SPECTRAL DECOMPOSITION AND SIEGEL-VEECH TRANSFORMS FOR STRATA: THE CASE OF MARKED TORI

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Abstract. Generalizing the well-known construction of Eisenstein series on the modular curves, Siegel-Veech transforms provide a natural construction of square-integrable functions on strata of differentials on Riemannian surfaces. This space carries actions of the foliated Laplacian derived from the $\mathrm{SL}_2(\mathbb{R})$-action as well as various differential operators related to relative period translations.

In the paper we give spectral decompositions for the stratum of tori with two marked points. This is a homogeneous space for a special affine group, which is not reductive and thus does not fall into well-studied cases of the Langlands program, but still allows to employ techniques from representation theory and global analysis. Even for this simple stratum exhibiting all Siegel-Veech transforms requires novel configurations of saddle connections. We also show that the contiunuous spectrum of the foliated Laplacian is much larger than the space of Siegel-Veech transforms, as opposed to the case of the modular curve. This defect can be remedied by using instead a compound Laplacian involving relative period translations.

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1. Introduction

For the modular surface or more generally for quotients of the upper half plane by a cofinite Fuchsian group Γ the space $\mathrm{L}^2(\Gamma\backslash\mathbb{H})$ is well-known to decompose into the cuspidal part, the space of Eisenstein transforms and the residual spectrum. The Laplace operator acts with discrete spectrum on the cuspidal part, while Eisenstein

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series provide the continuous spectrum. The fine structure of the cuspidal part, the size of the spectral gap and the description of the residual spectrum is the context of various open conjectures. There is a similar decomposition of $\mathrm{L}^2(\Gamma\backslash\mathrm{SL}_2(\mathbb{R}))$, after first decomposing the space into K-types, where $\mathrm{K} = \mathrm{SO}(2)$ is the standard maximal compact subgroup of $\mathrm{SL}_2(\mathbb{R})$.

There are two natural generalizations of this decomposition problem. First, we may replace $\mathrm{SL}_2(\mathbb{R})$ by any Lie group G of higher rank or even p-adic and study the decomposition of $\mathrm{L}^2(\Gamma\backslash G)$. Second, we may replace $\Gamma\backslash\mathrm{SL}_2(\mathbb{R})$ by a stratum $\mathcal{H}(\alpha)$ of area one flat surfaces with zeros of order $\alpha = (m_1, \ldots, m_n)$ with the Masur–Veech measure ν_{MV}. For instance the stratum $\mathcal{H}(0)$ of area one tori with one marked points can be identified with the unit tangent bundle $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$ to the modular surface $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. The first generalization has been studied intensively for semi-simple Lie groups, in particular in connection with the Langlands program, for example [Lan70; Lan89; Art13]. For the second generalization, the spaces $\mathrm{L}^2(\mathcal{H}(\alpha)) := \mathrm{L}^2(\mathcal{H}(\alpha); \nu_{\mathrm{MV}})$ and even more generally for linear submanifolds of $\mathcal{H}(\alpha)$, notably the existence of a spectral gap for the foliated Laplacian corresponding to the $\mathrm{SL}_2(\mathbb{R})$-action has been established in work of Avila–Gouëzel [AG13]. However their work explicitly avoids a decomposition of the spectrum as above ("since the geometry at infinity is very complicated"). Given recent progress towards understanding the boundary of strata [BCGGM] we aim to shed light on how the boundary relates with the continuous spectrum for strata.

In this paper we focus on the first non-classical case namely the stratum $\mathcal{H}(0,0)$ of area one tori with two marked points. At the same time this is an instance of a space $\mathrm{L}^2(\Gamma\backslash G)$ for a non-reductive group G, namely the quotient of the special affine group $\mathrm{SAff}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2$ by its integral lattice $\mathrm{SAff}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ minus the zero section, which is identified with $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$. Since the Masur–Veech measure ν_{MV} extends over this locus, we may and will use the identification

$$\mathrm{L}^2(\mathrm{SAff}_2(\mathbb{Z})\backslash\mathrm{SAff}_2(\mathbb{R})) \;=\; \mathrm{L}^2(\mathcal{H}(0,0))$$

throughout. We will rely on tools from representation theory, explain why simple-minded generalizations from the modular surface case might fail, and how these failures can be bridged.

The perspective of Siegel–Veech transforms. The Siegel–Veech transform is a method to construct functions in $\mathrm{L}^2(\mathcal{H}(\alpha); \nu_{\mathrm{MV}})$ based on the analogy between lattice vectors for homogeneous spaces and saddle connections on strata. It takes as input a function f on $\mathbb{R}^2$, often supposed smooth and compactly supported, and a 'configuration' and returns the function $\mathrm{SV}(f)$ associating with the flat surface (X, ω) the sum over $f(v)$ for all saddle connections vectors v that stem from the given configuration (see Section 6 for the precise definition). For the special case of the modular surface, i.e., the case of $\mathcal{H}(0)$, there is a unique configuration, which yields all primitive lattice vectors and the Siegel–Veech transform of the spherical function $f(v) = |v|^{2s}$ is just the usual (non-holomorphic) Eisenstein series. In general the range of Siegel–Veech transforms on the modular surface yields the spectral projection on the continuous spectrum of the Laplace operator. For general strata, examples of configurations are given by all saddle connections joining a simple zero to a triple zero or by all core curves of cylinders. The modular surface model case triggers the following questions.

(Q1) What is a complete set of configurations in the sense that their Siegel–Veech transforms account for all possible Siegel–Veech transforms?
(Q2) Are Siegel–Veech transforms responsible for all of the continuous spectrum of the foliated Laplacian $-\Delta^{\mathrm{fol}}$ (as defined below)?
(Q3) Is there a notion of cusp forms so that Siegel–Veech transforms are precisely the orthogonal complement of cusp forms? Is this notion of cusp forms related to boundary divisors in the multi-scale compactification from [BCGGM], as they do in the case of the modular surface?

We will answer these questions for $\mathcal{H}(0,0)$ at the end of the introduction. For each of the questions the answer is not quite the one we expected initially. For general $\mathcal{H}(\alpha)$ all three of them seem completely open.

The perspective of differential operators. The action of $\mathrm{SL}_2(\mathbb{R})$ on strata $\mathcal{H}(\alpha)$ gives rise to a Casimir element acting as a operator $\mathcal{D}^{\mathrm{fol}}$ on $\mathrm{L}^2(\mathcal{H}(\alpha))$. It is this operator or the corresponding Laplace operator $-\Delta_k^{\mathrm{fol}}$ acting on weight-k modular forms on the projectivized stratum $\mathcal{H}(\alpha)/\mathrm{SO}_2(\mathbb{R})$ that we are mainly interested in. See Section 2 for details.

Since we work in a homogeneous space for the group $\mathrm{SAff}_2(\mathbb{R})$ we have more differential operators at our disposal, which will also be the case for strata $\mathcal{H}(0^k)$ of tori with more than just one zero. Even though $\mathrm{SAff}_2(\mathbb{R})$ is not reductive, we show in Proposition 2.1 that the center of the universal enveloping algebra is a polynomial ring generated by a degree *three* 'Casimir' element, which acts as an operator that we call the total Casimir $\mathcal{D}^{\mathrm{tot}}$. Again we define the corresponding Laplace operators $-\Delta_k^{\mathrm{tot}}$ on the projectivized strata.

Another option is to incorporate the translation along torus fibers, i.e. the relative period foliation, into a degree two differential operator is to use an operator Δ^{vert}. The compound operator $\Delta_k^{\mathrm{cmp}(\varepsilon)} := \Delta_k^{\mathrm{fol}} + \varepsilon\Delta^{\mathrm{vert}}$ operator is elliptic if and only if $\varepsilon > 0$, invariant under $\mathrm{SAff}_2(\mathbb{R})$-translations but, contrary to Δ_k^{tot}, does not commute with most other covariant differential operators. We return to this operator at the end of the introduction in connection with (Q2).

The perspective of representation theory. Pullback via the map $\mathcal{H}(0,0) \to \mathcal{H}(0)$ forgetting the last point gives an inclusion $\mathrm{L}^2(\mathcal{H}(0)) \hookrightarrow \mathrm{L}^2(\mathcal{H}(0,0))$. We call its orthogonal complement the *genuine part*

$$\mathrm{L}^2(\mathcal{H}(0,0))^{\mathrm{gen}} = \mathrm{L}^2(\mathcal{H}(0))^{\perp}.$$

From now on we focus on this genuine part and discard the pullbacks of $\mathrm{SL}_2(\mathbb{R})$-representations. The irreducible representations of $\mathrm{SAff}_2(\mathbb{R})$ are classified by Mackey theory. As we recall in Theorem 3.7 they are pullbacks of $\mathrm{SL}_2(\mathbb{R})$-representations, which we discarded, and representations $\pi_{n,m}^{\mathrm{SAff}}$ induced from characters of a fixed Heisenberg subgroup of $\mathrm{SAff}_2(\mathbb{R})$, with $\pi_{n_1,m_1}^{\mathrm{SAff}}$ and $\pi_{n_2,m_2}^{\mathrm{SAff}}$ isomorphic if and only if $n_1 m_1^2 = n_2 m_2^2$. As a first step towards answering our main questions, we exhibit the decomposition of $\mathrm{L}^2(\mathcal{H}(0,0))$.

Theorem 1.1. *The genuine part of the* L^2*-space of the stratum* $\mathcal{H}(0,0)$ *admits a decomposition*

$$\mathrm{L}^2\big(\mathcal{H}(0,0)\big)^{\mathrm{gen}} = \mathrm{L}^2\big(\mathrm{SAff}_2(\mathbb{Z})\backslash\mathrm{SAff}_2(\mathbb{R})\big)^{\mathrm{gen}} \cong \bigoplus_{m=1}^{\infty}\bigoplus_{n\in\mathbb{Z}} \pi_{n,m}^{\mathrm{SAff}} \tag{1.1}$$

Explicitly, the representation $\pi^{\mathrm{SAff}}_{n,m}$ is the $\mathrm{SAff}_2(\mathbb{R})$-invariant subspace generated by the lifts of Eisenstein series $E_{k;m,\beta}$ for $n = 0$ and Poincaré series $P_{k;n,m,\beta}$ for $n \neq 0$ for any integrable function $\beta : \mathbb{R}^+ \to \mathbb{C}$, as defined in (4.13) and (4.12).

A main tool in the proof of Theorem 1.1 are Fourier expansions. The Fourier expansions along the translation subgroup $\mathbb{R}^2$ of $\mathrm{SAff}_2(\mathbb{R})$ plays only a minor role. More important is the Fourier expansion along a subgroup isomorphic to $\mathbb{R}^2$ inside a Heisenberg subgroup but with non-trivial intersection with $\mathrm{SL}_2(\mathbb{R})$. We name these the Fourier–Heisenberg coefficients $c^{\mathrm{H}}(\,\cdot\,, n, r; v, v/y)$, since we decompose the coefficient $r = 0$ even further, along a Heisenberg group, see Section 4. Here $(\tau, z) = (x+iy, u+iv)$ are the standard coordinates on the *Jacobi half-space* $\mathbb{H}\times\mathbb{C}$.

Next we aim for the decomposition of $\mathrm{L}^2(\mathcal{H}(0,0))$ into irreducible $\mathrm{SL}_2(\mathbb{R})$-representations. In general the problem of decomposing the restriction of representations into irreducible ones is known as the *branching* problem and discussed in many instances (e.g. [KKP16; GGP20] and the references therein). Our case might be known, but since we were not able to locate a proof in the literature we give the details of the following result, see Proposition 3.9 for the full statement including the case $n = 0$.

Proposition 1.2. *For any $m \in \mathbb{Z}^\times$ and $n \in \mathbb{Z}\setminus\{0\}$, the restrictions of the $\mathrm{SAff}_2(\mathbb{R})$-representations decompose as a direct integral*

$$\mathrm{Res}^{\mathrm{SAff}_2(\mathbb{R})}_{\mathrm{SL}_2(\mathbb{R})}\ \pi^{\mathrm{SAff}}_{n,m} \;\cong\; \bigoplus_{k=2}^{\infty} D^{\mathrm{SL}}_{\mathrm{sgn}(n)k} \,\oplus\, \int^{\oplus}_{\mathbb{R}^+} \big(I^{\mathrm{SL}}_{+,it} \oplus I^{\mathrm{SL}}_{-,it}\big)\,\mathrm{d}t, \tag{1.2}$$

where the discrete series $D^{\mathrm{SL}}_{\mathrm{sgn}(n)k}$ and the principal series representation $I^{\mathrm{SL}}_{\pm,it}$ are defined along with Theorem 3.4.

In particular the complementary series does not occur in the decompostion of $\mathrm{L}^2\big(\mathcal{H}(0,0)\big)^{\mathrm{gen}}$.

The decomposition into irreducible $\mathrm{SAff}_2(\mathbb{R})$-representations in Theorem 1.1 is fully discrete. This corresponds to the fact that there are square-integrable Eisenstein and Poincaré series that contribute to individual constituents $\pi^{\mathrm{SAff}}_{n,m}$. It contrasts the classical situation for $\mathrm{SL}_2(\mathbb{R})$ in which Eisenstein series contribute to the continuous spectrum and cannot be square-integrable and eigenfunctions for the Laplacian simultaneously. Proposition 1.2 recovers the classical situation in parts: There are some square-integrable Eisenstein series for $\mathrm{SAff}_2(\mathbb{R})$ that are eigenfunctions of the foliated Laplacian, but there are also others that behave like Eisenstein series for $\mathrm{SL}_2(\mathbb{R})$.

In the next result we clarify which Eisenstein and Poincaré series are generating the discrete and continuous pieces in which the representation breaks up according to Proposition 1.2. In the sequel we thus consider $\pi^{\mathrm{SAff}}_{n,m}$ as a subrepresentation of $\mathrm{L}^2\big(\mathrm{SAff}_2(\mathbb{Z})\backslash\mathrm{SAff}_2(\mathbb{R})\big)^{\mathrm{gen}}$ via the isomorphism of Theorem 1.1. The Γ-factor in the next result and the Whittaker function $\mathrm{W}_{\kappa,\mu}(y)$ are defined along with the complete statement of this result in Theorem 5.6. It also includes the corresponding statement for the representations $\pi^{\mathrm{SAff}}_{0,m}$.

Theorem 1.3. *For $k \in \mathbb{Z}\setminus\{0,\pm1\}$ and $n \in \mathbb{Z}$ with $nk > 0$ the representation $D^{\mathrm{SL}}_{\mathrm{sgn}(n)k}$ in (1.2) is generated by the Poincaré series for $\beta = e^{-2\pi|n|y}$ if $k > 1$ and $\beta = y^{-k}e^{-2\pi|n|y}$ if $k < -1$.*

Associating to $n \in \mathbb{Z} \setminus \{0\}$ and $\psi \in \mathrm{L}^2(\mathbb{R}^+, dt)$ the lifts of the Poincaré series $P_{k;n,m,\beta^{\mathrm{W}}_{k,n,\psi}}$ of the 'Whittaker transform'

$$\beta^{\mathrm{W}}_{k,n,\psi}(y) \;:=\; \frac{1}{4\pi\,|n|^{\frac{3}{2}}} \int_{t\in\mathbb{R}^+} \frac{\psi(t)}{(\Gamma^{\mathrm{W}}(t)\Gamma^{\mathrm{W}}(-t))^{\frac{1}{2}}}\, y^{-\frac{k}{2}}\, \mathrm{W}_{\frac{\mathrm{sgn}(n)k}{2},it}\big(4\pi|n|\,y\big)\;\mathrm{d}t$$

gives rise to isometric embeddings

$$\mathrm{P}^{\mathrm{W}}_{+} : \bigoplus_{k\in 2\mathbb{Z}} \mathrm{L}^2\big(\mathbb{R}^+, \mathrm{d}t\big) \to \pi^{\mathrm{SAff}}_{n,m}, \qquad \mathrm{P}^{\mathrm{W}}_{-} : \bigoplus_{k\in 1+2\mathbb{Z}} \mathrm{L}^2\big(\mathbb{R}^+, \mathrm{d}t\big) \to \pi^{\mathrm{SAff}}_{n,m}$$

whose images are $\int^{\oplus}_{\mathbb{R}^+} I^{\mathrm{SL}}_{+,it}\,\mathrm{d}t$ and $\int^{\oplus}_{\mathbb{R}^+} I^{\mathrm{SL}}_{-,it}\,\mathrm{d}t$ respectively, in the decomposition (1.2).

The proof has of course similarities with the way the Eisenstein transform identifies the continuous spectrum of the modular surface, see e.g. [Ber16, Section 4.2.5] for a textbook version. Note however that the principal series appear with infinite multiplicity which we accomodate by first restricting to individual $\pi^{\mathrm{SAff}}_{n,m}$. Further, as opposed to the classical case, Poincaré series associated with Whittaker functions contribute to the continuous spectrum, which requires a more delicate estimate.

The main results. In view of the next theorem we define the space of *cusp forms* to be the subspace of modular-invariant functions on projectivized strata where the Fourier coefficient $c^{\mathrm{H}}(\,\cdot\,,0,0;v,v/y)$ vanishes. We use the same terminology for the lifts of these functions to $\mathrm{L}^2\big(\mathrm{SAff}_2(\mathbb{Z})\backslash\mathrm{SAff}_2(\mathbb{R}))\big)^{\mathrm{gen}}$. Similarly, we focus on this genuine subspace by considering only Siegel–Veech transforms of mean-zero functions from now on.

Theorem 1.4. *Siegel–Veech transforms of compactly supported mean-zero functions are contained in the subspace of $\mathrm{L}^2\big(\mathrm{SAff}_2(\mathbb{Z})\backslash\mathrm{SAff}_2(\mathbb{R}))\big)^{\mathrm{gen}}$ which is annihilated by $\mathcal{D}^{\mathrm{tot}}$, which is the subspace $\oplus_{m=1}^{\infty}\pi^{\mathrm{SAff}}_{0,m}$. This space is the orthogonal complement of the space of cusp forms.*

In the case $\mathcal{H}(0,0)$ there are two obvious configurations, using the 'absolute periods', i.e. lattice vectors, and using 'relative periods' joining one zero to the other. We denote the corresponding Siegel–Veech transforms by $\mathrm{SV}_{\mathrm{abs}}(\,\cdot\,)$ and $\mathrm{SV}_{\mathrm{rel}}(\,\cdot\,)$ respectively. The absolute Siegel–Veech transforms only contribute to the well-studied non-genuine part of the L^2-space and will be disregarded in the sequel.

However the above is not a complete list of configurations! In fact, for a point $(\Lambda, z) \in \mathcal{H}(0,0)$ and any $M \in \mathbb{N}$ the set $z + \frac{1}{M}\Lambda$ of translates of the relative period by a $1/M$-th lattice vector also satisfies all properties of a 'configuration', and Theorem 1.4 also includes these. We denote the corresponding Siegel-Veech transform by $\mathrm{SV}_{\mathrm{rel},M}$ and let

$$\mathcal{SV}_{\mathrm{rel},M} \;=\; \overline{\mathrm{span}}\Big\{\,\mathrm{SV}_{\mathrm{rel},M}(f) : f \in \mathrm{C}^{\infty}_{c,0}(\mathbb{R}^2)\Big\} \tag{1.3}$$

Together with Theorem 1.4 the following result shows that we have found all configurations, thus answering (Q1).

Theorem 1.5. *There is an orthogonal decomposition*

$$\mathrm{L}^2\big(\mathcal{H}(0,0)\big)^{\mathrm{gen}} \;=\; \mathrm{L}^2\big(\mathcal{H}(0,0)\big)^{\mathrm{gen}}_{\mathrm{cusp}} \oplus \overline{\mathrm{span}}\Big(\bigcup_{M=1}^{\infty}\mathcal{SV}_{\mathrm{rel},M}\Big).$$

It also implies together with Proposition 1.2 and Theorem 1.1 that Siegel-Veech transforms do not account for the full continuous spectrum of $\mathcal{D}^{\mathrm{fol}}$ on $\mathcal{H}(0,0)$, since every $\pi_{n,m}^{\mathrm{SAff}}$ regardless of whether $n = 0$ or not contributes to its continuous spectrum. This answers (Q2) negatively for this stratum. Finally we observe that Theorem 1.5 is a positive answer to the first part of (Q3). Note, however, that vanishing of a single Fourier coefficient of an $\mathbb{R}^2$-action is a codimension two condition rather than a divisorial condition.

While (Q2) was answered negatively it makes sense to modify it to

(Q2') Is there an operator for which the Siegel–Veech transforms are responsible for all its continuous spectrum, and if so what is it?

The answer to (Q2') is that there is such an operator, and it is the compound Laplacian introduced earlier. With the given definition of cusp forms, the behaviour of this operator parallels the usual Laplacian on the modular surface, and as $\varepsilon \searrow 0$ its discrete spectrum converges to the part of the continuous spectrum of the foliated Laplacian missed by the Siegel–Veech transforms.

Theorem 1.6. *The compound Laplacian $-\Delta_k^{\mathrm{cmp}(\varepsilon)}$ has discrete spectrum on the space of genuine cusp forms of* K*-type k. As $\varepsilon \searrow 0$, the spectrum of $-\Delta_k^{\mathrm{fol}}$ is comprised of limit points from the spectra of $-\Delta_k^{\mathrm{cmp}(\varepsilon)}$. This remains true of the restriction of these operators to cusp forms or their orthogonal complement.*

All the differential operators considered here, Δ^{fol}, Δ^{tot}, and $\Delta^{\mathrm{cmp}(\varepsilon)}$, also exist for strata and linear manifolds therein provided they have a non-trivial relative period foliation. Among those, linear manifolds of rank one are the natural scope to extend the main results of this paper. We plan to explore this in a follow-up paper.

Notes and references. For a given hyperbolic surface $\Gamma\backslash\mathrm{SL}_2(\mathbb{R})/\mathrm{K}$ the interpretation of the Siegel-Veech transform as Eisenstein series has been used in a number of papers, starting with [Vee89]. See in particular [BNRW20] and the references there, for example for applications to counting problems of lattice vectors in star-shaped regions.

The compound differential operator and its spectral decomposition for the special case of Maass forms of weight zero appear in an unpublished manuscript of Balslev [Bal11] in the equivalent guise of Jacobi forms of weight and index 0. After adjusting to his set of coordinates one checks that his Laplacian equals our $-\Delta_0^{\mathrm{cmp}(4)}$. The Fourier expansions (Section 4) are also discussed in [Bal11] aiming to decompose the L^2-space into eigenspaces of his Laplacian. The first statement of Thereom 1.6 is also claimed without proof in [Bal11]. Balslev moreover computes explicitly a Weyl's law for the spectrum of his Laplacian (in weight 0) and briefly addresses the same question for the covering space given by replacing $\mathrm{SAff}_2(\mathbb{Z})$ with a subgroup of small finite index.

The reader might also view this paper as a complement to the book of Berndt–Schmidt [BS98], where representations of the Jacobi group are discussed from a perspective inspired by automorphic representation theory. They, however, restrict very early in their treatment to central character zero, which rules out precisely the case that we consider in the present work. The prominent role played by the Schrödinger–Weil representation in their setting reduces them to representations of the metaplectic group, which were intensively studied for instance by Waldspurger in prior work. Plenty of representation theoretic subtleties in the present work can only occur because of the lack of such a tight connection to any (covering of) a classical group.

The Siegel(−Veech) transform for affine lattices has been used for effective equidistribution results in [GKY22], see also [SV20].

There is a long history using Ratner's theory on the space $\mathcal{H}(0,0)$ to study saddle connection, notably their gap distributions, see e.g. [EM04; MS10; San22].

The analog of Selberg's conjecture (the size of the spectral gap or the non-existence of complementary series) for strata or its congruence covers is a question of Yoccoz. See [Mag19] and [MR19] for progress in this direction.

Organization of the paper. Section 2 sets the table by defining the various Casimir and Laplace differential operators relevant to our analysis from all perspectives. Section 3 is dedicated to the representation theory, building a decomposition of $\mathrm{L}^2(\mathrm{G}'(\mathbb{Z})\backslash\mathrm{G}'(\mathbb{R}))$ via Mackey theory. In order to study the spectral decomposition of $\mathrm{L}^2(\mathcal{H}(0,0))$ or the role of the Siegel–Veech transforms, we introduce the Fourier and Fourier–Heisenberg expansions in Section 4. We provide the spectral decomposition in Section 5. There, we note that Theorems 5.1 and 5.6 are refinements rather than restatements of Theorems 1.1 and 1.3 respectively. This is also where cusp forms are introduced. While they do not correspond to the discrete part of the spectrum for the foliated Laplacian, we show that they do for the compound Laplacian. Finally, in Section 6 we introduce the Siegel–Veech transform and show that they cover the complement of the cusp forms, giving a final decomposition and interpretation of $\mathrm{L}^2(\mathcal{H}(0,0))$.

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2. Differential operators for the special affine group

Each stratum $\mathcal{H}(\alpha)$ admits an action by $\mathrm{G}(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})$. For an introduction to strata and the dynamics of the $\mathrm{SL}_2(\mathbb{R})$-action, see, for example [AM24]. Up to measure zero, the stratum $\mathcal{H}(0,0)$ agrees with $\mathrm{SAff}_2(\mathbb{Z})\backslash\mathrm{SAff}_2(\mathbb{R})$ and is, contrary to other strata in higher genus, a homogeneous space. There are several interesting differential operators acting on this space. First, the Casimir element C of $\mathrm{SL}_2(\mathbb{R})$ induces a second order 'foliated' differential operator $\mathcal{D}^{\mathrm{fol}}$ that involves only the derivatives along the leaves of the foliation by $\mathrm{SL}_2(\mathbb{R})$-orbits. Second, we show in Proposition 2.1 that the group $\mathrm{G}'(\mathbb{R}) = \mathrm{SAff}_2(\mathbb{R})$, despite not being reductive, has a universal enveloping algebra, whose center is a polynomial ring in one variable. We call a generator of this polynomial ring a Casimir element C'. It induces an order *three* differential operator $\mathcal{D}^{\mathrm{tot}}$.

Just as in the classical case of the modular curve, we may pass between functions on $\mathcal{H}(0,0)$ of a given K-type and modular-invariant functions on the quotient $\mathcal{H}(0,0)/\mathrm{K}$, which is the quotient of the Jacobi half plane $\mathbb{H}'$ by $\mathrm{SAff}_2(\mathbb{Z})$. We state this correspondence in Section 2.3. Under this correspondence, the 'total' and 'foliated' differential operators $\mathcal{D}^{\mathrm{tot}}$ and $\mathcal{D}^{\mathrm{fol}}$ correspond to Laplace operators $-\Delta^{\mathrm{tot}}$ and $-\Delta^{\mathrm{fol}}$.

To complete the picture, we observe that besides these two operators there is a vertical Laplace operator $-\Delta^{\mathrm{vert}}$ which is $\mathrm{G}'(\mathbb{R})$-invariant. We call any linear combination $-\Delta^{\mathrm{cmp}(\varepsilon)} = -\Delta^{\mathrm{fol}} - \varepsilon\Delta^{\mathrm{vert}}$ with $\varepsilon > 0$ a *compound Laplace operator*, whose basic properties we discuss in Section 2.4.

We will write elements in $\mathrm{G}'(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2$ as (g, w) where $w = (w_1, w_2)$ is a row vector with composition law $(g, w) \cdot (\tilde{g}, \tilde{w}) = (g\tilde{g}, w\tilde{g} + \tilde{w})$. We need the compact subgroup $\mathrm{K} := \mathrm{SO}_2(\mathbb{R}) \subset \mathrm{G}(\mathbb{R}) \subset \mathrm{G}'(\mathbb{R})$. The Poincaré upper half plane and its affine extension, called the Jacobi upper half space in [EZ85], are

$$\mathbb{H} \;=\; \big\{\tau \in \mathbb{C} \,:\, \mathrm{Im}(\tau) > 0\big\} \;\cong\; \mathrm{G}(\mathbb{R})/\mathrm{K},$$
$$\mathbb{H}' \;=\; \mathbb{H} \times \mathbb{C} \;=\; \big\{(\tau, z) \in \mathbb{C}^2 \,:\, \mathrm{Im}(\tau) > 0\big\} \;\cong\; \mathrm{G}'(\mathbb{R})/\mathrm{K}.$$

Following the conventions for Jacobi forms, we use the coordinates

$$\tau \;=\; x + iy \quad \text{and} \quad z \;=\; u + iv \;=\; p\tau + q. \tag{2.1}$$

Let $\mathfrak{g}$ and $\mathfrak{g}'$ be the complexified Lie algebras of $\mathrm{G}(\mathbb{R})$ and $\mathrm{G}'(\mathbb{R})$ respectively. Given the elements of $\mathfrak{g}$

$$F \;=\; \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \quad H \;=\; \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \quad \text{and} \quad G \;=\; \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right),$$

we use as a basis of $\mathfrak{g}$

$$Z \;=\; -i(F - G), \quad \text{and} \quad X_\pm \;=\; \tfrac{1}{2}\big(H \pm i(F + G)\big). \tag{2.2}$$

Considering additionally the elements of $\mathfrak{g}'$

$$P \;=\; \big(\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), (1, 0)\big) \quad \text{and} \quad Q \;=\; \big(\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), (0, 1)\big)$$

we use as a basis for $\mathfrak{g}'$ the set

$$(Z, 0, 0), \quad (X_\pm, 0, 0), \quad \text{and} \quad Y_\pm = \tfrac{1}{2}(P \pm iQ). \tag{2.3}$$

In the sequel, we abuse notation and denote $Z = (Z, 0, 0)$ and $X_\pm = (X_\pm, 0, 0)$ when it is clear that we are considering them as elements of $\mathfrak{g}'$.

2.1. **A Casimir element for the special linear group.** By the general theory of reductive groups, a Casimir element C for $\mathfrak{g} = \mathfrak{sl}_2$ is any generator of the center $\mathfrak{z}$ of the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$. Such an element is given by $C = \sum_X X X^\vee$, where X runs through a basis of the Lie algebra and $X^\vee$ is the dual of X with respect to the Killing form. Explicitly

$$C = \tfrac{1}{4} X_+ X_- + \tfrac{1}{8} Z^2 + \tfrac{1}{4} X_- X_+ = \tfrac{1}{2} X_+ X_- + \tfrac{1}{8} Z^2 - \tfrac{1}{4} Z, \tag{2.4}$$

and we define a foliated differential operator as the left action

$$\mathcal{D}^{\mathrm{fol}} f \;:=\; 2C\, f \tag{2.5}$$

of the Casimir element, matching normalization used e.g. in the theory of elliptic modular forms.

2.2. **A Casimir element for the special affine group.** Since $G'(\mathbb{R}) = \mathrm{SAff}_2(\mathbb{R})$ is not reductive, we determine the center of $\mathrm{U}(\mathfrak{g}')$ in an ad hoc way. We nevertheless refer to C' below as a *Casimir element*. Similar computations of Casimir elements (that are also degree three) have appeared for the Jacobi group in [BCR12] and [CWR16]. The following proposition complements these computations (and also those in [BS98]) which were always restricted to representations of the Jacobi group with non-trivial central character.

Proposition 2.1. *The center $\mathfrak{z}'$ of the the universal enveloping algebra $\mathrm{U}(\mathfrak{g}')$ is a polynomial ring*

$$\mathfrak{z}' \;=\; \mathbb{C}[C'] \quad \textit{with generator} \quad C' \;=\; ZY_+Y_- - X_+Y_-^2 + X_-Y_+^2. \tag{2.6}$$

Before proving the proposition, we observe that it gives rise to a differential operator via the left action

$$\mathcal{D}^{\mathrm{tot}}\, f \;:=\; 2C'\, f. \tag{2.7}$$

Proof. Let $\mathfrak{A} = \mathrm{gr}\,\mathrm{U}(\mathfrak{g}')$ be the associated graded algebra and

$$\sigma : \mathfrak{A} \longrightarrow \mathrm{U}(\mathfrak{g}'), \quad m_1 \cdots m_n \longmapsto \sum_{\pi \in \mathrm{S}_n} m_{\pi(1)} \cdots m_{\pi(n)}, \quad m_i \in \mathfrak{g}' \text{ for } 1 \le i \le n,$$

be the linear symmetrization map (which is not an algebra homomorphism).

The leading term of any element of $\mathfrak{z}'$ yields a central element of $\mathfrak{A}$. Conversely, since commutators in the associative algebra $\mathrm{U}(\mathfrak{g}')$ strictly lower the degree filtration and by induction on the degree (cf. [Hel59, Theorem 10] and the lemmas used in its proof), we see that the symmetrization map yields a bijection

$$\ker\big(\mathfrak{g}' \circlearrowright \mathfrak{A}\big) \longrightarrow \mathfrak{z}'.$$

In particular, $\mathfrak{A}$ is commutative, but carries a non-trivial, degree preserving representation of $\mathfrak{g}'$. To determine the kernel of the $\mathfrak{g}'$-action on $\mathfrak{A}$, we record that

$$\begin{aligned}
\big[Y_+,\, Z^m X_+^{n_+} X_-^{n_-}\big] &\;=\; -mZ^{m-1} X_+^{n_+} X_-^{n_-} + n_- Z^m X_+^{n_+} X_-^{n_- -1}, \\
\big[Y_-,\, Z^m X_+^{n_+} X_-^{n_-}\big] &\;=\; mZ^{m-1} X_+^{n_+} X_-^{n_-} - n_+ Z^m X_+^{n_+ -1} X_-^{n_-}.
\end{aligned}$$

We conclude that any homogeneous element of $\mathfrak{A}$ that vanishes under $[Y_+,\,\cdot\,]$ is of the form

$$\sum_m \big(ZY_- + X_-Y_+\big)^m\, p_m^+(X_+, Y_+, Y_-),$$

and any homogeneous element of $\mathfrak{A}$ that vanishes under $[Y_-,\,\cdot\,]$ is of the form

$$\sum_m \big(ZY_+ - X_+Y_-\big)^m\, p_m^-(X_-, Y_+, Y_-),$$

for suitable polynomials $p_m^{\pm}$. By induction on the degree in Z, we conclude that an element that is annihilated by both $[Y_+,\,\cdot\,]$ and $[Y_-,\,\cdot\,]$ is of the form

$$\sum_m \big(ZY_+Y_- - X_+Y_-^2 + X_-Y_+^2\big)^m\, q_m(Y_+, Y_-)$$

for suitable polynomials q_m.

We next argue that every q_m is constant. To this end, note that q_m must vanish under the Lie action of $\mathfrak{g}'$, since $ZY_+Y_- - X_+Y_-^2 + X_-Y_+^2$ does. We have

$$\big[X_+,\, Y_+^{n_+} Y_-^{n_-}\big] = -n_- Y_+^{n_+ +1} Y_-^{n_- -1} \quad \text{and} \quad \big[X_-,\, Y_+^{n_+} Y_-^{n_-}\big] = -n_+ Y_+^{n_+ -1} Y_-^{n_- +1}.$$

By induction on the degree in Y_+, we find that

$$\ker\big(\mathfrak{g}' \circlearrowleft \mathfrak{A}\big) \;=\; \mathbb{C}\big[ZY_+Y_- - X_+Y_-^2 + X_-Y_+^2\big].$$

The image of the generator on the right hand side under the symmetrization map equals

$$\begin{aligned}&\big(ZY_+Y_- + ZY_-Y_+ + Y_+ZY_- + Y_-ZY_+ + Y_+Y_-Z + Y_-Y_+Z\big)\\ &\qquad - 2\big(X_+Y_-^2 + Y_-X_+Y_- + Y_-^2X_+\big) + 2\big(X_-Y_+^2 + Y_+X_-Y_+ + Y_+^2X_-\big).\end{aligned}$$

Using the commutator of Z and $Y_\pm$, we calculate that the expression in the first pair of parentheses simplifies to $6ZY_+Y_-$. For the two other expressions, we obtain

$$\begin{aligned} 2\big(X_+Y_-^2 + Y_-X_+Y_- + Y_-^2X_+\big) \;&=\; 2\big(X_+Y_-^2 + 2Y_-X_+Y_- + Y_-Y_+\big)\\ =\; 2\big(3X_+Y_-^2 + 2Y_+Y_- + Y_-Y_+\big) \;&=\; 2\big(3X_+Y_-^2 + 3Y_+Y_-\big),\end{aligned}$$

and similarly

$$\begin{aligned} 2\big(X_-Y_+^2 + Y_+X_-Y_+ + Y_+^2X_-\big) \;&=\; 2\big(X_-Y_+^2 + 2Y_+X_-Y_+ + Y_+X_-Y_-\big)\\ =\; 2\big(3X_-Y_+^2 + 2Y_-Y_+ + Y_+Y_-\big) \;&=\; 2\big(3X_-Y_+^2 + 3Y_+Y_-\big).\end{aligned}$$

Since the contributions of Y_+Y_- for these terms cancel each other, we recover $6C'$ and finish the proof. ☐

2.3. **Affine modular-invariant functions.** It will be convenient to pass back and forth between functions on $\mathrm{G}'(\mathbb{R})$ and functions on the Jacobi upper half plane $\mathbb{H}' = \mathrm{G}'(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$. For this we define the *slash action* on functions on $\mathbb{H}'$ parametrized by $k \in \mathbb{Z}$ by

$$\Big(\phi\big|'_k \big(\big(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\big), w_1, w_2\big)\Big)(\tau, z) \;=\; (c\tau+d)^{-k}\,\phi\Big(\frac{a\tau+b}{c\tau+d}, \frac{z+w_1\tau+w_2}{c\tau+d}\Big), \tag{2.8}$$

extending the usual slash action on the upper half plane $\mathbb{H}$. We say that $\phi : \mathbb{H}' \to \mathbb{C}$ is an *affine modular-invariant function of weight k* if

$$\phi\big|'_k(\gamma, w) \;=\; \phi \quad \text{for all} \quad (\gamma, w) \in \mathrm{G}'(\mathbb{Z}) = \mathrm{SAff}_2(\mathbb{Z}). \tag{2.9}$$

The first half of the correspondence is the *lift* of affine modular-invariant functions to functions on $\mathrm{G}'(\mathbb{R})$ by

$$\widetilde{\phi}(g) := \big(\phi\big|'_k\, g\big)(i, 0) \;=\; e^{ik\theta} y^{\frac{k}{2}}\, \phi(\tau, z) \tag{2.10}$$

for forms of weight k, where for the second expression $\tau = x + iy$, $z = u + iv$, and $g = (x, y, u, v, \theta)$ as in the Iwasawa decomposition in (3.8) below. Note that the notation $\widetilde{\phi}$ suppresses the weight k. We generalise the standard raising as lowering operators and define the operators L_k, R_k, $\mathrm{L}^{\mathrm{H}}_k$, and $\mathrm{R}^{\mathrm{H}}_k$ on affine modular invariant functions via the lifts

$$\widetilde{\mathrm{L}_k\,\phi} := X_-\widetilde{\phi}, \quad \widetilde{\mathrm{R}_k\,\phi} := X_+\widetilde{\phi}, \quad \widetilde{\mathrm{L}^{\mathrm{H}}_k\,\phi} := Y_-\widetilde{\phi}, \quad \widetilde{\mathrm{R}^{\mathrm{H}}_k\,\phi} := Y_+\widetilde{\phi}. \tag{2.11}$$

From

$$\begin{aligned} X_\pm\,\widetilde{\phi} \;&=\; \pm\tfrac{i}{2} e^{i(k\pm 2)\theta} y^{\frac{k}{2}} \big(2y(\partial_x\phi) + 2v(\partial_u\phi) \mp 2iy(\partial_y\phi) \mp 2iv(\partial_v\phi) \mp ik\phi - ik\phi\big),\\ Y_\pm\,\widetilde{\phi} \;&=\; \pm\tfrac{i}{2} e^{i(k\pm 1)\theta} y^{\frac{k+1}{2}} \big((\partial_u\phi) \mp i(\partial_v\phi)\big)\end{aligned}$$

we read off that the weight of the functions $\mathrm{L}_k\,\phi$, $\mathrm{R}_k\,\phi$, $\mathrm{L}^{\mathrm{H}}_k\,\phi$, and $\mathrm{R}^{\mathrm{H}}_k\,\phi$ in (2.11) is $k-2$, $k+2$, $k-1$, and $k+1$ respectively. Explicit calculations show

$$\begin{aligned} \mathrm{L}_k &= -2iy^2\,\big(\partial_{\overline{\tau}} + vy^{-1}\partial_{\overline{z}}\big), & \mathrm{R}_k &= 2i\,\big(\partial_\tau + vy^{-1}\partial_z\big) + ky^{-1}, \\ \mathrm{L}^{\mathrm{H}}_k &= -iy\,\partial_{\overline{z}}, & \mathrm{R}^{\mathrm{H}}_k &= i\partial_z. \end{aligned} \tag{2.12}$$

and yield the following lemma:

Lemma 2.2. *There are differential operators, $-\Delta^{\mathrm{fol}}_k$ and $-\Delta^{\mathrm{tot}}_k$ which we call the* foliated Laplacian *and* total Laplacian *of weight k, respectively, with the property that*

$$\widetilde{\Delta^{\mathrm{fol}}_k\,\phi} := \mathcal{D}^{\mathrm{fol}}\,\widetilde{\phi} \quad \textit{and} \quad \widetilde{\Delta^{\mathrm{tot}}_k\,\phi} := \mathcal{D}^{\mathrm{tot}}\,\widetilde{\phi} \tag{2.13}$$

for any affine modular-invariant function ϕ of weight k. In (x, y, u, v) coordinates

$$\begin{aligned} \Delta^{\mathrm{tot}}_k &= k\,\mathrm{R}^{\mathrm{H}}_{k-1}\mathrm{L}^{\mathrm{H}}_k - \mathrm{R}_{k-2}\mathrm{L}^{\mathrm{H}}_{k-1}\mathrm{L}^{\mathrm{H}}_k + \mathrm{L}_{k+2}\mathrm{R}^{\mathrm{H}}_{k+1}\mathrm{R}^{\mathrm{H}}_k \\ &= y\big(k\,\partial_{\overline{z}} + 2iv\,\partial_z\partial_{\overline{z}}\big)\big(\partial_{\overline{z}} + \partial_z\big) + 2iy^2\,\big(\partial_{\overline{\tau}}\partial_z^2 + \partial_\tau\partial_{\overline{z}}^2\big) \\ &= \tfrac{k}{2}y\,\partial_u(\partial_u + i\partial_v) + \tfrac{i}{2}y^2\,\partial_x(\partial_u^2 - \partial_v^2) + iy^2\,\partial_y\partial_u\partial_v + \tfrac{i}{2}yv\,\partial_u(\partial_u^2 + \partial_v^2), \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} \Delta^{\mathrm{fol}}_k &= \mathrm{R}_{k-2}\,\mathrm{L}_k \\ &= 4y^2\partial_\tau\partial_{\overline{\tau}} + 4yv\big(\partial_\tau\partial_{\overline{z}} + \partial_{\overline{\tau}}\partial_z\big) + 4v^2\partial_z\partial_{\overline{z}} - 2ik\big(y\partial_{\overline{\tau}} + v\partial_{\overline{z}}\big)1\big) \\ &= y^2(\partial_x^2 + \partial_y^2) + 2yv(\partial_x\partial_u + \partial_y\partial_v) + v^2(\partial_u^2 + \partial_v^2) \\ &\quad - iky(\partial_x + i\partial_y) - ikv(\partial_u + i\partial_v)\big) \end{aligned} \tag{2.15}$$

Furthermore, in (x, y, p, q) coordinates,

$$\Delta^{\mathrm{fol}}_k = y^2(\partial_x^2 + \partial_y^2) - iky(\partial_x + i\partial_y). \tag{2.16}$$

Note that there is no dependence on p, q in the definition of Δ^{fol}_k.

The converse of the correspondence (2.10) is stated for fixed K-type. Here a vector v in a representation ρ of K is said to be *of* K*-type* $k \in \mathbb{Z}$, if

$$\rho\big(\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)\big)v = e^{ik\theta}\,v. \tag{2.17}$$

In particular, a function on $\mathrm{G}'(\mathbb{R})$ transforming in this way under right shifts by K is said to be of K-type k.

Lemma 2.3. *Given an affine modular-invariant function ϕ, the function $\widetilde{\phi}$ defined in* (2.10) *is a $\mathrm{G}'(\mathbb{Z})$-left-invariant function.*

Conversely, if f is $\mathrm{G}'(\mathbb{Z})$-left-invariant and of K*-type k, then*

$$\phi(x + iy, u + iv) \;=\; f\big(\left(\begin{smallmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{smallmatrix}\right), uy^{-1/2}, vy^{-1/2}\big)$$

is an affine modular-invariant function of weight k with $\widetilde{\phi} = f$.

Proof. The modular invariance of ϕ implies that for $\gamma \in \mathrm{G}'(\mathbb{Z})$ and $g \in \mathrm{G}'(\mathbb{R})$

$$\widetilde{\phi}(\gamma g) \;=\; \big(\phi\big|'_k\,\gamma g\big)(i, 0) \;=\; \big(\phi\big|'_k\,g\big)(i, 0) \;=\; \widetilde{\phi}(g).$$

That is, we can view $\widetilde{\phi}$ as a function on $\mathrm{G}'(\mathbb{Z})\backslash\mathrm{G}'(\mathbb{R})$.

To determine the action of K by right-shifts on $\widetilde{\phi}$ in (2.10) we consider $\left(\begin{smallmatrix} d & -c \\ c & d \end{smallmatrix}\right) \in \mathrm{K}$ with $d = \cos\theta$, $c = -\sin\theta$ and $g \in \mathrm{G}'(\mathbb{R})$ and compute:

$$\begin{aligned} \widetilde{\phi}\big(g \left(\begin{smallmatrix} d & -c \\ c & d \end{smallmatrix}\right)\big) \;&=\; \big(\phi\big|_k' \, g \left(\begin{smallmatrix} d & -c \\ c & d \end{smallmatrix}\right)\big)(i,0) \\ =\; (ci+d)^{-k} \, \big(\phi\big|_k' \, g\big)(i,0) \;&=\; (ci+d)^{-k} \, \widetilde{\phi}(g) \;=\; e^{ik\theta} \, \widetilde{\phi}(g). \end{aligned}$$

In particular, $\widetilde{\phi}$ is K-finite. The converse is a direct computation, see also the Iwasawa decomposition in (3.8). □

Under this correspondence the usual L^2-scalar product on $\mathrm{G}'(\mathbb{Z})\backslash\mathrm{G}'(\mathbb{R})$ corresponds to the scalar product

$$\begin{aligned} \big\langle \psi_1, \psi_2 \big\rangle \;&:=\; \int_{\Gamma'\backslash\mathbb{H}'} \psi_1(\tau, z)\overline{\psi_2(\tau, z)} \, \frac{\mathrm{d}x\mathrm{d}y\mathrm{d}u\mathrm{d}v}{y^{3-k}} \\ &=\; \int_{\Gamma'\backslash\mathbb{H}'} \psi_1(\tau, z)\overline{\psi_2(\tau, z)} \, \frac{\mathrm{d}x\mathrm{d}y\mathrm{d}p\mathrm{d}q}{y^{2-k}}. \end{aligned} \tag{2.18}$$

We write the corresponding norm as $\|\cdot\|_{\mathbb{H}',k}$ or $\|\cdot\|_{\mathbb{H}'}$, suppressing the k-dependence. It follows from the definition that the measure on $\mathrm{G}'(\mathbb{Z})\backslash\mathrm{G}'(\mathbb{R})/\mathrm{K}$ induced by this scalar produce is the (push-forward to the K-quotient of the) Masur-Veech measure ν_{MV}.

2.4. **Invariant differential operators.** We define the vertical Laplace operator in analogy with the formula $\Delta_k^{\mathrm{fol}} = \mathrm{R}_{k-2}\mathrm{L}_k$ in (2.15) for the foliated one as

$$\Delta^{\mathrm{vert}} \;:=\; \mathrm{R}^{\mathrm{H}}\mathrm{L}^{\mathrm{H}} \;=\; y\partial_z\partial_{\bar{z}}. \tag{2.19}$$

Note that it does not depend on the weight. The vertical Laplace operator does not play a distinguished role by itself, but it is the foundation to define a one parameter family of *compound Laplace operators* perturbing the foliated one. For $\varepsilon > 0$, we set

$$-\Delta_k^{\mathrm{cmp}(\varepsilon)} := -\Delta_k^{\mathrm{fol}} - \varepsilon\Delta^{\mathrm{vert}}. \tag{2.20}$$

Lemma 2.4. *The foliated Laplace operator, the vertical Laplace operator and consequently the family of compound Laplace operators are equivariant with respect to the action of the special affine group, i.e.,*

$$\begin{aligned} \big(\Delta_k^{\mathrm{fol}}\phi\big)\big|_k' \, g \;&=\; \Delta_k^{\mathrm{fol}}\big(\phi\big|_k' \, g\big), \quad \big(\Delta^{\mathrm{vert}}\phi\big)\big|_k' \, g = \Delta^{\mathrm{vert}}\big(\phi\big|_k' \, g\big) \\ \big(\Delta_k^{\mathrm{cmp}(\varepsilon)}\phi\big)\big|_k' \, g \;&=\; \Delta_k^{\mathrm{cmp}(\varepsilon)}\big(\phi\big|_k' \, g\big). \end{aligned} \tag{2.21}$$

for all $g \in \mathrm{G}'(\mathbb{R})$.

Proof. One directly computes the covariance properties

$$\begin{aligned} \big(\mathrm{L}_k\phi\big)\big|_{k-2}' \, g = \mathrm{L}_k\big(\phi\big|_k' \, g\big), \qquad & \big(\mathrm{R}_k\phi\big)\big|_{k+2}' \, g = \mathrm{R}_k\big(\phi\big|_k' \, g\big), \\ \big(\mathrm{L}_k^{\mathrm{H}}\phi\big)\big|_{k-1}' \, g = \mathrm{L}_k^{\mathrm{H}}\big(\phi\big|_k' \, g\big), \qquad & \big(\mathrm{R}_k^{\mathrm{H}}\phi\big)\big|_{k+1}' \, g = \mathrm{R}_k^{\mathrm{H}}\big(\phi\big|_k' \, g\big) \end{aligned} \tag{2.22}$$

for any $\phi : \mathbb{H}' \longrightarrow \mathbb{C}$, any $k \in \mathbb{Z}$, and any $g \in \mathrm{G}'(\mathbb{R})$, see also [Bum97, Section 2.1] for the first two equalities. (This goes back to the general setup considered by Helgason [Hel59], or also [CWR16].) The claimed equivariance follows directly from this. □

The following proposition tells us that it is the compound Laplacian that has better chances to have a good spectral decomposition for $\mathrm{L}^2(\mathbb{H}')$.

Proposition 2.5. *For every $k \in \mathrm{N}$ and $\varepsilon > 0$, the compound Laplace operator $-\Delta_k^{\mathrm{cmp}(\varepsilon)}$ is a self-adjoint elliptic operator on $\mathrm{L}^2(\mathbb{H}')$, and the foliated Laplacian $-\Delta_k^{\mathrm{fol}}$ is hypoelliptic. Furthermore, the symmetric bilinear form on $\mathrm{L}^2(\mathcal{H}(0,0), \nu_{\mathrm{MV}})$ associated with the compound Laplacian $-\Delta_k^{\mathrm{cmp}(\varepsilon)}$ is*

$$
\begin{aligned}
Q_k^{(\varepsilon)}(\phi, \psi) = & \int_{\Gamma'\backslash\mathbb{H}'} y^k \, \nabla_{x,y}\phi \cdot \overline{\nabla_{x,y}\psi} \; \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}p \, \mathrm{d}q \\
& + \int_{\Gamma'\backslash\mathbb{H}'} iky^{k-1}\partial_x\phi\overline{\psi} \; \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}p \, \mathrm{d}q \\
& + \varepsilon \int_{\Gamma'\backslash\mathbb{H}'} y^{k-2}\nabla_{u,v}\phi \cdot \overline{\nabla_{u,v}\psi} \; \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}u \, \mathrm{d}v.
\end{aligned}
\tag{2.23}
$$

Remark 2.6. Note that the terms in the bilinear form are not all given in terms of the same coordinates. While unconventional, this greatly simplifies the proof that $\Delta^{\mathrm{cmp}(\varepsilon)}$ has compact resolvent when restricted to cusp forms.

Proof. We first observe that $iky\partial_x$ is self-adjoint as the product of commuting self-adjoint operators, and that the second term in the bilinear form is simply $\langle iky\partial_x\phi, \psi\rangle$. We turn our attention to the rest of the foliated Laplacian, and use indices in differential operator to represent the variables they are acting on:

$$
\begin{aligned}
\langle(-y^2\Delta_{x,y} - ky\partial_y)\phi, \psi\rangle \; = \; & \int_{\Gamma'\backslash\mathbb{H}'} y^2((-\Delta_{x,y} - ky^{-1}\partial_y)\phi)\overline{\psi} \, \frac{\mathrm{d}x \, \mathrm{d}y \, \mathrm{d}p \, \mathrm{d}q}{y^{2-k}} \\
= \; & \int_{\Gamma'\backslash\mathbb{H}'} ((-\Delta_{x,y} - ky^{-1}\partial_y)\phi)\overline{\psi}y^k \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}p \, \mathrm{d}q \\
= \; & \int_{\Gamma'\backslash\mathbb{H}'} \nabla_{x,y}\phi \cdot \nabla_{x,y}(y^k\overline{\psi}) \; \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}p \, \mathrm{d}q \\
& - k \int_{\Gamma'\backslash\mathbb{H}'} \partial_y\phi \cdot \psi \, y^{k-1} \; \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}p \, \mathrm{d}q \\
= \; & \int_{\Gamma'\backslash\mathbb{H}'} y^k \, \nabla_{x,y}\phi \cdot \overline{\nabla_{x,y}\psi} \; \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}p \, \mathrm{d}q \, ,
\end{aligned}
$$

which we recognise as the first term in the bilinear form. Finally, for the vertical Laplacian we compute in (x, y, u, v) coordinates:

$$
\begin{aligned}
\langle -\varepsilon\Delta^{\mathrm{vert}}\phi, \psi\rangle \; = \; & \varepsilon \int_{\Gamma'\backslash\mathbb{H}'} (-\Delta_{u,v}\phi)\overline{\psi}y^{k-2} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}u \, \mathrm{d}v \\
= \; & \varepsilon \int_{\Gamma'\backslash\mathbb{H}'} y^{k-2}\nabla_{u,v}\phi \cdot \overline{\nabla_{u,v}\psi} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}u \, \mathrm{d}v.
\end{aligned}
$$

Self-adjointness is now a consequence of the fact that the Laplacian is represented by a symmetric bilinear form.

For ellipticity, we express the vertical Laplacian also in (x, y, p, q) coordinates as

$$
-\varepsilon\Delta^{\mathrm{vert}} = -\frac{\varepsilon}{4}y\big(\partial_q^2 + y^{-2}(\partial_p - x\partial_q)^2\big)
\tag{2.24}
$$

Viewing derivatives as tangent vectors, consider the following local coordinates

$$
T^*\mathbb{H}' = \big\{(x, \xi, y, \eta, p, \zeta, q, \omega) : y > 0, \, \langle\xi, \partial_x\rangle = \langle\eta, \partial_y\rangle = \langle\zeta, \partial_p\rangle = \langle\omega, \partial_q\rangle = 1\big\}.
$$

for the cotangent bundle, where we use $\langle\cdot, \cdot\rangle$ to denote the pairing between the tangent and cotangent bundle. We can now read off the expression of the foliated and compound Laplacian in coordinates that their principal symbols are given by

$$\mathrm{symb}\big(-\Delta_k^{\mathrm{fol}}\big) \;=\; y^2(\xi^2+\eta^2)$$

and

$$\mathrm{symb}\big(-\Delta^{\mathrm{vert}}\big) \;=\; \frac{y}{4}\big(\omega^2 + y^{-2}(\zeta - x\omega)^2\big).$$

Note that these principal symbols do not depend on k, and that an operator P is hypoelliptic if $\mathrm{symb}(P)$ does not change sign and elliptic if $\mathrm{symb}(P) = 0$ implies that $\xi = \eta = \zeta = \omega = 0$. Hypoellipticity of the foliated Laplacian is directly observed. Since $\mathrm{symb}(-\Delta_k^{\mathrm{cmp}(\varepsilon)}) = \mathrm{symb}(-\Delta_k^{\mathrm{fol}}) + \mathrm{symb}(-\varepsilon\Delta^{\mathrm{vert}})$ and $y > 0$, it is easy to see that $\mathrm{symb}(-\Delta_k^{\mathrm{cmp}(\varepsilon)}) > 0$ whenever $(\xi,\eta,\omega) \neq (0,0,0)$. On the other hand, if $\omega = 0$, then any $\zeta \neq 0$ ensures the same thing, so that the operator is elliptic. □

3. The special affine group and its representation theory

The first goal of this section, Theorem 3.7 is to recall an application of Mackey theory and to classify the genuine representations of $\mathrm{G}'(\mathbb{R}) = \mathrm{SAff}_2(\mathbb{R})$ up to isomorphism. These are the representations $\pi^{\mathrm{SAff}}_{nm^2}$ defined in (3.18). The second goal of this section is to compute the restrictions of these representations as representations of $\mathrm{G}(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})$.

3.1. **The goal: decomposing the L^2-space.** The Haar measure on $\mathrm{G}'(\mathbb{R})$ gives rise to a right-invariant measure on $\mathrm{G}'(\mathbb{Z})\backslash\mathrm{G}'(\mathbb{R})$. We are interested in the space of square-integrable functions on this quotient, $\mathrm{L}^2\big(\mathrm{G}'(\mathbb{Z})\backslash\mathrm{G}'(\mathbb{R})\big)$ on this quotient. This is the same as understanding the space $\mathrm{L}^2(\mathcal{H}(0,0),\nu_{\mathrm{MV}})$ of square-integrable functions on the space of tori with two marked points, equipped with the Masur–Veech measure, as $\mathcal{H}(0,0)$ and $\mathrm{G}'(\mathbb{Z})\backslash\mathrm{G}'(\mathbb{R})$ differ by a set of measure zero.

By [Dix57, Théorème 1] the group $\mathrm{G}'(\mathbb{R})$ is of type I. In particular by [BH20, Theorem 6.D.7] we have a direct integral decomposition

$$\mathrm{L}^2\big(\mathrm{G}'(\mathbb{Z})\backslash\mathrm{G}'(\mathbb{R})\big) \;\cong\; \int_{\widehat{\mathrm{G}}'}^{\oplus} \pi \,\mathrm{d}\mu_{\mathrm{G}'}(\pi), \tag{3.1}$$

where $\widehat{\mathrm{G}}'$ is the unitary dual of $\mathrm{G}'(\mathbb{R})$. Restricting to $G(\mathbb{R})$-representations gives us another direct integral decomposition

$$\mathrm{Res}^{\mathrm{G}'(\mathbb{R})}_{\mathrm{G}(\mathbb{R})}\,\mathrm{L}^2\big(\mathrm{G}'(\mathbb{Z})\backslash\mathrm{G}'(\mathbb{R})\big) \;\cong\; \int_{\widehat{\mathrm{G}}}^{\oplus} \pi \,\mathrm{d}\mu_{\mathrm{G}}(\pi). \tag{3.2}$$

There is an embedding

$$\mathrm{L}^2\big(\mathrm{G}(\mathbb{Z})\backslash\mathrm{G}(\mathbb{R})\big) \hookrightarrow \mathrm{L}^2\big(\mathrm{G}'(\mathbb{Z})\backslash\mathrm{G}'(\mathbb{R})\big).$$

Its range is consists in functions invariant under the action of the translation subgroup $\mathbb{R}^2$ of $\mathrm{G}'(\mathbb{R})$.

Definition 3.1. We call

$$\mathrm{L}^2\big(\mathrm{G}'(\mathbb{Z})\backslash\mathrm{G}'(\mathbb{R})\big)^{\mathrm{gen}} \;:=\; \mathrm{L}^2\big(\mathrm{G}(\mathbb{Z})\backslash\mathrm{G}(\mathbb{R})\big)^{\perp} \tag{3.3}$$

the *genuine part* of $\mathrm{L}^2\big(\mathrm{G}'(\mathbb{Z})\backslash\mathrm{G}'(\mathbb{R})\big)$, so that

$$\mathrm{L}^2\big(\mathrm{G}'(\mathbb{Z})\backslash\mathrm{G}'(\mathbb{R})\big) \;=\; \mathrm{L}^2\big(\mathrm{G}(\mathbb{Z})\backslash\mathrm{G}(\mathbb{R})\big) \oplus \mathrm{L}^2\big(\mathrm{G}'(\mathbb{Z})\backslash\mathrm{G}'(\mathbb{R})\big)^{\mathrm{gen}}. \tag{3.4}$$

Since our goal is an explicit determination of the right-hand side of the decompositions (3.1) and (3.2), we may of course restrict attention to the genuine subspace. Integration along the torus fibers of the projection $\mathcal{H}(0,0) \to \mathcal{H}(0)$ defines an averaging map $\mathrm{av} : \mathrm{L}^2(\mathcal{H}(0,0)) \to \mathrm{L}^2(\mathcal{H}(0))$. Disintegrating the Haar measure of $\mathrm{G}'(\mathbb{R})$ along the torus fibers shows

$$\mathrm{L}^2\big(\mathrm{G}'(\mathbb{Z})\backslash\mathrm{G}'(\mathbb{R})\big)^{\mathrm{gen}} \;=\; \mathrm{Ker}(\mathrm{av})\,. \tag{3.5}$$

Standard subgroups of $\mathrm{G}'(\mathbb{R})$ **and coordinates**. We fix notation for the standard subgroups of $\mathrm{SL}_2(\mathbb{R})$ (left) and the special affine group (right), noting that we abuse notation and write again $g = (g,0,0)$ for the image of an element of G in G′.

$$\begin{aligned} \mathrm{A}(\mathbb{R}) &:= & \mathrm{A}'(\mathbb{R}) &:= \big\{ \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1}\end{smallmatrix}\right) : a \in \mathbb{R}^\times \big\}, \\ \mathrm{N}(\mathbb{R}) &:= \big\{ \left(\begin{smallmatrix} 1 & b \\ 0 & 1\end{smallmatrix}\right) : b \in \mathbb{R} \big\}, & \mathrm{N}'(\mathbb{R}) &:= \big\{ \big(\left(\begin{smallmatrix} 1 & b \\ 0 & 1\end{smallmatrix}\right), w_1, w_2\big) : b, w_1, w_2 \in \mathbb{R} \big\}, \\ & & \mathrm{H}'(\mathbb{R}) &:= \big\{ \big(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1\end{smallmatrix}\right), w_1, w_2\big) : w_1, w_2 \in \mathbb{R} \big\}. \end{aligned}$$

which gives rise to the Iwasawa decomposition

$$\mathrm{G}'(\mathbb{R}) \;=\; \mathrm{N}'(\mathbb{R})\,\mathrm{A}'(\mathbb{R})\,\mathrm{K}, \tag{3.6}$$

as well as two further decompositions

$$\mathrm{G}'(\mathbb{R}) \;=\; \mathrm{G}(\mathbb{R})\,\mathrm{H}'(\mathbb{R}) = \mathrm{H}'(\mathbb{R})\mathrm{G}(\mathbb{R}).$$

We denote the parabolic subgroups in the Iwasawa decomposition by

$$\begin{aligned} \mathrm{P}(\mathbb{R}) &= \mathrm{N}(\mathbb{R})\,\mathrm{A}(\mathbb{R}) = \mathrm{A}(\mathbb{R})\,\mathrm{N}(\mathbb{R}), \\ \mathrm{P}'(\mathbb{R}) &= \mathrm{N}'(\mathbb{R})\,\mathrm{A}'(\mathbb{R}) = \mathrm{A}'(\mathbb{R})\,\mathrm{N}'(\mathbb{R}). \end{aligned} \tag{3.7}$$

Given $(\tau, z) \in \mathbb{H}'$ as in (2.1), we let $a = \sqrt{y}$, $b = x$, $w_1 = v/y$, and $w_2 = u$. Then for all $\theta \in \mathbb{R}$ the Iwasawa decomposition corresponds to the identity

$$\begin{aligned} (\tau, z) &= \big(\left(\begin{smallmatrix} 1 & b \\ 0 & 1\end{smallmatrix}\right), w_1, w_2\big) \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1}\end{smallmatrix}\right) \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{smallmatrix}\right) (i,0) \\ &= \left(\left(\begin{smallmatrix} a & ba^{-1} \\ 0 & a^{-1}\end{smallmatrix}\right), w_1 a, w_2 a^{-1}\right) (i,0)\,. \end{aligned} \tag{3.8}$$

Alternatively, if we let $a = \sqrt{y}$, $b = x$, $w_1 = p$ and $w_2 = q$, then emphasizing the coordinates $z = p\tau + q$ for all $\theta \in \mathbb{R}$ we have

$$\big(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1\end{smallmatrix}\right), w_1, w_2\big) \left(\begin{smallmatrix} 1 & b \\ 0 & 1\end{smallmatrix}\right) \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1}\end{smallmatrix}\right) \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{smallmatrix}\right) (i,0) \;=\; (\tau, z). \tag{3.9}$$

The relation between these sets of coordinates is given by $p = v/y$ and $q = u - vx/y$.

In the coordinates of (3.8), the Haar measures on the groups $\mathrm{N}'(\mathbb{R})$, $\mathrm{A}'(\mathbb{R})$, K, and $\mathrm{N}'(\mathbb{R})\mathrm{A}'(\mathbb{R})\mathrm{K}$ are given respectively by

$$\mathrm{d}b\,\mathrm{d}w_1\,\mathrm{d}w_2, \quad \frac{\mathrm{d}a}{a}, \quad \frac{\mathrm{d}\theta}{2\pi}, \quad \text{and} \quad \frac{\mathrm{d}\theta\,\mathrm{d}b\,\mathrm{d}w_1\,\mathrm{d}w_2\,\mathrm{d}a}{2\pi\, a^3}. \tag{3.10}$$

Notation for L^2-induction of representations. We only consider the case of a locally compact group $\mathrm{G} = \mathrm{HL}$ for two subgroups H and L such that G/H is isomorphic to L as a measure space. Our notation is consistent with [Wal88, Sections 1.5 and 5.2] and [BS98, Section 2.1], which in turn follows [Kir76].

Given a representation σ of H on a Hilbert space $V(\sigma)$, its L^2-induction to G is given by right shifts on

$$V\big(\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}\,\sigma\big) \;:=\; \Big\{f : \mathrm{G} \longrightarrow V(\sigma) : f \text{ measurable}, \; f \text{ square integrable on } L, \\ f(hg) \;=\; \sqrt{\frac{\delta_{\mathrm{H}}(h)}{\delta_G(h)}}\,\sigma(h)f(g) \text{ for all } h \in \mathrm{H}, g \in \mathrm{G}\Big\}\,,,$$

where Δ_{G} and Δ_{H} are the modular functions on G and H.

3.2. **Representation theory of the upper triangular subgroup $\mathrm{P}(\mathbb{R})$.** Consider the representations of $\mathrm{P}(\mathbb{R})$ that factor through the quotient by $\mathrm{N}(\mathbb{R})$

$$\chi^{\mathrm{P}}_{+,s} : \left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right) \longmapsto |a|^s \quad \text{and} \quad \chi^{\mathrm{P}}_{-,s} : \left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right) \longmapsto \mathrm{sgn}(a)|a|^s, \quad s \in \mathbb{C}, \tag{3.11}$$

and abbreviate

$$\mathrm{sgn}^{\mathrm{P}} \;:=\; \chi^{\mathrm{P}}_{-,0}. \tag{3.12}$$

Further, through the paper, we set

$$e(x) \;:=\; e^{2\pi i\, x}.$$

Proposition 3.2. *The irreducible representations that are trivial on* $\mathrm{N}(\mathbb{R})$ *are given by* $\chi^{\mathrm{P}}_{-,s}$ *and* $\chi^{\mathrm{P}}_{+s,}$. *They are unitary if and only if* $s \in i\mathbb{R}$.

The irreducible unitary representations which are not trivial on $\mathrm{N}(\mathbb{R})$ *are given, up to unitary equivalence, by*

$$\pi^{\mathrm{P}}_{\pm} \;:=\; \mathrm{Ind}_{\mathrm{N}(\mathbb{R})}^{\mathrm{P}(\mathbb{R})}\,\big(\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \longmapsto e(\pm b)\big) \quad \textit{and} \quad \mathrm{sgn}^{\mathrm{P}}\,\pi^{\mathrm{P}}_{\pm}. \tag{3.13}$$

Proof. For the first statement we observe that those representations factor through the quotient $\mathrm{A}(\mathbb{R}) \cong \mathbb{R}^{\times}$ and are thus characters. For the second statement, consider the map

$$\left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right) \longmapsto \left(\begin{smallmatrix} a^2 & b \\ 0 & 1 \end{smallmatrix}\right),$$

which is surjective with central kernel ± 1 onto the connected component of the identity $\mathrm{SAff}_1(\mathbb{R})^0$ of the one-dimensional affine group $\mathrm{SAff}_1(\mathbb{R})$, see [BH20, Remark 3.C.6]. We can thus apply the classification given there and append the central character $\mathrm{sgn}^{\mathrm{P}}$ to obtain the desired statement. □

Proposition 3.3. *The regular representation of* $\mathrm{P}(\mathbb{R})$ *decomposes as*

$$\mathrm{L}^2\big(\mathrm{P}(\mathbb{R})\big) \;\cong\; \aleph_0\left(\pi^{\mathrm{P}}_{+} \oplus \pi^{\mathrm{P}}_{-} \,\oplus\, \mathrm{sgn}^{\mathrm{P}}\pi^{\mathrm{P}}_{+} \oplus \mathrm{sgn}^{\mathrm{P}}\pi^{\mathrm{P}}_{-}\right), \tag{3.14}$$

where the right hand side denotes a countably infinite direct sum of the representation in the brackets.

Proof. Let $\mathrm{P}(\mathbb{R})^0 \subset \mathrm{P}(\mathbb{R})$ be the connected component of the identity. We use induction by steps and first decompose the regular representation of $\mathrm{N}(\mathbb{R})$:

$$\mathrm{L}^2\big(\mathrm{P}(\mathbb{R})\big) \;=\; \mathrm{Ind}_1^{\mathrm{P}(\mathbb{R})}\,\mathbb{1} \;\cong\; \mathrm{Ind}_{\mathrm{P}(\mathbb{R})^0}^{\mathrm{P}(\mathbb{R})}\,\mathrm{Ind}_{\mathrm{N}(\mathbb{R})}^{\mathrm{P}(\mathbb{R})^0}\,\mathrm{Ind}_1^{\mathrm{N}(\mathbb{R})}\,\mathbb{1}.$$

Now by Fourier analysis, we have

$$\mathrm{L}^2\big(\mathrm{N}(\mathbb{R})\big) \;=\; \mathrm{Ind}_1^{\mathrm{N}(\mathbb{R})} \,\mathbb{1} \;\cong\; \int_{\mathbb{R}}^{\oplus} \big(\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \longmapsto e(nb)\big)\, \mathrm{d}n.$$

Using Fubini and the definition of induced representations, we see that

$$\mathrm{Ind}_{\mathrm{N}(\mathbb{R})}^{\mathrm{P}(\mathbb{R})^0} \int_{\mathbb{R}}^{\oplus} \big(\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \longmapsto e(nb)\big)\, \mathrm{d}n \;\cong\; \int_{\mathbb{R}}^{\oplus} \mathrm{Ind}_{\mathrm{N}(\mathbb{R})}^{\mathrm{P}(\mathbb{R})^0} \;\big(\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \longmapsto e(nb)\big)\, \mathrm{d}n.$$

Proceeding as in the proof of Proposition 3.2, we recognize the inductions on the right hand side. They are isomorphic to the restriction of $\pi^{\mathrm{P}}_{\mathrm{sgn}(n)}$ to $\mathrm{P}(\mathbb{R})^0$ if $n \neq 0$, while if $n = 0$ it is the regular representation of $\mathrm{P}(\mathbb{R})^0/\mathrm{N}(\mathbb{R})$. Since $\{0\} \subset \mathbb{R}$ has measure zero, we can discard its contribution to the direct integral. We conclude that

$$\begin{aligned} &\int_{\mathbb{R}}^{\oplus} \mathrm{Ind}_{\mathrm{N}(\mathbb{R})}^{\mathrm{P}(\mathbb{R})^0} \;\big(\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \longmapsto e(nb)\big)\, \mathrm{d}n \\ \cong\; &\int_{\mathbb{R}^+}^{\oplus} \mathrm{Res}_{\mathrm{P}(\mathbb{R})^0}^{\mathrm{P}(\mathbb{R})} \;\pi^{\mathrm{P}}_+ \,\mathrm{d}n \;\oplus\; \int_{\mathbb{R}^-}^{\oplus} \mathrm{Res}_{\mathrm{P}(\mathbb{R})^0}^{\mathrm{P}(\mathbb{R})} \;\pi^{\mathrm{P}}_- \,\mathrm{d}n. \end{aligned}$$

The direct integrals on the right hand side have constant integrand. The direct integral over $\mathbb{R}^{\pm}$ yields countably infinite multiplicity by the isomorphism of Hilbert spaces $\mathrm{L}^2(\mathbb{R}) \cong \mathrm{L}^2(\mathbb{Z}_{>0})$, so that we arrive at

$$\int_{\mathbb{R}}^{\oplus} \mathrm{Ind}_{\mathrm{N}(\mathbb{R})}^{\mathrm{P}(\mathbb{R})^0} \;\big(\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \longmapsto e(nb)\big)\, \mathrm{d}n \;\cong\; \aleph_0 \left(\mathrm{Res}_{\mathrm{P}(\mathbb{R})^0}^{\mathrm{P}(\mathbb{R})} \;\pi^{\mathrm{P}}_+ \;\oplus\; \mathrm{Res}_{\mathrm{P}(\mathbb{R})^0}^{\mathrm{P}(\mathbb{R})} \;\pi^{\mathrm{P}}_- \right).$$

Finally, induction to $\mathrm{P}(\mathbb{R})$ introduces the sign character $\mathrm{sgn}^{\mathrm{P}}$ confirming the proposition. □

3.3. **Representation theory of $\mathrm{SL}_2(\mathbb{R})$: a brief summary.** The following results are standard and appear in many text books, e.g. [Kna01; Wal88]. We set

$$I^{\mathrm{SL}}_{+,s} \;:=\; \mathrm{Ind}_{\mathrm{P}(\mathbb{R})}^{\mathrm{G}(\mathbb{R})} \;|a|^{s+1}, \quad I^{\mathrm{SL}}_{-,s} \;:=\; \mathrm{Ind}_{\mathrm{P}(\mathbb{R})}^{\mathrm{G}(\mathbb{R})} \;\mathrm{sgn}(a)|a|^{s+1}.$$

Note that the shift $s+1$ in the exponents is chosen in such a way that purely imaginary s correspond to unitary representation. We have a duality between $I^{\mathrm{SL}}_{\varepsilon,s}$ and $I^{\mathrm{SL}}_{\varepsilon,-s}$ via intertwining operators explained in [Wal88, Section 5.3]. If $k \in \mathbb{Z}_{>0}$, then $I^{\mathrm{SL}}_{-,k-1}$ is reducible with two infinite-dimensional constituents $D^{\mathrm{SL}}_{\pm k}$, which are discrete series if $k > 1$ and limits of discrete series if $k = 1$.

Theorem 3.4 (Bargmann)**.** *The irreducible unitary representations of* $\mathrm{SL}_2(\mathbb{R})$ *are given up to unitary equivalence by*

- *(a) the principal series* $I^{\mathrm{SL}}_{\varepsilon,s} \cong I^{\mathrm{SL}}_{\varepsilon,-s}$ *for* $\varepsilon = +$ *and* $s \in i\mathbb{R}$ *or* $\varepsilon = -$ *and* $s \in i\mathbb{R} \setminus \{0\}$,
- *(b) the complementary series* $I^{\mathrm{SL}}_{+,s}$ *for* $s \in \mathbb{R}$, $0 < |s| < 1$,
- *(c) the (limits of) discrete series representations* D^{SL}_k *for* $k \in \mathbb{Z} \setminus \{0\}$, *and*
- *(d) the trivial representation*

Section 5.6.4 of [Wal88] also provides a list of which of these representations are square-integrable or tempered, which allows us to deduce the Plancherel measure of regular representations of $\mathrm{SL}_2(\mathbb{R})$.

Theorem 3.5. *Among the representations in Theorem 3.4, the ones contained in the regular representation* $\mathrm{L}^2(\mathrm{SL}_2(\mathbb{R}))$ *are*

(a) *the discrete series representations* D_k^{SL} *for* $k \in \mathbb{Z} \setminus \{0, \pm 1\}$.

Beyond those, the ones that are weakly contained in $\mathrm{L}^2(\mathrm{SL}_2(\mathbb{R}))$ *are*

(b) *the principal series* $I_{\varepsilon,s}^{\mathrm{SL}} \cong I_{\varepsilon,-s}^{\mathrm{SL}}$ *for* $\varepsilon = +$ *and* $s \in i\mathbb{R}$ *or* $\varepsilon = -$ *and* $s \in i\mathbb{R} \setminus \{0\}$,

(c) *the limits of discrete series representations* D_k^{SL} *for* $k \in \{\pm 1\}$.

3.4. **Representation theory of** $\mathrm{SAff}_2(\mathbb{R})$**: Mackey theory.** We summarize the general Mackey theory for representations of semidirect products in our special case of $\mathrm{G}'(\mathbb{R}) = \mathrm{SAff}_2(\mathbb{R})$. As a prerequisite we need to understand representations of $\mathrm{N}'(\mathbb{R})$.

Proposition 3.6. *The irreducible unitary representations with nontrivial central character of* $\mathrm{N}'(\mathbb{R})$ *are given up to unitary equivalence by*

$$\pi_r^{\mathrm{N}} \;:=\; \mathrm{Ind}_{\mathrm{H}'(\mathbb{R})}^{\mathrm{N}'(\mathbb{R})} \big((w_1, w_2) \mapsto e(r w_2) \big), \quad r \in \mathbb{R}^\times. \tag{3.15}$$

The unitary irreducible representations of $\mathrm{N}'(\mathbb{R})$ *with trivial central character are the characters*

$$\chi_{n,m}^{\mathrm{N}} : \big(\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right), w_1, w_2 \big) \mapsto e(nb + m w_1), \quad n, m \in \mathbb{R}. \tag{3.16}$$

Proof. This is an instance the Stone–von Neumann theorem, see for example [Bum13, Exercise 32.5] or [Zim84, Example 7.3.3]. □

We can now state the result for the special affine group.

Theorem 3.7. *The unitary dual of* $\mathrm{G}'(\mathbb{R})$ *is exhausted by the pullback of* $\mathrm{SL}_2(\mathbb{R})$*-representations and by, for any fixed* $m \in \mathbb{R}^\times$*, the representations*

$$\pi_{n,m}^{\mathrm{SAff}} \;:=\; \mathrm{Ind}_{\mathrm{N}'(\mathbb{R})}^{\mathrm{G}'(\mathbb{R})} \, \chi_{n,m}^{\mathrm{N}}, \quad n \in \mathbb{R}. \tag{3.17}$$

The two unitary representations $\pi_{n_1,m_1}^{\mathrm{SAff}}$ *and* $\pi_{n_2,m_2}^{\mathrm{SAff}}$ *are isomorphic if and only if* $n_1 m_1^2 = n_2 m_2^2$.

We fix representatives of the isomorphism classes as the representations

$$\pi_s^{\mathrm{SAff}} \;:=\; \pi_{s,1}^{\mathrm{SAff}} \tag{3.18}$$

with a single index. The proof of Theorem 3.7 requires the next statement, which we will also need independently.

Proposition 3.8. *The total Casimir operator acts on the representation* $\pi_{n,m}^{\mathrm{SAff}}$ *with eigenvalue* $-4\pi^3\, n m^2$.

Proof. We observe that the correspondence in (2.10) allows us to view K-isotypical elements of $V(\pi_{n,m}^{\mathrm{SAff}})$ as functions on $\mathbb{H}'$, and then calculate with the total Laplace operator via (2.13). The K-spherical element of $V(\pi_{n,m}^{\mathrm{SAff}})$ that is constant on $\mathrm{A}'(\mathbb{R})$ corresponds to $e(nx + mv/y)$. We have

$$-\Delta_k^{\mathrm{tot}}\, e\big(nx + m\frac{v}{y}\big) \;=\; 4\pi^3 n m^2\, e\big(nx + m\frac{v}{y}\big), \tag{3.19}$$

by a straightforward calculation using the formula for the total Laplace operator in Lemma 2.2, once we observe that the given expression is independent of u. □

Proof of Theorem 3.7. The first statement is a reformulation of Theorem 2.4.2 of [BS98]. More precisely, the representations in part i) of loc. cit. theorem are pullbacks from $\mathrm{SL}_2(\mathbb{R})$. The representations in part ii) are the representations in our theorem. The character ψ in the notion of [BS98] is non-trivial and thus corresponds to $x \mapsto e(mx)$ for an arbitrary but fixed $m \in \mathbb{R}^\times$. The character

$$\big(\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix} \right), w_1, w_2 \big) \mapsto \psi(bn/m + w_1) \;=\; e(nb + mw_1)$$

is genuine for any $n \in \mathbb{R}$. Conversely every genuine character of $\mathrm{N}'(\mathbb{R})$ is of this form by Proposition 3.6. See also e.g. [Zim84, Example 7.3.4].

By the first statement, if $m_1 \neq m_2$ for each n_1 there is exactly one n_2 such that the two inductions are isomorphic. To determine this value we employ the eigenvalue under the total Casimir operator as given in Proposition 3.8. □

3.5. **Restrictions of $\mathrm{SAff}_2(\mathbb{R})$-representations.** Our goal is to prove the following branching of $\pi^{\mathrm{SAff}}_{n,m}$ to $\mathrm{SL}_2(\mathbb{R})$.

Proposition 3.9. *For any $m \in \mathbb{R}^\times$ and $n \in \mathbb{R}$, the restrictions of the $\mathrm{SAff}_2(\mathbb{R})$-representations to $\mathrm{SL}_2(\mathbb{R})$ decompose as direct integrals*

$$\mathrm{Res}^{\mathrm{G}'(\mathbb{R})}_{\mathrm{G}(\mathbb{R})}\; \pi^{\mathrm{SAff}}_{0,m} \;=\; \int^{\oplus}_{\mathbb{R}^+} 2\big(I^{\mathrm{SL}}_{+,it} \oplus I^{\mathrm{SL}}_{-,it}\big)\,\mathrm{d}t,$$

$$\mathrm{Res}^{\mathrm{G}'(\mathbb{R})}_{\mathrm{G}(\mathbb{R})}\; \pi^{\mathrm{SAff}}_{n,m} \;=\; \bigoplus_{k=2}^{\infty} D^{\mathrm{SL}}_{\mathrm{sgn}(n)k} \;\oplus\; \int^{\oplus}_{\mathbb{R}^+} \big(I^{\mathrm{SL}}_{+,it} \oplus I^{\mathrm{SL}}_{-,it}\big)\,\mathrm{d}t, \quad \textit{if } n \neq 0.$$

We prove this proposition at the end of the section. The proof features various intermediate inductions and restrctions, which we exhibit separately. It will depend on Lemmas 3.10, 3.11, 3.12, and 3.13, that we state and prove first.

Lemma 3.10. *Given any $m \in \mathbb{R}^\times$, we have the decomposition into characters*

(3.20)
$$\mathrm{Ind}^{\mathrm{P}(\mathbb{R})}_{\mathrm{N}(\mathbb{R})}\; \mathrm{Res}^{\mathrm{N}'(\mathbb{R})}_{\mathrm{N}(\mathbb{R})}\; \chi^{\mathrm{N}}_{0,m} \;\cong\; \int^{\oplus}_{\mathbb{R}} \big(\chi^{\mathrm{P}}_{+,it} \oplus \chi^{\mathrm{P}}_{-,it}\big)\,\mathrm{d}t \;=\; \int^{\oplus}_{\mathbb{R}} \big(\mathbb{1} \oplus \mathrm{sgn}^{\mathrm{P}}\big)\chi^{\mathrm{P}}_{+,it}\,\mathrm{d}t.$$

Proof. We write $\mathrm{A}(\mathbb{R})^0$ and $\mathrm{P}(\mathbb{R})^0$ for the connected components of the identity of $\mathrm{A}(\mathbb{R})$ and $\mathrm{P}(\mathbb{R})$ respectively, that is, the subgroups whose elements have positive diagonal entries. Transitivity of induction yields

$$\mathrm{Ind}^{\mathrm{P}(\mathbb{R})}_{\mathrm{N}(\mathbb{R})}\; \mathrm{Res}^{\mathrm{N}'(\mathbb{R})}_{\mathrm{N}(\mathbb{R})}\; \chi^{\mathrm{N}}_{0,m} \;\cong\; \mathrm{Ind}^{\mathrm{P}(\mathbb{R})}_{\mathrm{P}(\mathbb{R})^0}\; \mathrm{Ind}^{\mathrm{P}(\mathbb{R})^0}_{\mathrm{N}(\mathbb{R})}\; \mathrm{Res}^{\mathrm{N}'(\mathbb{R})}_{\mathrm{N}(\mathbb{R})}\; \chi^{\mathrm{N}}_{0,m}.$$

Functions in the representation space of the inner induction are left invariant under $\mathrm{N}(\mathbb{R})$. Since $\mathrm{N}(\mathbb{R}) \subset \mathrm{P}(\mathbb{R})^0$ is normal, we can identify them with square-integrable functions on

$$\mathrm{N}(\mathbb{R})\backslash \mathrm{P}(\mathbb{R})^0 \;\cong\; \mathrm{A}(\mathbb{R})^0 \;\cong\; \mathbb{R}^+.$$

That is, we have to determine the decomposition of $\mathrm{L}^2(\mathbb{R}^+)$ as a representation of $\mathbb{R}^+$. We use the map $\mathbb{R}^+ \longrightarrow \mathbb{R}$, $a \mapsto \log(a)$ and classical Fourier analysis to find

$$\mathrm{Ind}^{\mathrm{P}(\mathbb{R})}_{\mathrm{P}(\mathbb{R})^0}\; \mathrm{Ind}^{\mathrm{P}(\mathbb{R})^0}_{\mathrm{N}(\mathbb{R})}\; \mathrm{Res}^{\mathrm{N}'(\mathbb{R})}_{\mathrm{N}(\mathbb{R})}\; \chi^{\mathrm{N}}_{0,m} \;\cong\; \mathrm{Ind}^{\mathrm{P}(\mathbb{R})}_{\mathrm{P}(\mathbb{R})^0} \int^{\oplus}_{\mathbb{R}} \chi^{\mathrm{P}}_{+,it}\,\mathrm{d}t \;\cong\; \int^{\oplus}_{\mathbb{R}} \mathrm{Ind}^{\mathrm{P}(\mathbb{R})}_{\mathrm{P}(\mathbb{R})^0}\; \chi^{\mathrm{P}}_{+,it}\,\mathrm{d}t,$$

where for simplicity we identify $\chi^{\mathrm{P}}_{+,it}$ with its restrction to $\mathrm{P}(\mathbb{R})^0$. The remaining induction is central, contributing one copy of $\chi^{\mathrm{P}}_{+,it}$ and one of $\chi^{\mathrm{P}}_{-,it} = \mathrm{sgn}^{\mathrm{P}} \chi^{\mathrm{P}}_{+,it}$ as desired. □

We now turn to the case $n \neq 0$.

Lemma 3.11. *Given any $m \in \mathbb{R}^\times$ and $n \neq 0$, we have the decomposition into irreducible representations*

$$\mathrm{Ind}_{\mathrm{N}(\mathbb{R})}^{\mathrm{P}(\mathbb{R})}\ \mathrm{Res}_{\mathrm{N}(\mathbb{R})}^{\mathrm{N}'(\mathbb{R})}\ \chi_{n,m}^{\mathrm{N}}\ \cong\ \pi_{\mathrm{sgn}(n)}^{\mathrm{P}} \oplus \mathrm{sgn}^{\mathrm{P}} \pi_{\mathrm{sgn}(n)}^{\mathrm{P}}.$$

Proof. Inducing in steps to $\mathrm{P}(\mathbb{R})^0$ as in the proof of Lemma 3.10 we can apply Mackey's Irreducibility Criterion (Corollary 1.F.5 of [BH20]; compare also Remark 3.C.6 of op. cit.) to obtain the restriction of $\pi_{\mathrm{sgn}(n)}^{\mathrm{P}}$ to $\mathrm{P}(\mathbb{R})^0$. The central induction from $\mathrm{P}(\mathbb{R})^0$ to $\mathrm{P}(\mathbb{R})$ then yields two copies of it, one of which is twisted by the sign character. □

The proof of Lemma 3.13, the last auxiliar statement for the proof of Proposition 3.9, requires us to determine the restrictions of principal and discrete series of $\mathrm{SL}_2(\mathbb{R})$ to $\mathrm{N}(\mathbb{R})$. The statements of the next lemma are given, for instance, in Proposition 3.3.2 and 3.3.3 of Kobayashi's notes [Kob05].

Lemma 3.12. *For all $t \in \mathbb{R}$, we have the decomposition*

$$\mathrm{Res}_{\mathrm{N}(\mathbb{R})}^{\mathrm{G}(\mathbb{R})}\ I_{\pm,it}^{\mathrm{SL}}\ \cong\ \int_{\mathbb{R}}^{\oplus} \big(\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \longmapsto e(nb) \big)\, \mathrm{d}n.$$

For all $k \in \mathbb{Z} \setminus \{0\}$, we have the decomposition

$$\mathrm{Res}_{\mathrm{N}(\mathbb{R})}^{\mathrm{G}(\mathbb{R})}\ D_k^{\mathrm{SL}}\ \cong\ \int_{\mathbb{R}^{\mathrm{sgn}(k)}}^{\oplus} \big(\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \longmapsto e(nb) \big)\, \mathrm{d}n.$$

Lemma 3.13. *The inductions from the upper triangular subgroup $\mathrm{P}(\mathbb{R})$ to $\mathrm{G}(\mathbb{R})$ decompose as follows:*

$$\begin{aligned} \mathrm{Ind}_{\mathrm{P}(\mathbb{R})}^{\mathrm{G}(\mathbb{R})}\ \pi_{\pm}^{\mathrm{P}}\ &\cong\ \bigoplus_{\substack{k=2 \\ k\ \mathit{even}}}^{\infty} D_{\pm k}^{\mathrm{SL}}\ \oplus\ \int_{\mathbb{R}^+}^{\oplus} I_{+,it}^{\mathrm{SL}}\, \mathrm{d}t, \\ \mathrm{Ind}_{\mathrm{P}(\mathbb{R})}^{\mathrm{G}(\mathbb{R})}\ \mathrm{sgn}^{\mathrm{P}} \pi_{\pm}^{\mathrm{P}}\ &\cong\ \bigoplus_{\substack{k=3 \\ k\ \mathit{odd}}}^{\infty} D_{\pm k}^{\mathrm{SL}}\ \oplus\ \int_{\mathbb{R}^+}^{\oplus} I_{-,it}^{\mathrm{SL}}\, \mathrm{d}t. \end{aligned} \tag{3.21}$$

Proof. The strategy is to spell out Mackey's version of Frobenius reciprocity [Mac53] for the inclusion of groups $\mathrm{P}(\mathbb{R})$ and $\mathrm{G}(\mathbb{R})$ and then apply the same technique as in Section 7(b) of loc. cit. Namely, we determine the isomorphism class of all restrictions on the right hand side of equality (3.22) below, and consider this equation as representation of $\mathrm{P}(\mathbb{R}) \times 1 \subseteq \mathrm{P}(\mathbb{R}) \times \mathrm{G}(\mathbb{R})$. Since only finitely many, four in fact, isomorphism classes of $\mathrm{P}(\mathbb{R}) \times 1$-representations contribute, we thus derive an isomorphism of isotypical components for $\mathrm{P}(\mathbb{R}) \times 1$ on the left and right hand side as representations of the commutator of $\mathrm{P}(\mathbb{R}) \times 1$, thus of representations of the group $1 \times \mathrm{G}(\mathbb{R})$, as desired.

The Frobenius reciprocity formula involves all the representations weakly contained in the regular representations, which have been listed in Proposition 3.3 and Theorem 3.5 for the two groups under consideration. We deduce from Mackey's theorem that

$$
\begin{aligned}
&\big(\overline{\pi}^{\mathrm{P}}_{+} \otimes \mathrm{Ind}_{\mathrm{P}(\mathbb{R})}\ \pi^{\mathrm{P}}_{+}\big) \quad\oplus\quad \big(\overline{\pi}^{\mathrm{P}}_{-} \otimes \mathrm{Ind}_{\mathrm{P}(\mathbb{R})}\ \pi^{\mathrm{P}}_{-}\big)\\
\oplus\ &\big(\overline{\mathrm{sgn}}^{\mathrm{P}}\overline{\pi}^{\mathrm{P}}_{+} \otimes \mathrm{Ind}_{\mathrm{P}(\mathbb{R})}\ \overline{\mathrm{sgn}}^{\mathrm{P}}\pi^{\mathrm{P}}_{+}\big) \ \oplus\ \big(\overline{\mathrm{sgn}}^{\mathrm{P}}\overline{\pi}^{\mathrm{P}}_{-} \otimes \mathrm{Ind}_{\mathrm{P}(\mathbb{R})}\ \overline{\mathrm{sgn}}^{\mathrm{P}}\pi^{\mathrm{P}}_{-}\big)\\
\cong\ &\bigoplus_{k\in\mathbb{Z}\setminus\{0,\pm1\}} \mathrm{Res}^{\mathrm{G}(\mathbb{R})}_{\mathrm{P}(\mathbb{R})}\big(\overline{D}^{\mathrm{SL}}_{k} \otimes D^{\mathrm{SL}}_{k}\big)\\
&\oplus \int_{\mathbb{R}} \mathrm{Res}^{\mathrm{G}(\mathbb{R})}_{\mathrm{P}(\mathbb{R})}\big(\overline{I}^{\mathrm{SL}}_{+,it} \otimes I^{\mathrm{SL}}_{+,it}\big)\,\mathrm{d}t \ \oplus\ \int_{\mathbb{R}} \mathrm{Res}^{\mathrm{G}(\mathbb{R})}_{\mathrm{P}(\mathbb{R})}\big(\overline{I}^{\mathrm{SL}}_{-,it} \otimes I^{\mathrm{SL}}_{-,it}\big)\,\mathrm{d}t.
\end{aligned}
\tag{3.22}
$$

where the overline denotes the contragredient representation. These are easily computed to be $\overline{\mathrm{sgn}}^{\mathrm{P}} = \mathrm{sgn}^{\mathrm{P}}$ and $\overline{\chi}^{\mathrm{N}}_{n} = \chi^{\mathrm{N}}_{-n}$, hence the induction satisfies $\overline{\pi}^{\mathrm{P}}_{\pm} \cong \pi^{\mathrm{P}}_{\mp}$. By a similar argument, we find that for $t \in \mathbb{R}$

$$
\overline{I}^{\mathrm{SL}}_{\pm,it} \ \cong\ I^{\mathrm{SL}}_{\pm,\overline{it}} \ =\ I^{\mathrm{SL}}_{\pm,-it} \ \cong\ I^{\mathrm{SL}}_{\pm,it} \quad \text{and} \quad \overline{D}^{\mathrm{SL}}_{k} \ =\ D^{\mathrm{SL}}_{k}
$$

by inspection of K-types.

We next determine the multiplicities of $\pi^{\mathrm{P}}_{\pm}$ and $\mathrm{sgn}^{\mathrm{P}}\pi^{\mathrm{P}}_{\pm}$ in the restrictions on the right hand side. Since central characters are preserved by restriction, there are multiplicities $m_{\mathrm{D},k;\pm}$ and $m_{\mathrm{I},\pm,it;\pm}$, which are possibly infinite, such that

$$
\begin{aligned}
\mathrm{Res}^{\mathrm{G}(\mathbb{R})}_{\mathrm{P}(\mathbb{R})}\ D^{\mathrm{SL}}_{k} &\cong \big(\mathrm{sgn}^{\mathrm{P}}\big)^{k}\,\big(m_{\mathrm{D},k;+}\pi^{\mathrm{P}}_{+} \ \oplus\ m_{\mathrm{D},k;-}\pi^{\mathrm{P}}_{-}\big),\\
\mathrm{Res}^{\mathrm{G}(\mathbb{R})}_{\mathrm{P}(\mathbb{R})}\ I^{\mathrm{SL}}_{+,it} &\cong \big(m_{\mathrm{I},+,it;+}\pi^{\mathrm{P}}_{+} \oplus m_{\mathrm{I},+,it;-}\pi^{\mathrm{P}}_{-}\big),\\
\mathrm{Res}^{\mathrm{G}(\mathbb{R})}_{\mathrm{P}(\mathbb{R})}\ I^{\mathrm{SL}}_{-,it} &\cong \mathrm{sgn}^{\mathrm{P}}\,\big(m_{\mathrm{I},-,it;+}\pi^{\mathrm{P}}_{+} \oplus m_{\mathrm{I},-,it;-}\pi^{\mathrm{P}}_{-}\big).
\end{aligned}
$$

To find these multiplicities we restrict both sides to $\mathrm{N}(\mathbb{R})$ and note that

$$
\mathrm{Res}^{\mathrm{G}(\mathbb{R})}_{\mathrm{N}(\mathbb{R})}\,\pi^{\mathrm{P}}_{\pm} \ \cong\ \mathrm{Res}^{\mathrm{G}(\mathbb{R})}_{\mathrm{N}(\mathbb{R})}\,\mathrm{sgn}^{\mathrm{P}}\pi^{\mathrm{P}}_{\pm} \ \cong\ \int^{\oplus}_{\mathbb{R}^{\pm}} \big(\left(\begin{smallmatrix}1 & b\\ 0 & 1\end{smallmatrix}\right) \longmapsto e(nb)\big)\,\mathrm{d}n.
\tag{3.23}
$$

Comparing this with Lemma 3.12 below implies that

$$
m_{\mathrm{D},k,\mathrm{sgn}(k)} \ =\ 1, \quad m_{\mathrm{D},k,\mathrm{sgn}(-k)} \ =\ 0 \quad \text{and} \quad m_{\mathrm{I},\pm,it;+} \ =\ m_{\mathrm{I},\pm,it;-} \ =\ 1.
$$

Coming back to (3.22), viewed as a representation of $\mathrm{P}(\mathbb{R}) \times 1$, and using the multiplicities in (3.23), we deduce our statement. ☐

Proof of Proposition 3.9. Comparing the decompositions

$$
\mathrm{G}'(\mathbb{R}) \ =\ \mathrm{N}'(\mathbb{R})\mathrm{A}(\mathbb{R})\mathrm{K}, \quad \mathrm{G}(\mathbb{R}) \ =\ \mathrm{N}(\mathbb{R})\mathrm{A}(\mathbb{R})\mathrm{K}, \quad \text{and } \mathrm{N}'(\mathbb{R}) \ =\ \mathrm{H}'(\mathbb{R})\mathrm{N}(\mathbb{R})
$$

we claim that there is an isomorphism

$$
\mathrm{Res}^{\mathrm{G}'(\mathbb{R})}_{\mathrm{G}(\mathbb{R})}\ \pi^{\mathrm{SAff}}_{n,m} \ =\ \mathrm{Res}^{\mathrm{G}'(\mathbb{R})}_{\mathrm{G}(\mathbb{R})}\ \mathrm{Ind}^{\mathrm{G}'(\mathbb{R})}_{\mathrm{N}'(\mathbb{R})}\ \chi^{\mathrm{N}}_{n,m} \ \cong\ \mathrm{Ind}^{\mathrm{G}(\mathbb{R})}_{\mathrm{N}(\mathbb{R})}\ \mathrm{Res}^{\mathrm{N}'(\mathbb{R})}_{\mathrm{N}(\mathbb{R})}\ \chi^{\mathrm{N}}_{n,m}.
\tag{3.24}
$$

In fact, the left hand side of (3.24) consists of functions that transform like

$$
f(\tilde{n}g) \ =\ \chi^{\mathrm{N}}_{n,m}(\tilde{n})f(g) \quad \text{for all } \tilde{n} \in \mathrm{N}'(\mathbb{R}) = \mathrm{N}(\mathbb{R})\mathrm{H}'(\mathbb{R}) \text{ and } g \in \mathrm{G}'(\mathbb{R}).
\tag{3.25}
$$

In particular, such f is uniquely defined by its restriction to $G(\mathbb{R}) \cong \mathrm{G}'(\mathbb{R})/\mathrm{H}'(\mathbb{R})$, and this restriction satisfies again (3.25), now for all $\tilde{n} \in \mathrm{N}(\mathbb{R})$ and $g \in \mathrm{G}(\mathbb{R})$. Moreover, by the left covariance with respect to $\mathrm{H}'(\mathbb{R})$, the function f is measurable if and only if its restriction to $\mathrm{G}(\mathbb{R})$ is so. Square integrability on $\mathrm{A}(\mathbb{R})\mathrm{K} \subset \mathrm{G}(\mathbb{R})$ is preserved by the restriction to $\mathrm{G}(\mathbb{R})$.

We can perform the induction on the right hand of (3.24) side in steps, that is,

$$
\mathrm{Ind}^{\mathrm{G}(\mathbb{R})}_{\mathrm{N}(\mathbb{R})}\ \mathrm{Res}^{\mathrm{N}'(\mathbb{R})}_{\mathrm{N}(\mathbb{R})}\ \chi^{\mathrm{N}}_{n,m} \cong \mathrm{Ind}^{\mathrm{G}(\mathbb{R})}_{\mathrm{P}(\mathbb{R})}\ \mathrm{Ind}^{\mathrm{P}(\mathbb{R})}_{\mathrm{N}(\mathbb{R})}\ \mathrm{Res}^{\mathrm{N}'(\mathbb{R})}_{\mathrm{N}(\mathbb{R})}\ \chi^{\mathrm{N}}_{n,m}.
$$

In the case of $n = 0$, Lemma 3.10 implies that

$$\mathrm{Ind}_{\mathrm{N}(\mathbb{R})}^{\mathrm{G}(\mathbb{R})}\ \mathrm{Res}_{\mathrm{N}(\mathbb{R})}^{\mathrm{N}'(\mathbb{R})}\ \chi^{\mathrm{N}}_{0,m} \cong \mathrm{Ind}_{\mathrm{P}(\mathbb{R})}^{\mathrm{G}(\mathbb{R})} \int_{i\mathbb{R}}^{\oplus} \big(\chi^{\mathrm{P}}_{+,s} \oplus \chi^{\mathrm{P}}_{-,s}\big)\, \mathrm{d}s.$$

To obtain the result, it suffices to note that induction and direct integral decomposition intertwine by Fubini's theorem, and to use that $I^{\mathrm{SL}}_{\pm,it} \cong I^{\mathrm{SL}}_{\pm,-it}$.

In the case $n \neq 0$ the statement now follows directly by combining Lemma 3.11 and Lemma 3.13. □

4. Fourier expansions and Poincaré series

The first goal of this section is to examine and relate to each other two Fourier expansions of affine modular forms. One of them along the torus fiber is merely compatible with the action of $\mathrm{SL}_2(\mathbb{R})$, but exhibits better compatibility with the action of the foliated Laplace operator and with the corresponding notion of Eisenstein and Poincaré series. The other one arises from the Heisenberg subgroup $\mathrm{N}'(\mathbb{R}) \subset \mathrm{G}'(\mathbb{R})$ and is compatible with the description of $\mathrm{G}'(\mathbb{R})$-representations that appears in Theorem 3.7. In particular we give in Proposition 4.4 a criterion in terms of Fourier coefficients that ensures that some representation $\pi^{\mathrm{SAff}}_{nm^2}$ appears in the L^2-representation $\pi_{\mathrm{L}^2}(\phi)$ generated by a modular form ϕ.

In Section 4.4 we construct affine group versions of Eisenstein and Poincaré series. We compute their Fourier coefficients and show in Proposition 4.13 that they generate the representation π^{SAff}_n for all values $n \in \mathbb{Z}$. As a corollary we obtain the eigenvalues of the total Casimir acting on π^{SAff}_n.

Parts of the Fourier expansion along the torus fiber for affine modular-invariant functions of K-type 0 and, expressed differently, the related construction of Eisenstein and Poincaré series has appeared in unpublished work of Balslev [Bal11].

4.1. Fourier expansions on the torus fibers.

The subgroup of translations is an abelian subgroup isomorphic to $\mathbb{R}^2$ in $\mathrm{SAff}_2(\mathbb{R})$, whose dual yields a Fourier expansion of continuous affine modular-invariant function ϕ of weight k:

$$\phi(\tau, z) \;=\; \sum_{r,m \in \mathbb{Z}} c^{\mathrm{T}}\big(\phi;\, m, r;\, \tau\big)\, e(mp + rq)\,, \qquad (z \;=\; p\tau + q). \tag{4.1}$$

The Fourier coefficients are of course given by

$$c^{\mathrm{T}}(\phi; m, r; \tau) = \int_{\mathbb{Z}\backslash\mathbb{R}} \int_{\mathbb{Z}\backslash\mathbb{R}} \phi(\tau, p\tau + q) e(-mp - rq)\ \mathrm{d}p\, \mathrm{d}q. \tag{4.2}$$

The superscript T indicates that this is the Fourier expansion along the torus fibers of the projection $\mathcal{H}(0,0) \to \mathcal{H}(0)$.

Lemma 4.1. *The Fourier coefficients have the equivariance property*

$$c^{\mathrm{T}}\big(\phi\big|'_k\, \gamma;\, m, r;\, \tau\big) \;=\; c^{\mathrm{T}}\big(\phi;\, \widetilde{m}, \widetilde{r};\, \tau\big)\,\big|_k\, \gamma, \quad (\widetilde{m}, \widetilde{r}) = (m, r)\, {}^{\mathrm{t}}\gamma.$$

for a continous affine modular-invariant function ϕ of weight k and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

Proof. We apply γ to ϕ and eventually compare coefficients, using the uniqueness of the Fourier series. We get

$$\sum_{r,m\in\mathbb{Z}} c^{\mathrm{T}}\big(\phi;\, m,r;\, \tau\big)\, e(mp+rq) \;=\; \phi(\tau,z) \;=\; \big(\phi\big|_k'\,\gamma\big)(\tau,z)$$
$$=\; \sum_{\widetilde{m},\widetilde{r}\in\mathbb{Z}} (c\tau+d)^{-k} c^{\mathrm{T}}\big(\phi;\, \widetilde{m},\widetilde{r};\, \gamma\tau\big)\, e(\widetilde{m}\widetilde{p}+\widetilde{r}\widetilde{q}),$$

where $(\widetilde{p},\widetilde{q}) = (p,q)\gamma^{-1}$ and $\gamma = \big(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\big)$, since

$$\frac{z}{c\tau+d} \;=\; \widetilde{p}(\gamma\tau)+\widetilde{q} \quad \text{with } (\widetilde{p},\widetilde{q}) = (p,q)\gamma^{-1}\,.$$

Now to compare coefficients, we need to determine the $(\widetilde{m},\widetilde{r})$ for which $mp+rq = \widetilde{m}\widetilde{p}+\widetilde{r}\widetilde{q}$. This yields the equality

$$(p,q)\,{}^{\mathrm{t}}(m,r) \;=\; mp+rq \;=\; \widetilde{m}\widetilde{p}+\widetilde{r}\widetilde{q}$$
$$= (\widetilde{p},\widetilde{q})\,{}^{\mathrm{t}}(\widetilde{m},\widetilde{r}) \;=\; (p,q)\gamma^{-1}\,{}^{\mathrm{t}}(\widetilde{m},\widetilde{r}) \;=\; (p,q)\,{}^{\mathrm{t}}\big((\widetilde{m},\widetilde{r})\,{}^{\mathrm{t}}\gamma^{-1}\big),$$

which yields desired relation between (m,r) and $(\widetilde{m},\widetilde{r})$. □

Proposition 4.2. *Let ϕ be an affine modular-invariant function of weight k such that the Fourier coefficients $c^{\mathrm{T}}(\phi;\, m,0;\tau)$ vanishes for all $m\in\mathbb{Z}$. Then $\phi=0$.*

Proof. By modular invariance under $\mathrm{SL}_2(\mathbb{Z})$ and Lemma 4.1, every c^{T} with index $(m,0)\gamma$ for some γ and some m vanishes. This exhausts all terms in the Fourier expansion (4.1), implying $\phi=0$. □

4.2. **Fourier series along the Heisenberg group.** We now study a Fourier expansion of affine modular function that is compatible with the induction of characters that appear in Theorem 3.7. Specifically, for given $m\in\mathbb{Z}\setminus\{0\}, n\in\mathbb{Z}$ we show that the nonvanishing of specific Fourier coefficients implies that the $\mathrm{SAff}_2(\mathbb{R})$-representation of the representation generated by an affine modular-invariant function contains $\pi^{\mathrm{SAff}}_{nm^2}$.

The dual of the abelian group

$$\big\{\big(\big(\begin{smallmatrix} 1 & b \\ 0 & 1\end{smallmatrix}\big),0,w_2\big)\in \mathrm{G}'(\mathbb{R})\big\} \;\cong\; \mathbb{R}^2 \qquad \supset Z'(\mathbb{R}) \;:=\; \big\{\big(\big(\begin{smallmatrix} 1 & 0 \\ 0 & 1\end{smallmatrix}\big),0,w_2\big),\; w_2\in\mathbb{R}\big\}$$

yields the second Fourier expansion. On the constant term with respect to the w_2-action, i.e. on functions that are $Z'(\mathbb{R})$-invariant the factor group

$$\mathrm{G}'(\mathbb{R})/\mathrm{Z}'(\mathbb{R}) \;\cong\; \big\{\big(\big(\begin{smallmatrix} 1 & b \\ 0 & 1\end{smallmatrix}\big),w_1,\mathbb{R}\big)\big\} \;\cong\; \mathbb{R}^2$$

acts. This yields another two-variable expansion, that we call *Fourier–Heisenberg expansion.*

$$\phi(\tau,z) \;=\; \sum_{n,r\in\mathbb{Z}} c^{\mathrm{H}}\big(\phi;\, n,r;\, y,\frac{v}{y}\big)\, e(nx+ru), \tag{4.3}$$

and we refine this further by writing for any $n\in\mathbb{Z}$

$$c^{\mathrm{H}}\big(\phi;\, n,0;\, y,\frac{v}{y}\big) \;=\; \sum_{m\in\mathbb{Z}} c^{\mathrm{H0}}(\phi;\, n,m;\, y)\, e\big(m\frac{v}{y}\big)\,. \tag{4.4}$$

The Fourier–Heisenberg coefficients are again given by

$$c^{\mathrm{H}}\big(\phi;n,r;y,\frac{v}{y}\big) = \int_{\mathbb{Z}\backslash\mathbb{R}}\int_{\mathbb{Z}\backslash\mathbb{R}} \phi(x+iy,u+iv)e(-nx-ru)\;\mathrm{d}x\,\mathrm{d}u \tag{4.5}$$

and

$$c^{\mathrm{H0}}(\phi; n, m; y) = \frac{1}{y} \int_{y\mathbb{Z}\backslash\mathbb{R}} c^{\mathrm{H}}\big(\phi; n, 0; y, \frac{v}{y}\big) e\big(-m\frac{v}{y}\big)\, \mathrm{d}v. \tag{4.6}$$

Lemma 4.3. *The $r = 0$-part of the torus Fourier expansions coincides with the $r = 0$-part in the Fourier–Heisenberg expansion. That is, for any continuous affine modular-invariant function ϕ*

$$c^{\mathrm{T}}\big(\phi;\, m, 0;\, \tau\big) \;= \sum_{n \in \mathbb{Z}} c^{\mathrm{H0}}\big(\phi;\, n, m;\, y\big)\, e(nx).$$

In particular if for an affine modular-invariant function ϕ all the Fourier–Heisenberg coefficients $c^{\mathrm{H0}}(\phi;\, n, m; y)$ vanish for $n, m \in \mathbb{Z}$, then $\phi = 0$.

Proof. Comparing the coefficient expressions (4.2), (4.5) and (4.6) we see

$$\begin{aligned} c^{\mathrm{T}}\big(\phi;\, m, 0;\, \tau\big) \;&=\; \int_{\mathbb{Z}\backslash\mathbb{R}} \int_{\mathbb{Z}\backslash\mathbb{R}} \phi(\tau, q\tau + p) e(-mp)\, \mathrm{d}p\, \mathrm{d}q, \\ \sum_{n \in \mathbb{Z}} c^{\mathrm{H0}}\big(\phi;\, n, m;\, y\big) e(nx) \;&=\; \frac{1}{y} \int_{y\mathbb{Z}\backslash\mathbb{R}} \int_{\mathbb{Z}\backslash\mathbb{R}} \phi(\tau, u + iv) e\big(-m\frac{v}{y}\big)\, \mathrm{d}u\, \mathrm{d}v. \end{aligned} \tag{4.7}$$

The claim then follows from the change of variables $u + iv = p\tau + q$, which gives $\mathrm{d}u + i\mathrm{d}v = \tau\mathrm{d}p + \mathrm{d}q$, that is, $\mathrm{d}u = x\mathrm{d}p + \mathrm{d}q$ and $\mathrm{d}v = y\mathrm{d}p$. □

The following proposition gives us a criterion in terms of Fourier–Heisenberg expansions to show that the representations $\pi^{\mathrm{SAff}}_{nm^2}$ occur in $\mathrm{L}^2(\mathcal{H}(0,0))$. It will be used in the proof of Theorem 5.1. Recall that $\pi_{\mathrm{L}^2}(\phi)$ denotes the smallest $\mathrm{G}'(\mathbb{R})$-invariant subspace of $\mathrm{L}^2(\mathcal{H}(0,0))$ that contains ϕ.

Proposition 4.4. *Given a continuous square-integrable affine modular function ϕ, assume that the Fourier coefficient $c^{\mathrm{H0}}(\phi;\, n, m;\, y)$ in (4.3) does not vanish for some $n, m \in \mathbb{Z}$, $m \neq 0$. Then averaging over the subgroup $\mathrm{N}'(\mathbb{R})$ defines a surjective homomorphism*

$$\pi_{\mathrm{L}^2}(\phi) \longrightarrow \pi^{\mathrm{SAff}}_{nm^2}, \quad f \longmapsto \Big(g \mapsto \int_{\mathrm{N}'(\mathbb{Z})\backslash\mathrm{N}'(\mathbb{R})} f(hg)\, \overline{\chi}^{\mathrm{N}}_{n,m}(h)\, \mathrm{d}h\Big)\,.$$

Proof. Since $\pi_{\mathrm{L}^2}(\phi)$ consists of functions that are left invariant with respect to $\mathrm{G}'(\mathbb{Z})$, hence with respect to $\mathrm{N}'(\mathbb{Z})$, and since $\mathrm{N}'(\mathbb{Z})\backslash\mathrm{N}'(\mathbb{R})$ is compact, the integral is well defined. Given $\tilde{h} \in \mathrm{N}'(\mathbb{R})$, we have

$$\int_{\mathrm{N}'(\mathbb{Z})\backslash\mathrm{N}'(\mathbb{R})} f(h\tilde{h}g)\, \overline{\chi}^{\mathrm{N}}_{n,m}(h)\, \mathrm{d}h \;=\; \overline{\chi}^{\mathrm{N}}_{n,m}(\tilde{h}) \int_{\mathrm{N}'(\mathbb{Z})\backslash\mathrm{N}'(\mathbb{R})} f(h\tilde{h}g)\, \overline{\chi}^{\mathrm{N}}_{n,m}(h\tilde{h})\, \mathrm{d}h.$$

Since the integral is taken with respect to the right Haar measure, we can replace $h\tilde{h}$ in the integrand by h. Thus the image of the map in Proposition 4.4 is contained in the representation space $V(\pi^{\mathrm{SAff}}_{n,m})$. Since $\pi^{\mathrm{SAff}}_{n,m}$ is irreducible by Theorem 3.7, the statement follows once we establish that the integral does not vanish for some $f \in V(\pi_{\mathrm{L}^2}(\phi))$.

Recall that the function in $V(\pi_{\mathrm{L}^2}(\phi))$ corresponding to ϕ is given by

$$f(hg) \;=\; \big(\phi\big|_{k,0} hg\big)(i, 0).$$

We apply the Iwasawa decomposition to g and h, that is,

$$g \;=\; \big(\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right), w_1, w_2\big) \left(\begin{smallmatrix} a & \\ & a^{-1} \end{smallmatrix}\right) \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right), \quad h \;=\; \big(\left(\begin{smallmatrix} 1 & \tilde{b} \\ 0 & 1 \end{smallmatrix}\right), \tilde{w}_1, \tilde{w}_2\big)$$

and when inserting this into f, we obtain

$$\begin{aligned} f(hg) &= e^{ik\theta} a^k\, \phi\big(a^2 i + b + \tilde{b}, (w_1 + \tilde{w}_1)a^2 i + (w_2 + \tilde{w}_2 + b\tilde{w}_1)\big) \\ &= e^{ik\theta} a^k \Big(\sum_{n,m \in \mathbb{Z}} c^{\mathrm{H0}}(\phi;\, n, m;\, a^2)\, e\big(n(b+\tilde{b}) + m(w_1 + \tilde{w}_1)\big) \\ &\qquad + \sum_{\substack{n,r \in \mathbb{Z} \\ r \neq 0}} c^{\mathrm{H}}\big(\phi;\, n, r;\, a^2, w_1 + \tilde{w}_1\big)\, e\big(n(b+\tilde{b}) + r(w_2 + \tilde{w}_2 + b\tilde{w}_1)\big)\Big). \end{aligned}$$

We next insert this into the expression defining the map in the proposition. We replace n and m in the character $\chi^{\mathrm{N}}_{n,m}$ to distinguish them from the indices of the Fourier–Heisenberg expansion. We obtain

$$\begin{aligned} &\int_{\mathrm{N}'(\mathbb{Z})\backslash \mathrm{N}'(\mathbb{R})} f(hg) \overline{\chi}^{\mathrm{N}}_{\tilde{n},\tilde{m}}(h)\, dh \\ &= e^{ik\theta} a^k \int_{(\mathbb{R}/\mathbb{Z})^3} \Bigg(\sum_{n,m \in \mathbb{Z}} c^{\mathrm{H0}}(\phi;\, n, m;\, a^2)\, \frac{e\big(n(b+\tilde{b}) + m(w_1 + \tilde{w}_1)\big)}{e\big(\tilde{n}\tilde{b} + \tilde{m}\tilde{w}_1\big)} \\ &\quad + \sum_{\substack{n,r \in \mathbb{Z} \\ r \neq 0}} c^{\mathrm{H}}\big(\phi;\, n, r;\, a^2, w_1 + \tilde{w}_1\big)\, \frac{e\big(n(b+\tilde{b}) + r(w_2 + \tilde{w}_2 + b\tilde{w}_1)\big)}{e\big(\tilde{n}\tilde{b} + \tilde{m}\tilde{w}_1\big)}\, \mathrm{d}\tilde{w}_2 \mathrm{d}\tilde{b} \mathrm{d}\tilde{w}_1 \Bigg). \end{aligned}$$

We can interchange the summation over n, m, r and the integral over the compact set $\mathrm{Z}'(\mathbb{R})/\mathrm{Z}'(\mathbb{Z})$, which is parametrized by $\tilde{w}_2$. This allows us to discard all contributions from the second part of the Fourier–Heisenberg series. We are then left with the integral

(4.8)

$$\begin{aligned} &e^{ik\theta} a^k \sum_{n,m \in \mathbb{Z}} c^{\mathrm{H0}}(\phi;\, n, m;\, a^2) \int_{(\mathbb{R}/\mathbb{Z})^2} e\big(nb + (n - \tilde{n})\tilde{b} + m w_1 + (m - \tilde{m})\tilde{w}_1\big)\, \mathrm{d}\tilde{b} \mathrm{d}\tilde{w}_1 \\ &= e^{ik\theta} a^k\, c^{\mathrm{H0}}(\phi;\, \tilde{n}, \tilde{m};\, a^2)\, e(nb + m w_1). \end{aligned}$$

By the assumptions of the proposition the righthand side does not vanish for some choice of $\tilde{n}, \tilde{m}, a$. □

4.3. **Towards an L^2-isometry.** In this section we give an expression of the scalar product of affine modular-invariant functions in terms of Fourier–Heisenberg coefficients. It will be used for several orthogonality statements. It generalizes the classical 'unfolding' construction on the modular surface. To generalize the content of this paper to general strata one of the main challenges will be to find a replacement of this lemma.

Lemma 4.5. *The scalar product of two continuous affine modular-invariant functions ϕ_i can be expressed in Fourier–Heisenberg coefficients as*

$$\langle \phi_1, \phi_2 \rangle = \sum_{n,m \in \mathbb{Z}} \int_{\mathbb{R}^+} c^{\mathrm{H0}}\big(\phi_1;\, n, m;\, y\big)\, \overline{c^{\mathrm{H0}}\big(\phi_2;\, n, m;\, y\big)}\, \frac{\mathrm{d}y}{y^{2-k}}. \tag{4.9}$$

Proof. Starting with (2.18) and abbreviating $X = \mathrm{G}(\mathbb{Z})\backslash\mathrm{G}(\mathbb{R})/\mathrm{K}$ we find

$$\begin{aligned}
\langle \phi_1, \phi_2 \rangle &= \int_{\mathrm{G}'(\mathbb{Z})\backslash\mathrm{G}'(\mathbb{R})/\mathrm{K}} \phi_1(\tau, p\tau + q)\overline{\phi_2(\tau, p\tau + q)}\, \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}p\,\mathrm{d}q}{y^{2-k}} \\
&= \int_X \sum_{m \ge 0} \sum_{\substack{c,d \in \mathbb{Z} \\ \gcd(c,d)=1}} c^{\mathrm{T}}\big(\phi_1;\, m\cdot(d,-c);\, \tau\big) \cdot \overline{c^{\mathrm{T}}\big(\phi_2;\, m\cdot(d,-c);\, \tau\big)}\, \frac{\mathrm{d}x\,\mathrm{d}y}{y^{2-k}} \\
&= \int_X \sum_{m \ge 0} \sum_{\gamma \in \Gamma^+_\infty\backslash\mathrm{G}(\mathbb{Z})} c^{\mathrm{T}}\big(\phi_1;\, (m,0);\, \tau\big)\big|_k\gamma \cdot \overline{c^{\mathrm{T}}\big(\phi_2;\, (m,0);\, \tau\big)\big|_k\gamma}\, \frac{\mathrm{d}x\,\mathrm{d}y}{y^{2-k}} \\
&= \sum_{m \ge 0} \int_{\Gamma^+_\infty\backslash\mathrm{G}(\mathbb{R})/\mathrm{K}} c^{\mathrm{T}}\big(\phi_1;\, m, 0;\, \tau\big)\, \overline{c^{\mathrm{T}}\big(\phi_2;\, m, 0;\, \tau\big)}\, \frac{\mathrm{d}x\,\mathrm{d}y}{y^{2-k}}\,.
\end{aligned}$$

Here we used the orthogonality of the exponential terms $e(\ell p + rq)$ on $\mathrm{L}^2(\mathbb{Z}^2\backslash\mathbb{R}^2)$, and writing $m = \gcd(r, \ell)$ we used Lemma 4.3. We then combined the X-integral and the summation over $\Gamma^+_\infty\backslash\mathrm{G}(\mathbb{Z})$ to the integral over the strip $\Gamma^+_\infty\backslash\mathrm{G}(\mathbb{R})/\mathrm{K}$. The identity is then obtained by rewriting the c^{T}-coefficients in terms of c^{H0}-coefficients using Lemma 4.3 and performing the x-integration. Using orthogonality of exponential terms, only the diagonal terms remain, which gives the claimed formula. ☐

The following statement will help to prove injectivity in Theorem 5.1, compare with the disintegration of the Haar measure along the torus fibres in (3.5). It complements the vanishing statements in Proposition 4.2 and Lemma 4.3.

Lemma 4.6. *For a non-genuine affine-invariant modular function ϕ of weight k, we have*

$$c^{\mathrm{T}}(\phi; m, r; \tau) = 0 \quad \textit{for all } (m, r) \neq (0, 0).$$

An affine-invariant modular function ϕ is genuine if and only if $c^{\mathrm{T}}(\phi; 0, 0; \tau) = 0$.

Proof. Non-genuine modular functions are pullbacks from $\mathbb{H}$ to $\mathbb{H}'$, i.e. they are constant in z. In particular, the only Fourier coefficient with respect to p and q that appears in (4.1) is the one of index $(0, 0)$.

A genuine modular function ϕ of weight k by (3.4) is orthogonal with respect to the inner product (2.18) to all non-genuine ones of the same weight. The second statement thus follows from the first and Lemma 4.5. ☐

We get an immediate corollary, useful in proving discreteness of the spectrum of the compound Laplacian on cusp forms.

Proposition 4.7. *Let f be a genuine affine-invariant modular form of weight k. For all $n \in \mathbb{Z}$, the Fourier–Heisenberg coefficients*

$$c^{\mathrm{H0}}(f;\, n, 0;\, y) \;=\; 0. \tag{4.10}$$

Proof. Genuine affine-invariant modular forms have $c^{\mathrm{T}}(f; 0, 0; \tau) = 0$. It follows from Lemma 4.3 that

$$0 = |c^{\mathrm{T}}(f; 0, 0; \tau)|^2 = \sum_{n \in \mathbb{Z}} |c^{\mathrm{H0}}(f; n, 0; y)|^2 \tag{4.11}$$

which is only possible if every term in this sum vanishes. ☐

4.4. **Eisenstein and Poincaré series.** We now define the series that provides generators for the constituents of $\mathrm{L}^2(\mathrm{SAff}_2(\mathbb{Z})\backslash\mathrm{SAff}_2(\mathbb{R}))$. Given $n, m \in \mathbb{Z} \setminus \{0\}$ and moreover a function $\beta : \mathbb{R}^+ \longrightarrow \mathbb{C}$ with $|\beta(y)| \ll y^{1-\frac{k}{2}+\varepsilon}$ as $y \longrightarrow 0$, we define the *affine Poincaré series*

$$P_{k;n,m,\beta}(\tau, z) := 2^{-\frac{1}{2}} \sum_{\gamma \in \Gamma^+_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \beta(y)\, e\big(nx + m\frac{v}{y}\big)\big|'_k\, \gamma, \tag{4.12}$$

where $\Gamma^+_\infty = \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \rangle$. The case $n = 0$ in this description is what we call the *affine Eisenstein series*

$$E_{k;m,\beta}(\tau, z) := 2^{-\frac{1}{2}} \sum_{\gamma \in \Gamma^+_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \beta(y)\, e\big(m\frac{v}{y}\big)\big|'_k\, \gamma. \tag{4.13}$$

Lemma 4.8. *The righthand sides of* (4.13) *and* (4.12) *are absolutely and locally uniformly convergent.*

Proof. The summations in (4.13) and (4.12) are well-defined thanks to the periodicity of of $e(\cdot)$.

We identify a coset $\gamma \in \Gamma^+_\infty \backslash \mathrm{SL}_2(\mathbb{Z})$ with the entries $c, d \in \mathbb{Z}$, $\gcd(c, d) = 1$ in the bottom row of the matrix. To show convergence, we need an estimate for the right hand side of

$$\Big| \sum_{\gamma \in \Gamma^+_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \beta(y)\, e\big(nx + m\frac{v}{y}\big)\big|'_k\, \gamma \Big| \le \sum_{\substack{c,d \in \mathbb{Z} \\ \gcd(c,d)=1}} |c\tau + d|^{-k}\, \big|\beta\big(\mathrm{Im}(\gamma\tau)\big)\big|$$

both for $n = 0$ and $n \neq 0$. Using the bound $\beta(y) \ll y^{1-\frac{k}{2}+\varepsilon}$ as $y \longrightarrow 0$, we obtain the estimate

$$\begin{aligned} \sum_{\substack{c,d \in \mathbb{Z} \\ \gcd(c,d)=1}} |c\tau + d|^{-k}\, \big|\beta\big(\mathrm{Im}(\gamma\tau)\big)\big| &= y^{-\frac{k}{2}} \sum_{\substack{c,d \in \mathbb{Z} \\ \gcd(c,d)=1}} \mathrm{Im}(\gamma\tau)^{\frac{k}{2}}\, \big|\beta\big(\mathrm{Im}(\gamma\tau)\big)\big| \\ &\ll y^{-\frac{k}{2}} \sum_{\substack{c,d \in \mathbb{Z} \\ \gcd(c,d)=1}} \mathrm{Im}(\gamma\tau)^{1+\varepsilon}, \end{aligned}$$

which converges absolutely and locally uniformly as required. ☐

We determine the Fourier–Heisenberg expansions of affine Eisenstein and Poincaré series. This allows us to examine the $\mathrm{SAff}_2(\mathbb{R})$ representations that those series generate. It is also an important ingredient in the proof of the $\mathrm{SAff}_2(\mathbb{R})$ decomposition in Theorem 5.1.

Lemma 4.9. *The Fourier–Heisenberg coefficients of the affine Eisenstein and Poincaré series at* $r = 0$ *are*

$$\begin{aligned} c^{\mathrm{H}0}\big(E_{k;m,\beta}(\tau, z); \tilde{n}, \tilde{m}; y\big) &= \begin{cases} 2^{-\frac{1}{2}}\, \beta(y), & \text{if } \tilde{n} = 0 \text{ and } \tilde{m} = \pm m; \\ 0, & \text{otherwise.} \end{cases} \\ c^{\mathrm{H}0}\big(P_{k;n,m,\beta}(\tau, z); \tilde{n}, \tilde{m}; y\big) &= \begin{cases} 2^{-\frac{1}{2}}\, \beta(y), & \text{if } \tilde{n} = n \text{ and } \tilde{m} = \pm m; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

In particular, if $y^{\frac{k-1}{2}} \beta(y)$ *in* (4.12) *and* (4.13) *is square-integrable, then*

$$\big\| E_{k,\beta,m} \big\|^2_{\mathbb{H}'} = \big\| P_{k,\beta,m,n} \big\|^2_{\mathbb{H}'} = \big\| y^{\frac{k-1}{2}} \beta \big\|^2,$$

where on the right hand side the L^2*-norm is with respect to the Haar measure on* $\mathbb{R}^+$*.*

Note that these Fourier coefficients are independent of k.

Proof. By definition the coefficients $c^{\mathrm{H}0}$ are the 0-th Fourier coefficient with respect to u. We observe that $e(n\mathrm{Re}(\tau))$ will not contribute any dependency on u and neither will $(c\tau+d)^{-k}$. We thus have to examine only the contribution of $e(mv/y)$. We have

$$\frac{v}{y}\Big|_0' \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{z(c\overline{\tau}+d)-\overline{z}(c\tau+d)}{|c\tau+d|^2}\,\frac{|c\tau+d|^2}{y} = \frac{v}{y}(cx+d)-cu. \tag{4.14}$$

In particular, the only contributions to the 0-th Fourier coefficient with respect to u arise from $c=0$, which is the term of the identity matrix.

The second statement follows from the first and Lemma 4.5. □

The Fourier–Heisenberg expansion also allows us to determine the L^2-norm of Eisenstein and Poincaré series. This stands in stark contrast to the case of Maaß forms for $\mathrm{SL}_2(\mathbb{Z})$, where Maaß Eisenstein series are not square-integrable in general and Maaß Poincaré series have comparably inaccessible formulae for their Fourier expansion.

4.5. **Representations generated by Eisenstein and Poincaré series.** The main concern of this section is to determine the isomorphism class of the representations generated by lifts of the Eisenstein and Poincaré series, which we will achieve in Proposition 4.13. To prepare its proof, we first identify the pullback of Eisenstein and Poincaré series to $\mathrm{G}'(\mathbb{R})$ as the images of partially defined maps from $\pi^{\mathrm{SAff}}_{nm^2} \cong \pi^{\mathrm{SAff}}_{n,m}$ (allowing $n=0$ to include the case of Eisenstein series) to $\mathrm{L}^2(\mathrm{G}'(\mathbb{Z})\backslash\mathrm{G}'(\mathbb{R}))$. Second, we show that these maps are isometries on their range and equivariant with respect to $\mathrm{SAff}_2(\mathbb{R})$. In the proof of Proposition 4.13 this allows us to extend them to all of $\pi^{\mathrm{SAff}}_{n,m}$.

Recall that $\pi^{\mathrm{SAff}}_{n,m}$ is an induction from $\mathrm{N}'(\mathbb{R})$ to $\mathrm{G}'(\mathbb{R})$. The Iwasawa decomposition in (3.6) shows that $f \in V(\pi^{\mathrm{SAff}}_{n,m})$ of K-type k is uniquely defined by its values on $\mathrm{A}'(\mathbb{R})/(\mathrm{A}'(\mathbb{R})\cap\mathrm{K})$. To make the connection to the function β in (4.13) and (4.12), we identify this quotient with $\mathbb{R}^+$ via the section

$$\mathbb{R}^+ \longrightarrow \mathrm{A}'(\mathbb{R})\big/\big(\mathrm{A}'(\mathbb{R})\cap\mathrm{K}\big), \quad a \longmapsto \pm\begin{pmatrix} a & \\ & a^{-1}\end{pmatrix}.$$

The functions $\alpha(a)$ in this section correspond to $\beta(y)=\alpha(\sqrt{y})$ in (4.13) and (4.12).

Given a function $\alpha : \mathbb{R}^+ \longrightarrow \mathbb{C}$ and $k \in \mathbb{Z}$, we use the pullback construction in (2.10), which implicitly depends on k, to define the function

$$\widehat{\alpha}_k(g) := \Big(\big(\alpha(\sqrt{y})\,e\big(nx+m\frac{v}{y}\big)\big)\Big|_k' g\Big)(i,0), \tag{4.15}$$

where we suppress n and m from our notation.

Lemma 4.10. *If* $\alpha \in \mathrm{L}^2(\mathbb{R}^+, a^{2k-3}\mathrm{d}a)$*, then* $\widehat{\alpha}_k \in V(\pi^{\mathrm{SAff}}_{n,m})$*. More precisely, given the* K*-type decomposition*

$$V\big(\pi^{\mathrm{SAff}}_{n,m}\big) \;=\; \bigoplus_{k\in\mathbb{Z}} V\big(\pi^{\mathrm{SAff}}_{n,m}\big)_k,$$

we have

$$V\big(\pi^{\mathrm{SAff}}_{n,m}\big)_k \;=\; \mathrm{span}\big\{\widehat{\alpha}_k : \alpha \in \mathrm{L}^2\big(\mathbb{R}^+, a^{2k-3}\mathrm{d}a\big)\big\}. \tag{4.16}$$

Proof. The coordinates for the decomposition $\mathrm{G}'(\mathbb{R}) = \mathrm{N}'(\mathbb{R})\,\mathrm{A}'(\mathbb{R})\,\mathrm{K}$ given in (3.8) yield

$$\widehat{\alpha}_k\Big(\big(\big(\begin{smallmatrix}1 & b\\ & 1\end{smallmatrix}\big), w_1, w_2\big)\, \big(\begin{smallmatrix}a & \\ & a^{-1}\end{smallmatrix}\big)\, \big(\begin{smallmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{smallmatrix}\big)\Big) = e^{ik\theta} a^k\, \alpha(a) e\big(nb + mw_1\big).$$

From this we directly read off that $\widehat{\alpha}_k$ satisfies the transformation properties required for elements of $V(\pi^{\mathrm{SAff}}_{n,m})$, namely

$$\widehat{\alpha}_k(hg) \;=\; \chi^{\mathrm{N}}_{n,m}(h)\, \widehat{\alpha}_k(g) \quad \text{for all } h \in \mathrm{N}'(\mathbb{R}).$$

To verify that $\widehat{\alpha}_k \in V(\pi^{\mathrm{SAff}}_{n,m})$ it remains to verify that it is square integrable on $\mathrm{A}'(\mathbb{R})\,\mathrm{K}$, which follows from

$$\int_{\mathrm{A}'(\mathbb{R})\,\mathrm{K}} \Big|\widehat{\alpha}_k\Big(\big(\begin{smallmatrix}a & \\ & a^{-1}\end{smallmatrix}\big)\, \big(\begin{smallmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{smallmatrix}\big)\Big)\Big|^2 \frac{\mathrm{d}\theta\, \mathrm{d}a}{2\pi\, a^3} = \int_{\mathbb{R}^+} a^{2k-2} |\alpha(a)|^2\, \frac{\mathrm{d}a}{a}.$$

Notice that the correspondence between α and $\widehat{\alpha}_k$ is one-to-one.

In order to confirm the given K-type decomposition, note that by the Peter–Weyl theorem for the compact group K, any $f \in V(\pi^{\mathrm{SAff}}_{n,m})$ can be decomposed as a square-summable series $\sum_k f_k$, where f_k is square-integrable and of K-type k. By the previous argument and for fixed k, we find $f_k = \widehat{\alpha}_k$ for some $\alpha \in \mathrm{L}^2(\mathbb{R}^+, a^{2k-3}\mathrm{d}a)$ as desired. □

Consider $f \in V(\pi^{\mathrm{SAff}}_{n,m})$. We associate to f a $\mathrm{SAff}_2(\mathbb{Z})$-left-invariant function on $\mathrm{SAff}_2(\mathbb{R})$ via

$$E(f, \cdot) := \Big(g \mapsto \sum_{\gamma \in \Gamma^+_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} f(\gamma g)\Big), \qquad g \in \mathrm{SAff}_2(\mathbb{R})\,, \tag{4.17}$$

provided absolute convergence of the series.

Lemma 4.11. *Suppose $f \in V(\pi^{\mathrm{SAff}}_{n,m})$ has a finite decomposition $\sum \widehat{\alpha}_k$ for functions $\alpha_k \in \mathrm{L}^2(\mathbb{R}^+, a^{2k-3}\mathrm{d}a)$, $k \in \mathbb{Z}$ according to (4.16). Assume that*

$$\Big|\widehat{\alpha}_k\big(\big(\big(\begin{smallmatrix}1 & b\\ & 1\end{smallmatrix}\big), w_1, w_2\big)\, \big(\begin{smallmatrix}a & \\ & a^{-1}\end{smallmatrix}\big)\, \big(\begin{smallmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{smallmatrix}\big)\big)\Big| \ll a^{2-k+\varepsilon} \quad \text{as} \quad a \longrightarrow 0.$$

Then $E(f, \cdot)$ converges absolutely and locally uniformly and $\|f\|^2 = 2\|E(f, \cdot)\|^2$.

Proof. Since K is compact, every $\widehat{\alpha}_k$ in the decomposition $f = \sum \widehat{\alpha}_k$ satisfies the same growth condition as f. Therefore, it suffices to demonstrate convergence of $E(\widehat{\alpha}_k, g)$ with $\widehat{\alpha}_k$ as in (4.15). Further, since we average over $\Gamma^+_\infty \backslash \mathrm{SL}_2(\mathbb{Z})$ from the left and K-types are defined via right-shifts, the K-type of each summand in the definition (4.17) of $E(\widehat{\alpha}_k, g)$ is k as well. It thus suffices to consider $g \in \mathrm{N}'(\mathbb{R})\mathrm{A}'(\mathbb{R})$. Using (2.10), this allows us to perform the proof for functions on $\mathbb{H}'$ instead of $\mathrm{G}'(\mathbb{R})$.

Recall the expression $e^{ik\theta} a^k \alpha_k(a)\, e(nb + mw_1)$ from the proof of Lemma 4.10. Under the map in (2.10) using the coordinates of (3.8), we have to show the absolute and locally uniform convergence of

$$\sum_{\gamma \in \Gamma^+_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} e^{-ik\theta} y^{-\frac{k}{2}}\, e^{ik\theta} \sqrt{y}^k \alpha_k(\sqrt{y}) e\big(nx + m\frac{v}{y}\big)\big|'_k\, \gamma$$
$$= \sum_{\gamma \in \Gamma^+_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \alpha_k(\sqrt{y}) e\big(nx + m\frac{v}{y}\big)\big|'_k\, \gamma.$$

Since by assumptions $\beta_k(y) := \alpha_k(\sqrt{y}) < \sqrt{y}^{2-k+\varepsilon} = y^{1-k/2+\varepsilon/2}$, we can apply Lemma 4.8 to finish the proof of convergence.

To show the isometry property, we notice that we have the K-type decomposition

$$E(f,\,\cdot\,) = \sum_{k\in\mathbb{Z}} E(\widehat{\alpha}_k,\,\cdot\,),$$

and that the summands on the right hand side are mutually orthogonal.

$$\|E(f,\,\cdot\,)\|^2 = \sum_{k\in\mathbb{Z}} \|E(\widehat{\alpha}_k,\,\cdot\,)\|^2$$
$$= \sum_{k\in\mathbb{Z}} \left\|y^{\frac{k-1}{2}}\beta_k(y)\right\|^2 = \sum_{k\in\mathbb{Z}} \left\|a^{k-1}\alpha_k(a)\right\|^2 = \|f\|^2.$$

We employed the identification in (4.18) and then Lemma 4.9 to express the resulting norms in terms of the norms of first β_k and then α_k. □

Lemma 4.12. *Given a function f subject to the conditions of Lemma 4.11, the function $f_h(g) := f(gh)$ for $h \in \mathrm{SAff}_2(\mathbb{R})$ also satisfies the conditions of Lemma 4.11, and $E(f, gh) = E(f_h, g)$.*

Proof. The first statement is clear when applying the $\mathrm{N}'(\mathbb{R})\mathrm{A}'(\mathbb{R})\mathrm{K}$-decomposition to $g = nak$ and $khk^{-1} = \tilde{n}\tilde{a}\tilde{k}$ and multiplying out the result as $gh = na\tilde{n}\tilde{a}\,\tilde{k}k$. Now the second claim follows from the observation that $E(f_h, g)$ is defined by a sum over left-shifts, while h acts via right-shifts. □

We can now state the first main result of this section.

Proposition 4.13. *The Eisenstein series and Poincaré series for appropriate parameters generate all the genuine unitary irreducible representations of $\mathrm{G}'(\mathbb{R})$. More precisely, for $k \in \mathbb{Z}$, and $\beta : \mathbb{R}^+ \longrightarrow \mathbb{C}$ with $|\beta(y)| \ll y^{1-\frac{k}{2}+\varepsilon}$ as as $y \longrightarrow 0$*

$$\pi_{\mathrm{L}^2}\big(\widetilde{E_{k;m,\beta}}\big) \cong \pi_0^{\mathrm{SAff}} \quad \textit{and} \quad \pi_{\mathrm{L}^2}\big(\widetilde{P_{k;n,m,\beta}}\big) \cong \pi_{nm^2}^{\mathrm{SAff}}.$$

Proof. In the proof of Lemma 4.11 we have shown that if we let $\alpha(a) = \beta(a^2)$ then

$$E(\widehat{\alpha}_k,\,\dot{}\,) = \widetilde{E_{k;m,\beta}}, \quad \text{if } n = 0, \quad \text{and} \quad E(\widehat{\alpha}_k,\,\cdot\,) = \widetilde{P_{k;n,m,\beta}}, \quad \text{otherwise.} \tag{4.18}$$

The automorphic Eisenstein series $E(\,\cdot\,,\,\cdot\,)$ in (4.17) yields by Lemma 4.11 and Lemma 4.12 a map from a dense subspace of $V(\pi_{n,m}^{\mathrm{SAff}})$ to $\mathrm{L}^2(\mathrm{SAff}_2(\mathbb{Z})\backslash\mathrm{SAff}_2(\mathbb{R}))$ which is $\mathrm{SAff}_2(\mathbb{R})$-equivariant. As it is an isometry by Lemma 4.11, this map does not vanish and is a homomorphism of Hilbert space representations from $\pi_{n,m}^{\mathrm{SAff}}$ to $\mathrm{L}^2(\mathrm{SAff}_2(\mathbb{Z})\backslash\mathrm{SAff}_2(\mathbb{R}))$. Since $\pi_{n,m}^{\mathrm{SAff}} \cong \pi_{nm^2}^{\mathrm{SAff}}$ is irreducible by the classification in Theorem 3.7, the image of the map $E(\,\cdot\,,\,\cdot\,)$ in (4.18) is equal to the image of

$$E : \pi_{n,m}^{\mathrm{SAff}} \longrightarrow \mathrm{L}^2\big(\mathrm{SAff}_2(\mathbb{Z})\backslash\mathrm{SAff}_2(\mathbb{R})\big)\,.$$

as claimed in the proposition. □

5. Decomposition of the L^2-space

5.1. **Decomposition as $\mathrm{G}'(\mathbb{R})$-representations.** We restate and refine Theorem 1.1 from the introduction.

Theorem 5.1. *The genuine part of the L^2-space of the stratum $\mathcal{H}(0,0)$ admits the abstract decomposition*

$$\mathrm{L}^2\big(\mathrm{G}'(\mathbb{Z})\backslash\mathrm{G}'(\mathbb{R})\big)^{\mathrm{gen}} \;\cong\; \bigoplus_{m=1}^{\infty}\bigoplus_{n\in\mathbb{Z}} \pi_{n,m}^{\mathrm{SAff}}\,. \tag{5.1}$$

into irreducible $\mathrm{G}'(\mathbb{R})$*-representations. More precisely, it can be decomposed as*

$$\mathrm{L}^2\big(\mathrm{G}'(\mathbb{Z})\backslash\mathrm{G}'(\mathbb{R})\big)^{\mathrm{gen}} = \bigoplus_{m=1}^{\infty}\Big(\pi_{\mathrm{L}^2}\big(\widetilde{E_{k;m,\beta}}\big) \oplus \bigoplus_{n\in\mathbb{Z}\setminus\{0\}} \pi_{\mathrm{L}^2}\big(\widetilde{P_{k;n,m,\beta}}\big)\Big), \tag{5.2}$$

and the map E from (4.17) defines an isometry of $\mathrm{G}'(\mathbb{R})$*-representations*

$$\pi^{\mathrm{SAff}}_{0,m} \cong \pi_{\mathrm{L}^2}\big(\widetilde{E_{k;m,\beta}}\big) \quad \text{and} \quad \pi^{\mathrm{SAff}}_{n,m} \cong \pi_{\mathrm{L}^2}\big(\widetilde{P_{k;n,m,\beta}}\big)$$

for any $k \in \mathbb{Z}$ and $0 \neq \beta \in \mathrm{L}^2(\mathbb{R}^+, y^{k-2}\mathrm{d}y)$, if the left hand side is provided with the L^2*-norms from (4.10). Furthermore, for $\beta_i \in \mathrm{L}^2(\mathbb{R}^+, y^{k_i-2}\mathrm{d}y), i = 1, 2$*

$$\begin{aligned}\pi_{\mathrm{L}^2}\big(E_{k_1;m_1,\beta_1}\big) &\cong \pi_{\mathrm{L}^2}\big(E_{k_2;m_2,\beta_2}\big) && \text{for all } m_1 \text{ and } m_2, \text{ and}\\ \pi_{\mathrm{L}^2}\big(P_{k_1;n_1,m_1,\beta_1}\big) &\cong \pi_{\mathrm{L}^2}\big(P_{k_2;n_2,m_2,\beta_2}\big) && \text{if and only if } n_1m_1^2 = n_2m_2^2.\end{aligned}$$

Proof. The isomorphisms stated in the second part are a consequence of Theorem 3.7. Note that the weights k_1 and k_2 do not appear in these statements and hence do not distinguish representations.

By Proposition 4.13, Eisenstein and Poincaré series yield isomorphisms

$$\begin{aligned}\pi^{\mathrm{SAff}}_{0,m} &\longrightarrow \pi_{\mathrm{L}^2}\big(\widetilde{E_{k;m,\beta}}\big) \subseteq \mathrm{L}^2\big(\mathrm{G}'(\mathbb{Z})\backslash\mathrm{G}'(\mathbb{R})\big)^{\mathrm{gen}} && \text{and}\\ \pi^{\mathrm{SAff}}_{n,m} &\longrightarrow \pi_{\mathrm{L}^2}\big(\widetilde{P_{k;n,m,\beta}}\big) \subseteq \mathrm{L}^2\big(\mathrm{G}'(\mathbb{Z})\backslash\mathrm{G}'(\mathbb{R})\big)^{\mathrm{gen}} && \text{for } n \neq 0.\end{aligned}$$

Taking the direct sum, we obtain a map

$$\bigoplus_{m=1}^{\infty}\bigoplus_{n\in\mathbb{Z}} \pi^{\mathrm{SAff}}_{n,m} = \bigoplus_{m=1}^{\infty} \pi^{\mathrm{SAff}}_{0,m} \oplus \bigoplus_{m=1}^{\infty}\bigoplus_{n\in\mathbb{Z}\setminus\{0\}} \pi^{\mathrm{SAff}}_{n,m} \longrightarrow \mathrm{L}^2\big(\mathrm{G}'(\mathbb{Z})\backslash\mathrm{G}'(\mathbb{R})\big)^{\mathrm{gen}}. \tag{5.3}$$

The Fourier–Heisenberg expansions group provide us by Proposition 4.4 with a map

$$\mathrm{L}^2\big(\mathrm{G}'(\mathbb{Z})\backslash\mathrm{G}'(\mathbb{R})\big)^{\mathrm{gen}} \cap \mathrm{C}\big(\mathrm{G}'(\mathbb{Z})\backslash\mathrm{G}'(\mathbb{R})\big) \longrightarrow \bigoplus_{m=1}^{\infty}\bigoplus_{n\in\mathbb{Z}} \pi^{\mathrm{SAff}}_{n,m}. \tag{5.4}$$

By Lemma 4.8 Eisenstein and Poincaré series associated with continuous functions β that satsify the growth condition $\beta(y) \ll y^{1-\frac{k}{2}+\varepsilon}$ as $y \longrightarrow 0$ are continuous.

Let $V \subset \bigoplus_{m\geq 0}\bigoplus_{n\in\mathbb{Z}} \pi^{\mathrm{SAff}}_{n,m}$ be the dense subspace consisting in finite sums of continuous functions that satisfy the assumptions given in Lemma 4.11. By Lemma 4.9 the composition of the restriction of (5.3) to V and (5.4) is the multiplication by $2^{-1/2}$ map. Since V is a dense subspace, this shows that there is an injection

$$\bigoplus_{m=1}^{\infty}\bigoplus_{n\in\mathbb{Z}} \pi^{\mathrm{SAff}}_{n,m} = \bigoplus_{m=1}^{\infty}\bigoplus_{n\in\mathbb{Z}} \pi^{\mathrm{SAff}}_{nm^2} \hookrightarrow \mathrm{L}^2\big(\mathrm{G}'(\mathbb{Z})\backslash\mathrm{G}'(\mathbb{R})\big)^{\mathrm{gen}}. \tag{5.5}$$

We next investigate the kernel of (5.4). Lemma 4.3 shows that it consists of functions whose K-isotypical components, say ϕ_k, satisfy $c^{\mathrm{T}}(\phi_k; m, 0; \tau) = 0$ for all positive m. By Lemma 4.1 with γ equal the negative identity, this implies that $c^{\mathrm{T}}(\phi_k; m, 0; \tau) = 0$ for all $m \neq 0$. Since ϕ_k is genuine, we also have $c^{\mathrm{T}}(\phi_k; 0, 0; \tau) = 0$ by Lemma 4.6. This allows us to apply Proposition 4.2 to deduce that the kernel of (5.4) is trivial. Hence (5.5) is an isomorphism, finishing the proof. □

5.2. **Cusp forms.** Consider an affine modular-invariant function f of weight k. We call f a *cusp form* if the $r = n = 0$ Fourier–Heisenberg coefficients vanish, i.e. $c^{\mathrm{H}}(f;\, 0, 0; v, v/y) = 0$. By definition of the Fourier–Heisenberg expansion, this is equivalent to

$$c^{\mathrm{H}0}(f;\, 0, m;\, y) \;=\; 0 \quad \text{for all } m \in \mathbb{Z}. \tag{5.6}$$

We denote by $\mathrm{L}^2(\mathcal{H}(0,0))^{\mathrm{gen}}_{\mathrm{cusp}}$ the closure of the space of the lifts of genuine cusp forms. (Recall Lemma 4.6 for a characterization of these in terms of Fourier coefficients.)

Proposition 5.2. *In terms of the decomposition* (5.1) *the space of cusp forms in the genuine part coincides with* $\oplus_{m=1}^{\infty} \pi^{\mathrm{SAff}}_{0,m}$.

Proof. Proposition 3.8 implies that $\oplus_{m=1}^{\infty} \pi^{\mathrm{SAff}}_{0,m}$ is precisely the subspace where the total Casimir acts with eigenvalue zero and by (3.19) this is equivalent to the vanishing of the Fourier coefficients involved in the definition of a cusp form. □

5.3. **Spectral decomposition of the foliated Laplacian.** This section prepares for the explicit description of $\pi^{\mathrm{SAff}}_{n,m}$ Theorem 5.6 below. The idea is that the spectral data of the foliated Laplacian $-\Delta^{\mathrm{fol}}_k$ suffices to distinguish almost all unitary representations of $\mathrm{SL}_2(\mathbb{R})$, in particular, those that appear in the continuous and discrete part of $\pi^{\mathrm{SAff}}_{n,m}$ per Proposition 3.9. We thus compute here the solutions of the differential equations for functions of the form (4.15) that are generalized eigenfunctions of $-\Delta^{\mathrm{fol}}_k$. Since we already know the abstract decomposition of these representations thanks to Proposition 3.9 we only solve the generalized eigenvalue equation for the Casimir eigenvalues of the representations appearing there. Recall that the Casimir eigenvalue of the discrete series D^{SL}_k is equal to $\frac{|k|}{2}(\frac{|k|}{2} - 1)$ and for the complementary series $I^{\mathrm{SL}}_{+,s}$ it is equal to $s^2 - 1/4$.

Recall e.g. from [DLMF, Section 13.14] that the *Whittaker differential equation*

$$\frac{\mathrm{d}^2 f}{\mathrm{d}y^2} + \Big(\frac{-1}{4} + \frac{\kappa}{y} + \frac{\frac{1}{4} - \mu^2}{y^2}\Big) f \;=\; 0 \tag{5.7}$$

has two solutions, traditionally called the *Whittaker functions* $\mathrm{M}_{\kappa,\mu}$ *and* $\mathrm{W}_{\kappa,\mu}$ except if $2\mu \in \mathbb{Z}_{<0}$.

We have computed in (3.19) the action of Δ^{tot} on the summands of the Poincaré series aiming for the computation of Δ^{tot}-eigenvalues. Using Lemma 2.13 we similarly find (with an auxiliary factor $y^{-\frac{k}{2}}$ that simplifies the equation):

Lemma 5.3. *A smooth function* $\beta(y) : \mathbb{R}^\times \longrightarrow \mathbb{C}$ *is mapped under the foliated Laplace operator to*

$$\begin{aligned} &- \Delta^{\mathrm{fol}}_k \Big(y^{-\frac{k}{2}} \beta(y)\, e\big(nx + m\frac{v}{y}\big)\Big) \\ &=\; y^{-\frac{k}{2}} \Big(- y^2 \beta''(y) + 4\pi^2 n^2\, y^2 \beta(y) - 2\pi k n\, y \beta(y)\Big)\, e\big(nx + m\frac{v}{y}\big). \end{aligned}$$

We apply the preceding lemma to search for the eigenfunctions and generalized eigenfunctions we expect according to Proposition 3.9.

Lemma 5.4. *For fixed* $k \in \mathbb{Z} \setminus \{0, \pm 1\}$ *and* $n \neq 0$ *consider the differential equation*

$$-\Delta^{\mathrm{fol}}_k\, \beta(y)\, e\big(nx + m\frac{v}{y}\big) \;=\; \lambda \cdot \beta(y)\, e\big(nx + m\frac{v}{y}\big). \tag{5.8}$$

For $\lambda = \frac{|k|}{2}(1-\frac{|k|}{2})$ *it has a basis of solutions consisting of*

$$e^{-2\pi|n|y} \quad \text{and} \quad y^{-\frac{k}{2}}\,\mathrm{M}_{\frac{|k|}{2},\frac{|k|-1}{2}}(4\pi|n|y) \tag{5.9}$$

if $k, n > 0$, *and*

$$y^{-k}\,e^{-2\pi|n|y} \quad \text{and} \quad y^{-\frac{k}{2}}\,\mathrm{M}_{\frac{|k|}{2},\frac{|k|-1}{2}}(4\pi|n|y), \tag{5.10}$$

if $k, n < 0$.

For fixed $k \in \mathbb{Z}$, $n \neq 0$ *and* $\lambda = t^2 + 1/4$ *the differential equation* (5.8) *has a basis of solutions consisting of*

$$y^{-\frac{k}{2}}\,\mathrm{W}_{\frac{\mathrm{sgn}(n)k}{2},it}\big(4\pi|n|\,y\big) \quad \text{and} \quad y^{-\frac{k}{2}}\,\mathrm{M}_{\frac{\mathrm{sgn}(n)k}{2},it}\big(4\pi|n|\,y\big). \tag{5.11}$$

Finally, for $n = 0$ *and* $\lambda = t^2 + 1/4$, *the differential equation* (5.8) *has a basis of solutions*

$$y^{\frac{1-k}{2}+it} \quad \text{and} \quad y^{\frac{1-k}{2}-it}. \tag{5.12}$$

Proof. For $n = 0$ the function $\tilde{\beta}(y) = y^{\frac{k}{2}}\,\beta(y)$ is a solution of the differential equation $\tilde{\beta}''(y) = \lambda\,y^{-2}\tilde{\beta}(y)$, whose solutions are directly seen to yield (5.12).

For $n \neq 0$ the solutions in (5.11) follow from the observation that $y^{\frac{k}{2}}\,\beta(y/4\pi|n|)$ satisfies the Whittaker differential equation with parameters $\mathrm{sgn}(n)k/2$ and it.

In the special case that $|k| > 1$ is an integer with the same sign as n, the exponential solutions are equal to the W-Whittaker solutions in (5.11) by [DLMF, Equation 13.14.9]. □

We will need the following asyptotics estimates in the next section to verify integrability. We define

$$\Gamma^{\mathrm{W}}(t) := \frac{\Gamma(2it)}{\Gamma\big(\frac{1-\mathrm{sgn}(n)k}{2} + it\big)}\,, \tag{5.13}$$

Lemma 5.5. *The Whittaker functions* $\mathrm{W}_{\kappa,\mu}(y)$ *decay exponentially as* $y \to \infty$. *Moreover the asympotices of* $y \to 0$ *is*

$$\begin{aligned}&(4\pi|n|\,y)^{-\frac{k}{2}}\,\mathrm{W}_{\frac{\mathrm{sgn}(n)k}{2},it}\big(4\pi|n|\,y\big)\\ &= \Gamma^{\mathrm{W}}(t)\,\big(4\pi|n|y\big)^{\frac{1-k}{2}-it} + \Gamma^{\mathrm{W}}(-t)\,\big(4\pi|n|y\big)^{\frac{1-k}{2}+it} + \mathcal{O}\big(y^{\frac{3-k}{2}}\big).\end{aligned} \tag{5.14}$$

Proof. This follows from [DLMF, Equations 13.14.21 and 13.14.16]. □

5.4. **Decomposition as** $\mathrm{SL}_2(\mathbb{R})$**-representations.** We now state and prove Theorem 1.3 in the complete version, including the case $n = 0$. Recall that we gave in Lemma 4.10 an explicit L^2-structure on the representations $\pi^{\mathrm{SAff}}_{n,m}$.

Theorem 5.6. *For* $k \in \mathbb{Z} \setminus \{0, \pm 1\}$ *and* $n \in \mathbb{Z}$ *with* $\mathrm{sgn}(nk) = 1$ *the representation* $D^{\mathrm{SL}}_{\mathrm{sgn}(n)k}$ *in Proposition 3.9 is generated by the Poincaré series for* $\beta = e^{-2\pi|n|y}$ *if* $k > 1$ *and* $\beta = y^{-k}e^{-2\pi|n|y}$ *if* $k < -1$.

Associating for fixed $n \in \mathbb{Z} \setminus \{0\}$ *with* $\psi \in \mathrm{L}^2(\mathbb{R}^+, dt)$ *the lifts of the Poincaré series* $P_{k;n,m,\beta^{\mathrm{W}}_{k,n,\psi}}$ *of the Whittaker transform*

$$\beta^{\mathrm{W}}_{k,n,\psi}(y) := \frac{1}{4\pi\,|n|^{\frac{3}{2}}} \int_{t\in\mathbb{R}^+} \frac{\psi(t)}{(\Gamma^{\mathrm{W}}(t)\Gamma^{\mathrm{W}}(-t))^{\frac{1}{2}}}\,y^{-\frac{k}{2}}\,\mathrm{W}_{\frac{\mathrm{sgn}(n)k}{2},it}\big(4\pi|n|\,y\big)\ \mathrm{d}t$$

gives isometric embeddings

$$\mathrm{P}^{\mathrm{W}}_{+} : \bigoplus_{k \in 2\mathbb{Z}} \mathrm{L}^2\big(\mathbb{R}^+, \mathrm{d}t\big) \to \pi^{\mathrm{SAff}}_{n,m}\,, \qquad \mathrm{P}^{\mathrm{W}}_{-} : \bigoplus_{k \in 1+2\mathbb{Z}} \mathrm{L}^2\big(\mathbb{R}^+, \mathrm{d}t\big) \to \pi^{\mathrm{SAff}}_{n,m}$$

whose images are $\int^{\oplus}_{\mathbb{R}^+} I^{\mathrm{SL}}_{+,it}\,\mathrm{d}t$ *and* $\int^{\oplus}_{\mathbb{R}^+} I^{\mathrm{SL}}_{-,it}\,\mathrm{d}t$ *respectively, in the decomposition of Proposition 3.9.*

Associating with $\psi \in \mathrm{L}^2(\mathbb{R}^+, dt)$ *the lifts of the Eisenstein series* $E_{k;m,\beta^{\mathrm{W}}_{k,n,\psi}}$ *of the 'y-power transform'*

$$\beta^{\mathrm{c}\pm}_{k,\psi}(y) := \int_{t \in \mathbb{R}^+} \psi(t)\, \big(y^{\frac{1-k}{2}+it} \pm y^{\frac{1-k}{2}-it}\big)\,\mathrm{d}t \tag{5.15}$$

gives isometric embeddings

$$\mathrm{E}^{\mathrm{c}\pm}_{+} : \bigoplus_{k \in 2\mathbb{Z}} \mathrm{L}^2\big(\mathbb{R}^+, \mathrm{d}t\big) \to \pi^{\mathrm{SAff}}_{0,m}\,, \qquad \mathrm{E}^{\mathrm{c}\pm}_{-} : \bigoplus_{k \in 1+2\mathbb{Z}} \mathrm{L}^2\big(\mathbb{R}^+, \mathrm{d}t\big) \to \pi^{\mathrm{SAff}}_{0,m}$$

whose images are one of the two summands $\int^{\oplus}_{\mathbb{R}^+} I^{\mathrm{SL}}_{+,it}\,\mathrm{d}t$ *and* $\int^{\oplus}_{\mathbb{R}^+} I^{\mathrm{SL}}_{-,it}\,\mathrm{d}t$ *respectively, in the decomposition of Proposition 3.9. Moreover, the images of* $\mathrm{E}^{\mathrm{c}+}_{\pm}$ *and* $\mathrm{E}^{\mathrm{c}-}_{\pm}$ *are orthogonal.*

Proof. We start with the discrete series and observe that $y^{\frac{k-1}{2}}\beta(y) \in \mathrm{L}^2(\mathbb{R}^+, \mathrm{d}y/y)$. Therefore, the Poincaré series in the first statement is defined as an L^2-limit of the series in Lemma 4.8. Recall that the representation generated by $P_{k;n,m,\beta}$ is isomorphic by Theorem 5.1 to the one generated by $\widehat{\beta}$, defined as the lift of $\beta(y)\, e\big(nx + mv/y\big)$. This function is smooth and square-integrable on $\mathrm{SAff}_2(\mathbb{R})$. In particular, we can compute the action $\mathcal{D}^{\mathrm{fol}}$ pointwise. Furthermore, the definition of $\mathcal{D}^{\mathrm{fol}}$ via Δ^{fol} in Lemma 2.2 allows us to compute it on $\mathbb{H}'$. Now the first statement of Lemma 5.4 confirms the existence of the eigenspaces. (Note that the M-Whittaker function given as the second solution in that lemma has exponential growth as $y \to \infty$ and will not give an L^2-function.)

Our argument for the principal series follows the argument for the modular surface (e.g. [Ber16, Section 4.2.5]) with one major difference in Lemma 5.7. Because we cannot evaluate exactly the inner products of truncated W-Whittaker functions that we define in the proof, we need to estimate some of their contribution via asymptotic remainder terms.

Suppose $n \neq 0$. Thanks to Lemma 5.5 partial integration with respect to t in the defining equation for $\beta^{\mathrm{W}}_{k,n,\psi}$ shows that $y^{(k-1)/2}\,\beta^{\mathrm{W}}_{k,n,\psi}(y)$ is square-integrable with respect to the Haar measure on $\mathbb{R}^+$. Using Lemma 5.7 below and Lemma 4.11 we conclude that assigning with ψ the Poincaré series is an isometry as claimed. To finish the proof in this case we apply Weyl's criterion for essential spectrum membership for any $t_0 \in \mathbb{R}$. Let $D = -\Delta^{\mathrm{fol}}_k - (t_0^2 + 1/4)$ and $\psi_n(t)$ a sequence of bump functions limiting to t_0 with $\|\psi_n\| = 1$. Then G-invariance of the Laplacian, abolute convergence and isometry of the E-operator (by Lemma 4.8 and Lemma 4.9) and the eigenvalue property of the Whittaker function from Lemma 5.4 imply

$$\begin{aligned} \big\| D(P_{k,n,m,\beta^{W}_{k,n,\psi_n}}))\big\| \;&=\; \big\| E\big(D(\beta^{W}_{k,n,\psi_n}\exp(..)), \cdot\big)\big\| \;=\; \big\| D(\beta^{W}_{k,n,\psi_n}\exp(..))\big\| \\ = \;&\left| \frac{1}{4\pi\, |n|^{\frac{3}{2}}} \int_{t \in \mathbb{R}^+} \frac{\psi_n(t)(t^2 - t_0^2)}{(\Gamma^{\mathrm{W}}(t)\Gamma^{\mathrm{W}}(-t))^{\frac{1}{2}}}\, y^{-\frac{k}{2}}\, \mathrm{W}_{\frac{\mathrm{sgn}(n)k}{2}, it}\big(4\pi |n|\, y\big)\, \mathrm{d}t \right| \;\to 0\,, \end{aligned}$$

since on the support of ψ_n the factor $(t^2 - t_0^2)$ becomes small. This shows that $t_0^2 + 1/4$ is an approximative eigenvalue and since the discrete spectrum is associate with positive eigenvalues, our knowledge about the total spectral decomposition from Proposition 3.9 shows that we have covered everything.

The case $n = 0$ is similar. By partial integration, we see that $y^{(k-1)/2}\,\beta^{c\pm}_{k,\psi}(y)$ is square-integrable. Now Lemma 5.8 below and Lemma 4.11 show the isometry claim. To show the orthogonality one proceeds as in Lemma 5.8, but now the first and last term cancel and we get zero as $\varepsilon \to 0$. The spectral conclusion is similar as above using Weyl's criterion, Lemma 5.4 and Proposition 3.9, arguing separately for each of the two orthogonal summands. □

Lemma 5.7. *The Whittaker transform $\psi \mapsto \beta^{W}_{k,n,\psi}$ is an isometry $L^2\big(\mathbb{R}^+, dt\big) \to L^2(\mathbb{R}^+, y^{k-2}dy)$.*

Proof. We first verify the claim for smooth compactly supported functions. To this end, we introduce for $\varepsilon > 0$ the truncated W-Whittaker functions

$$\begin{aligned} &W^{\varepsilon}_{\frac{\operatorname{sgn}(n)k}{2},it}\big(4\pi|n|\,y\big) \\ :=\; &W_{\frac{\operatorname{sgn}(n)k}{2},it}\big(4\pi|n|\,y\big) \;-\; \mathbb{1}_{(0,\varepsilon)}(y)\,\Big(\Gamma^{W}(t)\,\big(4\pi|n|y\big)^{\frac{1}{2}-it} + \Gamma^{W}(-t)\,\big(4\pi|n|y\big)^{\frac{1}{2}+it}\Big) \end{aligned}$$

and denote by $\beta^{W\varepsilon}_{k,n,\psi}(y)$ the Whittaker transform with respect to the truncated Whittaker functions. Using partial integration with respect to t we see that

$$\big\|y^{\frac{k-1}{2}}\big(\beta^{W}_{k,n,\psi}(y) - \beta^{W\varepsilon}_{k,n,\psi}(y)\big)\big\|^2 \ll \int_0^{\varepsilon} \frac{1}{y\log(y)^2}\,dy \ll \log(\varepsilon)^{-1}.$$

Combining this estimate with the Cauchy-Schwartz inequality, we conclude that

$$\big\|y^{\frac{k-1}{2}}\,\beta^{W}_{k,n,\psi}(y)\big\|^2 \;=\; \big\|y^{\frac{k-1}{2}}\,\beta^{W\varepsilon}_{k,n,\psi}(y)\big\|^2 + \mathcal{O}\big(\log(\varepsilon)^{-\frac{1}{2}}\big).$$

We next expand the defining integral for the L^2-norm and interchange the integration with respect to y, t_1, and t_2, which is justified because all integrands are nonnegative:

$$\begin{aligned} &\big\|y^{\frac{k-1}{2}}\,\beta^{W\varepsilon}_{k,n,\psi}(y)\big\|^2 \\ =\;& \big(4\pi|n|\big)^{-k}\int_{t_1,t_2\in\mathbb{R}^+}\psi(t_1)\overline{\psi(t_2)}\,\big(\Gamma^{W}(t_1)\Gamma^{W}(-t_1)\,\Gamma^{W}(t_2)\Gamma^{W}(-t_2)\big)^{-\frac{1}{2}} \\ &\qquad\int_{\mathbb{R}^+} W^{\varepsilon}_{\frac{\operatorname{sgn}(n)k}{2},it_1}\big(4\pi|n|\,y\big)\,W^{\varepsilon}_{\frac{\operatorname{sgn}(n)k}{2},-it_2}\big(4\pi|n|\,y\big)\;y^{-2}dy\,dt_1dt_2. \end{aligned}$$

Using the asymptotic expansion of the Whittaker function in (5.14), we can determine the leading asymptotic with respect to ε of the inner integral. For $\varepsilon_1 > \varepsilon_2 > 0$, we have

$$\begin{aligned} &\int_{\mathbb{R}^+} W^{\varepsilon_2}_{\frac{\operatorname{sgn}(n)k}{2},it_1}\big(4\pi|n|\,y\big)\,W^{\varepsilon_2}_{\frac{\operatorname{sgn}(n)k}{2},-it_2}\big(4\pi|n|\,y\big)\;y^{-2}dy \\ &\quad-\int_{\mathbb{R}^+} W^{\varepsilon_1}_{\frac{\operatorname{sgn}(n)k}{2},it_1}\big(4\pi|n|\,y\big)\,W^{\varepsilon_1}_{\frac{\operatorname{sgn}(n)k}{2},-it_2}\big(4\pi|n|\,y\big)\;y^{-2}dy \\ =\;&\int_{\varepsilon_2}^{\varepsilon_1}\Big(\Gamma^{W}(t_1)\,\big(4\pi|n|y\big)^{\frac{1}{2}-it_1} + \Gamma^{W}(-t_1)\,\big(4\pi|n|y\big)^{\frac{1}{2}+it_1}\Big) \\ &\qquad\Big(\Gamma^{W}(-t_2)\,\big(4\pi|n|y\big)^{\frac{1}{2}+it_2} + \Gamma^{W}(t_2)\,\big(4\pi|n|y\big)^{\frac{1}{2}-it_2}\Big)\,y^{-2}dy \;+\; \mathcal{O}(\varepsilon_1 - \varepsilon_2) \end{aligned}$$

$$\begin{aligned}
&= \Gamma^{\mathrm{W}}(t_1)\Gamma^{\mathrm{W}}(-t_2)\,\big(4\pi|n|\big)^{1-it_1+it_2}\,\frac{\varepsilon_1^{-it_1+it_2}-\varepsilon_2^{-it_1+it_2}}{-it_1+it_2}\\
&\quad+\Gamma^{\mathrm{W}}(-t_1)\Gamma^{\mathrm{W}}(-t_2)\,\big(4\pi|n|\big)^{1+it_1+it_2}\,\frac{\varepsilon_1^{+it_1+it_2}-\varepsilon_2^{it_1+is_2}}{it_1+it_2}\\
&\quad+\Gamma^{\mathrm{W}}(t_1)\Gamma^{\mathrm{W}}(t_2)\,\big(4\pi|n|\big)^{1-it_1-it_2}\,\frac{\varepsilon_1^{-it_1-it_2}-\varepsilon_2^{-it_1-it_2}}{-it_1-it_2}\\
&\quad+\Gamma^{\mathrm{W}}(-t_1)\Gamma^{\mathrm{W}}(t_2)\,\big(4\pi|n|\big)^{1+it_1-it_2}\,\frac{\varepsilon_1^{+it_1-it_2}-\varepsilon_2^{it_1-it_2}}{it_1-it_2}\;+\;\mathcal{O}(\varepsilon_1-\varepsilon_2).
\end{aligned}$$

To evaluate the integral with respect to t_2, we can perform the same steps as in [Ber16, Proposition 4.15] towards the end of his proof. Since $t_1, t_2 \in \mathbb{R}^+$, the second and third term in the inner integral are regular with respect to t_1 and t_2 and thus yield contributions of order $\log(\varepsilon)^{-1}$ after partial integration with respect to either of them. It remains to consider the first and fourth term, which yield

$$\big\|y^{\frac{k-1}{2}}\,\beta^{\mathrm{W}\varepsilon}_{k,n,\psi}(y)\big\|^2 \;=\; 4\pi\big(4\pi|n|\,\big)^{1-k}\int_{t_1\in i\mathbb{R}^+}\psi(t_1)\overline{\psi(t_1)}\;\mathrm{d}t_1\;+\;\mathcal{O}\big(\log(\varepsilon)^{-1}\big).$$

This establishes the claimed isometry, when letting ε tend to 0. It also guarantees that the assigment from ψ to β extends to a map on the L^2-spaces as claimed. □

With similar arguments we show:

Lemma 5.8. *The y-power transform $\psi \mapsto \beta^{\mathrm{W}}_{k,n,\psi}$ is an isometry $\mathrm{L}^2\big(\mathbb{R}^+, \mathrm{d}t\big) \to \mathrm{L}^2(\mathbb{R}^+, y^{k-2}\mathrm{d}y)$.*

Proof. The main difference to the previous lemma is that we need to truncate both towards 0 and ∞. We thus define

$$\beta^{\mathrm{c}\pm\varepsilon}_{k,\psi}(y) := \int_{t\in\mathbb{R}^+}\psi(t)\,\mathbb{1}_{(\varepsilon,1/\varepsilon)}\,\big(y^{\frac{1-k}{2}+it}\pm y^{\frac{1-k}{2}-it}\big)\,\mathrm{d}t.$$

Similar calculations as above yield

$$\begin{aligned}
&\big\|y^{\frac{k-1}{2}}\,\beta^{\mathrm{c}\pm\varepsilon}_{k,\psi}(y)\big\|^2\\
&=\int_{t_1,t_2\in\mathbb{R}^+}\psi(t_1)\overline{\psi(t_2)}\int_{\varepsilon}^{1/\varepsilon}\big(y^{+it_1}\pm y^{-it_1}\big)\big(y^{-it_2}\pm y^{+it_2}\big)\,y^{-1}\mathrm{d}y\,\mathrm{d}t_1\mathrm{d}t_2.
\end{aligned}$$

The inner integral equals

$$\begin{aligned}
&\int_{\varepsilon}^{1/\varepsilon}\big(y^{+it_1}\pm y^{-it_1}\big)\big(y^{-it_2}\pm y^{+it_2}\big)\,y^{-1}\mathrm{d}y\\
&=\int_{\varepsilon}^{1/\varepsilon}\big(y^{+it_1-it_2}\pm y^{-it_1-it_2}\pm y^{+it_1+it_2}+y^{-it_1+it_2}\big)\,y^{-1}\mathrm{d}y\\
&=\frac{\varepsilon^{it_2-it_1}-\varepsilon^{it_1-it_2}}{it_1-it_2}\pm\frac{\varepsilon^{it_1+it_2}-\varepsilon^{-it_1-it_2}}{-it_1-it_2}\\
&\qquad\pm\frac{\varepsilon^{-it_1-it_2}-\varepsilon^{it_1+it_2}}{it_1+it_2}+\frac{\varepsilon^{it_1-it_2}-\varepsilon^{it_2-it_1}}{-it_1+it_2}.
\end{aligned}$$

The second and third term, which agree, contribute $\mathcal{O}(\log(\varepsilon)^{-1})$ to the final expression. From the first and fourth term, which are also equal, we obtain

$$\big\|y^{\frac{k-1}{2}}\,\beta^{\mathrm{c}\pm\varepsilon}_{k,\psi}(y)\big\|^2 = 4\pi\int_{t_1\in\mathbb{R}^+}\psi(t_1)\overline{\psi(t_1)}\;\mathrm{d}t_1\;+\;\mathcal{O}\big(\log(\varepsilon)^{-1}\big)$$

and the claim follows taking the limit $\varepsilon \to 0$. □

5.5. **The compound operator.** The goal of this subsection is to understand the spectral decomposition of $-\Delta_k^{\mathrm{cmp}(\varepsilon)}$ and prove Theorem 1.6. This decomposition is closer in nature to that of the Laplacian on the modular surface. The following theorem is claimed without proof for $k = 0$ and $\varepsilon = 4$ in [Bal11].

Theorem 5.9. *For every $k \in \mathrm{N}$ and $\varepsilon > 0$, the* K*-type k cusp forms are an invariant subspace of the compound Laplacian $-\Delta_k^{\mathrm{cmp}(\varepsilon)}$ on which it has discrete spectrum.*

Proof. That cusp forms are an invariant subspace can be seen by computing the compound Laplacian term by term in the Fourier expansion. We first consider the principal self-adjoint part of the compound Laplacian, that is the operator $L = -\Delta_k^{\mathrm{cmp}(\varepsilon)} - iky\partial_x$ and prove that L has compact resolvent. We can read from Proposition 2.5 that the quadratic form associated with L is

$$\begin{aligned} Q_L(\phi) \;&=\; \int_{\Gamma'\backslash\mathbb{H}'} y^k |\nabla_{x,y}\phi|^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}p\,\mathrm{d}q + \varepsilon \int_{\Gamma'\backslash\mathbb{H}'} y^{k-2} |\nabla_{u,v}\phi|^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}u\,\mathrm{d}v \\ &=\; \sum_{n,m\in\mathbb{Z}\backslash\{0\}} \Bigg(\int_{\mathbb{R}^+} |c^{\mathrm{H0}}(y\partial_x\phi; n,m;y)|^2 + |c^{\mathrm{H0}}(y\partial_y\phi; n,m;y)|^2 \frac{\mathrm{d}y}{y^{2-k}} \\ &\qquad + \varepsilon \int_{\mathbb{R}^+} |c^{\mathrm{H0}}(\partial_u\phi; n,m;y)|^2 + |c^{\mathrm{H0}}(\partial_v\phi; n,m;y)|^2 \frac{\mathrm{d}y}{y^{2-k}} \Bigg), \end{aligned}$$

where the second equality follows from the Parseval identity for the Fourier–Heisenberg series, Lemma 4.5. To prove discreteness of the spectrum, we need to prove that the set

$$A = \left\{ \phi \in \mathrm{L}^2(\mathcal{H}(0,0))^{\mathrm{gen}}_{\mathrm{cusp}} : Q_L(\phi) \le 1 \right\} \tag{5.16}$$

is compact in the L^2 topology. We adapt the proof strategy in [LP76, Lemma 8.7]. Note that for any $a, b > 0$, in the compact region of $\Gamma'\backslash\mathbb{H}'$ where $a < y < b$, the quadratic form defines a norm equivalent to the standard Sobolev norm on $\mathrm{W}^{1,2}(\Gamma'\backslash\mathbb{H}')$; since genuine affine-invariant modular form have mean zero, the Rellich–Kondrachov theorem [AF03, Theorem 6.3] tells us that the set of genuine cusp forms in A supported on $a < y < b$ is compact in the L^2 topology (note that the theorem is usually stated in euclidean space, but is a purely local statement and so holds in the bulk of $\Gamma'\backslash\mathbb{H}'$). Using Lemma 4.5, to prove compactness it suffices to prove that cusp forms in A satisfy uniformly

$$0 \;=\; \lim_{a\to 0} \int_0^a \sum_{m,n\in\mathbb{Z}\backslash\{0\}} \left| c^{\mathrm{H0}}(\phi; n,m;y) \right|^2 \frac{\mathrm{d}y}{y^{2-k}} \tag{5.17}$$

and

$$0 \;=\; \lim_{b\to\infty} \int_b^\infty \sum_{m,n\in\mathbb{Z}\backslash\{0\}} \left| c^{\mathrm{H0}}(\phi; n,m;y) \right|^2 \frac{\mathrm{d}y}{y^{2-k}}. \tag{5.18}$$

By definition of cusp forms, they have no $n = 0$ terms in their Fourier–Heisenberg series. As such, we can study (5.18) as in the case of the modular surface, using that

$$|c^{\mathrm{H0}}(\phi; n,m;y)|^2 \le 4\pi n^2 |c^{\mathrm{H0}}(\phi; n,m;y)|^2 \;=\; y^{-2} |c^{\mathrm{H0}}(y\partial_x\phi; n,m;y)|^2$$

we deduce

$$\int_b^\infty \sum_{m,n\neq 0} \left| c^{\mathrm{H0}}(\phi; n, m; y) \right|^2 \frac{\mathrm{d}y}{y^{2-k}} \leq b^{-2} \int_b^\infty \sum_{m,n\neq 0} \left| c^{\mathrm{H0}}(y\partial_x\phi; n, m; y) \right|^2 \frac{\mathrm{d}y}{y^{2-k}} \leq \frac{Q_L(\phi)}{b^2}.$$

For (5.17), we use this time the derivative in the v direction. It follows from Corollary 4.7 that there are no $m = 0$ terms in the Fourier expansion of a genuine cusp form we can use

$$y^{-2}|c^{\mathrm{H0}}(\phi; n, m; y)|^2 \leq 4\pi m^2 y^{-2} |c^{\mathrm{H0}}(\phi; n, m; y)|^2 \; = \; |c^{\mathrm{H0}}(\partial_v\phi; n, m; y)|^2$$

to deduce

$$\int_0^a \sum_{m,n\neq 0} \left| c^{\mathrm{H0}}(\phi; n, m; y) \right|^2 \frac{\mathrm{d}y}{y^{2-k}} \; \leq \; a^2 \int_0^a y^{k-2} \sum_{m,n\neq 0} \left| c^{\mathrm{H0}}(\partial_v\phi; n, m; y) \right|^2 \mathrm{d}y \leq \frac{a^2}{\varepsilon} Q_L(\phi).$$

All in all, this implies that A is compact, so that the L restricted to cusp forms has compact resolvent and therefore discrete spectrum with finite multiplicity.

Recall now that we defined $-\Delta_k^{\mathrm{cmp}(\varepsilon)} = L + iky\partial_x$, and we now aim to show that $iky\partial_x$ is relatively compact with respect to L. From [Kat95, Theorem IV.5.35] we know that a relatively compact perturbation does not change the essential spectrum, and the essential spectrum of L is empty. To show that $iky\partial_x$ is relatively compact with respect to L, it is sufficient to show that for some $\lambda \in \mathbb{C}$, $iky\partial_x(L-\lambda)^{-1}$ is compact. We just proved that if z is not an eigenvalue of L then $(L-\lambda)^{-1}$ is compact, so that by the functional calculus $(L-\lambda)^{-1/2}$ also is. It also follows from its definition that L has the same principal symbol as $-\Delta_k^{\mathrm{cmp}(\varepsilon)}$ and as such is also a second order elliptic operator, therefore $(L-\lambda)^{-1/2}$ is a pseudodifferential operator of order -1 [See67, Theorem 2], so that $iky\partial_x(L-\lambda)^{-1/2}$ is a pseudodifferential operator of order 0, and therefore bounded by the Calderón–Vaillancourt theorem [Hör07, Theorem 18.1.11]. Through this discussion, we obtain that

$$iky\partial_x(L-\lambda)^{-1} = (iky\partial_x(L-\lambda)^{-1/2})(L-\lambda)^{-1/2}$$

is the composition of a bounded and compact operator, and as such compact. Therefore $-\Delta_k^{\mathrm{cmp}(\varepsilon)}$ also has compact resolvent when restricted to cusp forms, and as such discrete spectrum. □

Proposition 5.10. *For every $\lambda \in \mathbb{C} \setminus \mathbb{R}$, $(\Delta_k^{\mathrm{cmp}(\varepsilon)} + \lambda)^{-1} \to (\Delta_k^{\mathrm{fol}} + \lambda)^{-1}$ in the strong operator topology as $\varepsilon \to 0$.*

Remark 5.11. Because cusp forms are an invariant subspace for both $\Delta_k^{\mathrm{cmp}(\varepsilon)}$ and Δ_k^{fol} it also means that the restriction of the resolvents to cusp forms or to their orthogonal complement also converge appropriately in the strong operator topology.

Proof. Recall that the strong operator topology is induced by pointwise convergence. Let $f \in \mathrm{L}^2(\mathcal{H}(0,0))$, we need to show $(\Delta_k^{\mathrm{cmp}(\varepsilon)} + \lambda)^{-1} f \to (\Delta_k^{\mathrm{fol}} + \lambda)^{-1} f$. Since

$$\|(-\Delta_k^{\mathrm{cmp}(\varepsilon)} + \lambda)^{-1}\| \; \leq \mathrm{dist}(\lambda, \mathrm{spec}(-\Delta_k^{\mathrm{cmp}(\varepsilon)}))^{-1} \leq |\mathrm{Im}(\lambda)|^{-1},$$

the family $(-\Delta_k^{\mathrm{cmp}(\varepsilon)}+\lambda)^{-1}$ is uniformly bounded; therefore it is sufficient to verify this pointwise convergence of $(-\Delta_k^{\mathrm{cmp}(\varepsilon)}+\lambda)^{-1} \to (-\Delta_k^{\mathrm{fol}}+\lambda)^{-1}$ on a dense subset of $\mathrm{L}^2(\mathcal{H}(0,0))$, and in particular to verify it on functions of Schwartz class in y, which we now assume f to be. From the second resolvent identity, we have that

$$(-\Delta_k^{\mathrm{cmp}(\varepsilon)}+\lambda)^{-1}f - (-\Delta_k^{\mathrm{fol}}+\lambda)^{-1}f \;=\; \varepsilon(-\Delta_k^{\mathrm{cmp}(\varepsilon)}+\lambda)^{-1}\Delta^{\mathrm{vert}}(-\Delta_k^{\mathrm{fol}}+\lambda)^{-1}f.$$

Since $-\Delta_k^{\mathrm{fol}}+\lambda$ is hypoelliptic, its inverse is a pseudodifferential operator [Hör61], in particular the Schwartz class of functions is stable under $\Delta^{\mathrm{vert}}(-\Delta_k^{\mathrm{fol}}+\lambda)^{-1}$. Furthermore, the Schwartz class embeds boundedly in $\mathrm{L}^2(\mathcal{H}(0,0))$, and we have previously indicated that the family $(-\Delta_k^{\mathrm{cmp}(\varepsilon)}+\lambda)^{-1}$ is uniformly bounded as operators on L^2. Consequently, for any Schwartz function f,

$$\|\varepsilon(-\Delta_k^{\mathrm{cmp}(\varepsilon)}+\lambda)^{-1}\Delta^{\mathrm{vert}}(-\Delta_k^{\mathrm{fol}}+\lambda)^{-1}f\|_{\mathrm{L}^2} = O(\varepsilon),$$

and we deduce the strong convergence of the resolvents. □

As a corollary, we get that the family $\{-\Delta_k^{\mathrm{cmp}(\varepsilon)}\}$ is spectrally inclusive, meaning that the spectrum of $\{-\Delta_k^{\mathrm{fol}}\}$ is comprised of limit points from the spectra of $\{-\Delta_k^{\mathrm{cmp}(\varepsilon)}\}$.

Corollary 5.12. *For every $\lambda \in \mathrm{spec}(-\Delta_k^{\mathrm{fol}})$, there is a family $\{\lambda_\varepsilon \in \mathrm{spec}(-\Delta_k^{\mathrm{cmp}(\varepsilon)})\}$ such that $\lambda_\varepsilon \to \lambda$. Furthermore, for every bounded continuous function $f : \mathbb{R} \to \mathbb{R}$, $f(-\Delta_k^{\mathrm{cmp}(\varepsilon)}) \to f(-\Delta_k^{\mathrm{fol}})$ in the strong operator topology.*

Proof. The second statement is a consequence of strong convergence of the resolvent and [Wei80, Theorem 9.17]. Suppose that $\lambda \in \mathrm{spec}(-\Delta_k^{\mathrm{fol}})$ is not a limit point of the spectra of $-\Delta_k^{\mathrm{cmp}(\varepsilon)}$, and take f to be a function supported near λ and so that $\mathrm{supp}(f) \cap \mathrm{spec}(-\Delta_k^{\mathrm{cmp}(\varepsilon)}) = \varnothing$ for all sufficiently small ε. Then, it would be impossible for $f(-\Delta_k^{\mathrm{cmp}(\varepsilon)})$ to converge to $f(-\Delta_k^{\mathrm{fol}})$, in the strong topology, proving the corollary. □

Remark 5.13. Let us make a few remarks about Proposition 5.10 and Corollary 5.12. First, since cusp forms are an invariant subspace of Δ_k^{fol} and $\Delta_k^{\mathrm{cmp}(\varepsilon)}$, the statements apply *mutatis mutandis* to the restriction of those operators to cusp forms or their orthogonal complement. Second, the resolvents do not converge in the operator norm topology, to see this it suffices to compare their action on a sequence $f(\tau)e(\lfloor\varepsilon^{-1}\rfloor q)$ for a fixed f. Finally, the convergence of $f(-\Delta_k^{\mathrm{cmp}(\varepsilon)}) \to f(-\Delta_k^{\mathrm{fol}})$ in the strong operator topology is a bit weaker than convergence of the spectral projections but for most intents and purposes can be used the same way. Note that by monotonicity of the involved operators ($-\Delta_k^{\mathrm{fol}} \leq -\Delta_k^{\mathrm{cmp}(\varepsilon)}$ for all $\varepsilon > 0$), the condition on the continuity of f can be relaxed to right-continuity.

6. Siegel–Veech transforms

In this section we briefly recall basic properties of the Siegel–Veech transforms for any configuration on any stratum. We then specialize to our case of the stratum $\mathcal{H}(0,0)$ and prove the main results, Theorem 1.4 and Theorem 1.5, in particular showing that Siegel–Veech transforms exhaust the complement of cusp forms. The main technical step is the computation of Fourier coefficients of Siegel–Veech transforms in Proposition 6.8. We complement this in Section 6.6 by computing adjoints and kernels of Siegel–Veech transforms.

6.1. **Basic properties.** A flat surface $(X, \omega) \in \mathcal{H}(\alpha)$ determines a *singular flat metric*, with cone points of angle $2\pi(\alpha_i + 1)$ where ω has a zero of order α_i. A *saddle connection* γ is a geodesic in the flat metric connecting two zeros, with none in its interior. We denote the set of saddle connections by $\mathrm{SC}(\omega)$. To each saddle connection γ we associate the *holonomy vector* $\mathrm{hol}(\gamma) = \int_\gamma \omega \in \mathbb{C}$.

A *configuration* $\mathcal{C}$ is a choice of subset $\mathcal{C}(\omega) \subset \mathrm{SC}(\omega)$ such that if we set

$$\Lambda_\omega^{\mathcal{C}} \;=\; \{\mathrm{hol}(\gamma) : \gamma \in \mathcal{C}(\omega)\},$$

the assignment

$$\omega \;\longmapsto\; \Lambda_\omega^{\mathcal{C}}$$

is $\mathrm{SL}_2(\mathbb{R})$-equivariant. Examples of configurations include the set of saddle connections joining two specified zeros, two zeros of specified orders, saddle connections that sit at the boundary of cylinders in a fixed homotopy class, etc. Given any configuration $\mathcal{C}$ and a function $f : \mathbb{R}^2 \to \mathbb{C}$, we define the *Siegel–Veech transform with respect to* $\mathcal{C}$ as

$$\mathrm{SV}_{\mathcal{C}}(f) : \mathcal{H}(\alpha) \longrightarrow \mathbb{C}, \qquad (X,\omega) \longmapsto \sum_{v \in \Lambda_\omega^{\mathcal{C}}} f(v).$$

By definition $\mathrm{SV}_{\mathcal{C}}(g \cdot f) = g \cdot \mathrm{SV}_{\mathcal{C}}(f)$ for any $g \in \mathrm{SL}_2(\mathbb{R})$.

If the function f is of K-type k, then the Siegel–Veech transform is the lift (in the sense of (2.10)) of an affine modular-invariant function of weight k on $\mathbb{H}'$. We indicate that we work with this function by writing a pair of variable $\mathrm{SV}_{\mathcal{C}}(f)(\tau, z)$ with $(\tau, z) \in \mathbb{H}'$ as the argument of the Siegel–Veech transform.

We now specialize to the stratum $\mathcal{H}(0,0)$ we are mainly interested in. In this case there are two obvious configurations. The first consists of *absolute periods*, the configurations of saddle connections joining (say) the first zero to itself. If the second zero is not a rational point with respect to the period lattice based at the first zero, then $\Lambda_\omega^{\mathrm{abs}} = \Lambda_\omega^{\mathrm{prim}}$ consists of the primitive lattice vectors in the period lattice underlying (X, ω). Since we consider Siegel–Veech transforms as L^2-functions, we may ignore the measure zero complementary set. By definition $\mathrm{SV}_{\mathrm{abs}}(f)$ factors through the projection to $\mathcal{H}(0)$, i.e., contributes to the non-genuine part of $L^2(\mathcal{H}(0,0))$. It in fact orthogonal to cusp forms and covers their orthogonal complement, i.e., the space of Eisenstein transforms in any weight k, since

$$E_k(\tau|\psi) \;=\; \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ \gcd(c,d)=1}} \frac{(c\tau+d)^k}{|c\tau+d|^k}\,\psi\Big(\frac{\mathrm{Im}(\tau)}{|c\tau+d|^2}\Big) \;=\; \mathrm{SV}_{\mathrm{abs}}(f)(\Lambda_\tau) \tag{6.1}$$

for $f(\lambda) = \frac{\lambda^k}{|\lambda|}\psi(1/|\lambda|^2)$ and $\Lambda(\tau) = \langle \frac{\tau}{\sqrt{y}}, \frac{1}{\sqrt{y}} \rangle$. Here we consider $\Lambda \subset \mathbb{R}^2 \cong \mathbb{C}$ when taking powers of elements in Λ.

The second case consists of *relative periods*, the configurations of saddle connections $\Lambda_\omega^{\mathrm{rel}}$ joining (say) the first zero to the second zero. We denote the corresponding Siegel–Veech transform by $\mathrm{SV}_{\mathrm{rel}}(f)$. Having decompositions of L^2-spaces in mind we may ignore the flat surfaces where the relative period is a real multiple of an absolute period, since this is a measure zero set. Consequently, the definition of $\Lambda_\omega^{\mathrm{rel}}$ does not involve any primitivity condition on lattice vectors. However, these two cases do not exhaust all configurations:

Lemma 6.1. *For any $M \in \mathbb{N}$ assigning with $(\Lambda, z) \in \mathcal{H}(0,0)$ the set $\mathcal{C}_M = z + \frac{1}{M}\Lambda$ of translates of the relative period by a $1/M$-th lattice vector is a configuration.*

Proof. Both independence of the choice of the relative period and $\mathrm{G}(\mathbb{R})$-equivariance are obvious. □

We let $\mathrm{SV}_{\mathrm{rel},M} := \mathrm{SV}_{\mathcal{C}_M}$ be the corresponding Siegel-Veech transformation. That is,

$$\mathrm{SV}_{\mathrm{rel},M}(f)(\Lambda, z) = \sum_{w\in\mathcal{C}_M} f(w). \tag{6.2}$$

Remark 6.2. In this homogeneous space setting, the relative Siegel–Veech transform can also be stated as follows. Let $\mathrm{S}'(\mathbb{R}) \subset \mathrm{G}'(\mathbb{R})$ be the stabilizer of $(1,0)$ with respect to the right action and $\mathrm{S}'(\mathbb{Z}) = \mathrm{S}'(\mathbb{R})\cap\mathrm{G}'(\mathbb{Z})$. Then $\mathrm{S}'(\mathbb{R})\backslash\mathrm{G}'(\mathbb{R}) \cong \mathbb{R}^2\setminus\{0\}$ as $\mathrm{G}'(\mathbb{R})$ spaces, and we may set for any $f:\mathbb{R}^2\to\mathbb{R}$

$$\widehat{f}_M(g,w_1,w_2) \;=\; f\big((\tfrac{1}{M},0)\cdot(g,w_1,w_2)\big) \;=\; f\big((\tfrac{1}{M},0)g+(w_1,w_2)\big)\,.$$

This function is left-invariant under $\mathrm{S}'(\mathbb{R})$, which contains the lower triangular subgoup $\mathrm{L}(\mathbb{R}) \subset \mathrm{G}(\mathbb{R})$. Since the $\mathrm{G}'(\mathbb{Z})$-orbit of $(\frac{1}{M},0)$ is obviously $\frac{1}{M}\mathbb{Z}^2$, we conclude that

$$\begin{aligned}\mathrm{SV}_{\mathrm{rel},M}(f)(g,w_1,w_2) \;&=\; \sum_{m,n\in\mathbb{Z}} f\Big((w_1,w_2)+\frac{1}{M}\big(m(a,b)+n(c,d)\big)\Big)\\ &=\; \sum_{\gamma\in\mathrm{S}'(\mathbb{Z})\backslash\mathrm{G}'(\mathbb{Z})} \widehat{f}_M(\gamma\cdot(g,w_1,w_2)), \qquad g=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right),\end{aligned} \tag{6.3}$$

Proposition 6.3. *For any configuration $\mathcal{C}$ and any compactly supported function f, the Siegel–Veech transform $\mathrm{SV}_{\mathcal{C}}(f)\in \mathrm{L}^2(\mathcal{H}(\alpha))$.*

For any configuration $\mathcal{C}$ there is a constant, the Siegel–Veech constant $c_{\mathcal{C}}$*, depending on the stratum $\mathcal{H}(\alpha)$ such that*

$$\int_{\mathcal{H}(\alpha)} \mathrm{SV}_{\mathcal{C}}(f)\,\mathrm{d}\mu \;=\; c_{\mathcal{C}}\int_{\mathbb{R}^2} f(x,y)\,\mathrm{d}x\mathrm{d}y$$

for any compactly supported f. In particular $\mathrm{SV}_{\mathcal{C}}$ is a bounded linear operator.

For the stratum $\mathcal{H}(0,0)$ the equivariance

$$\mathrm{SV}_{\mathrm{rel}}(g'\cdot f) \;=\; g'\cdot \mathrm{SV}_{\mathrm{rel}}(f) \tag{6.4}$$

holds for any $g'\in G'(\mathbb{R})$, where g' acts on $\mathbb{R}^2$ affine-linearly.

Proof. The first statement is the main result of [ACM19], but can be proven for $\mathcal{H}(0,0)$ directly. The second result is the main result of Veech in [Vee98]. The third follows from direct computation or from Remark 6.2, observing that the Siegel–Veech transform is a sum over left cosets and that $g'\in\mathrm{G}'(\mathbb{R})$ acts on function on the left by acting on the variable on the right. □

6.2. **Casimir elements acting on the Euclidean plane.** The continuity and $\mathrm{G}'(\mathbb{R})$-equivariance (6.4) imply that for any $X\in\mathfrak{g}'$ the Lie derivative of the action functions f on $\mathbb{R}^2$

$$Xf \;:=\; \lim_{t\to 0}\frac{1}{t}(e^{tX}f-f) \tag{6.5}$$

has the property that $X\,\mathrm{SV}_{\mathcal{C}}(f) = \mathrm{SV}_{\mathcal{C}}(Xf)$. We compute this action explicity for the Casimir operators. We work on $\mathbb{R}^2$ with coordinates (w_1,w_2) and use the differential operators $D_{w_i} = w_i\frac{\partial}{\partial w_i}$.

Lemma 6.4. *The differential operators*

$$8\mathcal{D}^{\mathrm{fol}}_{\mathrm{eucl}} \;=\; D^2_{w_1} + D_{w_1}D_{w_2} + D_{w_2}D_{w_1} + D^2_{w_2} + 2D_{w_1} + 2D_{w_2} \quad \text{and} \quad \mathcal{D}^{\mathrm{tot}}_{\mathrm{eucl}} \;=\; 0$$

on $\mathbb{R}^2$ *have the property that*

$$\mathrm{SV}_{\mathcal{C}}(\mathcal{D}^{\mathrm{fol}}_{\mathrm{eucl}} f) \;=\; \mathcal{D}^{\mathrm{fol}}\,\mathrm{SV}_{\mathcal{C}}\, f \quad \text{and} \quad \mathrm{SV}_{\mathcal{C}}(\mathcal{D}^{\mathrm{tot}}_{\mathrm{eucl}} f) \;=\; \mathcal{D}^{\mathrm{tot}}\,\mathrm{SV}_{\mathcal{C}}\, f \tag{6.6}$$

for any smooth compactly supported $f : \mathbb{R}^2 \longrightarrow \mathbb{C}$ *and any configuration* $\mathcal{C}$*.*

Proof. Direct computations give the Lie derivative action of the standard generators (see Section 2) on such functions, namely

$$Pf \;=\; \frac{\partial}{\partial w_1} f, \qquad Qf \;=\; \frac{\partial}{\partial w_2} f, \qquad\qquad Hf \;=\; \Big(w_1 \frac{\partial}{\partial w_1} + w_2 \frac{\partial}{\partial w_2}\Big) f,$$

$$(F+G)f \;=\; \Big(w_2 \frac{\partial}{\partial w_1} + w_1 \frac{\partial}{\partial w_2}\Big) f, \qquad\qquad (F-G)f \;=\; \Big(-w_2 \frac{\partial}{\partial w_1} + w_1 \frac{\partial}{\partial w_2}\Big) f\,.$$

The claim follows by combining using (2.3) the expressions (2.4) and (2.6) for C and C' in these generators. ☐

6.3. **The representation generated by Siegel–Veech transforms.** Theorem 1.4 is a consequence of the following proposition together with Proposition 5.2.

Proposition 6.5. *For the relative period Siegel–Veech transforms of a mean-zero compactly supported function* $f \neq 0$*, there is a multiplicity* $m \in \mathbb{Z}_{\geq 0} \cup \{\aleph_0\}$ *(depending on* M*) such that the representation it generates is*

$$\pi_{\mathrm{L}^2}\big(\mathrm{SV}_{\mathrm{rel},M}(f)\big) \;\cong\; m\,\pi^{\mathrm{SAff}}_0 \quad \in \mathrm{L}^2\big(\mathcal{H}(0,0)\big)$$

where π^{SAff}_0 *is the representation from* (3.18) *with index* $n = 0$*.*

Proof. By Lemma 6.4 the Casimir element of $\mathrm{G}'(\mathbb{R})$ acts trivially on the representation $\pi_{\mathrm{L}^2}\big(\mathrm{SV}_{\mathrm{rel},M}(f)\big)$. The classification of representations of $\mathrm{G}'(\mathbb{R})$ shows that we have a direct sum decomposition

$$\pi_{\mathrm{L}^2}\big(\mathrm{SV}_{\mathrm{rel},M}(f)\big) \;\cong\; m\,\pi^{\mathrm{SAff}}_0 \,\oplus\, \pi^{\mathrm{G}}$$

for a nonnegative integer m and the pullback π^{G} of a $\mathrm{G}(\mathbb{R})$ representation. Consider the averaging map $\mathrm{av} : \mathrm{L}^2(\mathrm{G}'(\mathbb{Z})\backslash\mathrm{G}'(\mathbb{R})) \to \mathrm{L}^2(\mathrm{G}(\mathbb{Z})\backslash\mathrm{G}(\mathbb{R}))$, given by the integral along the torus $\mathrm{H}'(\mathbb{Z})\backslash\mathrm{H}'(\mathbb{R})$. When applied to the right hand side, the averaging yields π^{G}. When applied to a Siegel–Veech transform we combine the summation over the period lattice with the integral over a fundamental parallelogram to obtain the $\mathbb{R}^2$-integral of f, which is zero by hypothesis. Hence π^{G} is zero. ☐

6.4. **Fourier–Heisenberg coefficients.** We determine some Fourier–Heisenberg coefficients of Siegel–Veech transforms as preparation for Theorem 1.5. Suppose $f = f_0(r)\exp(ik\theta)$ is of K-type k. Then using Lemma 2.3 we may view the Siegel–Veech transform as a function on $\mathbb{H}'$, writing abusively $\mathrm{SV}_{\mathrm{rel},M}(f)(\tau,z)$ to indicate this, which is affine modular-invariant of weight k and whose lift (2.10) is the honest Siegel–Veech transform $\mathrm{SV}_{\mathrm{rel},M}(f)$ on $\mathcal{H}(0,0)$.

The first statement will be used to conclude that they are orthogonal to cusp forms.

Proposition 6.6. *Let* $f : \mathbb{R}^2 \longrightarrow \mathbb{C}$ *be a* K*-isotypical Schwartz function of* K*-type* k*. Then, for any* $M \in \mathbb{N}$*, the* c^{H0}*-Fourier coefficients of the* M*-relative Siegel-Veech transforms vanish, i.e.*

$$c^{\mathrm{H0}}\big(\mathrm{SV}_{\mathrm{rel},M}(f);\, n, m;\, y\big) \;=\; 0$$

for any $m \in \mathbb{Z}$ and any $n \in \mathbb{Z} \setminus \{0\}$, where $\tau = x + iy$ as usual.

Proof. From Lemma 4.3, we want to show that the constant term in the Fourier expansion with respect to u (which gives the sum of the c^{H0}-terms) is independent of x, so that only the constant term remains. We may view the Siegel–Veech transform of a function as an affine modular-invariant function of weight k on $\mathbb{H}'$. Explicitly

$$\mathrm{SV}_{\mathrm{rel},M}(f)(\tau, z) \;=\; \sum_{a,b\in\mathbb{Z}} f\left(\frac{1}{\sqrt{y}}\Big(u + iv + \frac{a(x+iy)+b}{M}\Big)\right) \tag{6.7}$$

so that, unfolding in b,

$$\int_0^1 \mathrm{SV}_{\mathrm{rel},M}(f)(\tau, z)\,\mathrm{d}u \;=\; \int_{\mathbb{R}} \sum_{a\in\mathbb{Z}} \sum_{b=1}^{M} f\left(\frac{1}{\sqrt{y}}\Big(u + iv + \frac{a(x+iy)+b}{M}\Big)\right)\mathrm{d}u\,. \tag{6.8}$$

Now it is clear that for any fixed v and y translating x does not change this integral. □

The second proposition will later help us to show that enough Fourier–Heisenberg coefficients can be controlled by Siegel–Veech transforms, and that they thus span the space of Eisenstein series. Before stating the proposition we require a few definitions. One of the many ways to define *Bessel functions* for integer index k is via the *Hansen–Bessel integral formula* [GR07, Formula 8.411.1]:

$$J_k(z) \;:=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(-ik\theta + iz\sin\theta)\,\mathrm{d}\theta\,. \tag{6.9}$$

Definition 6.7. The *Hankel transform of order* $k \in \mathbb{Z}$ is the integral operator defined on functions $f_0 : \mathbb{R}^+ \to \mathbb{C}$ as

$$\big(\mathcal{H}_k f_0\big)(s) \;:=\; \int_0^\infty f_0(r) J_k(sr)\, r\;\mathrm{d}r, \qquad s \geq 0. \tag{6.10}$$

While we would denote $\mathcal{H}_k$ any realisation of the Hankel transform from one function space to another, we make the observation that $\mathcal{H}_k$ is an isometric involution on $\mathrm{L}^2(\mathbb{R}^+, r\mathrm{d}r)$, in the sense that it is norm preserving and that $\mathcal{H}_k^{-1} = \mathcal{H}_k$. This can be deduced immediately from the orthogonality relation enjoyed by Bessel functions [GR07, Formula 6.512.8]:

$$\int_0^\infty J_k(sr)J_k(tr)\,r\,\mathrm{d}r \;=\; s^{-1}\delta(s-t)\,, \tag{6.11}$$

where δ is the Dirac delta distribution.

For $j \in \mathbb{Z} \setminus \{0\}$, define the isometry $T_j : \mathrm{L}^2(\mathbb{R}^+, r\mathrm{d}r) \to \mathrm{L}^2(\mathbb{R}^+, y^{-3}\mathrm{d}y)$ and its inverse S_j by

$$T_j h(y) \;=\; y h\Big(\frac{2\pi j}{\sqrt{y}}\Big) \quad \text{and} \quad S_j h(r) \;=\; \frac{r^2}{(2\pi j)^2}\, h\Big(\frac{(2\pi j)^2}{r^2}\Big)\,. \tag{6.12}$$

Proposition 6.8. *For every $m \in \mathbb{Z} \setminus \{0\}$, $k \in \mathbb{Z}$, $M \in \mathbb{N}$ and $f_0 \in \mathrm{L}^2(\mathbb{R}^+, r\mathrm{d}r)$, the function $f = f_0(r)\exp(ik\theta)$ of* K*-type k has Fourier coefficients*

$$c^{\mathrm{H0}}\big(\mathrm{SV}_{\mathrm{rel},M}(f);\, 0, mM;\, y\big) \;=\; (mM)^2\,\big(T_M \mathcal{H}_k f_0\big)\big(\frac{y}{m^2}\big) \in \mathrm{L}^2(\mathbb{R}^+, y^{-3}\mathrm{d}y)\,,$$

and every other Fourier coefficient vanishes.

Conversely, given $h \in \mathrm{L}^2(\mathbb{R}^+, y^{-3}\mathrm{d}y)$, the M-relative Siegel–Veech transform of the function $\widetilde{f} = M^{-2}\,(\mathcal{H}_k S_M h)\exp(ik\theta)$ of K*-type k has h as its Fourier coefficients, that is*

$$c^{\mathrm{H0}}\big(\,\mathrm{SV}_{\mathrm{rel},M}(\widetilde{f});\,0, mM;\,y\big) \;=\; m^2 h(m^{-2}y)\,.$$

Proof. Let $\widetilde{m} \in \mathbb{Z} \setminus \{0\}$. We compute, starting with (6.8) that the coefficient $c_{\widetilde{m}} = c^{\mathrm{H0}}\big(\,\mathrm{SV}_{\mathrm{rel},M}(f);\,0,\widetilde{m};\,y\big)$ equals

$$\begin{aligned}
c_{\widetilde{m}} \;&=\; \int_0^y \int_0^1 \int_{\mathbb{R}} \sum_{a\in\mathbb{Z}} \sum_{b=1}^{M} f\Big(\frac{u+iv}{y^{1/2}} + \frac{a(x+iy)+b}{y^{1/2}M}\Big)\,\mathrm{d}u\;\mathrm{d}x\, e(-\widetilde{m}\frac{v}{y})\,\mathrm{d}v \\
(x\text{-invariance}) \;&=\; \int_0^y \int_{\mathbb{R}} \sum_{a\in\mathbb{Z}} \sum_{b=1}^{M} f\Big(\frac{i(Mv+ay)}{y^{1/2}M} + \frac{uM+b}{y^{1/2}M}\Big)\,\mathrm{d}u\;\exp(-2\pi i\widetilde{m}\frac{v}{y})\,\mathrm{d}v \\
(\text{unfolding in } a) \;&=\; \sum_{a,b=1}^{M} \int_{\mathbb{R}} \int_{\mathbb{R}} f\Big(y^{-1/2}\Big(u + i(v + \frac{ay}{M}) + \frac{b}{M}\Big)\Big)\, e(-\widetilde{m}\frac{v}{y})\,\mathrm{d}u\;\mathrm{d}v \\
&=\; M \sum_{a=1}^{M} e\big(-\frac{a\widetilde{m}}{M}\big) \int_{\mathbb{R}} \int_{\mathbb{R}} f\big(\widetilde{u} + i\widetilde{v}\big)\, e\big(-\widetilde{m}\frac{\widetilde{v}}{\sqrt{y}}\big) y\,\mathrm{d}\widetilde{u}\;\mathrm{d}\widetilde{v}\,,
\end{aligned}$$

where we set $\widetilde{u} = \frac{u+b/M}{\sqrt{y}}$ and $\widetilde{v} = \frac{v+ay/M}{\sqrt{y}}$. At this point, we see that the integrals are independent of a and the sum vanishes whenever $\widetilde{m} \notin M\mathbb{Z}$, otherwise the sum is equal to M. We therefore continue, assuming that $\widetilde{m} = mM \in M\mathbb{Z}$ and changing to polar coordinates to obtain

$$\begin{aligned}
c^{\mathrm{H0}}\big(\,\mathrm{SV}_{\mathrm{rel},M}(f);\,0, mM;\,y\big) \;&=\; yM^2 \int_0^\infty \int_{-\pi}^{\pi} f_0(r)\exp(ik\theta - 2\pi i mM\frac{r\sin\theta}{\sqrt{y}})\,\mathrm{d}\theta\, r\,\mathrm{d}r \\
(\text{Hansen–Bessel formula}) \;&=\; yM^2 \int_0^\infty f_0(r) J_k(2\pi mMy^{-1/2}r) r\mathrm{d}r \\
&=\; (mM)^2\,\big(T_M\mathcal{H}_k f_0\big)(m^{-2}y) \in \mathrm{L}^2(\mathbb{R}^+, y^{-3}\mathrm{d}y).
\end{aligned}$$

For the converse statement, given $h \in \mathrm{L}^2(\mathbb{R}^+, s^{-3}\mathrm{d}s)$ apply the previous reasoning to $f_0 = M^{-2}\mathcal{H}_k S_M h \in \mathrm{L}^2(\mathbb{R}^+, r\mathrm{d}r)$ and $f = f_0\exp(ik\theta)$ to obtain the desired identity in the end. □

6.5. **Orthogonality to cusp forms.** Recall from the introduction that we want to prove that the closure $\mathcal{SV}_{\mathrm{rel},\infty} = \overline{\mathrm{span}}(\cup_{M=1}^\infty \mathcal{SV}_{\mathrm{rel},M})$ of the union of the spaces

$$\mathcal{SV}_{\mathrm{rel},M} \;=\; \overline{\mathrm{span}}\Big\{\,\mathrm{SV}_{\mathrm{rel},M}(f) : f \in \mathrm{C}^\infty_{c,0}(\mathbb{R}^2)\Big\} \tag{6.13}$$

fills the orthogonal complement of cusp forms.

Proof of Theorem 1.5. To show orthogonality it suffices to show orthogonality to all Siegel–Veech transforms of fixed K-type k. We may thus decompose the cusp form also in K-types and it suffices to show orthogonality of the component of type k. We may thus work on $\Gamma'\backslash\mathbb{H}'$ by the correspondence in Lemma 2.3. There we use the expression for the scalar product in Lemma 4.5. Each of these summands under the integral vanishes, either by Proposition 6.6 or by definition of a cusp form.

Let now $\varphi \perp \mathrm{L}^2(\mathcal{H}(0,0))^{\mathrm{gen}}_{\mathrm{cusp}} \oplus \mathcal{SV}_{\mathrm{rel},\infty}$, we need to show that $\varphi = 0$. Without loss of generality, we assume by density that φ is smooth and has compact support in the y variable. Then again Lemma 4.5 shows by Proposition 6.6 that for every M and every $f \in \mathrm{C}^\infty_{c,0}(\mathbb{R}^2)$ of K-type k

$$0 \;=\; \int_{\mathbb{R}^+} \sum_{\ell \geq 1} c^{\mathrm{H0}}\big(\mathrm{SV}_{\mathrm{rel},M}(f);\, 0, \ell;\, y\big)\, c^{\mathrm{H0}}\big(\varphi;\, 0, \ell;\, y\big) \frac{\mathrm{d}y}{y^{2-k}}$$

By Proposition 6.8 this implies that for every $M \in \mathbb{N}$ and any $h : (0, \infty) \to \mathbb{C}$ smooth and compactly supported

$$0 \;=\; \int_{\mathbb{R}_+} \sum_{\ell \geq 1} \ell^2 h(\ell^{-2} y)\, c^{\mathrm{H0}}\big(\varphi;\, 0, \ell M;\, y\big)\, \frac{\mathrm{d}y}{y^{2-k}}\,. \tag{6.14}$$

In particular, since φ and h are compactly supported in the y variable, this is a finite sum and we do not have to worry about convergence, let L be the largest index in the sum.

Towards a contradiction, suppose there were some $M \in \mathbb{N}$ and some $h \in \mathrm{L}^2(\mathbb{R}^+, s^{-3}\mathrm{d}s)$ so that

$$0 \;\neq\; \int_{\mathbb{R}^+} h(y) c^{\mathrm{H0}}(\varphi; 0, M; y)\, \frac{\mathrm{d}y}{y^{2-k}}\,,$$

without loss of generality assume that it is equal to 1. But then, it follows from (6.14) that

$$-1 \;=\; \int_{\mathbb{R}^+} \sum_{\ell=2}^{L} \ell^2 h(\ell^{-2} y)\, c^{\mathrm{H0}}\big(\varphi;\, 0, \ell M;\, y\big)\, \frac{\mathrm{d}y}{y^{2-k}}\,. \tag{6.15}$$

However, using again (6.14) with $2M$ replacing M, and h replaced with $\widetilde{h}(y) = 4h(y/4)$ we have that

$$\begin{aligned} 0 \;&=\; \int_{\mathbb{R}^+} \sum_{\ell=1}^{\lfloor L/2 \rfloor} \ell^2 \widetilde{h}(\ell^{-2} y)\, c^{\mathrm{H0}}\big(\varphi;\, 0, 2\ell M;\, y\big)\, \frac{\mathrm{d}y}{y^{2-k}} \\ &=\; \int_{\mathbb{R}^+} \sum_{\substack{2 \leq \ell \leq L \\ 2 \mid \ell}} \ell^2 h(\ell^{-2} y)\, c^{\mathrm{H0}}\big(\varphi;\, 0, \ell M;\, y\big)\, \frac{\mathrm{d}y}{y^{2-k}}\,. \end{aligned}$$

so that (6.15) can be rewritten as

$$-1 \;=\; \int_{\mathbb{R}^+} \sum_{\substack{2 \leq \ell \leq L \\ 2 \nmid \ell}} \ell^2 h(\ell^{-2} y)\, c^{\mathrm{H0}}\big(\varphi;\, 0, \ell M;\, y\big)\, \frac{\mathrm{d}y}{y^{2-k}}$$

and the other, finitely many, arithmetic progressions can all be sieved out in the same way so that the righthand side in (6.15) is necessarily 0, a contradiction.

By density, we therefore have that necessarily $c^{\mathrm{H0}}(\varphi; 0, M; y) = 0$ for all $M \in \mathbb{N}$, making φ a cusp form; yet we also supposed that φ was orthogonal to cusp forms so that $\varphi = 0$. $\square$

6.6. **Kernels, adjoints, and norms of Siegel–Veech transforms.** Understanding the Siegel-Veech transform as a linear operator between L^2-spaces comprises determining its range (as we did in the previous section), its adjoint and its kernel. We address the last two items here. The type of answers differs even between the cases $\mathcal{H}(0)$ and $\mathcal{H}(0,0)$, leaving a coherent picture for general strata as an interesting future problem.

Adjoints. Formal adjoints to the Siegel-Veech transform can be computed using a standard integration trick based on Fubini's theorem. This is classical for $\mathcal{H}(0)$, see e.g. [Lan85, p. 242], and can be adapted to $\mathcal{H}(0,0)$ as follows.

Proposition 6.9. *The formal adjoint of the relative Siegel-Veech transform is given by assigning with $h \in \mathrm{L}^2(\mathrm{G}'(\mathbb{Z})\backslash\mathrm{G}'(\mathbb{R}))$ the function*

$$\mathrm{SV}^*_{\mathrm{rel}}(h)(g') \;=\; \int_{\mathrm{S}'(\mathbb{Z})\backslash\mathrm{S}'(\mathbb{R})} h(sg')\,\mathrm{d}\nu(s)\,. \tag{6.16}$$

on $\mathrm{S}'(\mathbb{R})\backslash\mathrm{G}'(\mathbb{R}) \cong \mathbb{R}^2 \setminus \{0\}$.

Proof. We abbreviate $\Gamma' = \mathrm{G}'(\mathbb{Z})$ and disintegrate the Haar measure of $\mathrm{G}'(\mathbb{R})$ as $\mathrm{d}\mu = \mathrm{d}\nu(s)\,\mathrm{d}\overline{\mu}(g')$ into the Haar measure on $\mathrm{S}'(\mathbb{R})$ and the measure $\overline{\mu}$ on $\mathrm{S}'(\mathbb{R})\backslash\mathrm{G}'(\mathbb{R})$. Now

$$\begin{aligned}
\big\langle \mathrm{SV}_{\mathrm{rel}}(f), h\big\rangle_{\Gamma'\backslash\mathrm{G}'(\mathbb{R})} \;&=\; \int_{\Gamma'\backslash\mathrm{G}'(\mathbb{R})} \sum_{\gamma\in\mathrm{S}'(\mathbb{Z})\backslash\Gamma'} \widehat{f}(\gamma g')\,\overline{h}(g')\mathrm{d}\mu(g') \\
(\Gamma'\text{-invariance of } h) \;&=\; \int_{\mathrm{S}'(\mathbb{Z})\backslash\mathrm{G}'(\mathbb{R})} \widehat{f}(g')\,\overline{h}(g')\,\mathrm{d}\mu(g') \\
(\mathrm{S}'(\mathbb{R})\text{-invariance of } \widehat{f}) \;&=\; \int_{\mathrm{S}'(\mathbb{R})\backslash\mathrm{G}'(\mathbb{R})} \widehat{f}(g') \int_{\mathrm{S}'(\mathbb{Z})\backslash\mathrm{S}'(\mathbb{R})} \overline{h}(sg')\mathrm{d}\nu(s)\,\mathrm{d}\overline{\mu}(g') \\
&=\; \big\langle f, \mathrm{SV}^*_{\mathrm{rel}}(h)\big\rangle_{\mathrm{L}^2(\mathbb{R}^2)}
\end{aligned} \tag{6.17}$$

verifies the claim. □

Kernels and norms. On $\mathcal{H}(0) = \mathrm{G}(\mathbb{Z}^2)\backslash\mathrm{G}(\mathbb{R}^2)$, the Siegel–Veech transform is not an L^2-isometry, since it has obviously a non-trivial kernel consisting of odd functions. However the functional equation for Eisenstein series provides more:

Proposition 6.10. *The kernel of the absolute Siegel–Veech transform on $\mathcal{H}(0)$ strictly contains the odd functions.*

Proof. Working formally, putting $k = 0$, $\psi(u) = u^s$ in (6.1), we obtain the classical Eisenstein series

$$E(\tau, s) \;=\; \mathrm{SV}_{\mathrm{abs}}(h_s)(\Lambda_\tau),$$

where $h_s(x) = \|x\|^{-2s}$. Following, for example, Bergeron [Ber16, Section 4.1], we put

$$E^*(\tau, s) \;=\; \pi^{-s}\Gamma(s)\zeta(2s)E(\tau, s)\,.$$

Then the functional equation states

$$E^*(\tau, s) \;=\; E^*(\tau, 1-s)\,. \tag{6.18}$$

Formally, then, putting $h^*_s(x) = \pi^{-s}\Gamma(s)\zeta(2s)\,\|x\|^{-2s}$, this implies

$$\mathrm{SV}_{\mathrm{abs}}\left(h^*_s - h^*_{1-s}\right) \;= 0\,.$$

To resolve the obvious integrability and convergence issues, we perform a standard trick. We define $-2s = -1 + it$, so that $s = \frac{1}{2} - i\frac{t}{2}$ and $1 - s = \frac{1}{2} + i\frac{t}{2}$. For a smooth function η of compact support on $\mathbb{R}^+$, we write x in polar coordinates as (r, θ) and obtain the desired kernel elements as

$$f_\eta(x) \;=\; \int_{\mathbb{R}^+} \eta(t)(h_s(x) - h_{1-s}(x))\,\mathrm{d}t \;=\; \int_{\mathbb{R}^+} \eta(t)(r^{-1+it} - r^{-1-it})\,\mathrm{d}t\,.$$

(This construction can be generalized to functions of other K-types by defining

$$f_{k,\eta}(r,\theta) \;=\; e^{ik\theta}\int_{\mathbb{R}^+}\eta(t)(r^{-1+it}-r^{-1-it})\,\mathrm{d}t$$

for nonzero integers k.) ☐

This is in contrast to the stratum $\mathcal{H}(0,0)$:

Proposition 6.11. *The M-relative Siegel–Veech transform is M times an isometry on the space of mean zero functions, i.e.*

$$\big\|\,\mathrm{SV}_{\mathrm{rel},M}(f)\big\|_2 \;=\; M\|f\|_2$$

for $f\in C_c(\mathbb{R}^2)$ of mean zero. More precisely, for $f\in C_c(\mathbb{R}^2)$, we have

$$\int_{\mathcal{H}(0,0)}\mathrm{SV}_{\mathrm{rel},M}(f)\,\mathrm{d}\mu \;=\; M^2\int_{\mathbb{R}^2}f(x)\,\mathrm{d}x$$

and

$$\int_{\mathcal{H}(0,0)}\mathrm{SV}_{\mathrm{rel},M}(f)^2\,\mathrm{d}\mu \;=\; M^4\left(\int_{\mathbb{R}^2}f(x)\,\mathrm{d}x\right)^2+M^2\int_{\mathbb{R}^2}f(x)^2\,\mathrm{d}x.$$

Proof. By the equivariance (6.4), the map

$$f\longmapsto\int_{\mathcal{H}(0,0)}\mathrm{SV}_{\mathrm{rel},M}(f)\,\mathrm{d}\mu$$

is a $\mathrm{G}'(\mathbb{R}^2)$-invariant functional on $C_c(\mathbb{R}^2)$, and so we must have

$$\int_{\mathcal{H}(0,0)}\mathrm{SV}_{\mathrm{rel},M}(f)\,\mathrm{d}\mu \;=\; c_M\int_{\mathbb{R}^2}f(x)\,\mathrm{d}x,$$

since the only $\mathrm{G}'(\mathbb{R}^2)$-invariant measure on $\mathbb{R}^2$ is Lebesgue measure. To find the constant c_M, note that if we take $f=\chi_{B(0,R)}$ to be the indicator function of the ball of radius R, with R sufficiently large, the Siegel–Veech transform $\mathrm{SV}_{\mathrm{rel},M}(f)$ will be approximately constant, with value $M^2\pi R^2$, so $c_M = M^2$. For the L^2 computation, we consider the configuration $\mathcal{C}_M^2\subset\mathbb{R}^2\times\mathbb{R}^2$, and for $h\in C_c(\mathbb{R}^2\times\mathbb{R}^2)$, we define (by abuse of notation) $\mathrm{SV}_{\mathrm{rel},M}(h)$ as the sum over $\mathcal{C}_M^2$. By the same proof as for $M=1$ (see [Ath15] for further details), the Siegel–Veech transform $\mathrm{SV}_{\mathrm{rel},M}(h)\in\mathrm{L}^1(\mathcal{H}(0,0))$. Consequently, by the equivariance (6.4) the map

$$h\longmapsto\int_{\mathcal{H}(0,0)}\mathrm{SV}_{\mathrm{rel},M}(h)\,\mathrm{d}\mu$$

is a $\mathrm{G}'(\mathbb{R}^2)$-invariant functional on $C_c(\mathbb{R}^2\times\mathbb{R}^2)$, and so we must have

$$\int_{\mathcal{H}(0,0)}\mathrm{SV}_{\mathrm{rel},M}(h)\,\mathrm{d}\mu \;=\; a_M\int_{\mathbb{R}^2\times\mathbb{R}^2}h(x,y)\,\mathrm{d}x\mathrm{d}y+b_M\int_{\mathbb{R}^2}h(x,x)\,\mathrm{d}x,$$

since the only $\mathrm{G}'(\mathbb{R}^2)$-invariant measures on $\mathbb{R}^2\times\mathbb{R}^2$ are Lebesgue measure and the measure supported on the diagonal Δ. A similar argument to above shows that with $h(x,y)=\chi_{B(0,R)}(x)\chi_{B(0,R)}(y)$, for $R\gg 1$, that $a_M=M^4$, and $b_M=M^2$. ☐

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