi_1 \cdot \phi_2(K) = \mathrm{vol}_{2n}(\Delta K + A_1 \times A_2).$$

Here $\Delta : V \to V \times V, v \mapsto (v, v)$ *is the diagonal embedding and* vol_{2n} *is the product measure on* $V \times V$*. The neutral element is the Euler characteristic. The convolution is compatible with McMullen's grading in the sense that if* $\phi_i \in \mathrm{Val}^\infty_{k_i}$*, then* $\phi_1 \cdot \phi_2 \in \mathrm{Val}^\infty_{k_1+k_2}$*. It satisfies a version of Poincaré duality: if* $\phi \neq 0 \in \mathrm{Val}^\infty_k$*, we find* $\psi \in \mathrm{Val}^\infty_{n-k}$ *such that* $\phi \cdot \psi \neq 0$*.*

In the second theorem, it is not obvious but true that the resulting valuation is smooth again: the singularities of ΔK and $A_1 \times A_2$ are in some sense transversal.

THEOREM 4.3 (Alesker). *There is a Fourier type transform* $\mathbb{F} : \mathrm{Val}^\infty \to \mathrm{Val}^\infty$ *such that*

$$\mathbb{F}(\phi_1 \cdot \phi_2) = \mathbb{F}\phi_1 * \mathbb{F}\phi_2.$$

Moreover, the following Plancherel type formula holds:

$$\mathbb{F}^2\phi(K) = \phi(-K).$$

5. Some integral geometric formulas

Some integral geometric formulas follow easily from Hadwiger's classification theorem. Sometimes they are taken as definitions of the intrinsic volumes, and then Steiner's formula becomes a theorem.

THEOREM 5.1 (Crofton formula). *Let* $k + l \leq n$*. Then*

$$\int_{\overline{\mathrm{Gr}}_{n-k}(V)} \mu_l(K \cap \bar{E}) d\bar{\phi}_{n-k}(\bar{E}) = \begin{bmatrix} k+l \\ k \end{bmatrix} \begin{bmatrix} n \\ k \end{bmatrix}^{-1} \mu_{k+l}(K),$$

where $\begin{bmatrix} n \\ k \end{bmatrix} := \binom{n}{k}\frac{\omega_n}{\omega_k\omega_{n-k}}$ *and* $\overline{\mathrm{Gr}}_{n-k}(V)$ *is the Grassmann manifold of affine* $(n-k)$*-planes in* V*, endowed with a conveniently normalized invariant measure.*

THEOREM 5.2 (Kubota's formula). *For* $0 \leq k \leq l \leq n$ *we have*

$$\int_{\mathrm{Gr}_l(V)} \mu_k(\pi_E K) d\phi_l(E) = \begin{bmatrix} n-k \\ l-k \end{bmatrix} \begin{bmatrix} n \\ l \end{bmatrix}^{-1} \mu_k(K).$$

THEOREM 5.3 (Kinematic formulas). *For* $K, L \in \mathcal{K}(V)$ *and* $m = 0, \ldots, n$ *we have*

$$\int_{\overline{\mathrm{SO}}(n)} \mu_m(K \cap \bar{g}L) d\bar{g} = \sum_{k+l=n+m} \begin{bmatrix} k \\ m \end{bmatrix} \begin{bmatrix} n \\ l \end{bmatrix}^{-1} \mu_k(K)\mu_l(L).$$

Here $\overline{\mathrm{SO}}(n)$ *is the group generated by translations and rotations, endowed with the product measure.*

CHAPTER 2

Lecture II: Smooth valuations and valuations on manifolds

1. The normal cycle construction

Let us assume for simplicity that V is a euclidean vector space.

Definition 1.1. *The normal cycle* $\mathrm{Nor}(K)$ *of* K *is the set of pairs* $(x, v) \in V \times S^{n-1}$ *with* $x \in \partial K$ *and* v *a unit outer normal vector. It is a Lipschitz submanifold of dimension* $(n-1)$ *in* $V \times S^{n-1}$.

We think of $\mathrm{Nor}(K)$ as an $(n-1)$-dimensional current, i.e. a functional on the space $\Omega^{n-1}(V \times S^{n-1})$ of differential $(n-1)$-forms on $V \times S^{n-1}$.

Lemma 1.2. *The normal cycle has the following properties:*

(1) $\mathrm{Nor}(K)$ *is an integer multiplicity rectifiable current.*
(2) $\mathrm{Nor}(K)$ *is indeed a cycle, i.e. vanishes on exact forms.*
(3) $\mathrm{Nor}(K)$ *vanishes on multiples of* α *and on multiples of* $d\alpha$.
(4) *If* K, L *are compact convex bodies such that* $K \cup L$ *is convex as well, then*

$$\mathrm{Nor}(K \cup L) + \mathrm{Nor}(K \cap L) = \mathrm{Nor}(K) + \mathrm{Nor}(L).$$

Definition 1.3. *A functional* $\mu : \mathcal{K}(V) \to \mathbb{R}$ *of the form*

$$\mu(K) = \int_K \phi + \int_{\mathrm{Nor}\, K} \omega, \quad \phi \in \Omega^n(V), \omega \in \Omega^{n-1}(V \times S^{n-1})$$

is called a smooth valuation.

From the last property in Lemma 1.2 one sees that a smooth valuation is indeed a valuation. Moreover, a translation invariant valuation is smooth in the sense of Definition 3.2 if and only if it is smooth in the sense of Definition 1.3, and in this case ϕ and ω can be chosen translation invariant. These facts are not easy to prove.

2. Extension to non-convex sets

A very important fact in the theory of valuations is that *smooth* valuations can be extended to certain non-convex sets.

Federer has introduced the sets of positive reach.

Definition 2.1. *Let* V *be a euclidean vector space of finite dimension. A set* $P \subset V$ *is called set of positive reach if there is some* $r > 0$ *such that* *each point $x \in V$ whose distance to P is less than r has a unique foot point in P. The supremum over all such r is called the reach of P.*

Although the reach of P depends on the choice of the euclidean scalar product on V, the property of being of positive reach does not. More generally, one can define this notion on a riemannian manifold (or even in any metric space). A theorem by Bangert [**7**] says that the image of a set of positive reach under a diffeomorphism is again of positive reach, in particular the notion "set of positive reach" is independent of the riemannian metric.

A manifold is locally modelled on $\mathbb{R}^n$: the charts are maps $\phi : U \to \phi(U)$, where $\phi(U) \subset \mathbb{R}^n$ is an open set. Moreover, the coordinate change is smooth. A manifold with boundary is locally modelled over the half space $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$. The boundary of M is then the set of points that get mapped to the boundary of $\mathbb{R}^n_+$, and one can show that this property is independent of the coordinate chart. A manifold with corners is locally modelled over the octant $\{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1, \ldots, x_n \geq 0\}$. It admits a stratification by types, where the type of a point is determined by the number of zero coordinates. Finally, a differentiable polyhedron is locally modelled over a polyhedron in $\mathbb{R}^n$. It also admits a stratification by types, where the type is determined by the dimension of the face of the polyhedron.

DEFINITION 2.2. *A semialgebraic subset of $\mathbb{R}^n$ is a finite union of sets of the form*

$$\{x \in \mathbb{R}^n | f_1(x) = \ldots = f_l(x) = 0, g_1(x) > 0, \ldots, g_m(x) > 0\}$$

where f_i, g_j are polynomials.

Example: the semialgebraic subsets of $\mathbb{R}$ are precisely the finite union of intervals.

By definition, finite unions of semialgebraic sets are semialgebraic and it is easy to see that finite intersections are semialgebraic again. The complement of a semialgebraic set is again semialgebraic: it is enough to check this for a set of the form

$$\{x \in \mathbb{R}^n | f_1(x) = \ldots = f_l(x) = 0, g_1(x) > 0, \ldots, g_m(x) > 0\},$$

whose complement is

$$\bigcup_{i=1}^{l} \{f_i(x) > 0\} \cup \{-f_i(x) > 0\} \cup \bigcup_{j=1}^{m} \{-g_j(x) > 0\} \cup \{g_j(x) = 0\}.$$

Semialgebraic sets are in general not of positive reach, and convex sets or sets of positive reach are in general not semialgebraic (however, there are important classes of sets that are convex and semialgebraic, like spectahedra).

THEOREM 2.3 (Tarski-Seidenberg). *Let $\pi : \mathbb{R}^n \to \mathbb{R}^k$ be the projection onto the first k coordinates. If $X \subset \mathbb{R}^n$ is semialgebraic, then $\pi(X) \subset \mathbb{R}^k$ is semialgebraic.*

Definition 2.4. *A semianalytic subset of $\mathbb{R}^n$ is a finite union of sets of the form*

$$\{x \in \mathbb{R}^n | f_1(x) = \ldots = f_l(x) = 0, g_1(x) > 0, \ldots, g_m(x) > 0\}$$

where f_i, g_j are real analytic functions (meaning that in a neighborhood of each point, they agree with their Taylor expansion).

This class is closed under finite unions, finite intersections, and complement. However, it is not closed under projection.

Definition 2.5. *A set $X \subset \mathbb{R}^n$ is subanalytic if there is some $N > n$ and a semianalytic set $Y \subset \mathbb{R}^N$ such that $X = \pi(Y)$.*

It is obvious that a semialgebraic set is semianalytic, and a semianalytic set is subanalytic. See [**11**] for more information on semianalytic and subanalytic sets.

A generalization is given by o-minimal structures. We refer to [**23**] or [**24**].

Theorem 2.6. *Let X be semialgebraic. Then*

(1) *X admits a Nash stratification, that is $X = \bigcup S_\alpha$, where each S_α is a semialgebraic subvariety such that if $S_\alpha \cap \bar{S}_\beta \neq \emptyset$ for $\alpha \neq \beta$, then $S_\alpha \subset \bar{S}_\beta$ and $\dim S_\alpha < \dim S_\beta$. We let $\mathcal{S}_d$ be the union of all strata of dimension d.*
(2) *X can be written as a finite disjoint union of semialgebraic sets that are (semialgebraically) homeomorphic to some open cube $(0,1)^d$.*

It follows that one can define the Euler characteristic of a compact semialgebraic set X by $\chi(X) = \sum_d (-1)^d n_d$, where n_d is the number of cubes of dimension d. In the non-compact case, one can define two different versions of the Euler characteristic (corresponding to cohomology or to cohomology with compact support). We will restrict to the compact case.

Important fact 2.7. *For each class of tame sets described above, there is a normal cycle construction. In particular, a smooth valuation μ can be extended to such tame sets by setting*

$$\mu(X) = \int_X \phi + \int_{\mathrm{Nor}\, X} \omega.$$

In the case of sets of positive reach, the construction is as for convex bodies. In the case of semialgebraic or subanalytic sets, one has to use stratified Morse theory to attach certain integer multiplicities to outer normal vectors in such a way that the resulting current is a cycle, see [**13**].

Corollary 2.8. *The Euler characteristic is finitely additive in the sense that if X, Y are compact semialgebraic sets, then*

$$\chi(X \cup Y) + \chi(X \cap Y) = \chi(X) + \chi(Y).$$

3. Valuations on manifolds

The theory of valuations on manifolds was mainly developed by Alesker, see [**1, 2, 6, 4, 3, 5, 10**].

Let M be smooth manifold. We assume for simplicity that M is endowed with a riemannian metric g. Let SM be the sphere bundle of M and $\pi : SM \to M$ the projection. There is a canonical 1-form α on SM, defined by

$$\alpha_{(x,v)}(w) := \langle v, d\pi(w) \rangle.$$

This makes M into a contact manifold. This means that $\ker \alpha$ is a completely non-integrable distribution in SM. Alternatively, one has that $\alpha \wedge d\alpha^{n-1}$ is a nowhere vanishing form on SM. A differential form that is a multiple of α is called *vertical*.

We let $\mathcal{P}(M)$ denote the set of compact differentiable polyhedra in M (sets of positive reach or semialgebraic/subanalytic subsets would also work). Then $X \in \mathcal{P}(M)$ admits a normal cycle, which is an $(n-1)$-dimensional current in SM.

LEMMA 3.1. *The normal cycle has the following properties:*

(1) $\operatorname{Nor}(X)$ *is an integer multiplicity rectifiable current.*
(2) $\operatorname{Nor}(X)$ *is a cycle, i.e. vanishes on exact forms.*
(3) $\operatorname{Nor}(X)$ *vanishes on vertical forms.*
(4) *If* $X, Y \in \mathcal{P}(M)$ *such that* $X \cup Y \in \mathcal{P}(M)$ *is convex as well, then*

$$\operatorname{Nor}(X \cup Y) + \operatorname{Nor}(X \cap Y) = \operatorname{Nor}(X) + \operatorname{Nor}(Y).$$

(5) $\pi_* \operatorname{Nor}(X) = \partial X$.

DEFINITION 3.2. *A smooth valuation on* M *is a functional* $\mu : \mathcal{P}(M) \to \mathbb{R}$ *of the form*

$$\mu(X) = \int_X \phi + \int_{\operatorname{Nor} X} \omega,$$

where $\phi \in \Omega^n(M), \omega \in \Omega^{n-1}(SM)$. *We also write* $\mu = [[\phi, \omega]]$ *in this case. The space of smooth valuations is denoted by* $\mathcal{V}(M)$.

A smooth curvature measure on M *is a functional* $\Phi : \mathcal{P}(M) \to \operatorname{Meas}(M)$ *of the form*

$$\Phi(X, U) = \int_{X \cap U} \phi + \int_{\operatorname{Nor} X \cap \pi^{-1}(U)} \omega,$$

where $\phi \in \Omega^n(M), \omega \in \Omega^{n-1}(SM)$. *We also write* $\Phi = [\phi, \omega]$ *in this case. The space of smooth curvature measures is denoted by* $\mathcal{C}(M)$. *There is an obvious surjective map* $\mathcal{C}(M) \to \mathcal{V}(M), \Phi \mapsto [\Phi]$, *where* $[\Phi](X) := \Phi(X, M)$.

If $M = V$ is a euclidean vector space, then $\mathcal{V}(M)$ coincides with the notion from Definition 1.3.

It is important to notice that, by the properties of the normal cycle, the forms ϕ, ω defining a valuation are not unique. To describe the kernel precisely, we need the Rumin operator.

PROPOSITION 3.3 (Rumin operator, [**20**]). *Let $\omega \in \Omega^{n-1}(SM)$. Then there exists a unique vertical form $\alpha \wedge \xi \in \Omega^{n-1}(SM)$ such that $d(\omega+\alpha\wedge\xi)$ is vertical. The Rumin operator is the second order differential operator $D : \Omega^{n-1}(SM) \to \Omega^n(SM)$ defined by $D\omega := d(\omega+\alpha\wedge\xi)$.*

To find ξ, we must solve for the equation

$$d\omega + d\alpha \wedge \xi \equiv 0 \mod \alpha,$$

which is possible by basic symplectic linear algebra ($d\alpha$ defines a symplectic form on $\ker \alpha$), and involves one derivative of ω. Since we take d once more, D is a second order differential operator.

LEMMA 3.4. *The Rumin operator has the following properties:*

(1) *D vanishes on vertical forms.*
(2) *D vanishes on exact forms.*
(3) *D vanishes on multiples of $d\alpha$.*

PROOF. If ω is vertical, we take $\alpha \wedge \xi := -\omega$. If ω is exact, we take $\alpha \wedge \xi := 0$. If $\omega = d\alpha \wedge \tau$, we take $\xi := -d\tau$. $\square$

We thus see that there is a strong connection between the forms that lie in the kernel of D and those that lie in the kernel of each $\mathrm{Nor}(X)$. This can be made precise as follows.

THEOREM 3.5 (B'-Bröcker 2007, [**10**]). *The pair (ϕ,ω) induces the trivial valuation if and only if*

(1) $D\omega + \pi^*\phi = 0$.
(2) $\pi_*\omega = 0$.

Here π_* denotes fiber integration, i.e. $\pi_*\omega(x) = \int_{S_xM} \omega$.

The proof uses a local variation argument and Stokes' theorem.

Remark: Chern has constructed in [**16**] a pair of differential forms such that $D\omega + \pi^*\omega = 0, \pi_*\omega = 1$. The corresponding valuation is the Euler characteristic, i.e. $\phi(X) = \chi(X)$.

An important fact in the theory of valuations on manifolds is the existence of a product structure on smooth valuations.

THEOREM 3.6. *There exists a natural commutative, associative product $\mathcal{V}(M) \times \mathcal{V}(M) \to \mathcal{V}(M)$ with neutral element given by the Euler characteristic. It satisfies a version of Poincaré duality: if $\phi \neq 0 \in \mathcal{V}(M)$, we find a compactly supported $\psi \in \mathcal{V}_c(M)$ such that $\int_M \phi \cdot \psi \neq 0$. This last notation means that we evaluate $\phi \cdot \psi$ at M (which is always possible if M is compact, and still possible if M is non compact but the support of $\phi \cdot \psi$ is compact).*

In the case of $M = V$ a euclidean vector space and translation invariant valuations, the product coincides with the product from Theorem 4.2.

CHAPTER 3

Lecture III: Weyl's principle

1. Alesker's approach

In the previous lectures we have seen that there is a family of nice valuations on a euclidean vector space, the intrinsic volumes. On the other hand, we also have valuations on manifolds, and one may ask whether there is some version of the intrinsic volumes on a riemannian manifold. A first partial answer was given by Weyl, who proved a version of Steiner's formula for manifolds.

THEOREM 1.1 (Weyl [**26**]). *Let $M \subset \mathbb{R}^N$ be a compact submanifold of dimension n, possibly with boundary. Then the volume of the r-tube around M is given, for small $r > 0$, by a polynomial*

$$\mathrm{vol}_N(M_r) = \sum_{k=0}^{n} \mu_k(M)\omega_{N-k} r^{N-k}.$$

The $\mu_k(M)$ do not depend on the embedding, but only on the inner geometry of the Riemannian manifold (M, g).

Examples: Suppose that $\partial M = \emptyset$.

- $\mu_0(M) = \chi(M)$.
- $\mu_n(M) = \mathrm{vol}_n(M)$.
- Important: $\mu_{n-2}(M) = \frac{1}{4\pi}\int_M s\, d\,\mathrm{vol}$, the total scalar curvature of M. Recall that it plays an important role in GR (Hilbert-Einstein functional).
- $\mu_k(M) = 0$ if $(n-k)$ is odd (if M has a boundary, there are boundary contributions, depending on the second fundamental form of the boundary).
- $\mu_k(M)$ can be written as the integral over some (rather complicated) polynomial in the Riemann curvature tensor of (M, g).
- $\mu_k(M)$ is a spectral invariant of the Laplacian acting on differential forms [**17**]).

Since any riemannian manifold can be isometrically embedded in some euclidean space by Nash's theorem, we can associate the intrinsic volumes $\mu_k(M)$ to any riemannian manifold by choosing an arbitrary isometric embedding. However, these are only the global invariant, not yet the valuations. However, Alesker noted that the same approach also works for valuations.

Let $X \in \mathcal{P}(M)$. We use Nash's embedding theorem to find an isometric embedding $\iota : (M, g) \hookrightarrow \mathbb{R}^N$ for some N. ιX is then in $\mathcal{P}(\mathbb{R}^N)$ and as such it has intrinsic volumes $\mu_k(\iota X)$. We set $\mu_k^M(X) := \mu_k(\iota X)$. One can then show that this is independent of the choice of ι. The functional $X \mapsto \mu_k^M(X)$ is a smooth valuation in the sense of the previous lecture. This follows from the fact that the restriction of a smooth valuation to a submanifold is again smooth.

THEOREM 1.2 (Enhanced Weyl principle).
- *If $(M, g) \to (\tilde{M}, \tilde{g})$ is an isometric embedding, then $\mu_k^{\tilde{M}}|_M = \mu_k^M$.*
- *Conversely, let μ be a functor that associates to each riemannian manifold a smooth valuation ϕ^M such that $\phi^{\tilde{M}}|_M = \phi^M$ whenever $(M, g) \to (\tilde{M}, \tilde{g})$ is an isometric embedding. Then there are constants c_k such that*
$$\mu = \sum_{k=0}^{\infty} c_k \mu_k.$$
- *With respect to the Alesker product of smooth valuations on manifolds, we have*
$$\mu_k^M \cdot \mu_l^M = \mu_{k+l}^M$$
In particular, the space spanned by the intrinsic volumes is closed under product, it is called Lipschitz-Killing algebra. (with the convention that $\mu_i(M) = 0$ if i is out of range).

In other words, there is a universal algebra of intrinsic volumes, isomorphic to $\mathbb{R}[[t]]$ (formal power series), that restricts to the Lipschitz-Killing algebra of (M, g) for every riemannian manifold (M, g).

2. Fu-Wannerer approach

Alesker's approach is simple and geometric. The disadvantage of this approach is that it uses Nash's embedding theorem (which may be difficult in practice) and does not give the differential forms (ϕ, ω) defining the intrinsic volumes on a riemannian manifold (M, g).

Fu and Wannerer have given a direct approach based on Cartan's apparatus. We simplify their construction slightly by using double forms. We refer to [**22**] for a recent account on double forms and their use in the proof of the Chern-Gauss-Bonnet theorem.

Let M be a Riemannian manifold of dimension n. We let $\Omega^{p,q}(M, M)$ be the space of double forms on M of bidegree (p, q), i.e. sections of $\Lambda^p T^*M \otimes \Lambda^q T^*M$. Then the metric $\boldsymbol{g} \in \Omega^{1,1}(M, M)$ and the Riemannian curvature tensor $\boldsymbol{R} \in \Omega^{2,2}(M, M)$. We let $\Omega^{p,q}(SM, M)$ be the space of double forms on SM of bidegree (p, q), i.e. sections of $\Lambda^p T^*SM \otimes \pi^* \Lambda^q T^*M$. Then the pull-back of the Levi-Civita connection with respect to the projection map $\pi : SM \to M$ can be used to define a form $\boldsymbol{\omega} \in \Omega^{1,1}(SM, M)$ by $\boldsymbol{\omega}_{(p,v)}(X, Y) := \langle (\pi^* D)_X v, Y \rangle$. There is an obvious multiplication (wedge product in each factor) for double forms. Moreover, a double form in $\Omega^{p,n}(M,M)$ (or in $\Omega^{p,n}(SM,M)$) can be identified, using the Riemannian volume, with a usual form in $\Omega^p(M)$ (or $\Omega^p(SM)$). Then

$$\psi_p = \boldsymbol{R}^p \wedge \boldsymbol{g}^{n-2p} \in \Omega^{n,n}(M,M) = \Omega^n(M)$$

$$\phi_{k,p} = \boldsymbol{\alpha}^\vee \wedge \boldsymbol{R}^p \wedge \boldsymbol{g}^{k-2p} \wedge \boldsymbol{\omega}^{n-k-1} \in \Omega^{n-1,n}(SM,M) = \Omega^{n-1}(SM).$$

Definition 2.1. *Define curvature measures*

$$C_{k,p} := \begin{cases} \frac{\omega_k}{\pi^k (n-k)! \omega_{n-k}} [0, \phi_{kp}] & k < n \\ \frac{\omega_k}{\pi^k} [\psi_p, 0] & k = n \\ 0 & k > n \end{cases} \in \mathcal{C}(M).$$

Finally we can define the Lipschitz-Killing curvature measures by

$$\Lambda_k := \frac{\pi^k}{k! \omega_k} \sum_j \binom{\frac{k}{2}+j}{j} \frac{1}{4^j} C_{k+2j,j} \in \mathcal{C}(M)$$

and valuations

$$\mu_k := [\Lambda_k] \in \mathcal{V}(M).$$

Theorem 2.2 (Weyl principle). *If $(M,g) \to (\tilde{M},\tilde{g})$ is an isometric embedding, then*

$$\Lambda_k^{\tilde{M}}|_M = \Lambda_k^M, \mu_k^{\tilde{M}}|_M = \mu_k^M.$$

If we want this theorem to hold, then the constants in the definition of Λ_k have to be what they are. A curious fact that is not fully understood is that this particular choice of constants has another remarkable property: if we denote by ω the corresponding form on SM (some linear combination of the $\phi_{k,p}$), then $d\omega$ is divisible by α, i.e. we do not need the correction term in the construction of the Rumin differential.

CHAPTER 4

Lecture IV: Curvature measures

In the previous lecture we have defined the Lipschitz-Killing curvature measures Λ_k on any riemannian manifold. In the final lecture we will relate it to more classical curvature notions in riemannian geometry and show how this approach can be used to define curvature notions on singular spaces.

1. The scalar curvature measure

Let (M, g) be a riemannian manifold of dimension n. For simplicity we assume that M is without boundary. Then $\Lambda_k(M, \bullet)$ is a signed measure on M. It follows from the Fu-Wannerer approach that this measure is given by integration of some polynomial expression in the curvature tensor. More precisely, $\Lambda_k(M, \bullet) \equiv 0$ if $k \not\equiv n \mod 2$, and $\Lambda_k(M, \bullet)$ is given by integrating a polynomial of degree $\frac{n-k}{2}$ in the curvature tensor with respect to the riemannian volume measure on M.

Examples:

(1) $\Lambda_n(M; \bullet)$ equals the n-dimensional riemannian volume measure on M.
(2) $\Lambda_{n-2}(M, U) = \frac{1}{4\pi} \int_U s d\,\mathrm{vol}$, where s is the scalar curvature of M.
(3) If M is compact, $\mu_0(M) = \Lambda_0(M, M)$ is the Euler characteristic of M (which is 0 if n is odd). The measure $\Lambda_0(M, \bullet)$ is precisely the Chern-Gauss-Bonnet measure, whose integral is the Euler characteristic.
(4) The measure $\Lambda_{n-4}(M, \bullet)$ is given by integration of some quadratic expression in the curvature tensor, but the geometric meaning is less clear.

The second item suggests that even for a non-smooth set X of dimension n, $\mathrm{scal}(X, \bullet) := 4\pi\Phi_{n-2}(X, \bullet)$ may be a candidate for a scalar curvature measure.

Let $X \subset \mathbb{R}^N$ be a closed semialgebraic subset. More generally, we could take (M, g) a real analytic riemannian manifold of dimension N and $X \subset M$ a closed subanalytic set of dimension n. Then X admits a Nash stratification, that is $X = \bigcup S_\alpha$, where each S_α is a semialgebraic connected subvariety such that if $S_\alpha \cap \bar{S}_\beta \neq \emptyset$ for $\alpha \neq \beta$, then $S_\alpha \subset \bar{S}_\beta$ and $\dim S_\alpha < \dim S_\beta$. We let $\mathcal{S}_d$ be the union of all strata of dimension d.

THEOREM 1.1 ([**8**]). *Let $X \subset \mathbb{R}^N$ be a closed n-dimensional semialgebraic set with a Nash stratification. Then for Borel subset $U \subset \mathbb{R}^N$*

$$\begin{aligned}\operatorname{scal}(X,U) = \sum_{S\in\mathcal{S}_n}\int_{S\cap U} s(x)d\operatorname{vol}_n(x) + 2\sum_{S\in\mathcal{S}_{n-1}}\int_{S\cap U}\operatorname{tr}(II_x)d\operatorname{vol}_{n-1}(x)\\ + 4\pi\sum_{S\in\mathcal{S}_{n-2}}\int_{S\cap U}\left(\frac{1}{2}+(-1)^n\frac{\chi_{loc}(X,x)}{2}-\theta_n(x)\right)d\operatorname{vol}_{n-2}\end{aligned}$$

Here $\operatorname{tr}(II_x)$ is the sum of the traces of the fundamental forms at x of all n-dimensional strata that contain S in their boundary, $\chi_{loc}(X,x) = \chi(X, X \setminus \{x\})$ is the local Euler characteristic, which is constant along the stratum, and $\theta(x) := \lim_{r\to 0}\frac{\operatorname{vol}_n(X\cap B(x,r))}{\omega_n r^n}$ is the density of X at a point x.

The signed measure $\operatorname{scal}(X,\bullet)$ is a good candidate for the scalar curvature of such a set X. Namely, it is compatible with curvature bounds in the sense of Alexandrov. Recall that on a riemannian manifold, if the sectional curvature is bounded from below (or above) by κ, then the Ricci curvature is bounded by below or above) by $(n-1)\kappa g$, which implies that the scalar curvature is bounded from below (or above) by $n(n-1)\kappa$. Now for metric spaces, there is notion of being of sectional curvature bounded below by κ (Alexandrov spaces), whereas being of scalar curvature bounded below will be interpreted in terms of the scalar curvature measure $\operatorname{scal}(X,\bullet)$.

THEOREM 1.2 ([**8**]). *Let X be a compact connected semialgebraic set which is an Alexandrov space with curvature bounded below by $\kappa \in \mathbb{R}$ for its inner metric. Then*

$$\operatorname{scal}(X,\bullet) \geq \kappa n(n-1)\operatorname{vol}.$$

An analogous theorem holds with upper curvature bounds, but is has some (necessary) additional topological assumptions on X.

Let us make some more comments. The three main curvature quantities on a riemannian manifold are the sectional curvature, the Ricci curvature and the scalar curvature. There have been different attempts to generalize such quantities to certain singular sets. A well known theory is the metric approach to (lower and upper) sectional curvature bounds by Alexandrov, Toponogov, Burago, Gromov, Perelman and many others [**12, 14, 15**]. The key idea is to compare triangles in a metric space with triangles in a model space of constant sectional curvature. Concerning Ricci curvature, the basic setting is that of a metric measure space, and (lower) Ricci curvature bounds are expressed using convexity properties of the entropy functional under optimal mass transport (see for instance [**19, 25**]). It is known that the lower sectional curvature bounds (in Alexandrov sense) implies a lower Ricci curvature bound. It is unknown whether a lower Ricci curvature bound implies a lower bound on the scalar curvature measure, as in the above theorem.

2. Tensor valued measures

Let M be a smooth manifold of dimension n. Let $C^\infty(M)$ be the Fréchet space of smooth complex valued functions on M. Let $\mathcal{M}^\infty(M)$ be the Fréchet space of smooth complex valued measures on M. By a subscript c we denote the elements with compact support. We then have a map

$$C_c^\infty(M) \times \mathcal{M}^\infty(M) \to \mathbb{C}, (f, \mu) \mapsto \int_M f\mu$$

that induces an injection with dense image

$$M^\infty(M) \hookrightarrow (C_c^\infty(M))^* =: M^{-\infty}(M).$$

Elements of the space on the right hand side are called generalized measures. For example, given $x \in M$, the Dirac delta $\delta_0(f) := f(0)$ is a generalized measure. A usual signed measure is also a generalized measure.

More generally, let $\mathcal{E} \to M$ be a vector bundle over M of finite rank. Let $|\omega_M|$ be the line bundle of densities on M (the fiber at a point x is the space of Lebesgue measures on T_xM). Then we have a pairing

$$C^\infty(M, \mathcal{E}) \times C_c^\infty(M, \mathcal{E}^* \otimes |\omega_x|) \to \mathbb{R},$$

just pair an element of $\mathcal{E}_x$ with an element of $\mathcal{E}_x^*$, and integrate the remaining density over M. The induced map

$$C^\infty(M, \mathcal{E}) \hookrightarrow (C_c^\infty(M, \mathcal{E}^* \otimes |\omega_x|))^* =: C^{-\infty}(M, \mathcal{E})$$

is continuous, injective, and has dense image. Elements of the space on the right hand side are called generalized sections.

Definition 2.1. *A generalized symmetric 2-tensor on M is an element of $C^{-\infty}(M, \mathrm{Sym}^2(T^*M))$.*

(The same definition would work for other tensor spaces, but we only need this one). If M is a riemannian manifold, then we can trivialize the bundle of densities, and we can identify $\mathrm{Sym}^2(T^*M)^*$ and $\mathrm{Sym}^2(T^*M)$, so that a generalized symmetric 2-tensor takes a compactly supported symmetric 2-tensor h on M and gives us a real number. If q is a smooth symmetric 2-tensor, then we may consider it as a generalized one by taking the map

$$h \mapsto \int_M \langle h, q \rangle d\,\mathrm{vol}\,.$$

3. Variational formulas

Let (M, g) be a compact riemannian manifold of dimension n with scalar curvature s and volume form vol. Let $g_t, t \in (-\epsilon, \epsilon)$ be a variation of the metric, with $h := \frac{d}{dt}|_{t=0} g_t$, which is a symmetric bilinear form, not necessarily positive definite. Let s_t and vol_t be the scalar curvature and volume form of (M, g_t). Then

$$\frac{d}{dt}|_{t=0} \int_M s_t d\,\mathrm{vol}_t = \int_M \left\langle h, \mathrm{ric} - \frac{s}{2} g \right\rangle d\,\mathrm{vol}\,.$$

The tensor $E = \mathrm{ric} - \frac{s}{2}g$ is the Einstein tensor. We want to generalize this formula to tame sets. For this, let (M, g) be a riemannian manifold of dimension m and $X \subset M$ be a fixed tame set of dimension n. Let g_t be a variation of the riemannian metric as above. Since $\mu_{n-2}^M(X, \bullet)$ is up to a multiple a good candidate for the scalar curvature measure of X, we can consider $\frac{d}{dt}\Big|_{t=0} \mu_{n-2}^{(M,g_t)}(X, \bullet)$.

THEOREM 3.1 (Variations of intrinsic volumes, [**9**]). *For each riemannian manifold (M, g) and each variation g_t with $h := \frac{d}{dt}|_{t=0}$, there exists a smooth valuation $\mu_k^{(M,g),h}$ on M such that for all compact tame sets $X \subset M$.*

$$\frac{d}{dt}|_{t=0}\mu_k^{(M,g+th)}(X) = \mu_k^{(M,g),h}(X).$$

For fixed X, the map $h \mapsto \mu_k^{(M,g),h}(X)$ defines a generalized symmetric bilinear form on M.

For example, if X is n-dimensional and if the support of h is contained in an n-dimensional stratum S of X, then

$$\mu_{n-2}^{(M,g),h}(X) = \frac{1}{4\pi}\int_S \langle h, \mathrm{ric} - \frac{s}{2}g\rangle d\,\mathrm{vol}\,.$$

However, at lower dimensional strata, there are contributions that are not smooth symmetric bilinear forms, but generalized ones.

4. Schlaefli's formula

Let Δ be a simplex in $\mathbb{R}^n$. Then, according to Theorem 1.1 (or more easily by looking at the contributions of each face in the Steiner polynomial) we have

$$\mathrm{scal}(\Delta) = 4\pi \sum_{F \in \mathfrak{F}_{n-2}} \mathrm{vol}_{n-2}(F)\theta_F,$$

where θ_F is the normalized outer angle at F.

Consider now a smooth family of simplices $\Delta_t, t \in (-\epsilon, \epsilon)$ in $\mathbb{R}^n$ and compute $\frac{d}{dt}|_{t=0}\,\mathrm{scal}(\Delta_t)$ in two ways. First it is clear that

$$\frac{d}{dt}|_{t=0}\,\mathrm{scal}(\Delta_t) = 4\pi \sum_{F \in \mathfrak{F}_{n-2}} (\mathrm{vol}_{n-2}(F)'\theta_F + \mathrm{vol}_{n-2}(F)\theta_F').$$

But we can also apply the variational formula Proposition 3.1. To do so, we realize that whenever we have two simplices Δ_1, Δ_2 in $\mathbb{R}^n$, there is an affine map $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\Phi(\Delta_1) = \Delta_2$. Since intrinsic volumes behave well with respect to isometric embeddings, we have

$$\mathrm{scal}(\Delta_2, g_{eukl}) = \mathrm{scal}(\Phi(\Delta_1), g_{eukl}) = \mathrm{scal}(\Delta_1, \Phi^* g_{eukl})$$

Hence if we have a family Δ_t and $\Delta := \Delta_0$, then

$$\frac{d}{dt}|_{t=0}\,\mathrm{scal}(\Delta_t) = \frac{d}{dt}|_{t=0}\,\mathrm{scal}(\Delta, \Phi_t^* g_{eukl}),$$

and this is precisely the situation of Proposition 3.1. We then get

$$\frac{d}{dt}|_{t=0}\,\mathrm{scal}(\Delta_t, \Delta_t) = 4\pi \sum_{F\in\mathfrak{F}_{n-2}} \mathrm{vol}_{n-2}(F)'\theta_F.$$

Comparing these two expressions, we obtain

$$\sum_{F\in\mathfrak{F}_{n-2}} \mathrm{vol}_{n-2}(F)\theta_F' = 0.$$

Since any polytope can be triangulated and the expression in the formula is finitely additive, we obtain for every smooth variation P_t of polytopes in $\mathbb{R}^n$

$$\sum_{F\in\mathfrak{F}_{n-2}(P)} \mathrm{vol}_{n-2}(F)\theta_F' = 0,$$

which is known as Schläfli's equation. The same proof also works for higher Schläfli formulas (just replace μ_{n-2} by any μ_k), and on hyperbolic and spherical spaces. See [**9**] for details.

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