UNIVERSITÀ DEGLI STUDI DI MILANO

Facoltà di Scienze Matematiche, Fisiche e Naturali
Corso di Laurea magistrale in Matematica

Algebraic cycles on K3 surfaces

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Anno Accademico 2012-2013

Acknowledgments

Foremost, I would like to express my sincere gratitude to my advisor Prof. van Geemen for the continuous support, for his patience, kindness, and vast knowledge. Without his guidance and persistent help this thesis would not have been possible.

I would also like to thank the rest of the professors I have met in these five years in Milan, who have helped me to discover how wonderful Mathematics is, how many beautiful landscapes one can see going deeper in the large varieties of fields that Mathematics offers, and how it can open our minds.

I wish to thank also my friend Luca for his help with courses and for all the useful recommended books, and all the other friends who are always there for me when I need them.

A special thanks goes to Arianna, who shared with me so many beautiful experiences in these years and helped me every day with her smiles.

Last but not least, I would like to thank my parents for their unconditioned support, both emotionally and financially throughout my studies. In particular Mum, thank you for everything, I would not have made it without you.

Introduction

In algebraic geometry, the Chow ring of a quasi-projective scheme X over a field is an algebraic-geometric analogue of the cohomology ring of X considered as a topological space. Before defining it, we must define the notion of rational equivalence. If Z and W are two k-dimensional subvarieties of X, we say that Z and W are rationally equivalent if there is a flat family parametrized by $\mathbb{P}^1$, contained in the product family $X \times \mathbb{P}^1$, such that two of whose fibres are Z and W. Let then $CH^k(X)$ (resp. $CH_k(X)$) be the quotient of the group generated by k-codimensional (resp. k-dimensional) subvarieties by the rational equivalence relation. This construction has good functorial properties, so that for any map $f : X \to Y$ between quasi-projective schemes, we can define a push-forward homomorphism $f_* : CH_k(X) \to CH_k(Y)$ and a pull-back homomorphism $f_* : CH^k(Y) \to CH^k(X)$.
Define $CH(X)$, the Chow ring of X, as the direct sum over k of the groups $CH^k(X)$ with the ring structure given by the intersection product of varieties. The intersection product, which will be defined in a precise way, gives to $CH(X)$ the structure of a graded ring.

If X is a complex manifold, for any subvariety V in X of codimension k, we can define a cohomological class $[V] \in H^{2k}(X, \mathbb{Z})$ determined by V. If we consider $H(X, \mathbb{Z}) = \bigoplus_k H^k(X, \mathbb{Z})$ as a ring with respect to the cup product, the map that associates to $V \in CH_k(X)$ the element $[V] \in H^{2k}(X, \mathbb{Z})$ is a ring homomorphism. Since this map is compatible with the operations of push-forward and pull-back, it gives a good translation from the algebraic language of the Chow ring to the topological language of cohomology and this is why we said that the Chow ring is an algebraic-geometric analogue of the cohomology ring.

In this thesis we will specialise to the study of the Chow ring of algebraic K3 surfaces. We call algebraic K3 surface any non-singular projective connected surface S with $H^1(S, \mathcal{O}_S) = 0$ and with trivial canonical bundle.
From now on let S be an algebraic K3 surface over an algebraically closed field k.
Since S is connected, $CH_2(S)$ is isomorphic to $\mathbb{Z}$, with generator given by the class determined by S; moreover, since it is smooth, $CH_1(S)$ is isomorphic to the Picard group of S. Hence the only interesting and mysterious object is $CH_0(S)$, the Chow group of zero cycles of S.

For our purposes, the most important property of K3 surfaces is that there are 'many' rational curves on them, where we call rational curve any curve birational to $\mathbb{P}^1$. This property, together with a deep theorem of Roitman about the fact that $CH_0(S)$ is a torsion-free group, allowed Beauville and Voisin to define a special class c_S in $CH_0(S)$ (see [2]). This is by definition the class determined by any point on any rational curve on S.

The next step is to generalise rational curves in this context, considering the curves on S such that all of their closed points determine the same element c_S in $CH_0(S)$. Huybrechts, in his article [17], refers to these curves as pointwise constant cycle curves. By definition of c_S, rational curves are clearly pointwise constant cycle curves, but in general they are not the only ones. The notion of pointwise constant cycle curve is not well-behaved under base changes of fields if we consider countable fields and moreover, if for example $k = \overline{\mathbb{Q}}$ and if a special case of the Bloch-Beilinson conjecture were true, then any curve would be a pointwise constant cycle curve in this case.

Hence for countable fields the notion of pointwise constant cycle curve is less interesting and so Huybrechts introduces the finer notion of constant cycle curves, which in the case of an uncountable field k coincides with that of pointwise constant cycle curve. The definition of constant cycle curve is more scheme-theoretic and we will give its details. We will also show that it is well-behaved under base change. Then we will present some concrete examples of constant cycle curves and we will show how Huybrechts, generalising one of them concerning constant cycle curves on elliptic K3 surfaces, proves that the union of all constant cycle curves is dense in S. This result must be considered with a view toward a folklore conjecture which states that the union of rational curves on a complex K3 surface is dense with respect to the classical or Zariski topology.

We then consider an approach to a different conjecture about complex K3 surfaces following the article [25] by Voisin. If X and Y are complex manifolds, by a generalisation of a theorem of Mumford we know that any correspondence $\Gamma \in CH_d(X \times Y)$ that induces the 0-map $\Gamma_* : CH_0(X)_{hom} \to CH_0(Y)_{hom}$, where $CH_0(-)_{hom}$ is the group of 0-cycle of degree zero, induces also the zero map on $[\Gamma]^* : H^{i,0}(Y) \to H^{i,0}(X)$ for any $i > 0$.

A special instance of the famous Bloch conjecture claims a sort of converse to this last result. In the case of surfaces it states that, if S is an algebraic complex surface, X is a complex manifold and $\Gamma \in CH_d(X \times S)$ is a correspondence such that the maps $[\Gamma]^* : H^{i,0}(S) \to H^{i,0}(X)$ vanish for $i > 0$, then $\Gamma_* : CH_0(X)_{alb} \to CH_0(S)$ vanishes, where $CH_0(X)_{alb} := Ker(alb_X : CH_0(X)_{hom} \to Alb(X))$.

We will see how a very special case of this conjecture would imply that finite order symplectic automorphisms on algebraic K3 complex surfaces should behave as the identity on the Chow group of zero cycles.

We will show how Voisin, in her article, proves this last statement in the case of symplectic involutions. Since the key of her proof makes use of the Prym varieties associated to the double covers of ample curves on S determined by the involution, it cannot be generalised to symplectic automorphisms of order bigger than two. However, even if we will not present it here, in these cases the problem was solved by Huybrechts using some results about a derived category of coherent sheaves on S. Hence it is indeed true that any symplectic automorphism of any finite order on a K3 surface S acts trivially on $CH_0(S)$.

In the first three chapters of this thesis we will give some background about the Chow ring of a variety, Riemann surfaces and K3 surfaces, underlying in particular the results that we use in the last three chapters, in which we present the results of the three articles mentioned above, resp. of Beauville-Voisin [2], Huybrechts [17] and Voisin [25].

Contents

1. Algebraic cycles and Chow ring

In the first two sections of this chapter we will introduce the Chow ring of a quasi-projective scheme over a field and some of its properties that will be used in the other chapters.
In the other sections we will set us in the complex projective setting and we will show the link between cohomology and the Chow ring and the compatibilities of the Abel-Jacobi maps with the rational equivalence relation. In the end we will present the definition of infinite dimensionality of the Chow group of 0-cycles of a complex projective manifold and we will state some important results and conjectures concerning also K3 surfaces.
We will mainly follow Voisin's book [27] and Murre's notes [21].

In this thesis a variety will be a reduced and irreducible scheme, and a subvariety of a scheme will be a closed subscheme which is a variety.
The local ring of a scheme X along a subvariety $V \subset X$, that is the localisation of $\mathcal{O}_X$ at the generic point $v \in V$, will be denoted by $\mathcal{O}_{X,V}$. The field of rational functions on a variety V will be denoted by $K(V)$.

1.1 Algebraic cycles

Let X be a quasi-projective scheme over a field K, i.e. a Zariski open set in a projective scheme, of dimension $d := dim(X)$.

Definition 1.1. Let $\mathcal{Z}_k(X)$ be the free abelian group generated by the closed k-dimensional subvarieties of X.

A *k-dimensional algebraic cycle*, or k-cycle, on X is an element of the the group $\mathcal{Z}_k(X)$.

We will denote by $\mathcal{Z}^k(X)$ the group of cycles of codimension k in X.

If $Y \subset X$ is a subscheme of dimension $\leq k$, possibly non-reduced and reducible, we can associate to Y a cycle

$$[Y] = \sum_W n_W W$$

where the sum is taken over the k-dimensional irreducible reduced components of Y and the multiplicity n_W is equal to the length $l(\mathcal{O}_{Y,W})$ of the ring $\mathcal{O}_{Y,W}$.

Remark 1.2. The group of codimension one cycles $\mathcal{Z}_{d-1}(X) = Div(X)$ is the group of Weil divisor on X, while $\mathcal{Z}_0(X)$ is the free abelian group generated by the closed points of X. If

$Z = \sum n_i P_i \in \mathcal{Z}_0(X)$, we call $deg(Z) := \sum_i n_i$ the degree of Z. In the next chapters we will focus in particular on $\mathcal{Z}_0(X)$, where X will be a surface.

We want now to define some basic operations on algebraic cycles.

1. **Push-forward** Let $f : Y \to X$ be a morphism between quasi-projective schemes. We can define a group homomorphism
$$f_* : \mathcal{Z}_k(Y) \to \mathcal{Z}_k(X)$$
called *push-forward* of f. By linearity it suffices to define this only for a subvarieties. If $Z \subset Y$ is a subvariety and $f(Z) \subset X$ is the set-theoretical image, then its Zariski closure $Z' := \overline{f(Z)}$ is an algebraic subvariety of X, irreducible if Z itself is irreducible, with $dim(Z') \leq dimZ$. If $K(Z)$ and $K(Z')$ are the function fields resp. of Z and Z', we define
$$f_*(Z) = deg[K(Z) : K(Z')]Z' \quad \text{ if } \ dim(Z') = dimZ$$
and 0 otherwise.
This homomorphism is well-defined since, if $dim(Z') = dimZ$, then $[K(Z) : K(Z')]$ is a finite extension of fields. Indeed $f : Z \to Z'$ is dominant, so that it induces an inclusion $K(Z) \subseteq K(Z')$ of finitely generated field over K; then, since $trdeg_K(K(Z)) = dimZ = dimZ' = trdeg_K(K(Z'))$, we have that $[K(Z) : K(Z')]$ is a finite algebraic extension.
Note that if we add the hypothesis that f is a proper morphism, so that f is closed map, we have $Z' = f(Z)$.

2. **Pull-back** Let $f : Y \to X$ be a *flat* morphism between quasi projective schemes of relative dimension $l := dimY - dimX$. If $Z \subset X$ is a k-dimensional subvariety, then $f^{-1}(Z)$ is a $(k+l)$-dimensional subscheme of Y, and thus it admits an associated cycle $[f^{-1}(Z)]$. We then define the *pull-back* of f to be the group homomorphism
$$f^* : \mathcal{Z}_k(X) \to \mathcal{Z}_{k+l}(Y), \quad f^*(Z) = [f^{-1}(Z)]$$
for any k-dimensional subvariety $Z \subset X$, extended to $\mathcal{Z}_k(X)$ by linearity.
Note that if we use the codimension notation we find $f^* : \mathcal{Z}^k(X) \to \mathcal{Z}^k(Y)$.
After the definition of intersection product, we will define the pull-back map also in the case of a morphism $f : Y \to X$ with X and Y smooth, but with no flatness condition on f.

3. **Intersection product** (only for proper intersections) Let X be a smooth d-dimensional variety. Let moreover Z and W be irreducible subvarieties of X of codimension resp. i and j. Then $Z \cap W$ is a finite union $\bigcup_l V_l$ of irreducible subvarieties $V_l \subset X$ of codimension $\leq i + j$ (this is because X is smooth; see [10, Ch. I, Th. 7.2]).
The intersection of Z and W at V_l is called *proper* if the codimension of V_l in X is exactly $i + j$. In this case we define the *intersection multiplicity* $i(Z, W; V_l)$ of Z and W at V_l as
$$i(Z, W; V_l) := \sum_{r=0}^{d} (-1)^r \ \text{length}(Tor_r^A(A/\mathcal{I}(Z), A/\mathcal{I}(W)))$$
where A is the local ring $\mathcal{O}_{X,V_l}$ of X at the generic point of V_l, and $\mathcal{I}(Z)$ and $\mathcal{I}(W)$ are the ideals defining Z and W in A (see [10, pag. 427]).

One can show that this definition of intersection multiplicity introduced by Serre coincides with the more geometric definitions of intersection multiplicity given classically. One could try only with the tensor product of $\mathcal{I}(Z)$ and $\mathcal{I}(W)$, but there are situations where the Tor-definition given here is necessary (see [10, Appendix A, Example 1.1.1]).
If the intersection is proper at every V_l, one defines the *intersection product* of Z and W as the cycle

$$V \cdot W := \sum_l i(Z, W; V_l) V_l \in \mathcal{Z}^{i+j}(X).$$

Extending it by linearity, we obtain the *intersection product*

$$\cdot : \mathcal{Z}^i(X) \times \mathcal{Z}^j(X) \to \mathcal{Z}^{i+j}(X)$$

for proper intersections.

Remark 1.3. Having the definition of intersection product, we can also define the pull-back map in the case of a morphism $f : Y \to X$ with X and Y smooth, but with no flatness condition on f. We define

$$f^* : \mathcal{Z}_k(X) \to \mathcal{Z}_{k+l}(Y), \quad f^*(Z) := pr_{Y*}(\Gamma_f \cdot pr_X{}^*(Z))$$

for any $Z \in \mathcal{Z}_k(X)$, where $\Gamma_f \subset Y \times X$ is the graph of f, $l := dimY - dimX$, and $pr_Y : Y \times X \to Y$, $pr_X : Y \times X \to X$ are the two projections ($pr_X{}^*(Z)$ is well-defined since pr_X is a flat morphism).
For now this new definition of pull-back morphism is defined only if $\Gamma_f \cdot pr_X{}^*(Z)$ is a proper intersection, but this is not a necessary condition when we work in the Chow group. This second definition of pull-back coincides with the first one in the cases where they are both well-defined (see [27, pag. 258]).

4. **Correspondences** Let X and Y be two smooth varieties with $d_X := dim(X)$ and $d_Y := dim(Y)$. A *correspondence* between X and Y is a cycle

$$\Gamma \in \mathcal{Z}_k(X \times Y), \quad k \geq 0.$$

Let $pr_X : X \times Y \to X$ and $pr_Y : X \times Y \to Y$ the projections. We define the homomorphisms induced by Γ as

$$\Gamma_* : \mathcal{Z}_l(X) \to \mathcal{Z}_{l+k-d_X}(Y), \qquad \Gamma_*(Z) = pr_{Y*}(pr_X{}^*(Z) \cdot \Gamma), \quad Z \in \mathcal{Z}_l(X)$$

and

$$\Gamma^* : \mathcal{Z}_l(Y) \to \mathcal{Z}_{l+k-d_Y}(X), \qquad \Gamma^*(Z) = pr_{X*}(pr_Y{}^*(Z) \cdot \Gamma), \quad Z \in \mathcal{Z}_l(Y).$$

In this context these homomorphism are defined only for those cycles Z such that the intersection products $pr_{Y*}(Z) \cdot \Gamma$ and $pr_{X*}(Z) \cdot \Gamma$ are well defined, but this is not a necessary condition when we work in the Chow group.

Remark 1.4. If we have a morphism $f : X \to Y$ and we call $\Gamma_f \subset X \times Y$ the graph of f, we get back the push-forward and the pull-back homomorphisms resp. as $f_* = \Gamma_{f*}$ and $f^* = \Gamma_f{}^*$.

The next step is the introduction of an equivalence relation $\sim$ on $\mathcal{Z}_k(X)$, for any k, defined in such a way that, in particular, the operations of push-forward and pull-back are compatible with it and such that the intersection product is always defined in the quotient $\mathcal{Z}_k(X)/\sim$.

1.2 Chow ring

Let V be a variety over a field K and let $W \subset V$ be a subvariety of codimension 1 in V. We want to extend, following Fulton's book [7, Section 1.2], the definitions of principal divisor and linear equivalence relation given in [10, pag. 131], to the case of V possibly singular along W.
The local ring $A := \mathcal{O}_{V,W}$ is a one dimensional local domain. Let $r \in K(V)^*$ be a rational invertible function on V. We will define a group homomorphism $ord_W : K(V)^* \to \mathbb{Z}$ and we will call $ord_W(r)$ the order of vanishing of r along W.
Any $r \in K(V)^*$ may be written as a fraction $r = a/b$, for $a, b \in A$. Since we want that ord_W is a homomorphism, we have $ord_W(a/b) = ord_W(a) - ord_W(b)$, so that it is enough to define $ord_W(r)$ only for $r \in A$.
In case V is a non-singular variety, the local ring of any point $x \in V$ is regular because it is the localisation of the local ring of any closed point in $\overline{\{x\}}$, which is a regular ring by [10, Ch. I, Th. 5.1]. In particular, if V is non-singular, then A is regular and so, since for one dimensional local domains regularity is equivalent to normality which is equivalent to be a discrete valuation ring, it is also a DVR. Hence in this case we can write $r = ut^m$, for u unit in A, t a generator of the maximal ideal of A and m an integer. For V non-singular we then set $ord_W(r) := \text{length}_A(A/(r)) = m$ (this is the classical definition used to define principal divisors).
In the general case, without the smoothness condition for V, we generalize this last definition and set, for $r \in A$,

$$ord_W(r) := l_A(A/(r))$$

where l_A denotes the length of the A-module in parentheses. One can prove that this definition determines a well-defined homomorphism $ord_W : K(V)^* \to \mathbb{Z}$ and moreover that, for a fixed $r \in K(V)^*$, there are only finitely many codimension one subvarieties W such that $ord_W(r) \neq 0$ (see [7, Section 1.2]).

Let now X be a quasi-projective scheme over a field K of dimension d and let V be a $(k+1)$-dimensional subvariety of X.
For any $r \in K(V)^*$, define the k-cycle $[div(r)]$ on X by

$$[div(r)] := \sum_W ord_W(r)[W]$$

where the sum is taken over all codimension one subvarieties W of V.

Definition 1.5 (Rational equivalence)**.** A k-dimensional cycle $\alpha \in \mathcal{Z}_k(X)$ is *rationally equivalent to zero*, written $\alpha \sim 0$, if there are a finite number of $(k+1)$-dimensional subvarieties V_i of X and $r_i \in K(V_i)^*$, $i = 1, \ldots, n$, such that

$$\alpha = \sum_{i=1}^{n} [div(r_i)].$$

Since $[div(r^{-1})] = -[div(r)]$, the k-cycles rationally equivalent to zero form a subgroup $\mathcal{Z}_k(X)_{rat} \subset$

$\mathcal{Z}_k(X)$. We denote by $CH_k(X)$ the group of k-cycles modulo rational equivalence

$$CH_k(X) := \mathcal{Z}_k(X)/\mathcal{Z}_k(X)_{rat}.$$

We also denote by $CH^k(X) := CH_{d-k}(X)$ the Chow group of cycles of codimension k in X. We define the *Chow group of* X to be the direct sum group

$$CH(X) := \bigoplus_{k=0}^{d} CH^k(X).$$

Remark 1.6. If X is a non-singular variety over K, we have that $CH_{d-1}(X) = Pic(X)$.
Indeed in general the group of codimension one cycles $\mathcal{Z}_{d-1}(X)$ is equal to the group of Weil divisors $Div(X)$ and, thanks to the smoothness condition, the definition of rational equivalence in $\mathcal{Z}_{d-1}(X)$ coincides with the definition of linear equivalence in $Div(X)$ (indeed for X smooth, $r \in K(X)^*$ and $V \subset X$ codimension one subvariety, we noted above that the definition of $ord_V(r)$ we gave is the same as the definition used to define principal divisor in [10, pag. 131]).
Moreover, again by the smoothness condition, all of local rings of X are regular local rings and so, in particular, they are UFD. Then by [10, Ch. II, Prop. 6.11] the group of Weil divisors of X is equal to the group of Cartier divisors. But, since we are in the case that X is a reduced irreducible scheme, by [10, Ch. II, Prop. 6.15], the group of Cartier divisors modulo linear equivalence is isomorphic to $Pic(X)$, the group of algebraic line bundles on X modulo isomorphism and so we are done.

The following lemma will be fundamental to show some equivalent definitions of the concept of rational equivalence and to prove that the push-forward homomorphism passes to the quotient under rational equivalence.

Lemma 1.7. *Let $f : X \to Y$ be a proper, surjective morphism of varieties with $dim(X) = dim(Y)$ and let $r \in K(X)^*$. Then*

$$f_*([div(r)]) = [div(N(r))]$$

where $N(r) = Norm_{K(X)/K(Y)}(r)$ is the norm of r, i.e. the determinant of the $K(Y)$-linear endomorphism of $K(X)$ given by multiplication by r.

Proof. See [7, Prop 1.4] or [27, Lem. 9.6]. □

We now show other characterisations of the rational equivalence relation.

Proposition 1.8 (Equivalent definitions of rational equivalence). *Let $\alpha \in \mathcal{Z}_k(X)$ be a k-dimensional cycle on X. The following are equivalent:*

(i) *α is rationally equivalent to zero.*

(ii) *There are a finite number of $(k+1)$-dimensional subvarieties V_i of X, $i = 1, \dots, n$, such that*

$$\alpha = \sum_{i=1}^{n} \nu_*[div(\widetilde{r_i})]$$

where $\nu_i : \widetilde{V_i} \to V_i$ is the normalisation of V_i and and $\widetilde{r_i} \in K(\widetilde{V_i})^*$ (this is the definition of rational equivalence given in Voisin's book [27] and Hartshorne's book [10]).

(iii) *There exists a correspondence $\Gamma \in \mathcal{Z}_{k+1}(X \times \mathbb{P}^1)$ and two points $a, b \in \mathbb{P}^1$ such that, if $\Gamma^* : \mathcal{Z}_0(\mathbb{P}^1) \to \mathcal{Z}_k(X)$ is the homomorphism induced by Γ, then*

$$\alpha = \Gamma^*(b) - \Gamma^*(a).$$

The geometric idea that gives definition (iii) will be often used in the next chapter. It says that if Z and W are two k-subvarieties of X, they are rationally equivalent if there is a flat family parametrized by $\mathbb{P}^1$, contained in the product family $X \times \mathbb{P}^1$, such that two of whose fibres are Z and W. For example, we have that every two distinct points of $\mathbb{P}^1$ are rationally equivalent. Indeed take $\Gamma := \Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$. Then if $a, b \in \mathbb{P}^1$, we have $\Gamma * (a) = a$ and $\Gamma * (b) = b$ and so a is rationally equivalent to b. If we consider then a rational curve $C \subset S$ on a surface S, i.e. if a curve C birationally equivalent to $\mathbb{P}^1$, we have that C is the image of a generically injective morphism $j : \mathbb{P}^1 \to S$. Then if we take any two points $p = j(a), q = j(b) \in C$, since we observed that $a = b \in CH_0(\mathbb{P}^1)$, we have that $p = j_*(a) = j_*(b) = q \in CH_0(S)$.

Proof of Proposition 1.8. **(i) ⇔ (ii)** The case (ii) implies (i) is clear using Lemma 1.7.

Now we want to prove that (i) implies (ii). Let V_i, $i = 1, \dots, n$, be $(k+1)$-dimensional subvarieties of X, and $r_i \in K(V_i)^*$, such that $\alpha = \sum_{i=1}^{n}[div(r_i)] \in \mathcal{Z}_k(X)$. If $\nu_i : \widetilde{V_i} \to V_i$ is the normalisation of V_i, then, by definition of normalisation of an integral scheme, the ring $\mathcal{O}_{\widetilde{V_i}}$ is the integral closure of $\mathcal{O}_{V_i}$ in its field of fractions $K(V_i)$. Hence the homomorphism induced by ν_i on the function fields $\nu_i^\sharp : K(V_i) \overset{\cong}{\to} K(\widetilde{V_i})$ is an isomorphism. Then, if $\widetilde{r_i} := \nu_i^\sharp(r_i)$, we have $\alpha = \sum_{i=1}^{n}[div(r_i)] = \sum_{i=1}^{n} \nu_*[div(\widetilde{r_i})]$ and thus we have (ii).

(i) ⇔ (iii) The key point here is that, if $V \subset X$ is an irreducible subvariety of dimension $k+1$, then any dominant morphism $r : V \to \mathbb{P}^1$ determines a rational function $r \in K(V)$ such that $[div(r)] = [r^{-1}(0)] - [r^{-1}(\infty)] \in \mathcal{Z}_k(X)$ (see [7, ex. 1.5.1]). Conversely any rational function $r \in K(V)$ defines a rational mapping from V to $\mathbb{P}^1$, i.e. a morphism $r_{|U} : U \to \mathbb{P}^1$ from some open set $U \subset V$ to $\mathbb{P}^1$. Recall also that, if $P \in \mathbb{P}^1$, the explicit form of Γ^* is given by

$$\Gamma^*(P) = pr_{X*}({pr_{\mathbb{P}^1}}^*(P) \cdot \Gamma) \in \mathcal{Z}_k(X).$$

For (iii) implies (i) we can assume, without loss of generality, that $\Gamma \subset X \times \mathbb{P}^1$ is irreducible and $b = 0$ and $a = \infty$ in $\mathbb{P}^1$.

Let $f := p_{\mathbb{P}^1|\Gamma} : \Gamma \to \mathbb{P}^1$ be the restriction of the second projection to Γ. For any $P \in \mathbb{P}^1$, we have that $f^{-1}(P) = \Gamma \cdot (X \times \{P\})$ and hence, by definition of Γ^*, we have that $\Gamma^*(P) =$

$p_{X*}[f^{-1}(P)]$ for any $P \in \mathbb{P}^1$.

The morphism $f : \Gamma \to \mathbb{P}^1$ determines a rational function $f \in K(\Gamma)^*$, with $[f^{-1}(0)] - [f^{-1}(\infty)] = [div(f)] \in \mathcal{Z}_k(\Gamma)$. Then, by applying p_{X*}, we obtain

$$\alpha = \Gamma^*(0) - \Gamma^*(\infty) = p_{X*}([div(f)]) \in \mathcal{Z}_k(X).$$

Then by Lemma 1.7, α is rationally equivalent to zero.

Now for the case (i) implies (iii), assume for simplicity that $\alpha = [div(r)]$, with $r \in K(V)$ and $V \subset X$ irreducible subvariety of dimension $k+1$. We have noted above that the rational function r induces a morphism $r_{|U} : U \to \mathbb{P}^1$ from some open set $U \subset V$ to $\mathbb{P}^1$. Let $\Gamma_r \subset V \times \mathbb{P}^1$ be the closure of the graph of $r_{|U}$ and consider the correspondence $\Gamma := (j \times id_{\mathbb{P}^1})(\Gamma_r) \subset X \times \mathbb{P}^1$, where $j : V \hookrightarrow X$ is the injection of V in X. If we let $f := p_{\mathbb{P}^1|\Gamma} : \Gamma \to \mathbb{P}^1$ be the restriction of the second projection to Γ, then as above we have $\Gamma^*(0) - \Gamma^*(\infty) = p_{X*}([div(f)]) \in \mathcal{Z}_k(X)$. But using Lemma 1.7 one can show that $p_{X*}([div(f)]) = [div(r)] = \alpha$ and so we are done. If $\alpha = \sum [div(r_i)]$, the needed correspondence Γ is the sum of the cycles $(j \times id_{\mathbb{P}^1})(\Gamma_{r_i})$.

□

The rational equivalence relation is a 'good' relation, in the sense that the operations on algebraic cycles introduced in Section 1.1 pass to the quotient and the intersection product in the Chow group is well defined for every pair of cycles. We summarize these arguments in the next proposition.

Proposition 1.9. *Let $f : X \to Y$ be a morphism of quasi-projective schemes of dimension d_X and d_Y respectively.*

(i) *If f is proper, the push-forward map $f_* : \mathcal{Z}_k(X) \to \mathcal{Z}_k(Y)$, for any k, sends $\mathcal{Z}_k(X)_{rat}$ to $\mathcal{Z}_k(Y)_{rat}$, and thus induces a group homomorphism*

$$f_* : CH_k(X) \to CH_k(Y).$$

Moreover, if $g : Y \to Z$ is another morphism of quasi-projective schemes, then $g_* \circ f_* = (g \circ f)_*$.

(ii) *If the pull-back map is defined, so that for example $f : X \to Y$ is flat or X and Y are smooth, then $f^* : \mathcal{Z}^k(Y) \to \mathcal{Z}^k(X)$ sends $\mathcal{Z}^k(X)_{rat}$ to $\mathcal{Z}^k(Y)_{rat}$ for any k, and thus induces a group homomorphism*

$$f^* : CH^k(Y) \to CH^k(X).$$

Moreover, if $g : Y \to Z$ is another morphism of quasi-projective schemes, then $f^* \circ g^* = (g \circ f)^*$.

(iii) *If X is smooth then $CH(X)$ is a commutative associative graded ring, called Chow ring, with respect to the intersection product.*

(iv) *If $\Gamma \in \mathcal{Z}_k(X \times Y)$ is a correspondence with X and Y smooth, then the induced homomorphism $\Gamma_* : \mathcal{Z}_l(X) \to \mathcal{Z}_{l+k-d_X}(X)$ and $\Gamma^* : \mathcal{Z}_l(Y) \to \mathcal{Z}_{l+k-d_Y}(X)$ pass to the quotient, so that Γ induces well defined group homomorphisms*

$$\Gamma_* : CH_l(X) \to CH_{l+k-d_X}(Y), \quad \Gamma^* : CH_l(Y) \to CH_{l+k-d_Y}(X)$$

depending only on the class of Γ in $CH_k(X \times Y)$.

We have also that the composition of correspondence is associative. Moreover, if Z is another smooth quasi-projective scheme and $\Gamma_* : CH(X) \to CH(Y)$, $\Gamma'_* : CH(Y) \to CH(Z)$ are the homomorphism induced by two correspondence Γ and Γ', then

$$(\Gamma' \circ \Gamma)_* = \Gamma'_* \circ \Gamma_*.$$

The analogue holds for Γ^* and Γ'^*.

Proof. **(i)** It follows immediately from 1.7 (see [27, Lem. 9.5]).

(ii) See [27, Lem. 9.7].

(iii) First one must show that the definition of intersection product for proper intersections given in Section 1.1 passes to the quotient. If $Z, Z' \in \mathcal{Z}_k(X)$ and $W \in \mathcal{Z}_l(X)$ are irreducible subvarieties such that $Z \cap W$ and $Z' \cap W$ are proper intersections and $Z \sim Z'$, we have then $Z \cdot W \sim Z' \cdot W$. Indeed, considering the definition of rational equivalence given by Proposition 1.8 (iii), we obtain that there exists a $\Gamma \in \mathcal{Z}_{k+1}(X \times \mathbb{P}^1)$ such that $\Gamma^*(a) = Z$ and $\Gamma^*(b) = Z'$, for some $a, b \in \mathbb{P}^1$. But then we find a correspondence $\Gamma_1 := \Gamma \cap (W \times \mathbb{P}^1)$ such that $\Gamma_1(a) = Z \cdot W$ and $\Gamma_1(b) = Z' \cdot W$, so that we have $Z \cdot W \sim Z' \cdot W$. Now one can show also, with a similar argument and using Lemma 1.7, that if $Z \sim 0$, then $Z \cdot W \sim 0$. We have thus that the definition of intersection product given only for proper intersections passes to the quotient.

It remains only to define the intersection product for every intersection of cycles and we do this using the *Chow's moving lemma* (see [23]). It says that if V and W are cycles on a smooth quasi-projective variety X, then there is a cycle Z', rationally equivalent to Z, such that Z' and W intersect properly. Moreover, if Z'' is another such, then $Z' \cdot W \sim Z'' \cdot W$. Hence the Chow's moving lemma allows us to define the intersection product for any two element of the Chow group, without the need of the condition of proper intersection.

We have then obtained a ring operation on the Chow group $CH(X)$ for X smooth quasi-projective variety. One can moreover show that the Chow ring is commutative and associative (see [7, Prop. 8.1.1]).

We mention that there is another way to define the intersection product for every intersection of cycles described by Fulton. It avoids the recourse to Chow's moving lemma and gives an explicit construction of the cycle $Z \cdot Z'$ for any intersecting cycles Z and Z' (see [27, Section 9.2.1]).

(iv) Since the intersection product, the push-forward and the pull-back pass to the quotient, then Γ_* and Γ^* , defined resp. as $\Gamma_*(-) = pr_{2*}(pr_{1*}(-) \cdot \Gamma)$ and $\Gamma_*(-) = pr_{1*}(pr_{2*}(-) \cdot \Gamma)$, must pass to the quotient and it is also clear that they depend only on the class of Γ in $CH_k(X \times Y)$. For the second part of the statement (iv) see [27, Lem. 9.17].

□

Remark 1.10. There is another 'good' equivalence relation called *algebraic equivalence*. It is a weaker relation than rational equivalence. We say that $\alpha \in \mathcal{Z}_k(X)$ is algebraically equivalent to zero if there exists a smooth curve C, a correspondence $\Gamma \in \mathcal{Z}_{k+1}(X \times C)$ and two points $a, b \in C$ such that, if $\Gamma^* = pr_{X*}(pr_{C*}(-) \cdot \Gamma) : \mathcal{Z}_0(C) \to \mathcal{Z}_k(X)$ is the homomorphism induced by Γ, then

$$\alpha = \Gamma^*(b) - \Gamma^*(a).$$

Note that the rational equivalence definition is given by taking $C = \mathbb{P}^1$. If we set $\mathcal{Z}_{k\,alg} := \{\alpha \in \mathcal{Z}_k(X) : \ \alpha \text{ is algebraically equivalent to zero}\}$, we have then $\mathcal{Z}_k(X)_{rat} \subseteq \mathcal{Z}_{k\,alg}$ and in general this is not an equality.

Now we show some compatibilities between the intersection product and the group homomorphisms f_* and f^* induced by a morphism f.

Proposition 1.11. *Let X and Y be smooth quasi-projective schemes and let $f : X \to Y$ be a morphism.*

(i) *The pull-back $f^* : CH(Y) \to CH(X)$ is a graded ring homomorphism, with $f^* : CH^k(Y) \to CH^k(X)$, i.e. for any $Z,\ Z' \in CH(Y)$ we have*

$$f^*(Z \cdot Z') = f^*(Z) \cdot f^*(Z') \in CH(X).$$

(ii) *(Projection formula) If f is proper, then for any $Z \in CH(Y)$ and $Z' \in CH(X)$, we have*

$$f_*(f^*Z \cdot Z') = Z \cdot f_*Z' \in CH(Y).$$

(iii) *If $Z, Z' \in CH(X)$ and $\Delta : X \to X \times X$ is the diagonal morphism with $\Delta_X := \Delta(X)$, then the intersection product of Z and Z' is equal to*

$$Z \cdot Z' = \Delta^*(\Delta_X \cdot (Z \times Z')) \in CH(X).$$

Proof. See [7, Section 8.1]. □

The following corollary is a consequence of the projection formula.

Corollary 1.12. *If f is proper and $dim(X) = dim(Y)$, for any $Z \in CH(Y)$, we have*

$$f_*(f^*(Z)) = deg(f)Z$$

where $deg(f)$ is defined to be equal to zero if f is not dominant, and to be equal to the degree of the field extension $K(Y) \subset K(X)$ otherwise.

Proof. With the notation as in Proposition 1.11 (ii), apply the projection formula with $Z' = X \in CH_d(X),\ d = dim(X)$. Then we have $f_*(f^*Z \cdot X) = Z \cdot f_*X \in CH(Y)$. But by definition of f_*, we have that $f_*(X) = deg(f)Y \in CH(Y)$. Hence we obtain

$$f_*(f^*Z) = f_*(f^*Z \cdot X) = Z \cdot f_*X = deg(f)(Z \cdot Y) = deg(f)Z \in CH(Y).$$

□

We conclude the section mentioning some other important properties of $CH(X)$, for X quasi-projective scheme, which we will use in the next chapters.

Proposition 1.13 (Localisation exact sequence)**.** *Let $i : F \to X$ be the inclusion of a closed subscheme F in X. Let $j : U := X - F \to X$ be the inclusion of the complement.*
Since i is proper (because it is a closed immersion), the push-forward homomorphism i_* is well defined. Moreover, since j is flat (because it is open), the pull-back homomorphism j^* is well defined too.
Then, for any $k \in \mathbb{N}$, the following sequence is exact:

$$CH_k(F) \stackrel{i_*}{\to} CH_k(X) \stackrel{j^*}{\to} CH_k(U) \to 0.$$

Proof. See [27, Lem. 9.12]. □

Proposition 1.14 (Homotopy property)**.** *Let $\mathbb{A}^1_k = SpecK[T]$ denote the affine line. Then the map*

$$p_X{}^* : CH_k(X) \to CH_{k+1}(\mathbb{A}^1_K \times X)$$

is a group isomorphism for any k.

Proof. See [27, Prop. 9.11]. □

Proposition 1.15 (Chow group of a vector bundle)**.** *Let $\pi : E \to X$ be a vector bundle of rank r. Then the map*

$$\pi^* : CH_k(X) \to CH_{k+r}(E)$$

is a group isomorphism for any k.

Proof. See [27, Th. 9.13]. □

Although we do not present them here, one can also compute the Chow group of a projective bundle or the Chow group of blow-ups (see resp. [27, Th. 9.25] and [27, Th. 9.27]). We only mention that, if $X = S$ is a smooth projective surface and $\nu : \widetilde{S} \to S$ is the blow-up of S at a point $x \in S$, the formula for the Chow group of a blow up implies that $\nu_* : CH_0(\widetilde{S}) \stackrel{\cong}{\to} CH_0(S)$ is an isomorphism.

Remark 1.16. This last result in particular shows that, if $X = S$ is a surface, then $CH_0(S)$, the Chow group of zero cycles of S, is a birational invariant. Indeed if $f : S \dashrightarrow T$ is a birational map then, by the factorisation property for birational map between surfaces (see [1, Cor. II.12]), we can use successive blow-ups of S to construct a variety $\nu : \widetilde{S} \to S$ and a morphism $g : \widetilde{S} \to T$ such that $f = g \circ \nu^{-1}$ on a Zariski open set of S. Then, as f is birational, g is of degree one and the map $g^* : CH_0(T) \overset{\cong}{\to} CH_0(\widetilde{S})$ is injective because admits g_* as a left inverse. But we have mentioned above that $\nu_* : CH_0(\widetilde{S}) \overset{\cong}{\to} CH_0(S)$ is an isomorphism, and so we obtain that $\nu_* \circ g^* : CH_0(T) \overset{\cong}{\to} CH_0(S)$ is injective. Using the fact that ν is birational, so that $\nu_* \circ \nu^* = id$, we obtain also that $g_* \circ \nu^* : CH_0(S) \to CH_0(T)$ is surjective. Exchanging the roles of S and T, and noting that the morphisms above do not depend on the choice of the desingularisation of f, we conclude that $\nu_* \circ g^*$ is an isomorphism.

1.3 Cycle class map

In this section we want to show the relation between the Chow group and cohomology.
We set us in the case of X projective smooth variety over $\mathbb{C}$ of complex dimension d, so that we can consider X_{an}, the underlying analytic complex manifold(see [10, Appendix B, Section 1] for the construction of X_{an}).
By Chow's Lemma every compact analytic subvariety of a complex projective space is algebraic. Moreover the G.A.G.A. principle by Serre describes an equivalence of category between the category of coherent sheaves on X and the category of coherent analytic sheaves on X_{an} (see [10, Appendix B, Section 2]). These results assure us that we can treat projective varieties over $\mathbb{C}$ equivalently as algebraic or analytic objects. Then we can work on X_{an} instead of X and consider holomorphic and meromorphic functions instead of regular and rational functions.

From now on we will work over the complex manifold X_{an}, which by abuse of notation we will denote by X. Since we are working in the category of complex analytic manifolds, we have both singular cohomology theory and the de Rham cohomology theory (note that one could indeed work in the category of smooth quasi-projective scheme over any field K using the étale-cohomology theory, cf. [21, pag. 11]).

For the singular homology and cohomology background we refer to the book by Hatcher [12], while for the de Rham cohomology theory we refer to the book by Bott and Tu [5].
We will denote the k-th de Rham cohomology group of X as $H^k_{DR}(X)$ and the k-th singular cohomology (resp. homology) group with coefficient in a group R as $H^k(X, R)$ (resp. $H_k(X, R)$).

Before stating the result about the relation between the Chow group and cohomology, we need some results on singular cohomology and de Rham cohomology, in particular about Poincaré duality, in order to define the cohomology class defined by a subvariety of X.

Theorem 1.17. *Let X be a projective complex manifold of complex dimension $d \in \mathbb{Z} > 0$. Since X is a projective complex manifold then it is a compact and orientable manifold, so that we have an isomorphism $H^{2d}(X, R) \cong R$ for every group R (cf. [12, Th. 3.26]).*

1. *(Universal coefficient theorem) If R is a field (for example $\mathbb{R}$ or $\mathbb{C}$), we have isomorphisms*

$$H_k(X, R) \cong H_k(X, \mathbb{Z}) \otimes_{\mathbb{Z}} R \quad \textit{and} \quad H^k(X, R) \cong Hom_R(H_k(X, R), R).$$

2. *(de Rham theorem) We have a group isomorphisms*

$$DR_k : H^k_{DR}(X) \xrightarrow{\cong} H^k(X, \mathbb{R}), \quad \omega \mapsto \alpha_\omega = \int_{-} \omega'$$

where $\omega \in H^k_{DR}(X)$ *is represented by the real closed* k*-form* ω' *and* $\alpha_\omega \in H^k(X, \mathbb{R}) \cong Hom_{\mathbb{R}}(H_k(X, \mathbb{R}), \mathbb{R})$ *is the linear map*

$$\alpha_\omega = \int_{-} \omega' : H_k(X, \mathbb{R}) \to \mathbb{R}, \quad Z = \int_Z \omega'$$

for any $Z \in H_k(X, \mathbb{R})$.
Moreover, considering $H^k_{DR}(X)$ *and* $H^k(X, \mathbb{R})$ *as rings with wedge product* $\wedge$ *and cup product* $\cup$ *respectively, the de Rham isomorphism is an isomorphism of rings.*

3. *(Poincaré duality) Considering the singular homology and cohomology groups over a ring* R*, we have isomorphisms*

$$PD_k : H^k(X, R) \xrightarrow{\cong} H_{2d-k}(X, R).$$

Moreover the pairing $\cup : H^k(X, R) \otimes H^{2d-k}(X, R) \to H^{2d}(X, R) \cong R$ *given by the cup product is non-degenerate where* R *is a field or where* $R = \mathbb{Z}$ *and torsion in* $H^*(X, R)$ *is factored out. Poincaré duality for the de Rham cohomology theory states that the pairing* $\int_X : H^k_{DR}(X) \otimes H^{2d-k}_{DR}(X) \to \mathbb{R}$ *given by the integral of the wedge product of two forms, is non-degenerate. Hence we have isomorphisms*

$$H^k_{DR}(X) \xrightarrow{\cong} H^{2d-k}_{DR}(X)^*, \quad \omega \to \int_X \omega' \wedge (-)$$

where $\omega \in H^k_{DR}(X)$ *is represented by the real closed* k*-form* ω' *and* $\int_X \omega' \wedge (-) \in H^{2d-k}_{DR}(X)^*$ *is the linear map*

$$\int_X \omega' \wedge (-) : H^{2d-k}_{DR}(X) \to \mathbb{R}, \quad \phi \mapsto \int_X \omega' \wedge \phi'$$

where $\phi \in H^{2d-k}_{DR}(X)$ *is represented by the real closed* $(2d-k)$*-form* ϕ'.
Using 1. and 2., one can show that these two versions of Poincaré duality agree.

Proof. 1.) The two isomorphisms in 1. can be proved, for example, using the Künneth formulas obtained by the degeneration of the Künneth spectral sequence (cf. [28, Section 3.6]).

2.) One can prove the de Rham theorem in an abstract way using sheaf cohomology and the abstract de Rham theorem (see [29, Th. 3.13]), or using spectral sequences (see [5]). For the proof that the de Rham isomorphism is a ring homomorphism see [5, Th. 14.28].

3.) For the singular homology and cohomology result see [12, Prop. 3.35] and [12, Prop. 3.38]. For the proof of the fact that the cup product is non-degenerate where $R = \mathbb{Z}$ and torsion in $H^*(X, R)$ is factored out see also [9, pag. 53].
For the de Rham version of Poincaré duality see [5], where one can prove it using Mayer-Vietoris arguments or spectral sequences. □

We are now ready to give the definition of cohomology class defined by a subvariety, possibly singular, $V \subset X$.
Let $k := dim_{\mathbb{C}}(V)$. We may define a linear functional on $H^{2k}_{DR}(X)$ given by $\omega \to \int_V \omega$, for any $2k$-form ω. Since, by Poincaré duality $H^{2k}_{DR}(X)^* \cong H^{2d-2k}_{DR}(X)$, we have that the linear functional $\int_V \in H^{2k}_{DR}(X)^*$ corresponds to an element $\eta_V \in H^{2d-2k}_{DR}(X)$ such that $\int_V \omega = \int_X \omega \wedge \eta_V$ for every $\omega \in H^{2k}_{DR}(X)$.

Definition 1.18. We call $\eta_V \in H^{2d-2k}_{DR}(X)$ the *Poincaré dual* of V.

One can show that η_V is also the Thom-class of the normal bundle of V in X (see [5, Prop 6.24]).
Using the de Rham theorem one can view η_V as an element $DR_{2d-2k}(\eta_V) \in H^{2d-2k}(X, \mathbb{R})$ called *real cohomology class* associated to V. Moreover, using the version of Poincaré duality for singular cohomology, one can consider $PD_{2d-2k}(DR_{2d-2k}(\eta_V)) \in H_{2k}(X, \mathbb{R})$. We call $PD_{2d-2k}(DR_{2d-2k}(\eta_V)) \in H_{2k}(X, \mathbb{R})$ the *real homology class* associated to V.

If we consider the singular cohomology groups with coefficients in $\mathbb{Z}$, one can construct a cohomology class defined by a subvariety V as an element of $H^{2d-2k}(X, \mathbb{Z})$. There is in fact the following exact sequence, with $U = X - V$:

$$\cdots \to H^{2d-2k-1}(U, \mathbb{Z}) \to H^{2d-2k}(X, U; \mathbb{Z}) \xrightarrow{\sigma} H^{2d-2k}(X, \mathbb{Z}) \to H^{2d-2k}(U, \mathbb{Z}) \to \dots$$

Assume for simplicity that V is smooth. By a theorem of Thom (see [26, pag. 167]) we have an isomorphism

$$T : H^{2d-2k}(X, U; \mathbb{Z}) \xrightarrow{\cong} H^0(V, \mathbb{Z}) = \mathbb{Z}.$$

If V is not smooth we replace it by $V - V_{sing}$ in the above sequence.

Definition 1.19. We define the *cohomology class associated to* V the element $[V] := \sigma(T^{-1}(1_{\mathbb{Z}})) \in H^{2d-2k}(X, \mathbb{Z})$.

Using the version of Poincaré duality for singular cohomology, one can consider also $PD_{2d-2k}([V]) \in H_{2k}(X, \mathbb{Z})$ and we call it *homology class associated to* V.

Using 1. and 2. of Theorem 1.17, one can show that eventually the definitions of Poincaré dual of V, real cohomology class associated to V and cohomology class associated to V are consistent (for more details see [27, Section 11.1.2]).

Now we can state the proposition that relates the algebraic cycle determined by a subvariety V and the cohomology class defined by V.

Proposition 1.20. *If V is rationally equivalent to zero then the cohomology class $[V] \in H^{2d-2k}(X, \mathbb{Z})$ associated to V is zero in $H^{2d-2k}(X, \mathbb{Z})$. Thus there is well defined map, called cycle class map,*

$$cl_k : CH_k(X) \to H^{2d-2k}(X, \mathbb{Z}), \quad V \to [V].$$

Its kernel will be denoted by $CH_k(X)_{hom} := Ker(cl_k)$.

Proof. See [27, Lem 9.18]. □

Remark 1.21. For divisors, so for $k = d-1$, the class map of a codimension one submanifold $V \subset X$ is the first Chern class $c_1(\mathcal{O}(V))$ of the line bundle $\mathcal{O}(V)$ associated to V (see [9, pag. 141]), i.e.

$$cl_{d-1}(V) = c_1(\mathcal{O}(V)).$$

In this case $CH_{d-1}(X)_{hom}$ is called $Pic^0(X)$, the Picard variety of X. One can show that it is an abelian variety.

For cycles of dimension zero the class map is anything else but the degree map.

Recall that if $f : X \to Y$ is a continuous map between topological spaces of dimension d_X and d_Y respectively, then we have a push-forward map on singular homology $f_* : H_k(X,\mathbb{Z}) \to H_k(Y,\mathbb{Z})$ and pull-back map on singular cohomology $f^* : H^k(Y,\mathbb{Z}) \to H^k(X,\mathbb{Z})$. Moreover, if X and Y are compact and orientable, then we can use Poincaré duality to define a push-forward map also on singular cohomology

$$f_* : H^k(X,\mathbb{Z}) \overset{\cong}{\to} H_{d_X-k}(X,\mathbb{Z}) \overset{f_*}{\to} H_{d_X-k}(Y,\mathbb{Z}) \overset{\cong}{\to} H^{k+d_Y-d_X}(Y,\mathbb{Z}).$$

The next result shows how the class map defined in Proposition 1.20 is compatible with all of the operations defined on the Chow group in Proposition 1.9 and the operations of push-forward, pull-back and cup product defined on singular cohomology.

Proposition 1.22. *Let $f : X \to Y$ be a morphism of smooth complex projective varieties of complex dimension d_X and d_Y respectively.*

(i) *If f is proper and $V \in CH_k(X)$, then*

$$cl(f_*(V)) = f_*(cl(V)) \in H^{2d-2d_Y}(Y,\mathbb{Z}).$$

(ii) *If $V \in CH_k(Y)$ then*

$$f^*(cl(V)) = cl(f^*(V)) \in H^{2d_Y-2k}(X,\mathbb{Z}).$$

(iii) *If we consider $CH(X) := \bigoplus_k CH^k(X)$ and $H(X,\mathbb{Z}) := \bigoplus_i H^i(X,\mathbb{Z})$ as graded rings with respect to intersection product product $\cdot$ and cup product $\cup$ respectively, the class map is a graded ring homomorphism, so that for $V \in CH^k(X)$ and $V' \in CH^l(X)$ we have*

$$cl(V \cdot V') = cl(V) \cup cl(V') \in H^{2k+2l}(X,\mathbb{Z}).$$

(iv) *Using to i), ii) and iii) we obtain that the class map is compatible with correspondences, so that if $\Gamma \in CH_k(X \times Y)$, then for any $Z \in CH_l(Y)$ we have*

$$cl(\Gamma^*(Z)) = [\Gamma]^*(cl(Z)) \in H^{2(d_X-d_Y-k-l)}(Y,\mathbb{Z})$$

where $[\Gamma]^* : H^{2r}(Y,\mathbb{Z}) \to H^{2t}(X,\mathbb{Z})$, $t = r + d_X - k$ is defined by

$$[\Gamma]^*(\alpha) = pr_{X*}(pr_Y{}^*(\alpha) \cup [\Gamma]).$$

Proof. See [27, Prop. 9.20, Prop. 9.21]. □

Thanks to the existence of this class map we can relate algebraic cycles to Hodge classes. Consider the Hodge decomposition(see [9, pag. 116])

$$H^{2k}(X,\mathbb{C}) \cong \bigoplus_{r+s=2k} H^{r,s}(X), \quad k = 1,\dots,d.$$

We define

$$Hdg^k(X) := H^{2k}(X,\mathbb{Z}) \cap j^{-1}(H^{k,k}(X))$$

where $j : H^{2k}(X,\mathbb{Z}) \to H^{2k}(X,\mathbb{C}) = H^{2k}(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$ is the natural map.

One can easily show that the image of the class map cl_k is contained in $Hdg^k(X)$ (see [9, pag. 162-163]).

The famous Hodge conjecture states that the image of the class map is all of $Hdg^k(X)$.
The only case this conjecture is proved is for divisors (this is the Lefschetz (1,1) Theorem, cf. [9, pag. 163]) and then, using the duality given by the Hard Lefschetz Theorem, also for curves.

Remark 1.23. In the case of divisors we have the map

$$cl_{d-1} = c_1 : CH_{d-1}(X) = Pic(X) \to H^2(X,\mathbb{Z})$$

and the group $Hdg^{d-1}(X) = Im(cl_{d-1}) = Im(c_1)$ is called *Néron-Severi* group $NS(X)$.
Since X is a complex projective variety then $H^2(X,\mathbb{Z})$ is a finitely generated abelian group (see [12, pag. 527]) and so also $NS(X) \subseteq H^2(X,\mathbb{Z})$ is finitely generated. The rank of its free part is called *Picard number*.

The cycle map allows us to introduce another 'good' equivalent relation for algebraic cycles called *homological equivalence*. We say that a cycle $\alpha \in \mathcal{Z}_k(X)$ is homologically equivalent to zero if $cl_k(\alpha) = 0 \in H^{2d-2k}(X,\mathbb{Z})$ and we denote by $\mathcal{Z}_k(X)_{hom}$ the subgroup of cycles homologically equivalent to zero. By Proposition 1.20 we have that $\mathcal{Z}_k(X)_{rat} \subseteq \mathcal{Z}_k(X)_{hom}$ and then we have

$$CH_k(X)_{hom} = \mathcal{Z}_k(X)_{hom}/\mathcal{Z}_k(X)_{rat} \subset CH_k(X).$$

For completeness we conclude the section mentioning another 'good' equivalent relation called *numerical equivalence*. We say that a cycle $\alpha \in \mathcal{Z}_k(X)$ is numerically equivalent to zero if $deg(\alpha \cdot \beta) = 0$ for every $\beta \in \mathcal{Z}_{d-k}(X)$.

Remark 1.24. The integer $deg(\alpha \cdot \beta) \in \mathbb{Z}$ is called the intersection number of α and β and it is sometimes denoted as $\sharp(\alpha \cdot \beta)$.

We call $\mathcal{Z}_k(X)_{num}$ the subgroup of k-cycles numerically equivalent to zero. Because of the compatibility of intersection product with the cup product of the corresponding cohomology classes, we have that $\mathcal{Z}_k(X)_{hom} \subseteq \mathcal{Z}_k(X)_{num}$.

Remark 1.25. In the case of divisors, so for $k = d-1$, the group $\mathcal{Z}_k(X)/\mathcal{Z}_k(X)_{num} = CH_{d-1}/\mathcal{Z}_k(X)_{num}$ is called $Num(X)$ and is isomorphic to the free part of the Néron-Severi group $NS(X)$, so that it is a free group of rank equal to the Picard number of X. In the case of $X = S$ smooth projective surface, considering the bilinear *non-degenerate* (by definition of Num) symmetric product $Num(S) \times Num(S) \to \mathbb{Z}$ given by the intersection number $\sharp(\alpha \cdot \beta)$, with $\alpha, \beta \in Num(S)$, one has an important theory called cone theory (see for example [15]).

1.4 Intermediate Jacobians and Abel-Jacobi maps

In this section we want to recall some background about the concepts of intermediate Jacobians and Abel-Jacobi maps and show the compatibility between rational equivalence and the Abel-Jacobi maps.

Let X be a smooth projective complex manifold of dimension d. Recall the Hodge decomposition (see [9, pag. 116])

$$H^k(X,\mathbb{C}) \cong \bigoplus_{r+s=k} H^{r,s}(X), \quad H^{r,s}(X) = \overline{H^{s,r}}$$

and the corresponding Hodge filtration

$$F^i H^k(X,\mathbb{C}) = \bigoplus_{r\geq i} H^{r,k-r} = H^{k,0}(X) + H^{k-1,1}(X) + \cdots + H^{i,k-i}(X)$$

where we denoted by $H^{r,s}(X) := H^s(X,\Omega^r_X)$ the sheaf cohomology groups of Ω^r_X, which is the sheaf of holomorphic r-forms on X.
Recall that these groups, by the abstract de Rham theorem (see [29, Th. 3.13]), can be computed using the Dolbeault resolution of Ω^r_X, so that an element of $H^s(X,\Omega^r_X)$ can be represented by a $\overline{\partial}$-closed form of (r,s)-type. This implies also that $H^{r,s}(X) = 0$ for r or s greater than d since the space of (r,s)-type forms is clearly empty if r or s are greater than d.

Definition 1.26. The *p-th intermediate Jacobian of X* is defined as

$$J^{2p-1}(X) := H^{2p-1}(X,\mathbb{C})/(F^pH^{2p-1}(X,\mathbb{C}) + H^{2p-1}(X,\mathbb{Z})).$$

So writing

$$V := H^{2p-1}(X,\mathbb{C})/F^pH^{2p-1}(X,\mathbb{C}) \cong H^{p-1,p}(X) + \cdots + H^{0,2p-1}$$

we have that

$$J^{2p-1}(X) = V/H^{2p-1}(X,\mathbb{Z}).$$

Lemma 1.27. *$J^{2p-1}(X)$ is a complex torus of dimension $dimJ^{2p-1}(X) = \frac{1}{2}b_{2p-1}(X)$, where $b_{2p-1}(X)$ is the $(2p-1)$-th Betti number of X.*

Proof. First note that, since $H^{r,s}(X) = \overline{H^{s,r}}$. Then $b_{2p-1}(X)$ is even and so $\frac{1}{2}b_{2p-1}(X)$ is an integer number.
To prove the lemma we have to show that the image of $H^{2p-1}(X,\mathbb{Z})$ is a lattice in the complex vector space V. The Hodge filtration on $H^{2p-1}(X,\mathbb{C})$ gives the decomposition as a direct sum

$$H^{2p-1}(X,\mathbb{C}) = F^pH^{2p-1}(X,\mathbb{C}) \oplus \overline{F^pH^{2p-1}(X,\mathbb{C})} = \overline{V} \oplus V.$$

Thus $F^pH^{2p-1}(X,\mathbb{C}) \cap H^{2p-1}(X,\mathbb{R}) = \{0\}$ and the composition map

$$H^{2p-1}(X,\mathbb{R}) \to H^{2p-1}(X,\mathbb{C})/F^pH^{2p-1}(X,\mathbb{C})$$

is an isomorphism of $\mathbb{R}$-vector spaces.

Therefore the lattice

$$H^{2p-1}(X,\mathbb{Z}) \subset H^{2p-1}(X,\mathbb{R})$$

gives a lattice in the $\mathbb{C}$-vector space $H^{2p-1}(X,\mathbb{C})/F^pH^{2p-1}(X,\mathbb{C})$.

$\square$

Remark 1.28. By Poincaré duality we have that $H^{2p-1}(X,\mathbb{C}) \cong H^{2d-2p+1}(X,\mathbb{C})^*$ and by Serre duality (cf. [9, pag. 153]) we have that $H^{r,s}(X) \cong H^{d-r,d-s}(X)^*$. We have then that V is the dual of $F^{d-p+1}H^{2d-2p+1}(X,\mathbb{C})$. Indeed

$$V = H^{p-1,p}(X)+\cdots+H^{0,2p-1}(X) \cong H^{d-p+1,d-p}(X)^*+\cdots+H^{d,d-2p+1}(X)^* = F^{d-p+1}H^{2d-2p+1}(X,\mathbb{C})^*.$$

Then we can write $J^{2p-1}(X)$ as

$$J^{2p-1}(X) = F^{d-p+1}H^{2d-2p+1}(X,\mathbb{C})^*/H^{2p-1}(X,\mathbb{Z}).$$

One can show that the injection of the lattice $H^{2p-1}(X,\mathbb{Z}) \hookrightarrow F^{d-p+1}H^{2d-2p+1}(X,\mathbb{C})^*$ is given by the integration of forms. Explicitly, as in Definition 1.18 of the Poincaré dual, if $\Lambda \in H^{2p-1}(X,\mathbb{Z})$, it induces a well-defined linear functional $\int_\Lambda(-) \in F^{d-p+1}H^{2d-2p+1}(X,\mathbb{C})^*$ given by $\omega \mapsto \int_\Lambda \omega'$, where $\omega \in F^{d-p+1}H^{2d-2p+1}(X,\mathbb{C})$ is represented by a closed $\mathcal{C}^\infty$-differential form ω' of degree $2d-2p+1$ (see [26, Section 12.1.2] for more details).

Even if the complex torus $J^{2p-1}(X)$ is in general not an Abelian variety, in the case $p=1$ and $p=d$ it certainly is. Indeed if $p=1$ one can show, using the long exact sequence induced by the exponential sequence, that $J^1(X)$ is the Picard variety $Pic^0(X)$ of X which is an abelian variety. We can then write

$$Pic^0(X) = J^1(X) = H^{0,1}(X)/H^1(X,\mathbb{Z}) = H^1(X,\mathcal{O}_X)/H^1(X,\mathbb{Z}).$$

In the case $p=d$ we obtain an Abelian variety called $Alb(X) := J^{2d-1}(X)$ the Albanese variety of X (see [26, Cor. 12.12]).

Since the concept of Albanese variety will be fundamental for Chapter 6, we now focus on the Albanese variety of a projective complex manifold X.

Recall that $H^{r,s}(X) = H^s(X,\Omega^r_X)$, where Ω^r_X is the sheaf of holomorphic r-form on X, so that $H^{r,s}(X)=0$ for r or s greater than $d=dim(X)$. Then, for $p=d$, we have that

$$H^{2d-1}(X,\mathbb{C}) = H^{d,d-1}(X) + H^{d-1,d}(X).$$

Hence we can write $Alb(X) = J^{2d-1}(X)$ as

$$Alb(X) = H^{2d-1}(X,\mathbb{C})/(H^{d,d-1}(X) + H^{2d-1}(X,\mathbb{Z})) = H^{d-1,d}(X)/H^{2d-1}(X,\mathbb{Z}).$$

Note now that, by Remark 1.28, we have $H^{d-1,d}(X) \cong H^{1,0}(X)^*$, and, by Poincaré duality, we have also that $H^{2d-1}(X,\mathbb{Z}) \cong H_1(X,\mathbb{Z})$. Then we can view the Albanese variety of X also as

$$Alb(X) = H^{1,0}(X)^*/H_1(X,\mathbb{Z}) = H^0(X,\Omega^1_X)^*/H_1(X,\mathbb{Z})$$

where we have explicitly, as in Remark 1.28, that the injection of the lattice $i: H_1(X,\mathbb{Z}) \hookrightarrow H^0(X,\Omega^1_X)^*$ is the map defined by $i(\gamma): \omega \mapsto \int_\gamma \omega$, for $\gamma \in H_1(X,\mathbb{Z})$ and $\omega \in H^0(X,\Omega^1_X)$.

Remark 1.29. In the case of a K3 surface S, by definition of K3 surface we have that the irregularity $q = h^{1,0} := dim(H^{1,0})$ is zero, so that $Alb(S) = 0$. This will be a key point in the proof of Theorem 6.1, the central theorem of Chapter 6. It is also a fundamental point for the proof of Theorem 4.1 about the torsion freeness of $CH_0(S)$ (though we will not present a proof here).

We define now the Abel-Jacobi maps AJ_X^{2p-1}. Recall that $\mathcal{Z}_k(X)_{hom}$ is the group of k-algebraic cycles homologically equivalent to zero.

Proposition 1.30. *There exists a homomorphism, for any $p = 1, \dots, d$,*

$$AJ_X^{2p-1} : \mathcal{Z}_{d-p}(X)_{hom} \to J^{2p-1}(X)$$

called Abel-Jacobi map and it factors through $\mathcal{Z}_{d-p}(X)_{rat}$, so that we obtain a map, which we call in the same way,

$$AJ_X^{2p-1} : CH_{d-p}(X)_{hom} \to J^{2p-1}(X).$$

Proof. We now give an outline of the construction of the map AJ_X^{2p-1} (see [26, Section 12.1.2] for more details). As in Remark 1.28, consider $J^{2p-1}(X) = F^{d-p+1}H^{2d-2p+1}(X, \mathbb{C})^*/H^{2p-1}(X, \mathbb{Z})$. Let $Z \in \mathcal{Z}_{d-p}(X)_{hom}$ be a cycle homologically equivalent to zero, so that $cl_{d-p}(Z) = 0 \in H^{2p}(X, \mathbb{Z})$. By Poincaré duality,

$$PD_{2p} : H^{2p}(X, \mathbb{R}) = H^{2p}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} \overset{\cong}{\to} H_{2d-2p}(X, \mathbb{R})$$

and then, since $cl_{d-p}(Z) = 0 \in H^{2p}(X, \mathbb{Z})$, we have that there exists a differentiable $(2d - 2p + 1)$-chain Λ such that

$$\partial\Lambda = PD_{2p}(cl_{d-p}(Z)).$$

We can now view Λ as a functional on $F^{d-p+1}H^{2d-2p+1}(X, \mathbb{C})$.
Indeed, if $\omega \in F^{d-p+1}H^{2d-2p+1}(X, \mathbb{C})$, then it is represented by a closed $\mathcal{C}^\infty$-differential form ω' of degree $2d - 2p + 1$, and so we can define the functional

$$AJ_X^{2p-1}(Z) : \omega \mapsto \int_\Lambda \omega'.$$

The choice of Λ determines then an element of $F^{d-p+1}H^{2d-2p+1}(X, \mathbb{C})^*$ and hence also of the intermediate Jacobian $J^{2p-1}(X)$. One can show that this element does not depend on the choice of ω' in the cohomology class ω and so the Abel-Jacobi map is a well-defined map. The only thing left is to show that the map is independent from the choice of Λ. Now if we have another Λ' such that $\partial\Lambda' = PD_{2p}(cl_{d-p}(Z))$, then $\partial(\Lambda - \Lambda') = 0$ and so the induced functional $\int_\Lambda - \int_{\Lambda'}$ is in the image of i, where $i : H^{2p-1}(X, \mathbb{Z}) \hookrightarrow F^{d-p+1}H^{2d-2p+1}(X, \mathbb{C})^*$ is the injection of the lattice. Then the functionals induced by Λ and Λ' define the same element in $J^{2p-1}(X)$.

For the fact that this map factors through $CH_{d-p}(X)_{hom}$ see [27, Lem. 9.19]. □

In the case of $p = d$, so when we are considering the Albanese variety $Alb(X) = J^{2d-1}(X)$, we call the Abel-Jacobi map *Albanese map*

$$alb_X := AJ_X^{2d-1} : CH_0(X)_{hom} \to Alb(X)$$

where $CH_0(X)_{hom}$ is the group generated by 0-cycles of degree zero modulo rational equivalence. The explicit form of this map, recalling that $Alb(X) = H^0(X, \Omega_X^1)^*/H_1(X, \mathbb{Z})$, is given by integration, so that on a generator of $CH_0(X)_{hom}$ of the form $x - x_0 \in CH_0(X)_{hom}$, for $x, x_0 \in X$, the map is given by

$$Alb_X(x - x_0) : \omega \mapsto \int_{x_0}^{x} \omega$$

where $\omega \in H^0(X, \Omega_X^1)$ and $\int_{x_0}^{x} \omega$ is well-defined up to elements of the form $\int_\delta \omega$, where δ is the topological chain representing an element of $H_1(X, \mathbb{Z})$.

Remark 1.31. In the case of $X = C$ curve, the Albanese variety is also called *Jacobian variety* and denoted by $\mathcal{J}(C)$. In this case the Albanese map, called Jacobi map,

$$alb_C : CH_0(C)_{hom} = Pic^0(C) \to Alb(C) = \mathcal{J}(C)$$

is an isomorphism (cf. Theorem 2.10).

If we fix a point $x_0 \in X$, we obtain a map

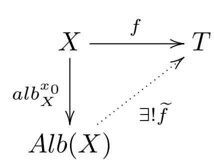
$$alb_X^{x_0} : X \to Alb(X), \quad x \mapsto AJ_X^d(x - x_0)$$

that we call again Albanese map. One can show that this map is holomorphic (cf. [26, Th. 12.4]). Moreover one can show, using the fact that the Albanese variety is a torus, that the map induced by $alb_X^{x_0}$ at level of 1-forms $alb_X^{x_0 *} : H^0(Alb(X), \Omega^1_{Alb(X)}) \to H^0(X, \Omega_X^1)$ is an isomorphism (see [1, Th. V.13]).

We have the following characterisation of the Albanese morphism $alb_X^{x_0}$.

Proposition 1.32. *Let X be a smooth projective complex variety. The Albanese morphism satisfies the following universal property:*

for any complex torus T and any morphism $f : X \to T$, there exists a unique morphism of complex tori $\widetilde{f} : Alb(X) \to T$ such that the following diagram commutes



From this universal property it follows immediately that the Albanese variety is functorial, i.e. if $f : X \to Y$ is a morphism of smooth projective varieties, there exists a unique morphism $alb(f) : Alb(X) \to Alb(Y)$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle alb_X^{x_0}}\downarrow & & \downarrow{\scriptstyle alb_Y^{y_0}} \\ Alb(X) & \xrightarrow{alb(f)} & Alb(Y) \end{array}$$

is commutative.

Proof. See [1, Th. V.13]. □

Note that if $X = S$ is a K3 surface, so $Alb(S) = 0$ by Remark 1.29, the universal property of the Albanese morphism implies that every morphism from S to a complex torus is trivial, i.e. the image is reduced to a point.

We want to conclude this section mentioning a result on the algebraic group $Alb(X)$.

Proposition 1.33. *The image $alb_X^{x_0} : X \to Alb(X)$ generates the torus $Alb(X)$ as a group. More precisely, for sufficiently large k, the morphism*

$$alb_X^{x_0,k} : X^k \to Alb(X), \quad (x_1, \ldots, x_k) \mapsto \sum_i alb_X^{x_0}(x_i),$$

where the right hand sum is taken using the group structure of the torus $Alb(X)$, is surjective.

Proof. See [26, Lem. 12.11]. □

1.5 Finite dimensionality of the Chow group and Mumford's theorem

In this section we will introduce the concept of infinite dimensionality for the Chow group of zero cycles of a complex projective variety. The rough idea is that we define the Chow group of zero cycles of a variety to be infinite dimensional if it cannot be parametrized by a variety.

We will also state an important result by Mumford which says that the Chow group of zero cycles of a surface S with $H^{2,0}(S) \neq 0$ is infinite dimensional. In particular, this result implies that the Chow group of zero cycles of a K3 surface, for which $H^{2,0}(S) \cong \mathbb{C}$, is infinite dimensional. This is one of the reasons why it is hard to study it.

In the end we will mention the existence of an important conjecture which claims the converse of Mumford's theorem.

Let X be a smooth complex projective variety of dimension d.

Assume that X is connected, so that $H_0(X, \mathbb{Z}) = \mathbb{Z}$. Thus the class map cl_0 is equal to the degree map

$$cl_0 = deg : CH_0(X) \to H_0(X, \mathbb{Z}), \quad Z = \sum_i n_i p_i \mapsto \sum_i n_i$$

where $p_i \in X$ and $n_i \in \mathbb{Z}$.

Consider now the k-symmetric product of X, denoted by $X^{(k)}$, defined by the quotient of X^k by the action of the symmetric group Σ_k. One can show that $X^{(k)}$ is a complex singular variety of dimension dk and it parametrizes the unordered sets of d points of X. We can then consider a

0-cycle $Z = \sum_{i=1}^{k} p_i \in \mathcal{Z}_0(X)$ of degree k as an element of $X^{(k)}$. We will denote by $[Z] \in CH_0(X)$ the class of Z modulo rational equivalence.

Definition 1.34. We say that $CH_0(X)$ is representable if the map

$$\sigma_k : X^{(k)} \times X^{(k)} \to CH_0(X)_{hom}, \quad (Z_1, Z_2) \mapsto [Z_1] - [Z_2]$$

is surjective for sufficiently large $k \in \mathbb{N}$, where $CH_0(S)_{hom} = Ker(deg) \subset CH_0(S)$ is the subgroup of cycles of degree 0.

Remark 1.35. In the above definition we have considered the subgroup $CH_0(X)_{hom}$ for the image of the σ_d because it is the 'big' part of the group $CH_0(X)$. Indeed, after fixing a point $x_0 \in X$, we can consider the decomposition

$$CH_0(X) = CH_0(X)_{hom} \oplus \mathbb{Z}$$

induced by the map $cl_0 = deg$. It is given by writing any $x \in CH_0(X)$ as $x = (x - deg(x)x_0) + deg(x)x_0$, with $x - deg(x)x_0 \in Ker(deg)$ and $deg(x)x_0 \in \mathbb{Z}x_0$. Hence we have

We now state a Lemma about the map σ_k, that will be useful in the proof of Proposition B in Chapter 6.

Lemma 1.36. *The fibres of σ_k are countable unions of closed algebraic subsets of $X^{(k)} \times X^{(k)}$. Moreover there exists a countable union B of proper algebraic subsets of $X^{(k)} \times X^{(k)}$ such that, for $x \in X^{(k)} \times X^{(k)} - B$, the dimension of the fibre $r := dim(\sigma_k^{-1}(\sigma_k(x)))$ is constant.*

Proof. See [27, Lem 10.7].

□

We define the dimension of $Im\sigma_k$ to be equal to $dim(Im\sigma_k) := dim(X^{(k)} \times X^{(k)}) - r = 2kd - r$, where r is defined above. Now we can state the precise definition of finite dimensionality for $CH_0(X)$.

Definition 1.37. We say that $CH_0(X)$ is infinite dimensional (in Roitman's sense) if

$$lim_{k \to \infty} dim(Im\sigma_k) = \infty$$

and finite dimensional otherwise.

We now state some equivalent definitions of the the concept of finite dimensionality of the Chow group of 0-cycles of a smooth complex projective variety X.

Proposition 1.38. *The following are equivalent:*

(i) *The group $CH_0(X)$ is finite dimensional.*

(ii) *The group $CH_0(X)$ is representable.*

(iii) *For every smooth curve $C = Y_1 \cap \cdots \cap Y_{d-1}$ which is a complete intersection of ample hypersurfaces $Y_i \subset X$, letting $j : C \hookrightarrow X$ be the inclusion of C in X, the map*

$$j_* : CH_0(C)_{hom} = Pic^0(C) \to CH_0(X)_{hom}$$

is surjective.

(iv) *There exists a smooth projective variety W and a correspondence $\Gamma \in CH_l(W \times X)$, with $l = dim(W)$, such that $\Gamma_* : CH_0(W) \to CH_0(X)_{hom}$ and*

$$CH_0(X)_{hom} = \{\Gamma_*(w),\ w \in W\}.$$

Proof. **(i)**$\Leftrightarrow$ **(ii)** See [27, Prop. 10.10].

(ii)$\Leftrightarrow$ **(iii)** See [27, Prop. 10.12].

(ii)$\Leftrightarrow$ **(iv)** For (ii) implies (iv), let $k \in \mathbb{N}$ be such that

$$\sigma_k : X^{(k)} \times X^{(k)} \to CH_0(X)_{hom}, \quad (x_1 + \cdots + x_k, x_{k+1} + \cdots + x_{2k}) \mapsto \sum_{i=1}^{k} x_i - \sum_{i=k+1}^{2k} x_i$$

is surjective.
Take then $W = X^{2k}$ and consider, for $i = 1, \ldots, 2k$, the maps

$$(p_i, p) : W \times X \to X \times X, \quad ((x_1, \ldots, x_{2k}), x) \mapsto (x_i, x)$$

where $((x_1, \ldots, x_{2k}), x) \in W \times X$.
Denote now by $\Gamma \in CH_{2kd}(W \times X)$ the correspondence

$$\Gamma := \sum_{i=1}^{k} (p_i, p)^*(\Delta) - \sum_{i=k+1}^{2k} (p_i, p)^*(\Delta)$$

where $\Delta \subset X \times X$ is the diagonal. Then

$$\Gamma_*(x_1, \ldots, x_{2k}) = p_*(p_W^*(x_1, \ldots, x_{2k}) \cdot \Gamma) = \sum_{i=1}^{k} x_i - \sum_{i=k+1}^{2k} x_i \in CH_0(X)_{hom}$$

where $(x_1, \ldots, x_{2k}) \in W = X^{2k}$ and $p : W \times X \to X$ is the projection. Hence $Im(\sigma_k) = Im(\Gamma_*)$. But σ_k is surjective and so we have that $\{\Gamma_*(w),\ w \in W\}$ is equal to all of $CH_0(X)_{hom}$.

Now we want to prove that (iv) implies (ii). Let W be a smooth projective variety and let $\Gamma \in CH_k(W \times X)$ such that $CH_0(X)_{hom} = \{\Gamma_*(w),\ w \in W\}$. Since Γ is an l-cycle in $W \times X$ we can write

$$\Gamma = \Gamma^+ - \Gamma^- = \sum_{i=1}^{h^+} n_i V_i - \sum_{j=1}^{h^-} n_j V_j$$

where $n_i, n_j \in \mathbb{Z} > 0$ and V_i, V_j are subvarieties of dimension l in $W \times X$.
If $w \in W$, then $p_W^*(w) = [X \times \{w\}] \in CH_d(X \times W)$ and so the cohomology classes associated to $p_W^*(w_1)$ and $p_W^*(w_2)$ are equal for any $w_1, w_2 \in W$. Hence the degrees of the 0-cycles in $\{p_W^*(w) \cdot V,\ w \in W\} \in CH_0(W \times X)$ are all equal to an integer $s_V < \infty$, for any l-dimensional variety $V \subset W \times X$.
We have thus that the degrees of the 0-cycles in $\{p_W^*(w) \cdot \Gamma^+,\ w \in W\} \in CH_0(W \times X)$ are equal to the integer $k := \sum_{i=1}^{h^+} n_i \cdot s_{V_i} < \infty$. Hence also the degrees of the 0-cycles in

$$\{p_{X*}(p_W^*(w) \cdot \Gamma^+),\ w \in W\} = \{\Gamma_*^+(w),\ w \in W\}$$

are equal to k.
Now, since $Im(\Gamma_*)$ is contained in $CH_0(X)_{hom}$, we have that $deg(\Gamma_*(w)) = 0$ for any $w \in W$, and so $deg(\Gamma_*^+(w)) = deg(\Gamma_*^-(w))$ for any $w \in W$. Thus also the degree of the 0-cycles in $\{\Gamma_*^-(w),\ w \in W\}$ are equal to k.
We have then the map

$$\sigma_k : X^{(k)} \times X^{(k)} \twoheadrightarrow \{\Gamma_*^+(w) - \Gamma_*^-(w),\ w \in W\} = \{\Gamma_*(w),\ w \in W\} \subseteq CH_0(X)_{hom}$$

is surjective.
But $\{\Gamma_*(w),\ w \in W\} = CH_0(X)_{hom}$ and so $CH_0(X)$ is representable.

□

The definition of finite dimensionality given by Proposition 1.38 (iv) will be the one used in Chapter 6.

The term 'representability' can be justified by the following theorem of Roitman, which shows that if $CH_0(X)$ is representable then it is an algebraic group.

Theorem 1.39. *Let X be a smooth complex projective variety.*
If $CH_0(X)$ is representable, then the Albanese map

$$alb_X : CH_0(X)_{hom} \to Alb(X)$$

is injective and, since alb_X is always surjective, it is an isomorphism.
Without the representability condition we have only that the Albanese map induces an isomorphism on torsion points.

Proof. See [27, Th. 10.11] and [27, Th. 10.14]. □

When $CH_0(X)$ is representable, it is easier to study the Chow group of zero cycles of X because in this case it is enough to study the Albanese variety of X. Unfortunately there are many varieties X with infinite dimensional Chow group of zero cycles. Indeed the next result implies that, using the definition of finite dimensionality of Proposition 1.38 (iii), every smooth complex projective variety X with $H^0(X, \Omega_X^k) \neq 0$ for some $k \geq 2$, have an infinite dimensional $CH_0(X)$.

Theorem 1.40. *Let X be a smooth d-dimensional complex variety. If there exists a subvariety $j : X' \hookrightarrow X$, with $dimX' < k$, such that the map $j_* : CH_0(X') \to CH_0(X)$ is surjective, then $H^0(X, \Omega^k_X) = 0$.*

Proof. The proof of Theorem 1.40 is quite elaborate and it will not be presented here. We refer to [27, Section 10.2.2].
We only mention that the key of the proof is an important result by Bloch and Srinivas. It establishes that, if X is as in the statement of Theorem 1.40, i.e. it is a smooth complex variety and there exists a subvariety $j : X' \hookrightarrow X$ such that the map $j_* : CH_0(X') \to CH_0(X)$ is surjective, then there exists a decomposition of the diagonal $\Delta \subset X \times X$:

$$m\Delta = Z_1 + Z_2 \in CH^d(X \times X)$$

where Z_2 is supported in $T \times X$ for $T \subset X$ proper algebraic subset and $m \in \mathbb{Z} > 0$. □

Remark 1.41. A direct consequence of Theorem 1.40 is that, if X is a a smooth projective variety of dimension d with $CH_0(X)_{hom} = 0$, then $H^0(X, \Omega^k_X) = 0$ for all $k > 0$.
One can prove a generalisation of this last statement which establish that a correspondence $\Gamma \in CH_d(X \times Y)$, which induces the 0-map $\Gamma_* : CH_0(X)_{hom} \to CH_0(Y)_{hom}$, has the property that the maps $[\Gamma]^* : H^i(Y, \mathbb{C}) \to H^i(X, \mathbb{C})$ vanish on $H^{i,0}(Y) \subset H^i(Y, \mathbb{C})$ for any $i > 0$, where $[\Gamma]^*$ is defined as in Proposition 1.22 (iv) (see [27, Lem. 11.26] for a more general result or [27, Prop. 11.18] for the case of surfaces).

Theorem 1.40 is a generalisation of a famous result by Mumford about the infinite dimensionality of the Chow group of zero cycles of surfaces that have at least one non-zero global 2-form.

Theorem 1.42 (Mumford)**.** *Let S be a smooth projective complex surface such that $H^{2,0}(S) \neq 0$. Then the group $CH_0(S)$ is infinite dimensional.*

Proof. If we use Theorem 1.40 with $X = S$ and $k = 2$, we obtain that, if $H^0(X, \Omega^2_X) \neq 0$ then there does not exist any subvariety of $j : C \hookrightarrow S$, with C of dimension one, such that the map $j_* : CH_0(C) \to CH_0(S)$ is surjective. Then using the definition of finite dimensionality given in Proposition 1.38 (iii), we obtain that $CH_0(S)$ is infinite dimensional. □

In particular Mumford's theorem applies to a K3 surface S, for which we have the simplest case of $H^{2,0}(S)$ different from zero since $H^{2,0}(S) \cong \mathbb{C}$. Thus in this case $CH_0(S)$ cannot be parametrized by a variety and so it is a difficult object to study. In fact in Chapters 4, 5 and 6 we will present some methods of approach to its study.

We conclude the section mentioning the famous Bloch conjecture. Since in the next chapters we will work on surfaces, we state a version of this conjecture for surfaces, which is the converse to Mumford's theorem.

Conjecture 1.43. *Let S be a smooth projective complex surface such that $H^{2,0}(S) = 0$. Then the Albanese map*

$$alb_S : CH_0(S)_{hom} \to Alb(S)$$

is an isomorphism.

This conjecture is proved for surfaces which are not of general type (see [27, Ch. 11]).
We only show here that Conjecture 1.43 holds for the simplest example of surfaces S satisfying $H^{2,0}(S) = 0$, which are ruled surfaces, i.e. surfaces birationally equivalent to $C \times \mathbb{P}^1$, where C is a smooth curve. For such surfaces we have more generally that $H^0(S, K_S^{\otimes n}) = 0$ for any $n > 0$ (see [1, Prop. III.21]).
Ruled surfaces clearly satisfy Conjecture 1.43 because they are covered by rational curves. Indeed if S is a ruled surface, every point $x \in S$ belongs to a rational curve $l \cong \mathbb{P}^1 \subset S$ which intersects a fixed ample curve D in a point $y \in D \cap l$. Then by the definition of rational equivalence given in Proposition 1.8 (iii), we have that x is rationally equivalent to y, and so the map $j_* : CH_0(D) \to CH_0(S)$ is surjective, where $j : D \hookrightarrow S$ is the inclusion.
Hence by Proposition 1.38 (ii) we have that $CH_0(S)$ is finite dimensional and so by Theorem 1.39 the Albanese map alb_S is an isomorphism for the ruled surface S.

Remark 1.44. For ruled surfaces, or more in general for any surface S with $H^{1,0}(S) = 0$ (so that $Alb(S) = 0$) which satisfies Conjecture 1.43, we have that $CH_0(S)_{hom} = 0$.

One can generalise Conjecture 1.43 considering correspondences. In Remark 1.41 we mentioned a generalisation of a consequence of Theorem 1.40, which states that if X and Y are smooth projective complex varieties with $d := dim(X)$ and there exists a correspondence $\Gamma \in CH_d(X \times Y)$ which induces the 0-map $\Gamma_* : CH_0(X)_{hom} \to CH_0(Y)_{hom}$, then $[\Gamma]^* : H^i(Y, \mathbb{C}) \to H^i(X, \mathbb{C})$ vanishes on $H^{i,0}(Y) \subset H^i(Y, \mathbb{C})$ for any $i > 0$. A more general version of Bloch conjecture is a sort of converse to the above statement. The version with Y of dimension two is the following.

Conjecture 1.45. *Let S be a smooth projective complex surface, let X be a smooth projective complex variety of dimension d and let $\Gamma \in CH_d(X \times S)$ be a correspondence such that the maps $[\Gamma]^* : H^{i,0}(S) \to H^{i,0}(X)$ vanish for $i > 0$, where $[\Gamma]^*$ is defined as in as in Proposition 1.22 (iv). Then*

$$\Gamma_* : CH_0(X)_{alb} \to CH_0(S)$$

vanishes, where $CH_0(X)_{alb} := Ker(alb_X : CH_0(X)_{hom} \to Alb(X))$.

In Chapter 6 we will present Voisin's article [25] in which it is proved that symplectic involutions on K3 surfaces act trivially on the Chow group of zero cycles. We will show that indeed it is a particular case of Conjecture 1.45.

We conclude by mentioning that Conjecture 1.43 and 1.45 are only particular cases of a more general Bloch conjecture. If X is a smooth projective variety, it claims the existence of a decreasing filtration on $CH_0(X)_{\mathbb{Q}} = CH_0(X) \otimes \mathbb{Q}$ which satisfies some functorial properties (cf. [27, Ch. 11]).

2. Linear systems on complex manifolds and results about curves

In this chapter we want to fix the notation for linear systems on complex manifolds and recall some basic results about compact Riemann surfaces that we will frequently use in the next chapters.
We will also introduce the Prym variety associated to a degree two covering of a curve, which will be a key element for the proof of Proposition A in Chapter 6.
We will then show some results about the Chow group of a curve, among which there will be the famous theorem of Abel, which states that the Albanese morphism $alb_C : Pic^0(C) \to Alb(C)$ is an isomorphism.

2.1 Linear systems on complex projective manifolds

In this section we recall some concepts and results about linear systems on a smooth complex projective manifolds in order to fix the notation that will be the same of the rest of the paper.

Let X be a smooth complex projective manifold. Let $D \in Div(X)$ be a divisor and let $[D] \in Pic(X)$ be the image of D in the Picard group $Pic(X) = H^1(X, \mathcal{O}_X^*)$ of X.
First of all recall that for a smooth complex projective manifold X, Weil divisors can be identified with Cartier divisors (see [10, Ch. II, Prop. 6.11] or [9, pag.132]). Moreover, thanks to the projectivity condition for X, we have also that the space of divisors modulo linear equivalence fills all of the Picard group of X, i.e. $Pic(X) = Div(X)/\sim$ ([cf. [9, pag. 161]), where $\sim$ is the linear equivalence relation.
Using Čech cohomology, one can show that $Pic(X)$ can be identified with the space of line bundles on X modulo line bundle isomorphisms (see [9, pag.133]). Thus we can consider the sheaf of sections $\mathcal{O}_X(D)$ of the line bundle $[D] \in Pic(X)$ associated to D.
If $H \subset \mathbb{P}^n$ is the hyperplane section of $\mathbb{P}^n$, we will denote by $\mathcal{O}_{\mathbb{P}^n}(m) := \mathcal{O}_{\mathbb{P}^n}(mH)$ the sheaf of sections of the line bundle associated to the divisor mH.

Let $L(D) := \{f \in \mathcal{M}(X) : \ D + (f) \geq 0\}$ be the Riemann-Roch space associated to D, where $\mathcal{M}(X)$ is the space of meromorphic functions on X and (f) is the divisor associated to f. Then we have an isomorphism $L(D) \cong H^0(X, \mathcal{O}_X(D))$ (see [9, pag. 136]).
Moreover, if $|D| := \{D' \sim D, \ D' \geq 0 \in Div(X)\} \subset Div(X)$ is the the set of all effective divisors linearly equivalent to D, we have isomorphisms

$$|D| \cong \mathbb{P}(L(D)) \cong \mathbb{P}(H^0(X, \mathcal{O}_X(D))$$

(see [9, pag. 137]). In general, the family of effective divisors on X corresponding to a linear subspace of $\mathbb{P}(H^0(X, \mathcal{O}_X(D)))$ is called *linear system*; a linear system is called complete linear system if it is of the form $|D|$. When we speak of dimension of a linear system, we will refer to the dimension of the projective space parametrizing it, thus in particular $dim|D| = h^0(X, \mathcal{O}_X(D)) - 1$. The common intersection of the divisors in a linear system is called *base locus* of the system; in particular, if the base locus contains a divisor F, we call F a *fixed component* of the linear system. A linear system that has an empty base locus is called a *base point free* linear system.

If $E \subset H^0(X, \mathcal{O}_X(D))$ defines a base point free linear system $|E| = \mathbb{P}(E)$ of dimension N, we call

$$\phi_E : X \to \mathbb{P}(E)^* \cong \mathbb{P}^N$$

the map induced by E, that is the map that sends a point $x \in X$ to the set of all sections $s \in E \subset H^0(X, \mathcal{O}_X(D))$ that vanish at x (or equivalently to the set of divisors $D \in |E| \subset |D|$ containing x), which is an hyperplane $H_x \in \mathbb{P}(E)^*$ in $|E|$. One can describe the map ϕ_E more explicitly. In terms of the identification $\mathbb{P}(E)^* \cong \mathbb{P}^N$ corresponding to a choice of a basis $s_0, \dots, s_N \in H^0(X, \mathcal{O}_X(D))$ for E, the map ϕ_E is given by

$$\phi_E(x) = [s_0(x) : \cdots : s_N(x)] \in \mathbb{P}^N$$

for $x \in X$.

From this representation we see that ϕ_E is holomorphic. Moreover we have that $[D] = \phi_E^*(\mathcal{O}_{\mathbb{P}^N}(1))$ and $E = \phi_E^*(H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)))$. Then we have find a 1:1 correspondence between the the set

$$\{[D] \in Pic(X),\ E \subset H^0(X, \mathcal{O}_X(D)) \text{ with } |E| \text{ base point free}\}$$

and the set

$$\{\text{non degenerate maps } f : X \to \mathbb{P}^N,\ \text{modulo projective transformation}\}.$$

Note that the degree of $\phi_E(X)$, that is the intersection of X with $m := dim(\phi_E(X))$ general hyperplanes, is just $deg(\phi_E(X)) = c_1([D])^m$.

We say that $[D] \in Pic(X)$ is *very ample* if $\phi_{|D|}$ is an embedding, while we say that $[D]$ is *ample* if exists $n \in \mathbb{N}$ such that $[D]^{\otimes n}$ is very ample.

The next result we want to recall is the Bertini's theorem, which states that the generic element of a linear system is smooth away the base locus of the system. We say that a property holds for the generic element of a linear system if it holds for every element in a non-empty Zariski open subset of the system.

Theorem 2.1 (Bertini)**.** *Let X be a smooth projective variety and let $|E|$ be a linear system on X. Then the generic element of $|E|$ is smooth away the base locus of $|E|$. Moreover if the induced morphism $\phi_E : X \to \mathbb{P}^N$ is such that $dim(\phi_E(X)) \geq 2$, then the generic element of $|E|$ is also irreducible.*

Proof. See [10, Ch. II, Th. 8.18] and [10, Ch. III, Cor. 10.9]. □

Thanks to Bertini's theorem we can now prove a proposition that we will use in Proposition 4.2 and Proposition 2.15.

Proposition 2.2. *Let X be a smooth projective variety of dimension $d > 1$. Then for every two points $x, y \in X$ pass an irreducible curve $C \subset X$. Moreover C can be singular only in x and y.*

Proof. Consider the blow-up $\nu : \widetilde{X} \to X$ of X in x and y. Call $E_x \cong \mathbb{P}^{d-1}$ and $E_y \cong \mathbb{P}^{d-1}$ the exceptional divisors. Since X is projective then also $\widetilde{X}$ is projective (see [26, Prop. 3.24]). Let then H be a very ample line bundle on $\widetilde{X}$ and let $\phi_H : \widetilde{X} \hookrightarrow \mathbb{P}^N$ be the induced embedding.
By Bertini's theorem 2.1, if $D \in |H|$ is a generic hyperplane section of $\widetilde{X}$, then D is smooth and irreducible. Moreover, since D and the exceptional divisors are of codimension one in $\mathbb{P}^N$, we have that $D \cap E_x$ and $D \cap E_y$ are non-empty.
Hence we have that $D' := \nu(D) \subset X$ is an irreducible hypersurface of X containing x and y. Besides, since D is smooth and ν is an isomorphism outside E_x and E_y, $\nu(D)$ can be singular only in the points $\nu(E_x) = x$ and $\nu(E_y) = y$.

Applying the previous argument inductively we obtain an irreducible curve $C \subset X$ with $x, y \in C$ and smooth outside x and y. □

We conclude the section recalling the definition of the Euler characteristic of a sheaf.
If $\mathcal{F}$ is a sheaf of $\mathcal{O}_S$-modules on X, we will denote by

$$\chi(\mathcal{F}) := \sum_{i=0}^{dim(X)} (-1)^i h^i(X, \mathcal{F})$$

the Euler characteristic of $\mathcal{F}$, where $h^i(X, \mathcal{F}) := dim(H^i(X, \mathcal{F}))$. Note that this is a finite number since we have $H^i(X, \mathcal{F}) = 0$ for $i > dim(X)$ (cf. [10, Ch. III, Th. 2.7]) and $h^i(X, \mathcal{F}) < \infty$ (cf. [10, Ch. II, Th. 5.19]).

2.2 Results about Riemann surfaces

In this section we recall two classical results about compact Riemann surfaces, i.e. smooth complex projective irreducible curves, that we will often use in the next chapters. We will only cite them and we refer, for the proofs and a more complete treatment of the subject, to the book by Griffiths and Harris [9, *Ch.*2] and to Hartshorne's book [10, Ch. IV].

The following result is the Riemann-Hurwitz formula, which gives a way to 'count' the ramification points of a map between Riemann surfaces. It is also generalizable to the case of maps between varieties of dimension greater then one (see [3, pag. 41]).

Theorem 2.3 (Riemann-Hurwitz)**.** *Let X and Y be compact Riemann surfaces and let $f : X \to Y$ be a holomorphic map of degree n. Then*

$$2g(X) - 2 = n \cdot (2g(Y) - 2) + degR_f$$

where $g(X)$ and $g(Y)$ are the genera of X and Y, R_f is the ramification divisor of degree

$$degR_f = \sum_{p \in X} mult_p(f) - 1,$$

where $mult_p(f)$ is the multiplicity of f at p.

Proof. See [10, Ch. IV, Cor. 2.4]. ☐

Using the Riemann-Hurwitz formula, one can directly prove the following.

Corollary 2.4 (Clebsch Formula)**.** *If $C \subset \mathbb{P}^2_{\mathbb{C}}$ is a non-singular curve then $g(C) = \frac{(d-1)(d-2)}{2}$, where $d = deg(C)$ is the degree of C.*

Proof. See [9, pag. 220]. ☐

Now we want to look at linear systems on curves and the maps induced by them.

For a curve C, one has the famous Riemann-Roch theorem in order to compute the dimension of a complete linear system $|D|$ given by a divisor D on C.

Theorem 2.5 (Riemann-Roch)**.** *Let C be a compact Riemann surface of genus $g := g(C)$ and let D be a divisor on C of degree $d := deg(D)$. Then*

$$\chi(\mathcal{O}_C(D)) = h^0(C, \mathcal{O}_C(D)) - h^1(C, \mathcal{O}_C(D)) = d + 1 - g$$

where $h^i(C, \mathcal{O}_C(D)) = dim(H^i(C, \mathcal{O}_C(D)))$.

Proof. See [10, Ch. IV, Th. 1.3]. ☐

One of the many consequences of this theorem is the following result about rational curves.

Corollary 2.6. *If C is of genus equal to zero, then it is biholomorphic to $\mathbb{P}^1$.*

Proof. See [10, Ch. IV, Example 1.3.5]. ☐

2.3 Prym varieties.

We want here to give a quick view of Prym varieties following [13, Ch. 6.1]. We will meet them in a key point of the proof of Proposition A in Chapter 6.

Let $\pi : \widetilde{C} \to C$ be an unbranched irreducible double covering of Riemann surfaces. We call $g(C) := g + 1$ the genus of C.

By the Riemann-Hurwitz formula $2 - 2g(\widetilde{C}) = 2(2 - g(C))$ so we have

$$g(\widetilde{C}) = 2g(C) - 1.$$

Lemma 2.7. *The étale double coverings of C are in 1:1 correspondence with the elements of $H_1(C, \mathbb{F}_2) - \{0\}$. Moreover $\beta_\pi \in H_1(C, \mathbb{F}_2) - \{0\}$ is the element that corresponds to the étale double covering $\pi : \widetilde{C} \to C$ if and only if, denoting again by β_π a topological loop representing the element $\beta_\pi \in H_1(C, \mathbb{F}_2) - \{0\}$, we have that the loop $\pi^{-1}(\beta_\pi)$ disconnects $\widetilde{C}$.*

Proof of Lemma 2.7. Fix $p_0 \in C$.
Then the étale double coverings of C are in 1:1 correspondence with the subgroups of index 2 of $\pi_1(C, p_0)$. We associate to an étale double covering π, the subgroup of index two $\pi_*(\pi_1(\widetilde{C}, \widetilde{p_0})) \subset \pi_1(C, p_0)$, where $\widetilde{p_0} \in \pi^{-1}(p_0)$ (cf. [12, Th. 1.38]).

Since the subgroup $\pi_*(\pi_1(\widetilde{C}, \widetilde{p_0})) \subset \pi_1(C, p_0)$ contains the commutator subgroup of $\pi_1(C, p_0)$, as well as the subgroup of squares of $\pi_1(C, p_0)$, then the étale double coverings of C are indeed in 1:1 correspondence with the subgroups of index two of $H_1(C, \mathbb{F}_2)$.

Being $\mathbb{F}_2$ a field, by Poincarè duality (cf. Theorem 1.17, 3.), the intersection pairing is non degenerate on $H_1(C, \mathbb{F}_2)$; then the subgroups of index 2 of $H_1(C, \mathbb{F}_2)$ are in 1:1 correspondence with the elements of $H_1(C, \mathbb{F}_2) - \{0\}$. We associate to $\beta \in H_1(C, \mathbb{F}_2) - \{0\})$ the index two subgroup given by $\beta^{\perp} = \{\alpha \in H_1(C, \mathbb{F}_2) : \ \beta \cdot \alpha = 0\}$.

So we have found that the étale double coverings of C are in 1:1 correspondence with the elements of $H_1(C, \mathbb{F}_2) - \{0\}$.

Moreover $\beta_\pi \in H_1(C, \mathbb{F}_2) - \{0\}$ is the element that corresponds to the étale double covering $\pi : \widetilde{C} \to C$ if and only if $\beta_\pi^{\perp} \subset H_1(C, \mathbb{F}_2)$ is equal to the subgroup of index 2 given by $\pi_*(H_1(\widetilde{C}, \mathbb{F}_2))$.

This happens if and only if $\pi^{-1}(\beta_\pi)$ disconnects $\widetilde{C}$, where we denoted again by β_π a loop representing the element $\beta_\pi \in H_1(C, \mathbb{F}_2) - \{0\}$. □

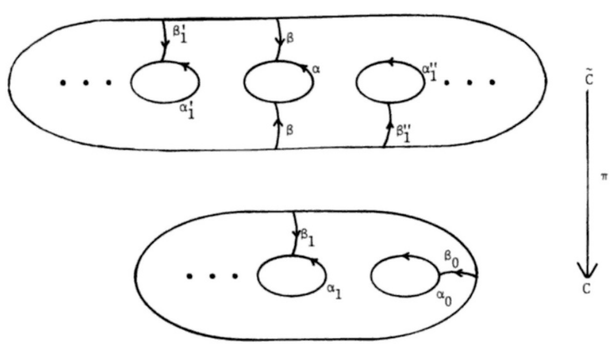
Let $\alpha_0, \dots, \alpha_g, \beta_0, \dots, \beta_g$ be a symplectic base of $H_1(C, \mathbb{Z})$ such that β_π is the image of β_0 under the map $H_1(C, \mathbb{Z}) \to H_1(C, \mathbb{F}_2)$, where β_π corresponds to the given double covering π. Then by Lemma 2.7, denoting again by β_0 the loop representing $\beta_0 \in H_1(C, \mathbb{Z})$, we have that $\pi^{-1}(\beta_0)$ disconnects $\widetilde{C}$.

We can now build a symplectic base of cardinality $2g(\widetilde{C}) = 4g(C) - 2 = 4g + 2$ on $H_1(\widetilde{C}, \mathbb{Z})$ (see [13] for more details):

$$\beta,\ \alpha,\ \beta'_1, \dots,\ \beta'_g,\ \beta''_1, \dots,\ \beta''_g,\ \alpha'_1, \dots,\ \alpha'_g,\ \alpha''_1, \dots,\ \alpha''_g$$

such that

$$\pi_*(\beta) = \beta_0,\ \pi_*(\alpha) = 2\alpha_0,\ \pi_*(\beta'_j) = \pi_*(\beta''_j) = \beta_j,\ \pi_*(\alpha'_j) = \pi_*(\alpha''_j) = \alpha_j.$$



Then we have that $\{\alpha'_j - \alpha''_j,\ \beta'_j - \beta''_j\}_{j=1,\dots,g}$ is a basis for $H_1(\widetilde{C},\mathbb{Z})^-$, the skew symmetric part of $H_1(\widetilde{C},\mathbb{Z})$ with respect to the involution i_*.

Let now $i : \widetilde{C} \to \widetilde{C}$ be the covering involution with respect to the étale double covering $\pi : \widetilde{C} \to C$. Then $i^* : H^{1,0}(\widetilde{C}) \to H^{1,0}(\widetilde{C})$ has eigenvalues ± 1.

Let $H^{1,0}(\widetilde{C})^-$ be the -1-eigenspace of $H^{1,0}(\widetilde{C})$ with respect to i_*.

Definition 2.8. The Prym variety of the double covering $\pi : \widetilde{C} \to C$ is defined to be

$$\mathcal{P}(\widetilde{C}/C) := \frac{H^{1,0}(\widetilde{C})^-}{H_1(\widetilde{C},\mathbb{Z})^-}.$$

Since $H^{1,0}(\widetilde{C})^+ \cong H^{1,0}(C)$, we have $dim(H^{1,0}(\widetilde{C})^+) = g(C)$.
We have also that $dim(H^{1,0}(\widetilde{C})) = g(\widetilde{C}) = 2g(C) - 1$.
So $dim(H^{1,0}(\widetilde{C})^-) = dim(H^{1,0}(\widetilde{C})) - dim(H^{1,0}(\widetilde{C})^+) = g(C) - 1 = g$.
Hence we have

$$dim(\mathcal{P}(\widetilde{C}/C)) = g(C) - 1 = g.$$

Remark 2.9. Consider the decomposition in eigenspaces of $H^{1,0}(\widetilde{C}) = H^{1,0}(\widetilde{C})^- \oplus H^{1,0}(\widetilde{C})^+$ and the injection $H_1(\widetilde{C},\mathbb{Z})^- \hookrightarrow H_1(\widetilde{C},\mathbb{Z})$, we obtain an injective morphism

$$\mathcal{P}(\widetilde{C}/C) \hookrightarrow J(\widetilde{C})$$

i.e. the Prym variety is a subvariety of $J(\widetilde{C})$.

2.4 Abel's theorem and Chow group for curves

In the case of smooth curves, the Chow group is not a complicated object. If C is a smooth curve of genus $g := g(C)$, one has clearly $CH_1(X) \cong \mathbb{Z}$ and, by the smoothness condition, we have also $CH_0(X) \cong Pic(X)$ (see Remark 1.6).
The cycle map cl_0 defined in Proposition 1.20 (that is also the first Chern class map c_1 for divisors) is the degree map

$$cl_0 = c_1 = deg : CH_0(X) = Pic(X) \to \mathbb{Z} \cong H^2(C,\mathbb{Z}), \quad Z = \sum_i n_i p_i \mapsto \sum_i n_i$$

where $p_i \in C$ and $n_i \in \mathbb{Z}$. Recall the kernel of this map $CH_0(C)_{hom} = Ker(cl_0) = Ker(deg)$ is called Picard variety $Pic^0(C)$ of C.

In Proposition 1.30 we have defined the Abel-Jacobi maps AJ^{2p-1}. In the case of curves the only non-zero intermediate Jacobian variety is the Albanese variety $Alb(C)$ so that the only non-zero Abel-Jacobi map is the Albanese morphism

$$alb_C : CH_0(C)_{hom} = Pic^0(C) \to Alb(C).$$

Recall that in Section 1.4 we have noted that we can write

$$Alb(C) = H^0(C,\Omega^1_C)^*/H_1(C,\mathbb{Z}).$$

In the case of curves, the Albanese variety is also called Jacobian variety $\mathcal{J}(C)$ for historical reasons. More specifically, the Jacobian variety $\mathcal{J}(C)$ is the Albanese variety after the choice of an isomorphism $Alb(C) \cong \mathbb{C}^g/\Gamma$ determined by the choice of a basis for $H^0(C, \Omega^1_C)$.
If $\omega_1, \dots, \omega_g \in H^0(C, \Omega^1)$ is a basis for the space of holomorphic 1-forms on C and $\delta_1, \dots, \delta_{2g} \in H_1(X, \mathbb{Z})$ is a canonical basis for $H_1(X, \mathbb{Z})$, one can consider the elements

$$\Pi_i := (\int_{\delta_i} \omega_1, \dots, \int_{\delta_i} \omega_g) \in \mathbb{C}^g$$

for every $i = 1, \dots, 2g$, which are called *periods*.

One can show that they are linearly independent over $\mathbb{R}$ so that they generate a lattice

$$\Gamma = \{m_1\Pi_1 + \dots + m_{2g}\Pi_{2g}, m_i \in \mathbb{Z}\} \subset \mathbb{C}^g.$$

If we define the Jacobian variety

$$\mathcal{J}(C) := \mathbb{C}^g/\Gamma,$$

it is clear that it is exactly the Albanese variety after an isomorphism induced by the choice of a basis.

In Section 1.4 we have found the explicit form of the Albanese map

$$Alb_C : CH_0(C)_{hom} \to Alb(C) = H^0(C, \Omega^1_C)^*/H_1(C, \mathbb{Z}), \quad p - q \mapsto (Alb_C(p-q) : \omega \mapsto \int_q^p \omega)$$

for $p, q \in C$ and $\omega \in H^0(C, \Omega^1_C)$.

One can show that the Albanese morphism, after the choice of a basis $\omega_1, \dots, \omega_g \in H^0(C, \Omega^1)$ that gives an isomorphism $Alb(C) \cong \mathcal{J}(C)$, assumes the form

$$alb_C : CH_0(C)_{hom} \to \mathcal{J}(C) = \mathbb{C}^g/\Gamma, \quad p - q \mapsto (\int_q^p \omega_1, \dots, \int_q^p \omega_g)$$

for $p, q \in C$.

The famous theorem by Abel claims that the Albanese morphism alb_C is indeed an isomorphism.

Theorem 2.10 (Abel)**.** *The albanese morphism*

$$alb_C : CH_0(C)_{hom} = Pic^0(C) \to \mathcal{J}(C) \cong Alb(C)$$

is an isomorphism.

Proof. See [9, Ch. 2, Section 2] for the more analytic and historical proof or [26, Cor. 12.8] for a more algebraic approach.

□

To conclude the section we want to show that the Chow group of 0-cycles on a curve C of genus $g := g(C)$ is representable, in the sense of Section 6.3. This result will be useful in order to prove a property of $CH_0(C)$ whose generalisation will be needed in the proofs of Proposition A and B in Chapter 6.

Let Σ_k be the k-th symmetric group and $C^{(k)} = C^k/\Sigma_k$ be the k-th symmetric product of C, i.e. the variety that parametrizes the effective divisors of C of degree k.

Then, if we fix a point $q_0 \in C$, we have the map

$$\alpha : C^{(k)} \to Pic^0(C), \qquad \alpha(p_1 + \cdots + p_k) = \sum_{i=1}^{k} p_i - kq_0.$$

Lemma 2.11. *If $k \geq g$ then α is surjective. Moreover, if $k \geq 2g-1$ and D is an effective divisor of degree k, then $\alpha^{-1}(\alpha(D)) = |D|$ and $dim(|D|) = k - g$, so the dimension of the fibre does not change varying D.*

Proof of Lemma 2.11. Let $x \in Pic^0(C)$, i.e. x is a divisor in $Pic(C)$ with $deg(x) = 0$. Then $D + kq_0$ has degree equal to k.

In order to prove that α is surjective, we need $x = D - kq_0 \in Pic^0(C)$, or equivalently $D = x + kq_0 \in Pic^0(C)$, for some effective divisor $D \in C^{(k)}$. This is equivalent to say that the linear system of effective divisors linearly equivalent to $x + kq_0$, has dimension greater or equal to 0. That is to say that $h^0(C, \mathcal{O}_C(x + kq_0)) > 0$. But by Riemann-Roch theorem, $h^0(C, \mathcal{O}_C(x + kq_0)) \geq k + 1 - g$ and so if $k \geq g$ then $h^0(C, \mathcal{O}_C(x + kq_0)) \geq 1$. Hence we have found that, for $k \geq g$, α is surjective. Now look at the fibres of α for $k \geq g$. Let $x \in Pic^0(C)$. Since α is surjective, we have $x = \alpha(D)$ for some $D \in C^{(k)}$ and so

$$\alpha^{-1}(x) = \alpha^{-1}(\alpha(D)) = \{D' \in C^{(k)} : \; D' - kq_0 \sim D - kq_0\} = |D| = \mathbb{P}^{h^0(C, \mathcal{O}_C(D)) - 1}.$$

Now if $deg(D) \geq 2g - 1$ then $deg(K_C - D) < 0$ and so $h^1(C, \mathcal{O}_C(D)) = h^1(C, \mathcal{O}_C(K_C - D)) = 0$. Then by Riemann-Roch, $h^0(C, \mathcal{O}_C(D)) = deg(D) + 1 - g = k + 1 - g$. So if we take $k \geq 2g - 1$ we have that the dimension of the fibres of α is constant and equal to $k - g$.

□

Remark 2.12. A direct consequence of Lemma 2.11 is that $CH_0(C)$ is representable and so finite dimensional. Because of this fact, we have that Abel's theorem can be obtained as a particular case of Theorem 1.39.

In order to prove the next proposition we need to state a Baire category result. It will be used also in the proof of Proposition B in Chapter 6.

Lemma 2.13 (Baire category argument)**.** *If Y is a locally compact Hausdorff space space then it is a Baire space and so if $Y = \bigcup_i Y_i$ is a countable union of closed subsets, then there is i such that Y_i is dense in Y.*

Now we are ready to prove an interesting property of the Chow group of zero cycles of a curve. It states that every 0-cycle on a curve is linearly equivalent to a 0-cycle with support outside of a countable union of closed subset.

Proposition 2.14. *Let C be a curve and let $N = \bigcup_{i \in I} \overline{p}_i \subseteq C$ be a countable union of points. Then, for any $\overline{p} \in N$, there exists some $p_i \in C - N$ such that $\overline{p} = \sum_i n_i p_i$ in $CH_0(C)$, i.e. any 0-cycle in $CH_0(C)$ is linearly equivalent to a 0-cycle with support in $C - N$.*

Proof. Let $\alpha : C^{(k)} \to Pic^0(C)$ be defined as above with $q_0 \in C - N$ and $k = 2g$. By Lemma 2.11 we have that $dim(\alpha^{-1}(x)) = g$ is constant for any $x \in Pic^0(C)$.

Let $\overline{p}+C^{(2g-1)} \subset C^{(2g)}$ be the image in $C^{(2g)}$ of the set $\{(p_1, \ldots, p_k) \in C^{2g} : \ p_i = \overline{p} \text{ for some } i\}$.

Note that $\alpha(\overline{p} + C^{(2g-1)}) = Pic^0(C)$ because $2g - 1 \geq g$ and so $\alpha_{|C^{(2g-1)}}$ is still surjective. Denote by

$$|D|_{\overline{p}} := (\overline{p} + C^{(2g-1)}) \cap \alpha^{-1}(\alpha(D)) \subset \alpha^{-1}(\alpha(D))$$

the intersection of $\overline{p} + C^{(2g-1)}$ with the fibre over D.

Then $|D|_{\overline{p}} = \{D' \in C^{(2g-1)} : \ \overline{p} + D' \sim D\} \cong |D - \overline{p}|$ is of dimension $g - 1$ by Riemann-Roch.

Summarizing we have

$$\alpha^{-1}(\alpha(D)) \cong |D| \cong \mathbb{P}^g \ \text{ and } \ |D|_{\overline{p}} \cong |D - \overline{p}| \cong \mathbb{P}^{g-1}.$$

The key point here is that $\alpha^{-1}(\alpha(D))$ cannot be equal to $\bigcup_{i\in I} |D|_{\overline{p_i}}$ with I countable set. Indeed if they were equal then, by a Baire category argument (see Lemma 2.13), one of the $|D|_{\overline{p_i}} \cong \mathbb{P}^{g-1}$ would be dense in $\alpha^{-1}(\alpha(D)) \cong \mathbb{P}^g$ but this is a contradiction because $\mathbb{P}^{g-1}$ cannot be dense in $\mathbb{P}^g$.

Therefore $\alpha^{-1}(\alpha(D)) - \bigcup_{i\in I} |D|_{\overline{p_i}} = \{D' \in C^{(2g)} : \ D' \sim D, \ D' \notin \overline{p_i} + C^{(2g-1)} \text{ for any } i\} = \{D' \in C^{(2g)} : \ D' \sim D, \ \overline{p_i} \notin D' \text{ for any } i\}$ is non-empty.

So we have proved that

$$\text{for any } x \in Pic^0(C), \text{ there exists } D' \in \alpha^{-1}(x) \text{ such that } \overline{p_i} \notin D' \text{ for any } i.$$

Then, taking $x = \overline{p} - q_0 \in Pic^0(C)$, we have that there exists an effective divisor D' on C of degree $2g$ with $\alpha(D') = \overline{p} - q_0 \in Pic^0(C)$ and $\overline{p_i} \notin D'$. But by definition of α we have also $\alpha(D') = D' - 2gq_0 \in Pic^0(C)$. Hence $\overline{p} = D' - (2g - 1)q_0 \in Pic^0(C)$, with $q_0 \notin N$ and $\overline{p_i} \notin D'$ for any i.

Thus we have proved that $\overline{p}$ is linearly equivalent to a cycle supported on $C - N \subset U$ and then Proposition 2.14 is proven.

□

As we mentioned above, a generalisation of this result will be used in the proof of Proposition A and B in Chapter 6. Since it is a direct corollary of Proposition 2.14, we state it here.

Proposition 2.15. *Let Y be a connected complex projective variety. Let $U \subset Y$ be the complement of a countable union of proper closed algebraic subsets Z_i. Then any 0-cycle of Y is rationally equivalent to a 0-cycle supported on U.*

Proof of Proposition 2.15. If Y is of dimension one then the proposition follows by Proposition 2.14. Let then $dim(Y) > 1$. Let

$$U = Y - \bigcup_{i\in I} Z_i$$

be the complement of a countable union of proper closed algebraic subsets Z_i and let $\overline{p} \in Y - U$. Fix moreover a point $q \in U$.

By Proposition 2.2 we can find an irreducible curve $C \subset Y$ such that $\overline{p} \in C$ and $q \in C \cap U \neq \emptyset$. Moreover C is smooth outside $\overline{p}$ and q.

Call

$$N := C \cap (\bigcup_{i \in I} Z_i) = \bigcup_{i \in I} (C \cap Z_i)$$

where I is a countable set. Then N consists of a countable union of points

$$\overline{p_i} := C \cap Z_i$$

and $\overline{p} \in N$.

Now consider the normalisation $\nu : \widetilde{C} \to C$ of C obtained by blowing-up x and y (see [19, Sec. 8.1.4] for the process of normalisation with blow-ups). If we call $\widetilde{N} := \nu^{-1}(N) \subset \widetilde{C}$, we have that $\widetilde{N}$ is a countable set of points.

By Proposition 2.14 we know that every 0-cycle in $\widetilde{C}$ is linearly equivalent to a cycle with support in $\widetilde{C} - \widetilde{N}$. Hence, since the push-forward homomorphism $\nu_* : CH_0(\widetilde{C}) \to CH_0(C)$ is compatible with the rational equivalence relation (see Proposition 1.9 (i)), we obtain that every 0-cycle in C is linearly equivalent to a cycle with support in $C - N = U$. So let $\sum_i n_i y_i \in CH_0(Y)$ a 0-cycle of Y. Then, for any $y_i \in Y - U$, we have shown that there exists a curve C_i such that $p_i \in C_i$ is linearly equivalent to a cycle with support in $C_i \cap U$. Hence we have that the 0-cycle $\sum_i n_i y_i \in CH_0(Y)$ is indeed linearly equivalent to a 0-cycle with support in U. □

3. K3 surfaces

In this chapter we want to give some background about algebraic K3 surfaces. We will state some results that we will use in Chapters 4, 5 and 6.
We say that S is an algebraic surface over an algebraically closed field k if it is a separated variety of finite type over k of dimension two.
For general results about algebraic surfaces over an algebraically closed field we refer to Hartshorne's book [10, Ch. V], while for results about complex surfaces we refer to Beauville's book [1].

We now give the definition of algebraic K3 surfaces.

Definition 3.1. An *algebraic K3 surface* over an algebraically closed field k is a projective non-singular surface S such that the canonical bundle of S is trivial and $H^1(S, \mathcal{O}_S) = 0$, where $\mathcal{O}_S$ is the sheaf of regular functions on S.

In this chapter we will consider only surfaces over $k = \mathbb{C}$. We will need however the more general setting in Section 3.4, where we will state some results that we will need in Chapter 5, in which we will work indeed on algebraic K3 surfaces over any algebraically closed field k.
If $k = \mathbb{C}$, by the Chow's Lemma and by the G.A.G.A. principle (look at the beginning of Section 1.3) we can consider a surface over $k = \mathbb{C}$ as a complex projective surface, i.e. a complex connected projective manifold of dimension two. We then restate Definition 3.1 in the complex analytic setting, which will be the one used in this chapter.

Definition 3.2. A *complex projective K3 surface* is a smooth projective complex surface S such that $\Omega^2_S \cong \mathcal{O}_S$ and $H^1(S, \mathcal{O}_S) = 0$, where $\mathcal{O}_S$ is the sheaf of holomorphic functions on S.

One can also give another characterisation of complex projective $K3$ surfaces, the historical one, defining a complex projective K3 surface S as a surface such that every non-singular hyperplane section of S is a canonical curve, i.e. is the image of a canonical embedding. For the equivalence of the two definitions see [22, Section 3.3].

One can moreover show that every complex projective K3 surface is minimal and that K3 surfaces are one of the four types of surfaces with Kodaira dimension equal to zero aside from Enriques surfaces, bielliptic surfaces and Abelian surfaces (see [1, Th. VIII.2]).

3.1 Properties and linear systems

In this next section we will mention some peculiar properties of complex projective K3 surfaces and of linear systems on them. Even if some of the properties presented in this section hold in the more

general case of algebraic K3 surfaces over an algebraically closed field k, we will not specify it but we refer to [15].
Let first fix some notation. Let S be a complex projective surface.

Consider the Hodge decomposition(cf. [9, pag. 116])

$$H^k(S,\mathbb{C}) \cong \bigoplus_{r+s=k} H^{r,s}(S), \quad H^{r,s}(S) = \overline{H^{s,r}(S)}, \quad k = 0,\dots,4.$$

Let $h^{r,s}(S) := dim(H^{r,s}(S))$ be the Hodge numbers of S. By the Hodge decomposition and Serre duality we have that $h^{r,s}(S) = h^{s,r}(S)$ and $h^{r,s}(S) = h^{2-r,2-s}(S)$.

We denote $q_S := h^{1,0}(S) = h^{0,1}(S) = h^{2,1}(S) = h^{1,2}(S)$ and we call it *irregularity* of S. Note that

$$q_S = h^1(\mathcal{O}_S) = dim(H^0(S,\Omega^1_S))$$

and so for a K3 surface S we have that $q_S = 0$.

We call $p_g(S) = h^{2,0} = h^{0,2}$ the *geometric genus* of S. Note that

$$p_g(S) = h^2(\mathcal{O}_S) = dim(H^0(S,\Omega^2_S))$$

and hence for a K3 surface S we have $p_g(S) = 1$.

Note now that

$$\chi(\mathcal{O}_S) = h^0(S,\mathcal{O}_S) - h^1(S,\mathcal{O}_S) + h^2(S,\mathcal{O}_S) = 1 - q_S + p_g(S)$$

so that for a K3 surface we have $\chi(\mathcal{O}_S) = 2$.

Recall also that the Hodge diamond of a surface is given by

$$\begin{array}{ccccccc} b_4 & & & & 1 & & & \\ b_3 & & & q & & q & & \\ b_2 & & p_g & & h^{1,1} & & p_g \\ b_1 & & & q & & q & & \\ b_0 & & & & 1 & & & \end{array}$$

where we denote by $b_i := dim_{\mathbb{C}}(H^i(S,\mathbb{C}))$ the i-th Betti number of S.

We will denote by

$$e(S) := \sum_{i=1}^{4} (-1)^i b_i(S) = 2 - 4q_S + 2p_g(S) + h^{1,1}(S)$$

the topological Euler characteristic of S.

Let now $\cdot : CH_1(S) \otimes CH_1(S) \to CH_0(S)$ be the intersection product defined in Chapter 1. Since S is a surface, we have a symmetric bilinear application, also called intersection product,

$$. : Pic(S) \otimes Pic(S) \to \mathbb{Z}, \quad [D_1].[D_2] = \sharp([D_1]\cdot[D_2]) = deg([D_1]\cdot[D_2])$$

where $[D_1],[D_2] \in Pic(S) \cong CH_1(S)$ (recall that we defined the intersection number $\sharp([D_1]\cdot[D_2])$ of $[D_1],[D_2]$ in Remark 1.24).

One can prove that this intersection product can be given explicitly as

$$[D_1].[D_2] = \chi(S,\mathcal{O}_S) - \chi(S,[D_1]^{-1}) - \chi(S,[D_2]^{-1}) + \chi(S,[D_1]^{-1}\otimes[D_2]^{-1})$$

and one can prove also that, if D_2 is a smooth irreducible curve, we have $[D_1].[D_2] = deg([D_1]_{|D_2})$ (see [1, Ch. I]).

The next lemma is a key useful property of ample line bundles with respect to the intersection product.

Lemma 3.3. *If $H \in Pic(S)$ is ample then $H.C > 0$ for any line bundle C associated to an irreducible curve $C \subset S$. Moreover $H.H > 0$.*

Proof. Let $n \in \mathbb{N}$ such that $H^{\otimes n}$ is very ample. Then if $\phi_{|H^{\otimes n}|} : S \hookrightarrow \mathbb{P}^N$ is the embedding induced by $H^{\otimes n}$, we have that $0 < deg(\phi_{|H^{\otimes n}|}(C)) = nC.H$ so that $C.H > 0$.
Moreover if we denote by nH an element of the complete linear system $|H^{\otimes n}|$, we have $0 < deg(\phi_{|H^{\otimes n}|}(nH)) = n^2 H.H$ and so also $H.H > 0$. □

From now on we will use indistinctly the multiplicative and the additive notation for the group operation in $Pic(S)$ since we will not distinguish the line bundle $[D]$ and the divisor D associated to $[D]$ (we can do this because, since S is projective, $Pic(S) \cong Div(S)/\sim$, where $\sim$ is the linear equivalence relation). We will moreover not distinguish a linear bundle L from its sheaf of sections $\mathcal{O}_S(L)$.

Given this new intersection product, one can formulate the following theorem which collects some key results about surfaces.

Theorem 3.4. *Let S be a projective complex surface.*

1. (Riemann-Roch formula) *For $L \in Pic(S)$ one has*

$$\chi(L) = h^0(S, L) - h^1(S, L) + h^2(S, L) = \chi(\mathcal{O}_S) + \frac{1}{2} L.(L - K_S)$$

where $K_S \in Pic(S)$ is the canonical line bundle.

2. (Noether formula) *We have*

$$12\chi(\mathcal{O}_S) = K_S.K_S + e(S)$$

where $e(S)$ is the topological Euler characteristic of S.

3. (Genus formula) *If $C \subset S$ is an irreducible smooth curve of genus $g(C)$ then*

$$g(C) = 1 + \frac{1}{2}(C.C + C.K_S).$$

4. (Hodge index theorem) *Let H be an ample line bundle on S and suppose that $L \neq 0 \in Pic(S)$ with $L.H = 0$. Then $D^2 < 0$.*
More generally the same holds also if H has only the property of being big, i.e. such that $H.H > 0$.

Proof. For the proofs of 1. and 2. see [9, pag. 472], while 3. is a direct consequence of 1. or of the adjunction formula $K_C = ([C] + K_S)_{|C}$ (see [9, pag. 147]). For the proof of the first part of 4. see [10, Ch. V, Th. 1.9]. The second part of 4. is a direct consequence of a corollary of the first part of the statement (see [15, pag. 13]), which says that, if $L_1, L_2 \in Pic(S)$ with $L_1.L_2 \geq 0$, then $(L_1)^2(L_2)^2 \leq (L_1L_2)^2$ and it is $=$ if and only if $L_1^{\otimes r} = L_2^{\otimes s} \in Pic(S)$ for some $r, s \in \mathbb{Z}$. □

Corollary 3.5 (Formulas of Theorem 3.4 for K3 surfaces)**.** *In the case of projective complex K3 surfaces the first three formulas of Theorem 3.4 have a simpler description. Let then S be a projective complex K3 surface.*

(i) *(Riemann-Roch formula) For* $L \in Pic(S)$ *one has*

$$h^0(S, L) - h^1(S, L) + h^0(S, -L) = \frac{1}{2}L.L + 2.$$

Moreover if L is an ample line bundle with $|L| > 0$ *we have*

$$h^0(S, L) = \frac{1}{2}L.L + 2.$$

(ii) *(Noether formula) We have* $e(S) = 12\chi(\mathcal{O}_S) = 24$.

(iii) *(Genus formula) If* $C \subset S$ *is an irreducible smooth curve of genus* $g(C)$*, we have*

$$g(C) = 1 + \frac{1}{2}C.C.$$

We have then also that for any irreducible smooth curve C on S the autointersection C.C is even and ≥ -2*. The curves C on S such that* $C.C = -2$ *are called* -2*-curves and they are smooth rational curves.*

Proof. (i) Recall that $\chi(\mathcal{O}_S) = 2$. Moreover we know that $K_S \cong \mathcal{O}_S \in Pic(S)$ and so we have $L.(L - K_S) = L.L$. Moreover by Serre duality $h^2(S, L) = h^0(S, K - L) = h^0(S, -L)$ and so we are done for the first part of (i).

For the second part of the statement recall that a line bundle L is trivial if and only if $h^0(S, L) \neq 0$ and $h^0(S, -L) \neq 0$, so that if $L \neq 0 \in Pic(S)$ then we cannot have $h^0(S, L) > 0$ and $h^0(S, -L) > 0$ at the same time. Moreover, by the Kodaira vanishing theorem (cf. [9, pag. 154]), if L is ample then $h^i(S, L + K) = h^i(S, L) = 0$ for any $i \geq 1$. Then, for L is an ample line bundle with $|L| > 0$, we have that the Riemann-Roch formula assumes the form $h^0(S, L) = \frac{1}{2}L.L + 2$.

(ii) Clear since $K_S \cong \mathcal{O}_S \in Pic(S)$ and $\chi(\mathcal{O}_S) = 2$.

(iii) The first part of the statement is clear since $C.K_S = 0$ being K_S trivial. For the second part note that if $C.C = -2$ we have that $g(C) = 0$ and so, by Corollary 2.6, we have that C is biholomorphic to $\mathbb{P}^1$. Hence any -2-curve is a rational curve. □

In Corollary 3.5 (ii), we noted that for a projective complex K3 surface S the topological Euler characteristic $e(S) = 2 - 4q_S + 2p_g(S) + h^{1,1}(S) = 4 + h^{1,1}(S)$ is equal to 24. Then we have

$$h^{1,1}(S) = 20,$$

so that the Hodge diamond for a projective complex K3 surface is given by

$$\begin{array}{lccccc} b_4 & & & 1 & & \\ b_3 & & 0 & & 0 & \\ b_2 & 1 & & 20 & & 1 \\ b_1 & & 0 & & 0 & \\ b_0 & & & 1 & & \end{array}$$

The next result asserts that in a projective complex K3 surface the relations of linear equivalence, homological equivalence and numerical equivalence coincide.
Recall that the Néron-Severi group of S is defined as

$$NS(S) = Pic(S)/Pic^0(S) = CH_1(S)/CH_1(S)_{hom}$$

(see Section 1.3) while $Num(S)$ is defined as $Num(S) = Pic(S)/Pic(S)_{num}$, where we denoted

$$Pic(S)_{num} := CH_1(S)_{num} = \{L \in Pic(S) : \ L.L' = 0 \text{ for any } L' \in Pic(S)\}$$

(see Remark 1.25).

Proposition 3.6. *If S is a K3 surface, then $Pic(S) = NS(S) = Num(S)$.*

Proof. Let $L \in Pic(S)$. It is enough to prove that if $L \neq \mathcal{O}_S \in Pic(S)$ then $L \notin Pic(S)_{num}$. Indeed this is equivalent to have $Pic(S)_{num} = 0$ and, since by the compatibility of intersection product the cup product (see Proposition 1.22 (iii)) we have that $Pic^0(S) \subseteq Pic(S)_{num}$, it is also equivalent to have $Pic^0(S) = 0$.

Let now $L' \in Pic(S)$ be an ample line bundle. If $L \neq \mathcal{O}_S$ but $L.L' = 0$ then we have $h^0(S, L) = 0$. Indeed if would exist an effective divisor linearly equivalent to L, then by Lemma 3.3 we had $L'.L > 0$.
Because of the same argument, we have that also $h^0(S, -L) = 0$. Then, by the Riemann-Roch formula, we obtain $0 \geq -h^1(S, L) = \frac{1}{2}L.L + 2 = 2$ that is a contradiction.

We have then found that $Pic(S)_{num} = Pic^0(S) = 0$ and so $Pic(S) = NS(S) = Num(S)$. □

Now we want to mention two theorems which describe some properties of linear systems on a projective complex K3 surface S.
The first result considers complete linear systems associated to smooth curves on S and gives a description of the morphisms induced by these linear systems.

Proposition 3.7. *Let S be a projective complex K3 surface and $C \subset S$ a smooth irreducible curve of genus g. Then we have:*

(i) $h^0(C) = g + 1$.

(ii) *If $g \geq 1$ the system $|C|$ is base point free. It defines a morphism $\phi_{|C|} : S \to \mathbb{P}^g$ such that the restriction of $\phi_{|C|}$ to C is the canonical morphism*

$$\phi_{|C|_{|C}} = \phi_{|K_C|} : C \to \mathbb{P}^g.$$

(iii) *If $g = 2$ then $\phi_{|C|} : S \to \mathbb{P}^2$ is a morphism of degree 2, whose branch locus is a sextic of $\mathbb{P}^2$.*

(iv) *If $g \geq 3$ then either $\phi_{|C|}$ is a birational morphism or it is a 2:1 morphism to a rational surface of degree $g-1$ in $\mathbb{P}^g$. In this second case a generic curve of $|C|$ is hyperelliptic.*

(v) *If $g \geq 3$ (resp. $g = 2$) then the morphism $\phi_{|2C|}$ (resp. $\phi_{|3C|}$) is birational.*

Proof. See [1, Prop. VIII.13]. □

One can use Proposition 3.7 in order to prove that, for any $g \geq 3$, there exists a K3 surface of degree $2g-2$ in $\mathbb{P}^g$ (see [1, Prop. VIII.15]). We will give examples of such surfaces in Section 3.2, where we will present K3 surfaces of degrees 4, 6 and 8 given as complete intersection of hypersurfaces resp. in $\mathbb{P}^3$, $\mathbb{P}^4$ and $\mathbb{P}^5$.

With the last result of this section, we want to state some properties of linear systems not necessarily determined by a single irreducible curve. We say that a divisor D is *nef* (numerically eventually free) if $D.C \geq 0$ for every curve $C \subset S$ and we say that it is *big* if its autointersection $D.D > 0$ is greater than zero.

Theorem 3.8. *Let S be a projective complex K3 surface.*

(i) *If D is an effective divisor on S then we can subtract an effective sum of -2-curves $F = \sum_i n_i \Gamma_i$ to get an effective and nef (possibly zero) divisor $M = D - F$ with $H^0(S, \mathcal{O}(D)) = H^0(S, \mathcal{O}(M))$.*

(ii) *If D is nef and effective such that $D.D = 0$, then $D = nE$, where $|E|$ is a base point free linear system of dimension one and $n \in \mathbb{N}$ is such that $h^0(D) = n + 1$.*

(iii) *If D is big and nef then $H^1(S, D) = 0$ (this generalises the Kodaira vanishing theorem extending it over big and nef divisors on a K3 surface).*

(iv) *If D is big and nef then either $|D|$ has no fixed component, i.e. it does not exist a divisor contained in the base locus of $|D|$, or $D = nE + \Gamma$, where $|E|$ is a base point free linear system of dimension one and Γ is an irreducible -2-curve such that $E.\Gamma = 1$.*

Proof. See [22, Section 3.8]. □

3.2 Examples of K3 surfaces

In this section we want to show some standard examples of projective complex K3 surfaces.

The simplest examples of surfaces one can think of, concern surfaces given as a complete intersection of hypersurfaces in a projective space. Denote by $S_{n_1,\ldots,n_h}$ a surface in $\mathbb{P}^{h+2}$ given as as a complete intersection of h hypersurfaces of degrees $n_1, \ldots, n_h$.

Proposition 3.9. *The only complete intersection projective complex surfaces with trivial canonical divisor are $S_4 \subset \mathbb{P}^3$, $S_{2,3} \subset \mathbb{P}^4$ and $S_{2,2,2} \subset \mathbb{P}^5$. Moreover they are K3 surfaces.*

Proof. If $S = S_{n_1,\ldots,n_h} \subset \mathbb{P}^{h+2}$ then, by the adjunction formula (see [9, pag.147]), we have that

$$K_S = (\sum_{i=1}^{h} \mathcal{O}_{\mathbb{P}^{h+2}}(n_i) + K_{\mathbb{P}^{h+2}})_{|S} = (\mathcal{O}_{\mathbb{P}^{h+2}}(\sum_{i=1}^{h} n_i - h - 3))_{|S}.$$

Then, since we obviously consider only $n_i > 1$, we have that the only possibilities to have $\sum_{i=1}^{h} n_i = h+3$ give the three surfaces $S_4 \subset \mathbb{P}^3$, $S_{2,3} \subset \mathbb{P}^4$ and $S_{2,2,2} \subset \mathbb{P}^5$.

Now we want to show that they are in particular K3 surfaces, so that we need to find that their irregularity is zero. But in general we have the following:

Lemma 3.10. *If $X \subset \mathbb{P}^N$ is a d-dimensional complete intersection then $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < d$.*

Proof of Lemma 3.10. We proceed by induction on the number of equations defining X.
If $X = \mathbb{P}^N$ the lemma is clear since $H^i(\mathbb{P}^N, \mathcal{O}^N_{\mathbb{P}}) = 0$ for any $i > 0$ (see [9, pag. 118]).
Now let the lemma hold for a $d+1$-dimensional complete intersection $X \subset \mathbb{P}^N$. We want to prove that it holds also for $X' := X \cap V$, where $V \subset \mathbb{P}^N$ is a hypersurface of positive degree.
By the Lefschetz theorem on hyperplane sections (cf. [9, pag. 156]), since X' is an ample hypersurface of X, the map $H^p(X, \Omega^q_X) \to H^p(X', \Omega^q_{X'})$ induced by the inclusion of X' in X is an isomorphism for $p + q < d$ and injective for $p + q = d$. Hence, since by induction hypothesis $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < d+1$, we have that $H^i(X', \mathcal{O}_{X'}) \cong H^i(X, \mathcal{O}_X) = 0$ for $0 < i < d$ and so we are done.

□

Then we have that the irregularity $q = h^{0,1}$ of any surface given as a complete intersection of hypersurfaces in a projective space is zero and so the proposition is proved. □

Another example of projective complex K3 surfaces is given by Kummer surfaces.

Let A be a complex projective Abelian surface, so that it is a projective surface isomorphic to a torus. If we choose an origin, A acquires then a group structure. Let i be the involution on A given by $a \to -a$, for $a \in A$. The non-zero fixed points of i are the points of order two of the group $A \cong (\mathbb{R}/\mathbb{Z})^4$. There are thus 16 fixed points of i in $A \cong (\mathbb{R}/\mathbb{Z})^4$, namely $p_1 = (0,0,0,0), p_2 = (1/2,0,0,0), \ldots, p_{16} = (1/2,1/2,1/2,1/2)$.
Let $\epsilon : \widetilde{A} \to A$ be the blow-up of these 16 points. One can show that the involution i extends to an involution τ of $\widetilde{A}$. Let

$$S := \widetilde{A}/\{1, \tau\}.$$

Then one can show (see [1, Prop. VIII.11]) that S is a K3 surface, called Kummer surface. The proof of the fact that Kummer surfaces are K3 surfaces is analogous to the proof given below in Section 3.3, where we have indeed adapted the proof of [1, Prop. VIII.11] to the case of a symplectic involution on a K3 surface instead of that on an Abelian surface.

Another standard example of projective complex K3 surface is given by a surface S expressed as a double covering $\pi : S \to \mathbb{P}^2$ of the plane ramified over a sextic curve.

Indeed if $C \subset \mathbb{P}^2$ is a smooth curve of degree six, one can construct a double covering of $\mathbb{P}^2$ ramified over C given as a 'cyclic covering' (cf. [3, Ch. I, Section 17]). In general, if $C \subset \mathbb{P}^2$ is of even degree $2d := deg(C)$, fix an isomorphism of line bundles

$$\mathcal{O}_{\mathbb{P}^2}(d) \otimes \mathcal{O}_{\mathbb{P}^2}(d) \to \mathcal{O}_{\mathbb{P}^2}(2d), \text{ locally given by } ((p, \xi_1), (p, \xi_2)) \mapsto (p, \xi_1 \otimes \xi_2)$$

where $p \in \mathbb{P}^2$ and $\xi_1, \xi_2 \in \mathbb{C}$.
Let $s \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$ be a section defining C, so that $(s) = C$, where (s) is the zero locus of s. Then let

$$S := \{(p, \xi) : \xi \otimes \xi = s(p)\} \subset \mathcal{O}_{\mathbb{P}^2}(d).$$

We have that S is a submanifold of the total space of $\mathcal{O}_{\mathbb{P}^2}(d)$ and the projection map $\pi : \mathcal{O}_{\mathbb{P}^2}(d) \to \mathbb{P}^2$ given by the line bundle structure of $\mathcal{O}_{\mathbb{P}^2}(d)$, expresses S as a double covering of $\mathbb{P}^2$ branched exactly along C.

For the example of K3 surface we consider $d = 3$, so that C is of degree six. Now we want to prove that S is a K3 surface.

By the formula for the canonical divisor of a cyclic covering ([3, Ch. I, Lem. 17.1, (iii)]), we have that

$$K_S = \pi^*(K_{\mathbb{P}^2} + \mathcal{O}_{\mathbb{P}^2}(3)) \cong \mathcal{O}_S$$

and so K_S is trivial. Now we need to find that the irregularity $q_S = 0$ is equal to zero. By the Clebsch formula (Corollary 2.4) we have that the genus of C is equal to ten and so the topological Euler characteristic of C is equal to $e(C) = -18$. Then we have that $e(S) = 2e(\mathbb{P}^2) - e(C) = 24$ and by Noether formula (Theorem 3.4,2.)

$$2 - q_S = 1 - q_S + p_g(S) = \frac{1}{12}e(S) = 2.$$

We have then $q_S = 0$ and hence S is a K3 surface.

Note that if, conversely, S is a K3 surface with a double covering $\pi : S \to \mathbb{P}^2$, then by Proposition 3.7 (iii), the branch locus of π is a sextic curve.

To conclude the section we mention that many K3 surfaces can be described as elliptic surfaces $\pi : S \to \mathbb{P}^1$, where π is a surjective morphism and its generic fibre is a smooth elliptic curve. We will meet again this type of K3 surfaces in Section 5.5 of Chapter 5.

3.3 A more involved example: quotient under a symplectic involution

In this section we want to show a more involved example consisting of a projective complex K3 surface obtained from another K3 surface provided with a symplectic involution.

Let S be a projective complex K3 surface and let $i : S \to S$ be a symplectic involution of S. Denote by $\omega \in H^0(S, \Omega^2)$ the nowhere zero 2-form of S.
Let $\pi : S \to \Sigma = S/i$ be the canonical projection and let $b : \widetilde{\Sigma} \to S/i$ be the blow up of the singular points of Σ.

In this section we want to show that $\widetilde{\Sigma}$ is a K3 surface. This fact will be used in particular in the proof of Proposition A in Chapter 6.

Proposition 3.11. *$\widetilde{\Sigma}$ is a smooth K3 surface.*

Proof. We adapt the proof of Proposition VIII.11 in [1], which is about Kummer surfaces, to our case.

First we will show a better way to construct $\widetilde{\Sigma}$.

It is known that there are 8 fixed points p_i for the symplectic involution i (cf. [8]).

Let $\epsilon : \widetilde{S} \to S$ be the blow up of S in $p_1, \dots, p_8$, the 8 fixed points for i, and $E_i := \epsilon^{-1}(p_i)$ the exceptional curves. The involution i extends to an involution τ on $\widetilde{S}$.

Call $\pi' : \widetilde{S} \to \widetilde{S}/\tau$ the projection.

Step 1: $\widetilde{S}/\tau = \widetilde{\Sigma}$.

We are in the following situation:

$$\begin{array}{ccccc} & & \widetilde{S} & \xrightarrow{\epsilon} & S \\ & \swarrow^{\pi'} & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \widetilde{S}/\tau & \overset{\cong ?}{\dashrightarrow} & \widetilde{\Sigma} & \xrightarrow{b} & \Sigma \end{array}$$

We now show that E_i are divisors whose points are fixed points for τ.

Theorem 3.12 (Cartan)**.** *In general, if $\phi : X \to X$ is an automorphism of a complex manifolds X such that $\phi^N = 1$ and $x \in X$ is a fixed point for ϕ, then there exists a local chart $z = (z_1, \dots, z_n)$ centred in x, i.e. with $z(x) = 0$, such that $\phi(z_1, \dots, z_n) = (\zeta_N^{a_1} z_1, \dots, \zeta_N^{a_n} z_n)$ where ζ_N is a primitive N-root of unity and $a_i \in \mathbb{Z}$.*

Proof. See [6]. □

If we consider $p \in \{p_1, \dots, p_8\} \subset S$ one of the fixed points for i, by Theorem 3.12 there exists a local chart (z, U), $z = (x, y)$, centred in p, such that we are in one of the following situations: (i) $i(x, y) = (x, y)$ or (ii) $i(x, y) = (x, -y)$ or (iii) $i(x, y) = (-x, y)$ or (iv) $i(x, y) = (-x, -y)$.

We now prove we are in case (iv).

Case (i) is not our case since, by the identity principle on S compact, if $i = id_S$ on the open set U given by the chart, then $i = id_S$ on all S and this is not true.

Recall now that $\omega \in H^0(S, \Omega^2)$ is the 2-form that is nowhere 0. Let then $\omega = f(x, y)dx \wedge dy$ be the local description of ω in the chart (z, U) around p, where $f(x, y) = a_0 + O(x, y)$ with $a_0 \neq 0$ since ω, and then f, is everywhere non zero on U. If we were in case (ii), then we had $i^*(w) = -f(x, -y)dx \wedge dy = (-a_0 + O(x, y))dx \wedge dy$ that is different from $\omega = (a_0 + O(x, y))dx \wedge dy$ since $a_0 \neq 0$. This is a contradiction since $i^*(\omega) = \omega$ being i a symplectic involution.

Case (iii) is similar and so we are indeed in case (iv).

Then for every $p \in S$ fixed point for i, there exists a local chart (z, U), $z = (x, y)$ centred in p such that $i(x, y) = (-x, -y)$. If we denote $x' = \epsilon^*(x)$, $y' = \epsilon^*(y)$ and consider a point

$q \in E := \epsilon^{-1}(p)$ on an exceptional divisor, we can suppose x' and $t = y'/x'$ are local coordinates near q.

Note now that we have

$$\tau^*(x') = -x' \quad \text{and} \quad \tau^*(t) = (-y')/(-x') = t.$$

Then, since E near q is defined by the points $z = (0, t)$, we have that $i(0, t) = (0, t)$ and so every exceptional divisor E is pointwise fixed by τ.

Thanks to this argument, one can show that $\widetilde{S}/i = \widetilde{\Sigma}$.

Step 2: $\widetilde{\Sigma}$ is a smooth surface.

In order to prove the smoothness of $\widetilde{\Sigma} \cong \widetilde{S}/\tau$, it is enough to prove the smoothness of $\widetilde{\Sigma}$ in the image of the points of $\pi'(E_i)$. Indeed, outside the exceptional divisors, the map π' is étale and so, outside $\pi'(E_i)$, the surface $\widetilde{\Sigma}$ is smooth since $\widetilde{S}$ is smooth.

Let the notation be the as in Step 1. So, if we denote $x' = \epsilon^*(x)$, $x' = \epsilon^*(x)$ and consider a point $q \in E_i$ on an exceptional divisor, we can suppose x' and $t = y'/x'$ are local coordinates near q.

Since $\tau^*(x') = -x'$ and $\tau^*(t) = (-y')/(-x') = t$, then

$$u = {x'}^2 \quad \text{and} \quad t' = t \quad \text{form a system of local coordinates on } \widetilde{\Sigma} \text{ around } \pi'(q).$$

Then we have that $\widetilde{\Sigma}$ is a smooth surface.

Step 3: $\Omega^2_{\widetilde{\Sigma}} \cong \mathcal{O}_{\widetilde{\Sigma}}$

Since $i^*(\omega) = \omega$ then, ϵ being a blow up, also $\epsilon^*(\omega)$ is invariant under τ and then $\epsilon^*(\omega) = {\pi'}^*(\alpha)$, where α is a meromorphic 2-form on $\widetilde{\Sigma}$ (see [1, Lem VI.11]).

Since ω is nowhere zero on S, the zeroes of α must be concentrated in $\pi'(E_i)$.

We set us in the same notation as in Step 1, so that we have a local chart (x, y) around p_i, fixed point for i, where $i(x, y) = (-x, -y)$ and ω has a local description $\omega = f(x, y)dx \wedge dy$ in this chart.

Then, if $q \in E_i$, we have a local chart around q given by (x', t) where $x' = \epsilon^*(x)$, $y' = \epsilon^*(y)$ and $t = y'/x'$.

We noted in Step 2 that $u = {x'}^2$ and $t' = t$ form a system of local coordinates on $\widetilde{\Sigma}$ around $\pi'(q)$.

So

$$\epsilon^*(\omega) = f(x', y')dx' \wedge dy' = x' f(x', tx')dx' \wedge dt = \frac{1}{2} f(u, t')du \wedge dt'$$

that is non zero in the chart since f is non zero.

So we have found a nowhere non zero holomorphic 2-form on $\widetilde{\Sigma}$ and then the canonical divisor of $\widetilde{\Sigma}$ is trivial.

Step 4: $h^{1,0}(\widetilde{\Sigma}) = 0$**.**

The last thing to prove is that the irregularity $q_{\widetilde{\Sigma}} = h^{1,0}(\widetilde{\Sigma}) = 0$.

We know that the irregularity is invariant under blow up (cf. [1, pag.36]), so $q_{\widetilde{S}} = q_S$ and $q_S = 0$ since S is a K3 surface. This means that $H^0(\widetilde{S}, \Omega^1) = 0$.

We have also that $H^0(\widetilde{S}, \Omega^1)^{\{1,i\}} \cong H^0(\widetilde{\Sigma}, \Omega^1)$ (cf. [1, pag.78]), and hence $H^0(\widetilde{\Sigma}, \Omega^1) = 0$.

We have then proved that $\widetilde{\Sigma}$ is a K3 surface. □

3.4 Rational curves

In this section we will state some results about rational curves on K3 surfaces, where we say that C is a rational curve if it is birationally equivalent to $\mathbb{P}^1$.
Let S be an projective complex K3 surface.

Definition 3.13. A *polarized K3 surface* (S, H) consists of a projective complex K3 surface S and an ample line bundle $H \in Pic(S)$.

The existence of sufficiently many rational curves on polarised K3 surfaces is a key property that will be frequently used in the next chapters and in particular it will be fundamental in order to define a special class in the Chow group of 0-cycles of K3 surfaces (cf. Theorem 4.3). This class is the beginning point of the two works of Huybrechts and Voisin presented in Chapters 5 and 6.

The next theorem is a deep and important result whose proof is beyond the level of this thesis. We refer to the notes of Huybrechts [15, Ch. 11] for a more complete treatment of the subject.

Theorem 3.14 (Bogomolov-Mumford)**.** **(i)** *Every polarized complex K3 surface (S, H) contains at least one rational curve $C \in |H|$.*

(ii) *The generic polarized complex K3 surface (S, H) contains a nodal integral rational curve $C \in |H|$.*

(iii) *For fixed $n > 0$, the generic polarized complex K3 surface (S, H) contains an integral rational curve $C \in |nH|$.*

Proof. The idea of the proof consists in constructing an explicit example of a K3 surfaces S_0, in particular S_0 is a Kummer surface, which contains the rational curves we need. Then one can use deformation theory to deform S_0 to a generic K3 surface S and in this way, the deformation of some rational curves on S_0 become the rational curves we need on S.

For the proof see [15, Ch.11, Th. 1.1]. □

By Theorem 3.14 (iii), it is clear that the generic polarized K3 surface (X, H) contains infinitely many integral rational curves.

Remark 3.15. One can show that this result holds indeed for S over any algebraically closed field k (see [15, pag. 165]).

Since in Chapter 5 we will work in this general setting, we state the next result in the general case of S over any algebraically closed field k. It is a corollary of Theorem 3.14, though it is not an immediate consequence and one has to do some non-trivial work to prove it.

Corollary 3.16. *Let S be an algebraic projective K3 surface over an algebraically closed field k. For any non-trivial effective line bundle L on S, there exists a curve in $|L|$ which can be written as an effective sum of rational curves.*

Corollary 3.16 is a key result frequently used in the next chapters.

4. Chow ring of a K3 surface

We want to specialize the study of the Chow ring in the case of a K3 surface S over $\mathbb{C}$ or possibly over an algebraically closed field k.

Since we are considering a surface, the Chow group has this decomposition:

$$CH(S) = CH_0(S) \oplus CH_1(S) \oplus CH_2(S).$$

Clearly $CH_2(S) \cong \mathbb{Z}$ is generated by $[S]$ since S is connected.
Moreover $CH_1(S) \cong Pic(S)$ since S is smooth. Recall also that, by Proposition 3.6, $Pic(S) = NS(S) \cong \mathbb{Z}^{\rho(S)}$ where $\rho(S) = rk(NS(S))$ is the Picard number of S.

Summarizing, we have found that

$$CH_2(S) \cong \mathbb{Z} \text{ and } CH_1(S) \cong Pic(S) \cong \mathbb{Z}^{\rho(S)}.$$

Note that these arguments work for any K3 surface S over any algebraically closed field k.

From now on we will go deep in the study of $CH_0(S)$, the only mysterious part of the Chow group of a K3 surface.

By Mumford's Theorem (Theorem 1.42), if S is a complex projective surface with $H^{2,0}(S) \neq 0$, we have that $CH_0(S)$ is infinite dimensional, that is to say that it cannot be parametrized by an algebraic variety.
Then for complex projective K3 surfaces, for which $H^{2,0}(S) \cong \mathcal{O}_S$, we have that

$$CH_0(S) \text{ is infinite dimensional.}$$

The next deep result we want to present about the Chow group of zero cycles of a K3 surface is originally due to Roitman.
The proof is really tough and so it will not be presented here. We only say that the key of the proof is the fact that the Albanese variety of S is zero (cf. Remark 1.29).

Theorem 4.1. *If S is a K3 surface over an algebraically closed field k, then $CH_0(S)$ is torsion free.*

Proof. See [4]. □

Theorem 4.1 is a key theorem that will be frequently used later on.

We now state another property of $CH_0(S)_{hom}$ that will be used in the next chapters.

Proposition 4.2. *If S is K3 surface over an algebraically closed field k, then $CH_0(S)_{hom}$ is a divisible group.*

Proof of Proposition 4.2. It is enough to show the divisibility of $CH_0(S)_{hom}$ at the level of generators, which are of the form $[x]-[y] \in CH_0(S)_{hom}$ with $x, y \in S$. Pick then $x, y \in S$. By Proposition 2.2 we can find an irreducible curve $j : C \hookrightarrow S$ such that $x, y \in C$ and C is smooth outside x and y. Let then $\nu : \widetilde{C} \to C$ be the normalisation of C obtained by blowing-up x and y (see [19, Sec. 8.1.4] for the process of normalisation with blow-ups) and $f := j \circ \nu : \widetilde{C} \to S$ be the composition with the inclusion of C in S. Let $\widetilde{x} \in \nu^{-1}(x)$ and $\widetilde{y} \in \nu^{-1}(y)$.
Since $\widetilde{C}$ is smooth we can consider the line bundle $\mathcal{O}(\widetilde{x}-\widetilde{y}) \in Pic^0(\widetilde{C})$. Since $Pic^0(\widetilde{C}) \cong Alb(\widetilde{C})$ is an Abelian variety, it is divisible (because every torus is a divisible group). So, for any $n \in \mathbb{Z}$, there exists an element $z \in Pic^0(\widetilde{C})$ such that $nz = \mathcal{O}_C(\widetilde{x}-\widetilde{y}) \in Pic^0(\widetilde{C})$.
In order to prove Proposition 4.2 we need to find, for any $n \in \mathbb{Z}$, an element $z' \in CH_0(S)_{hom}$ such that $nz' = [x]-[y]$. Then, if we consider $z' = f_*(z) \in CH_0(S)_{hom}$, we have that $nz' = nf_*(z) = f_*(nz) = [x]-[y]$ in $CH_0(S)_{hom}$. Thus we have proved that $CH_0(S)_{hom}$ is divisible.

□

Unfortunately, aside from the infinite dimensionality in the complex case and these two results, very little is known about $CH_0(S)$.

However, in the case of a K3 surface S over $\mathbb{C}$, Beauville and Voisin in [2] found out that there is a special class c_S in $CH_0(S)$ such that this class represents every point on every rational curve on the surface.

Recall that by rational curve R on S, we mean an irreducible (but possibly singular) curve R on S of geometric genus zero, that is to say that there exists a generically injective map $j : \mathbb{P}^1 \to S$ such that $R = j(\mathbb{P}^1)$ (i.e. R is birationally equivalent to $\mathbb{P}^1$).

Theorem 4.3 (Beauville-Voisin, cf. [2])**.** *Let S be a complex projective K3 surface.*

(i) *All points of S which lie on some (possibly singular) rational curve have the same class c_S in $CH_0(S)$.*

(ii) *The image of the intersection product*

$$Pic(S) \otimes Pic(S) \to CH_0(S)$$

is contained in $\mathbb{Z}c_S$.

(iii) *The second Chern class $c_2(S) \in CH_0(X)$ is equal to $24c_S$.*

The proofs of points (i) and (ii) are based on the existence of sufficiently many rational curves on S (see Corollary 3.16) and will be presented here.

The proof of point (iii) is more elaborate and we will not show it. We only say that it makes use of the existence of sufficiently many elliptic curves on K3 surfaces (see [2]).

Proof of points (i) and (ii). (i) Let R be a rational curve on S, i.e. $R = j(\mathbb{P}^1)$ for $j : \mathbb{P}^1 \to S$ generically injective map.

For $p \in \mathbb{P}^1$ arbitrary point, we define

$$c_R := j_*(p) \in CH_0(S).$$

Note now that, for any divisor D on S, we have $R \cdot D = j_*(j^*(R \cdot D)) \in CH_0(S)$ (this follows using Corollary 1.12 with j of degree 1). Since $j^* : CH_0(S) \to CH_0(\mathbb{P}^1)$ is a ring map (cf. Proposition 1.11 (i)), we have also that $j^*(R \cdot D) = j^*(R) \cdot j^*(D) = j^*(D)$. But $CH_0(\mathbb{P}^1) \cong p\mathbb{Z}$ and so we have $j^*(D) = np$, with $n = deg(R \cdot D)$. We have then found that, for any rational curve R on S and for any divisor D on S, the image of their intersection product in $CH_0(S)$ is given by

$$R \cdot D = j_*(j^*(D)) = j_*(np) = nc_R \in CH_0(S), \text{ with } n = deg(R \cdot D).$$

Now let T be another rational curve on S.
If $deg(R \cdot T) = n \neq 0$ then the above equality gives $nc_R = R \cdot T = nc_T \in CH_0(S)$. Then, since $CH_0(S)$ is torsion free (cf. Theorem 4.1), we have that $c_R = c_T$. Note that the torsion freeness is not necessary. It is enough to note that any point of R defines the class c_R and any point of T defines the class c_T. Hence if the two rational curves have a common point, it follows that $c_R = c_S$.

If $deg(R \cdot T) = 0$ choose a very ample divisor H on S. By Corollary 3.16 we have that $H \sim \sum_i R_i$ is linearly equivalent to a sum of rational curves R_i. Since $R_i \cong \mathbb{P}^1$ is a rational curve, by the definition of rational equivalence given in Proposition 1.8 (iii), every point of R_i defines the same element c_{R_i} in $CH_0(S)$.
For any intersecting R_i and R_j, we have that the class of $x_{ij} \in R_i \cap R_j$ is rational equivalent both to c_{R_i} and to c_{R_j}, so that $c_{R_i} = c_{R_j} \in CH_0(S)$.
Since H is connected then also $\sum_i R_i$ is connected, and so we have that for every i, there exists a j such that $R_i \cap R_j \neq \emptyset$. Thus we obtain that $c_{R_i} = c_{R_j} \in CH_0(S)$ for every i and j.
Since H is ample, we have moreover that $H \cdot R > 0$ and $H \cdot T > 0$, so that $R_i \cap R \neq \emptyset$ for some i and $R_j \cap T \neq \emptyset$ for some j. Then, using the same argument of above on the rational curves $R \cong \mathbb{P}^1$ and $T \cong \mathbb{P}^1$, we have $c_R = c_{R_i}$ for some i and $c_T = c_{R_j}$ for some j in $CH_0(S)$. But, since $c_{R_i} = c_{R_j}$ for every i and j, we conclude that $c_R = c_T \in CH_0(S)$.
Thus the class c_R does not depend on the choice of the rational curve R and hence the class c_S representing any point on any rational curve is well defined.

(ii) We have shown above that $R \cdot D = deg(R \cdot D)c_S \in CH_0(S)$ for any divisor D and any rational curve R.

Then it is enough to note that, by Corollary 3.16, the Picard group of S is generated by classes of rational curves. So, if D' is a divisor on S, then $D' = \sum_i n_i R_i \in Pic(S)$ with R_i rational curves. But then,

$$D' \cdot D = (\sum_i n_i R_i) \cdot D = \sum_i n_i (R_i \cdot D) = \sum_i n_i deg(R_i \cdot D) c_S = deg(D' \cdot D) c_S \in CH_0(S).$$

So we have not only proved that the image of the intersection product $Pic(S) \otimes Pic(S) \to CH_0(S)$ is contained in $\mathbb{Z}c_S$, but also that, for $D, D' \in Pic(S)$, the coefficient of c_S in $D' \cdot D$ is exactly $deg(D' \cdot D)$.

□

Remark 4.4. In the proof of Theorem 4.3 we have not used the fact that S is a K3 surface, but only that K3 surfaces have some specific properties. In particular, if S is a surface for which there exists a very ample line bundle that is linearly equivalent to a sum of rational curves and such that $Pic(S)$ is spanned by classes of rational curves, then this class c_S remains well defined.

The class c_S is therefore well defined also for K3 surfaces over any algebraically closed field k, since all the results we need hold in that case (see Remark 3.15 and Corollary 3.16).

Moreover statements (i) and (ii) of Theorem 4.3 hold also, for example, for surfaces S that admit a non-trivial elliptic fibration $S \to \mathbb{P}^1$ with a section.

This class c_S will be fundamental for the results presented in the next chapters.
In Chapter 5 we will show some results appearing in the paper [14] of Huybrechts, in which, using the existence of this class c_S for any K3 surface S over any algebraically closed field k, an appropriate generalisation of rational curve will be defined.

In Chapter 6, we will show how Voisin in [25] used the existence of this class in order to prove the main theorem of Chapter 6, which states that symplectic involutions act trivially on the 0-cycles in the Chow group.

5. Constant cycle curves on K3 surfaces

5.1 Introduction

In this chapter we will present some results of the paper [14] by Huybrechts.
In the following, if not stated otherwise, S will be an algebraic K3 surface over an algebraically closed field k (see Definition 3.1). We will work in the scheme-theoretic setting, so that S will be considered as a projective scheme of dimension two over k. We need this point of view because we have to consider the generic points of the curves on S in order to introduce the notion of constant cycle curve.

If $k = \mathbb{C}$, by Theorem 1.42 we have that $CH_0(S)$ is infinite dimensional and so, by one of the equivalent definitions of infinite dimensionality (see Proposition 1.38 (iii)), there is no curve $j : C \hookrightarrow S$ such that the push-forward map $j_* : CH_0(C) \to CH_0(S)$ is surjective.
We call pointwise constant cycle curves the curves $j : C \hookrightarrow S$ such that the image of j_* is as small as possible, i.e. zero.

This definition is generalizable to the case of any arbitrary algebraically closed field k, and so in general we call pointwise constant cycle curve any curve $C \subset S$ such that all of its closed point $x \in C$ define the same class $[x] \in CH_0(S)$. Due to the existence of ample rational curves on S (see Theorem 3.14), this class is the class c_S, i.e. is the class introduced in Theorem 4.3 of all points on every rational curves on S. Note that pointwise constant cycle curves are clearly a generalisation of rational curves.

However the key and more significant definition of the chapter is the notion of constant cycle curve, which is a weaker notion than that of being pointwise constant cycle curve (cf. Remark 5.15). Anyway we will see that this two notions coincide for k uncountable (cf. Proposition 5.14).

We say that $C \subset S$ is a constant cycle curve if a certain class $\kappa_C \in CH_0(S_{k(\eta_C)})$ is torsion, where $S_{k(\eta_C)} := S \times_k k(\eta_C)$ is the base change over the function field of C. The order of κ_C is called the order of C.
Note that for S over an algebraically closed field, we have that $CH_0(S)$ is torsion free (see Theorem 4.1) and so the information about the order of κ_C is contained in the kernel of $CH_0(S \times k(\eta_C)) \to CH_0(S \times \overline{k(\eta_C)})$.

Alternatively (cf. Proposition 5.13) we can define $C \subset S$ to be a constant cycle curve if the class of its generic $\eta_C \in C$ viewed as a closed point in $S_{k(\eta_C)}$ satisfies

$$n \cdot [\eta_C] = n \cdot (c_S)_{k(\eta_C)} \in CH_0(S_{k(\eta_C)})$$

for some positive integer n, where $(c_S)_{k(\eta_C)}$ is the image of the class $c_S \in CH_0(S)$ under the pull-back homomorphism induced by the base change $S_{k(\eta_C)} \to S$. The minimal n which satisfy the condition of above is the order of the constant cycle curve.

To get a better idea of the notion of constant cycle curve, we will give many concrete examples. We will see that rational curves are constant cycle curves of order one (cf. Proposition 5.17) and we will prove that every fixed curve of a non-symplectic automorphism is a constant cycle curve (cf. Proposition 5.20).

We will also show how we can find constant cycle curves on K3 surfaces provided with an elliptic fibration with a section. A generalisation of this example will lead us to prove an interesting result related to the following conjecture about rational curves on complex polarised K3 surfaces (whose motivation stems from Theorem 3.14).

Conjecture 5.1. *If S is a polarised complex K3 surface, the union $\bigcup C \subset S$ of all rational curves $C \subset S$ on a complex polarised K3 surface is dense.*

An evidence for this conjecture is given by a result of MacLean in [20] which shows that, if S is a polarised complex K3 surface, then the set $S_{c_S} := \{x \in S : [x] = c_S \in CH_0(S)\} \subset S$ is dense in the classical topology. To prove Conjecture 5.1 would be then enough to prove that any point x on a polarised complex K3 surface S, with $[x] = c_S \in CH_0(S)$, is contained in a constant cycle curve.

Since rational curves are constant cycle curve of order one, one can consider a weaker version of this conjecture, generalizable to a K3 surface S over any algebraically closed field.

Conjecture 5.2. *There exists $n > 0$ such that the union $\bigcup C \subset S$ of all constant cycle curves $C \subset S$ of order $\leq n$ is dense.*

We will present the proof by Huybrechts of a weaker statement than Conjecture 5.2, which states that the set given by the union of all the constant cycle curves of any order on a K3 surface S, is dense in S (cf. Corollary 5.24).

We will conclude the chapter presenting a more involved example (also introduced in [14]) describing some constant cycle curves on a quartic surface in $\mathbb{P}^3$.

5.2 Constant cycle curves

From now on let S be a projective K3 surface over an algebraically closed field k. As anticipated above, we take the scheme theoretic point of view, so that we consider S as a projective scheme over k.

Definition 5.3. A curve $C \subset S$ is a *pointwise constant cycle curve* if all closed points $x \in C$ have the same class $[x] \in CH_0(S)$.

Remark 5.4. The condition for C to be a pointwise constant cycle curve is equivalent to require that $[x] = c_S$ for all closed points $x \in C$ (recall that the class c_S is well defined over any algebraically closed field by Remark 4.4). Indeed, if we choose an ample divisor H on S, by Corollary 3.16, H is linearly equivalent to a sum of rational curves. Then, given a curve $C \subset S$, we have that there exists a rational curve intersecting C and then there is a always a point on C whose class is c_S.

Note that this definition is a generalisation of rational curves, since, by Proposition 1.8 (iii), the most trivial example of pointwise constant cycle curves are clearly rational curves.

A more compact way to express the definition of pointwise constant cycle curve is the follows. Let

$$\nu : \widetilde{C} \to C \text{ be the normalisation and } j : C \hookrightarrow S \text{ the inclusion.}$$

Then a curve $C \subset S$ is a pointwise constant cycle curve if and only if $(j \circ \nu)_* : Pic(\widetilde{C}) \to CH_0(S)$ takes image in $\mathbb{Z}c_S$. This is also equivalent to say that $(j \circ \nu)_* : Pic^0(\widetilde{C}) \to CH_0(S)$ is the zero map.

Now we introduce a finer scheme theoretic version of this notion.
Let first introduce the class κ_C associated to an integral curve $C \subset S$.
Let

$$\Delta_C := \{(p, j(p)) \in C \times S,\ p \in C\} \subset C \times S$$

be the graph of the inclusion of C in S.

Denote by η_C the generic point $\eta_C \in C$ and $S_{k(\eta_C)} = k(\eta_C) \times_k S := Spec(k(\eta_C)) \times_{Spec(k)} S$ the surface S after the base change over the function field of C. Let $b_S : S_{k(\eta_C)} \to S$ be the base change map.
Then $S_{k(\eta_C)}$ is also the scheme theoretic fibre over η_C of the morphism given by the first projection $p_1 : C \times S \to C$. So, if we call $\alpha' : Spec(k(\eta_C)) \to C$ the map representing the generic point $\eta_C \in C$, we have the commutative square

$$\begin{array}{ccc} S_{k(\eta_C)} & \xrightarrow{i} & C \times_k S \\ \downarrow & & \downarrow{\scriptstyle p_1} \\ Spec(k(\eta_C)) & \xrightarrow{\alpha'} & C \end{array}$$

We want to show that the generic point $\eta_C \in C$ can be viewed as a closed point in $S_{k(\eta_C)}$. This is indeed a general fact, i.e. it is true that if we have any variety X and a closed irreducible subvariety $Y \subset X$ whose generic point is η_Y, then η_Y can be viewed as a closed point in $X_{k(\eta_Y)}$. The key ingredient in order to understand the construction of the closed point of $X_{k(\eta_Y)}$ defined by η_Y is the following.

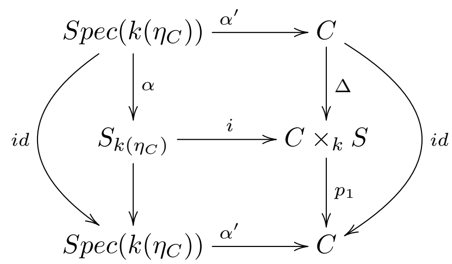
Lemma 5.5. *Let X be a variety over a field K. Then giving $\alpha \in Hom_{Sch_{\downarrow SpecK}}(Spec(K), X)$ is equivalent to give a closed point of X, where $Sch_{\downarrow SpecK}$ is the category of schemes over K.*

Proof of Lemma 5.5. We can reduce to the case of an affine variety $X = SpecA$, with $A \in K$-algebras. A morphism $f : Spec(K) \to X$ over $Spec(K)$ is equivalent to a K-algebra homomorphism $f^\sharp : A \to K$ and the point $f(Spec(K))$ is equal to the point given by the prime ideal $Ker(f^\sharp)$.
We have that $f(Spec(K))$ is a closed point if and only if $Ker(f^\sharp)$ is a maximal ideal, but this is clear since we have an injection $A/Ker(f^\sharp) \hookrightarrow K$ that is also a K-algebra homomorphism, so that it is indeed an isomorphism. □

We now show how to view $\eta_C \in C$ as a closed point in $S_{k(\eta_C)}$, so that we specialise to the case of a surface S and a curve $C \subset S$ although, as we said above, this is a general statement and we could replace S with a variety X of any dimension and C with any subvariety $Y \subset X$.

Lemma 5.6. *The generic point $\eta_C \in C$ can be viewed as a closed point $i^*(\Delta_C)$ in the K3 surface $S_{k(\eta_C)}$.*

Proof. Let $\Delta = (id_C, j) : C \to C \times S$, so that $\Delta(C) = \Delta_C$. Since $p_1 \circ \Delta = id_C$, we have that $\alpha' \circ id_{Spec(k(\eta_C))} = p_1 \circ (\Delta \circ \alpha')$. Then by the universal property of fibred product, there exists a unique map $\alpha : Spec(k(\eta_C)) \to S_{k(\eta_C)}$ such that all commutes. In summary we have the following commutative squares



Since $S_{k(\eta_C)}$ is a scheme over $k(\eta_C)$, by Lemma 5.6 we have that giving a map of schemes $\alpha \in Hom_{Sch_{\downarrow Spec(k(\eta_C))}}(Spec(k(\eta_C)), S_{k(\eta_C)})$ is equivalent to give a closed point of $S_{k(\eta_C)}$. Hence we have that $\alpha(Spec(k(\eta_C))) \in S_{k(\eta_C)}$ is a closed point.
By the commutativity of the upper square, this closed point is equal to $i^*(\Delta(\eta_C))$. But, since $i(S_{k(\eta_C)}) \cap \Delta_C = \Delta(\eta_C)$, we have that $i^*(\Delta(\eta_C)) = i^*(\Delta_C) \in S_{k(\eta_C)}$.
Hence $i^*(\Delta_C) = \alpha(Spec(k(\eta_C))) \in S_{k(\eta_C)}$ is a closed point.

□

In order to understand better the situation, we now give a concrete example considering a simple case.

Example 5.3. Let S be an affine surface defined by $Spec(A)$, with $A = k[x, y, z]/I$. For example, if $I = (g)$ where g is an irreducible polynomial of degree 4, S can be viewed as the affine part of a quartic surface in $\mathbb{P}^3$, which is a K3 surface.

Let $C \subset S$ be an irreducible curve on S, so that $C = V(\mathcal{P})$, where $\mathcal{P}$ is a prime ideal in $k[x, y, z]/I$. We can take for example $\mathcal{P} = (f)$, where $(f) \in k[x, y, z]/I$ is the image in $k[x, y, z]/I$ of an irreducible polynomial of $k[x, y, z]$.

We need to introduce some notation. Let

$$\pi : A \to A/\mathcal{P} \quad \text{and} \quad s : A/\mathcal{P} \to k(\eta_C)$$

be respectively the projection of A to the quotient $A/\mathcal{P}$, which is a domain, and the canonical map from $A/\mathcal{P}$ into its field of fraction.

The residue field at the point $\eta_C = \mathcal{P} \in Spec(A)$, is given by

$$k(\eta_C) = Frac(A/\mathcal{P}) = A_{\mathcal{P}}/\mathcal{P}A_{\mathcal{P}} = k[a, b, c]$$

where we denoted $a := s(\pi(x))$, $b := s(\pi(y))$ and $c := s(\pi(z))$. In particular $a,\ b$ and c are related by the condition $f(a, b, c) = 0$.

The surface $S_{k(\eta_C)}$ is given by

$$S_{k(\eta_C)} = k(\eta_C) \times_k S = Spec(A \otimes_k k(\eta_C)) = Spec(k(\eta_C)[x,y,z]/\overline{I}) = Spec(k(\eta_C)[\overline{x},\overline{y},\overline{z}])$$

where we denoted $\overline{I} := I \cdot k(\eta_C)[x,y,z]$ and $\overline{x},\overline{y},\overline{z}$ the images of x,y,z under the projection map $k(\eta_C)[x,y,z] \twoheadrightarrow k(\eta_C)[x,y,z]/\overline{I}$.

The variety $C \times_k S$ is given by

$$C \times_k S = Spec(A \otimes_k (A/\mathcal{P})) = Spec((A/\mathcal{P})[x,y,z]/\widetilde{I}) = Spec((A/\mathcal{P})[\widetilde{x},\widetilde{y},\widetilde{z}])$$

where we denoted $\widetilde{I} := I \cdot (A/\mathcal{P})[x,y,z]$ and $\widetilde{x},\widetilde{y},\widetilde{z}$ the images of x,y,z under the projection map $(A/\mathcal{P})[x,y,z] \twoheadrightarrow (A/\mathcal{P})[x,y,z]/\widetilde{I}$.

Finally the curve $C_{k(\eta_C)} := k(\eta_C) \times_k S$ is given by

$$C_{k(\eta_C)} = Spec(A/\mathcal{P} \otimes_k k(\eta_C)) = Spec(A/(f) \otimes_k k(\eta_C)) = Spec(k(\eta_C)[\overline{x},\overline{y},\overline{z}]/(\overline{f}) = V(\overline{f})$$

where we denoted $\overline{f}$ the image of f under the map $A \twoheadrightarrow k(\eta_C)[\overline{x},\overline{y},\overline{z}]$.

Note that, since the inclusion $C \subset S$ is given by the surjective map $\pi : A \to A/\mathcal{P}$, then the inclusion $C_{k(\eta_C)} \subset S_{k(\eta_C)}$ is given by the projection map $k(\eta_C)[\overline{x},\overline{y},\overline{z}] \twoheadrightarrow k(\eta_C)[\overline{x},\overline{y},\overline{z}]/(\overline{f})$.

In this example, the point η_C viewed as a closed point in $S_{k(\eta_C)}$ has the most natural possible description. Consider indeed the closed point $(\overline{x}-a,\overline{y}-b,\overline{z}-c) \in S_{k(\eta_C)} = Spec(k(\eta_C)[\overline{x},\overline{y},\overline{z}])$. Since $f(a,b,c)=0$, then also $\overline{f}(a,b,c)=0$ and so $(\overline{x}-a,\overline{y}-b,\overline{z}-c) \in V(\overline{f}) = C_{k(\eta_C)}$.

Claim: $(\overline{x}-a,\overline{y}-b,\overline{z}-c) \in C_{k(\eta_C)} \subset S_{k(\eta_C)} = Spec(k(\eta_C)[\overline{x},\overline{y},\overline{z}])$ *is the closed point representing the generic point* η_C *in* $k(\eta_C) \times_k S$.

Indeed consider the the commutative upper square of Lemma 5.6 and its equivalent at the level of rings using the contravariant equivalence of category from affine schemes to rings.

We have

$$\begin{array}{ccc} Spec(k(\eta_C)) & \xrightarrow{\alpha'} & C \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \Delta} \\ S_{k(\eta_C)} & \xrightarrow{i} & C \times_k S \end{array} \qquad \begin{array}{ccc} k[a,b,c] & \xleftarrow[\alpha'^\sharp]{} & A/\mathcal{P} \\ \uparrow{\scriptstyle \alpha^\sharp} & & \uparrow{\scriptstyle \Delta^\sharp} \\ k(\eta_C)[\overline{x},\overline{y},\overline{z}] & \xleftarrow[i^\sharp]{} & A \otimes_k A/\mathcal{P} \end{array}$$

As we have seen in the proof of Lemma 5.6, the closed point $i^*(\Delta_C)$ representing the generic point η_C in $S_{k(\eta_C)}$, is exactly the kernel of $\alpha^\sharp$.

To find the explicit form of the morphism $\alpha^\sharp$, we have to look at the explicit form of the other morphisms of the square.

Clearly, by the definition of Δ, if $u \in A$ and $v \in A/\mathcal{P}$, we have that $\Delta^\sharp(u \otimes v) = \pi(u) \cdot v \in A/\mathcal{P}$.

We have also that $\alpha'^\sharp$ is the map s defining the injection of $A/\mathcal{P}$ in its field of fractions $k(\eta_C)$, so that

$$\Delta^\sharp(\alpha'^\sharp(u \otimes v)) = s(\pi(u) \cdot v) \in k(\eta_C)$$

for any $u \in A$ and $v \in A/\mathcal{P}$.
Then if we consider the isomorphism $A \otimes_k A/\mathcal{P} \cong (A/\mathcal{P})[\widetilde{x}, \widetilde{y}, \widetilde{z}]$, we see that $\Delta^\sharp \circ \alpha'^\sharp$ is the only map that sends the element of the form $\sum_{ijk} v_{ijk} \cdot \widetilde{x}^i \widetilde{y}^j \widetilde{z}^k \in (A/\mathcal{P})[\widetilde{x}, \widetilde{y}, \widetilde{z}]$, for $v_{ijk} \in (A/\mathcal{P})$, to the element $\sum_{ijk} s(v_{ijk}) \cdot s(\pi(x))^i s(\pi(y))^j s(\pi(z))^k = \sum_{ijk} s(v_{ijk}) \cdot a^i b^j c^k \in k(\eta_C)$, i.e. $\Delta^\sharp \circ \alpha'^\sharp : (A/\mathcal{P})[\widetilde{x}, \widetilde{y}, \widetilde{z}] \to k[a, b, c]$ is the evaluation map.
Now, passing to the other route of the square, the map $i^\sharp$ is the unique ring map that sends the elements of the form $\sum_{ijk} v_{ijk} \cdot \widetilde{x}^i \widetilde{y}^j \widetilde{z}^k \in (A/\mathcal{P})[\widetilde{x}, \widetilde{y}, \widetilde{z}] \cong A \otimes_k A/\mathcal{P}$, for $v_{ijk} \in (A/\mathcal{P})$, to $\sum_{ijk} s(v_{ijk}) \cdot \overline{x}^i \overline{y}^j \overline{z}^k \in k(\eta_C)[\overline{x}, \overline{y}, \overline{z}]$.
By the commutativity of the square, we have then that the map $\alpha^\sharp$ is the evaluation map that sends $\overline{x} \mapsto a$, $\overline{y} \mapsto b$ and $\overline{z} \mapsto c$, so that $Ker(\alpha^\sharp)$ is exactly the maximal ideal $(\overline{x} - a, \overline{y} - b, \overline{z} - c) \in S_{k(\eta_C)} \cong k(\eta_C)[\overline{x}, \overline{y}, \overline{z}]$ and so Claim is proved. Note that being a maximal ideal, indeed it is a closed point.

Now we return to the general case and consider the cycle

$$\Delta_C - C \times \{x_0\}, \text{ where } x_0 \in S \text{ is an arbitrary point such that } [x_0] = c_S.$$

We are finally ready to define the 0-cycle

$$\kappa_C := i^*(\Delta_C - C \times \{x_0\}) \in CH_0(k(\eta_C) \times_k S).$$

Note that κ_C can be seen as the difference between $\eta_C \in C$ viewed as a closed point $[\eta_C] := i^*(\Delta_C) \in S_{k(\eta_C)}$ (cf. Lemma 5.6) and the 'constant point' $i^*(C \times \{x_0\})$.
Now we introduce the key definition of the chapter.

Definition 5.7. An integral curve $C \subset S$ is a *constant cycle curve* if $\kappa_C \in CH_0(k(\eta_C) \times_k S)$ is a torsion class. We call an arbitrary curve $C \subset S$ constant cycle curve if every integral component has this property.
The *order* of an integral constant cycle curve $C \subset S$ is the order of the torsion class κ_C. The order of an arbitrary constant cycle curve C is the maximal order of its integral components.

Remark 5.8. The class $\kappa_C \in CH_0(S_{k(\eta_C)})$ can also be described in another way.
Call $C_{k(\eta_C)} := k(\eta_C) \times_k C$ and consider the commutative squares

$$\begin{array}{ccccc} S & \xleftarrow{b_S} & S_{k(\eta_C)} & \xrightarrow{i} & S \times_k C \\ \uparrow{\scriptstyle j} & & \uparrow{\scriptstyle j_{k(\eta_C)}} & & \uparrow{\scriptstyle j\times j} \\ C & \xleftarrow{b_C} & C_{k(\eta_C)} & \xrightarrow{i} & C \times_k C \end{array}$$

Using the arguments of the proof of Lemma 5.6, we consider $[\eta_C] = i^*(\Delta_C) \in CH_0(C_{k(\eta_C)})$, where $\Delta_C \subset C \times_k C$ is the diagonal, as the generic point $\eta_C \in C$ viewed as a closed point in $C_{k(\eta_C)}$. Hence by the definition of κ_C and by the commutativity of the above left square we have that

$$\kappa_C = {j_{k(\eta_C)}}_*([\eta_C] - [x_0]) \in CH_0(S_k(\eta_C))$$

where $[x_0] = b_C^*(x_0') \in CH_0(C_{k(\eta_C)})$ for $x_0' \in C \subset S$ such that $j_*[x_0'] = c_S \in CH_0(S)$.
Note however that we cannot compute the order of C considering only the element $[\eta_C] - [x_0] \in CH_0(C_{k(\eta_C)})$ because it is never a torsion element except for C rational.
Indeed consider the base change map to the algebraic closure of $k(\eta_C)$ given by $b^*_{k(\eta_C)} : CH_0(C_{k(\eta_C)}) \to CH_0(C_{\overline{k(\eta_C)}})$. Since the base change map $\overline{b}_C^* = b^*_{k(\eta_C)} \circ b_C^* : CH_0(C) \to CH_0(C_{\overline{k(\eta_C)}})$ is an isomorphism on torsion (see [18]), if $[\eta_C] - [x_0] \in CH_0(C_{k(\eta_C)})$ were a torsion element, then $b^*_{k(\eta_C)}([\eta_C] - [x_0]) \in Im(\overline{b}_C^*)$ because $b^*_{k(\eta_C)}([\eta_C] - [x_0])$ would be torsion too. Hence, since $b^*_{k(\eta_C)}([x_0]) = \overline{b}_C^*([x_0]) \in Im(\overline{b}_C^*)$, we had also that $b^*_{k(\eta_C)}([\eta_C]) \in Im(\overline{b}_C^*)$. But this is a contradiction because η_C is not a closed point of C and so it does not belong to $CH_0(C)$.

We now state a result that gives a characterization of $CH_0(k(\eta_C) \times_k S)$ that will be useful later in many proofs in the chapter.

Lemma 5.9. *We have that*

$$CH^2(k(\eta_C) \times_k S) \cong \varinjlim_{U \subset C} CH^2(U \times_k S)$$

where the limit is taken over the open sets U of C.

Hence, κ_C is a torsion class if and only if the class $\kappa_{C,U} \in CH^2(U \times_k S)$, given by the restriction of $[\Delta_C - \{x_0\} \times C]$ to $U \times S$ for some non-empty open subset $U \subset C$, is torsion.
Moreover by shrinking $U \subset C$ enough, one can always assume that the order of κ_C and $\kappa_{C,U}$ coincide.

Proof of Lemma 5.9. There is a general result, Lemma 1A.1 in [4], which states that, if X is a smooth variety over an algebraically closed field k, and Y is any k-variety, then, for $n \geq 0$,

$$CH^n(X_{k(Y)}) \cong \varinjlim_{U \subset Y} CH^n(X \times_k U)$$

where the limit is taken over the open sets U of Y.

By taking $X = S$ and $Y = C$ we have then Lemma 5.9. □

Consider now a field extension K/k. Call $S_K := S \times_k Spec(K)$ and consider the base change map b given by

$$\begin{array}{ccc} S_K & \xrightarrow{b} & S \\ \downarrow & & \downarrow \\ Spec(K) & \longrightarrow & Spec(k) \end{array}$$

Then we have a base change map for 0-cycles

$$CH_0(S) \xrightarrow{b^*} CH_0(S_K).$$

We denote $(c_S)_K := b^*(c_S)$.

Remark 5.10. If K is algebraically closed then, since in this case a rational curve in S remains a rational curve under base change, we have $(c_S)_K = c_{S_K}$.

We now state an important result that we will need in the next propositions.

Lemma 5.11. *Let S be an algebraic K3 surface over any field k, this time possibly not algebraically closed, and K/k a field extension. Then the base change map $CH_0(S) \to CH_0(S \times_k Spec(K))$ has torsion kernel.*

Proof. Consider first a finite field extension K/k .

Then the base change map $b : S \times_k Spec(K) \to S$ is a finite morphism and thus, by a corollary of the projection formula (Corollary 1.12), $b_*(b^*(x)) = [K : k] \cdot x$ for any $x \in CH_0(S)$.

Hence, if $b^*(x) = 0 \in CH_0(S \times_k K)$ then $x \in CH_0(S)$ is torsion.

In the case of an arbitrary field extension we refer to the proof of Lemma 2.6 in [15]. □

We now use the previous lemma to show that the notion of constant cycle curve is compatible with the base change map.

Proposition 5.12. *Let $C \subset S$ be a curve and let K/k be a base field extension. Then C is a constant cycle curve of order n if and only if $C_K := b^*(C) \subset S_K$ is a constant cycle curve of order n.*

Proof. Consider the pull-back map

$$b^*_{k(\eta_C)} : CH_0(S \times_k k(\eta_C)) \to CH_0(S_K \times_K k(\eta_{C_K}))$$

induced by the base change map $b_{k(\eta_C)} : S_K \times_K k(\eta_{C_K}) = (S \times_k k(\eta_C)) \otimes_k K \to S \times_k k(\eta_C)$. Since by Lemma 5.11 $b^*_{k(\eta_C)}$ has torsion kernel and clearly maps k_C to k_{C_K}, we have that C is a constant cycle curve if and only if $C_K \subset S_K$ is a constant cycle curve.

For the proof that the order of the constant cycle curve is maintained after base change see [14, Lem. 3.6]. □

We now give other characterisations of constant cycle curves, observing what happens after base changes.

Proposition 5.13. *For an integral curve $C \subset S$ the following conditions are equivalent:*

(i) *The curve is a constant cycle curve.*

(ii) *If the generic point η_C is viewed as a closed point in $k(\eta_C) \times_k S$, then there exists a positive integer n, that will be the order of C, such that*

$$n \cdot [\eta_C] = n \cdot (c_S)_{k(\eta_C)} \in CH_0(k(\eta_C) \times_k S).$$

(iii) *If the generic point η_C is viewed as a point in the geometric generic fibre $\overline{k(\eta_C)} \times_k S$, then*

$$[\eta_C] = (c_S)_{\overline{k(\eta_C)}} \in CH_0(\overline{k(\eta_C)} \times_k S).$$

Proof. The equivalence of (i) and (ii) is a direct consequence of Lemma 5.11 which states that the base change map has torsion kernel, while the equivalence of (ii) and (iii) is given by Theorem 4.1 which states that the Chow group of zero cycles of a K3 surface over an algebraically closed field is torsion free. □

With the next result we will note how, in the case of k uncountable, the concepts of constant cycle curve and pointwise constant cycle curve coincide.

Proposition 5.14. *Assume that k is uncountable.*

Then $C \subset S$ is a constant cycle curve if and only if it is a pointwise constant cycle curve.

Proof of Proposition 5.14. We will prove the proposition for C integral and, for simplicity, smooth.

First of all assume C is a constant cycle curve, so that κ_C is torsion; say $n\kappa_C = 0$. Then, by Lemma 5.9, there exists an open subset $U := C - \{p_1, \dots, p_m\} \subset C$ such that $n[(\Delta_C - \{x_0\} \times C)_{|U\times S}] = 0 \in CH^2(U \times S)$.

By the localisation exact sequence given by Proposition 1.13

$$CH^1(\{p_1, \dots, p_m\} \times S) \to CH^2(C \times S) \to CH^2(U \times C) \to 0,$$

we can assume that the cycle $0 = n[(\Delta_C - \{x_0\} \times C)_{|U\times S}] \in CH_0(U \times S)$ is rationally equivalent to a cycle Z with support in $\{p_1, \dots, p_m\} \times S$.

Since any cycle on C is linearly equivalent to one disjoint to a finite set of points $\{p_1, \dots, p_m\}$ (see Proposition 2.14), then the map $[Z]_* : CH_0(C) \to CH_0(S)$, given by the correspondence Z contained in $\{p_1, \dots, p_m\} \times S$, is the zero map.

But $Z \sim n[\Delta_C - \{x_0\} \times C]$, so we have proved that $n[\Delta_C - \{x_0\} \times C]_* : CH_0(C) \to CH_0(S)$ is the zero map. Since, by Theorem 4.1 we have that $CH_0(S)$ is torsion free, then $[\Delta_C - \{x_0\} \times C]_*$ is the zero map.

But $[\Delta_C - \{x_0\} \times C]_*([y]) = [y] - [x_0]$ for any y closed point in C. Hence we have proved that $[y] = [x_0] = [c_S]$ for any closed point y in C and so C is a pointwise constant cycle curve.

Conversely, let C be a pointwise constant cycle curve. We need here the hypothesis of k uncountable in order to use Corollary 10.20 of [26] (it is stated for $k = \mathbb{C}$, but holds of any uncountable field k). It implies that $n[\Delta_C - \{x_0\} \times C] \sim Z$ for some $n > 0$ and with Z supported on a closed set of the form $\{p_1, \dots, p_m\} \times S$.

Then, if $i : k(\eta_C) \times_k S \to C \times_k S$ is the map given by the fibred product commutative square, we have $n\kappa_C = ni^*(Z)$. But, since Z supported on a closed set of the form $\{p_1, \dots, p_m\} \times S$, we have that $i^*(Z) = 0$ and hence κ_C is torsion.

So C is a constant cycle curve.

□

Remark 5.15. We haven't used the hypothesis of k uncountable in the proof of the first implication, so a constant cycle curve is always a pointwise constant cycle curve, for every algebraically closed field k. Hence the notion of constant cycle curve is weaker than the notion of pointwise constant cycle curve.

Note that the notion of pointwise constant cycle curve is not a priori well-behaved under base change and so it has good functorial properties only when it coincides with the notion of constant cycle curve. In particular, by Proposition 5.14, this holds for uncountable algebraically closed fields. Hence when we consider countable fields, it is better to consider constant cycle curves since they are more interesting than pointwise constant cycle curves thanks to their functorial properties. Moreover, according to a special case of the general set of the Bloch-Beilinson conjectures, every curve in a K3 surface over the countable field $\overline{\mathbb{Q}}$ should be a pointwise constant cycle curve, and so if the conjecture were true the notion of pointwise constant cycle curve would be clearly of even less interest in this case.

We will use Proposition 5.14 to prove the next result.
It will give us equivalent definitions of the concept of constant cycle curves avoiding the mention of the class c_S, so that we will have a definition for constant cycle curve that has sense for other type of surfaces aside from K3 surfaces.
Note indeed that the next result is a version of Proposition 5.13 without the mention of c_S.

Proposition 5.16 (Equivalent definitions of constant cycle curves generalizable to every surfaces)**.** *For an integral curve $C \subset S$ the following are equivalent:*

i) *The curve C is a constant cycle curve.*

ii) *There exists $n \in \mathbb{N}$ such that*

$$n \cdot [\eta_C] \in Im(b_S^* : CH_0(S) \to CH_0(k(\eta_C) \times_k S))$$

where the generic point η_C is viewed as a closed point in $k(\eta_C) \times_k S$.

iii) *If the generic point η_C is viewed as a point in the generic fibre $\overline{k(\eta_C)} \times_k S$, then*

$$[\eta_C] \in Im(CH_0(S) \to CH_0(\overline{k(\eta_C)} \times_k S)).$$

Proof. Clearly i) implies ii) and iii).
Moreover, since by Lemma 5.11 the base change map $CH_0(k(\eta_C) \times_k S) \to CH_0(\overline{k(\eta_C)} \times_k S)$ has torsion kernel, then iii) implies ii). Besides, since by Theorem 4.1 the group $CH_0(\overline{k(\eta_C)} \times_k S)$ is torsion free , then ii) implies iii).

It remains ii) implies i). We need now to the define, for any closed point $x \in C$, the specialisation map

$$s_x : CH_0(S \times k(\eta_C)) \to CH_0(S).$$

This map is the right inverse of the pull-back map, i.e. $b_S^* \circ s_x = id$ in $CH_0(S \times k(\eta_C))$ for any closed point $x \in C$.
Recall that $i : S \times k(\eta_C) \to S \times_k C$ is the map induced by the fibred product diagram defined in Lemma 5.6. Let $S \times_k \{x\} \subset S \times_k C$ be the subvariety isomorphic to S defined by the closed point $x \in C$. Then $i^{-1}(S \times_k \{x\}) \subset S \times k(\eta_C)$ is a subvariety isomorphic to S.

Example 5.4. We set us in the same setting and notations of Example 5.3, so that in particular $S = k[x, y, z]/I$and $k(\eta_C) = k[a, b, c]$, where a, b, c are the images of x, y, z under a suitable projection. If $x = (x - x_0, y - x_1, z - x_2) \in Spec(A)$ is the closed point determined by $x_0, x_1, x_2 \in k$, then the inclusion $i^{-1}(S \times_k x) \hookrightarrow S \times k(\eta_C)$ is defined by the surjective ring homomorphism $k(\eta_C)[x, y, z]/I = (k[a, b, c])[x, y, z]/I \twoheadrightarrow k[x, y, z]/I$ given by $a \mapsto x_0, b \mapsto x_1, c \mapsto x_2$.

Define now the map s_x as

$$s_x(D) := D \cdot i^{-1}(S \times_k x) \in CH_0(S \times_k x) = CH_0(S)$$

for any $D \in CH_0(S \times k(\eta_C))$ (for a general definition of specialisation map see [7, Ch. 20.3]).
If $n \cdot [\eta_C] \in Im(b_S^* : CH_0(S) \to CH_0(k(\eta_C) \times_k S))$, say $n \cdot [\eta_C] = \alpha_{k(\eta_C)}$ with $\alpha_{k(\eta_C)} = b_S^*(\alpha)$ for some $\alpha \in CH_0(S)$, then $n \cdot [x] = \alpha \in CH_0(S)$, as η_C specialise to $[x]$.
Thus we have proved that here exists an element $\alpha \in CH_0(S)$ such that, for any closed point $x \in C$, we have that $n \cdot [x] = \alpha \in CH_0(S)$, and so C is a pointwise constant cycle curve.
If k is uncountable, C is also a constant cycle curve by Proposition 5.14. If k is countable, use base change to an uncountable extension K/k. Since $n \cdot [\eta_{C_K}] = n \cdot [\eta_C]_K$ is contained in the image of $CH_0(S) \to CH_0(X_K \times_K k(\eta_{C_K}))$ then, thanks to our previous argument, C_K is a constant cycle curve and so, by Proposition 5.12, also C is. □

5.5 Examples of constant cycle curves

Let S be a K3 surface over an algebraically closed field.

The simplest example of a pointwise constant cycle curve is clearly given by a rational curve. We have in fact that they are also constant cycle curve.

Proposition 5.17. *If $C \subset X$ is a rational curve then it is a constant cycle curve of order one.*

Proof. First C is a constant cycle curve because, if we consider the point $\eta_C \in C$ as a closed point $[\eta_C] \in S \times_k \overline{k(\eta_C)}$, then clearly it belongs to $C \times_k \overline{k(\eta_C)}$ which is still a rational curve, so that $[\eta_C] = (c_X)_{\overline{k(\eta_C)}}$. Then by Proposition 5.13 (iii), C is a constant cycle curve.

Now we want to prove that the order of C is one. We will prove a stronger result, i.e. that $CH_0(S \times k(\eta_C))$ is torsion free if C is a rational curve.
By the formula for the Chow group of a projective bundle (cf. [27, Th. 9.25]) we have

$$CH_0(S) \bigoplus CH_1(S) \xrightarrow{\cong} CH_1(S \times \mathbb{P}^1), \quad (x, C) \to [x \times \mathbb{P}^1] + [C \times \{p\}]$$

for $x \in S$, $C \in CH_1(S)$ and $p \in \mathbb{P}^1$.
Considering in this case the exact sequence given by Proposition 1.13, for any $p \in \mathbb{P}^1$ and $U = \mathbb{P}^1 - \{p\}$ we have

$$CH_1(S \times \{p\}) \xrightarrow{i_*} CH_1(S \times \mathbb{P}^1) \cong CH_0(S) \bigoplus CH_1(S) \xrightarrow{j^*} CH_1(S \times U) \to 0$$

and hence, by the exactness of the sequence and by the explicit form of the isomorphism of above, we obtain that $ker(j^*) = Im(i_*) = CH_1(S) \subset CH_0(S) \bigoplus CH_1(S) \cong CH_1(S \times \mathbb{P}^1)$.
Hence we obtain $CH_0(S) \cong CH_1(S \times U)$ for any $U = \mathbb{P}^1 - \{p\}$. Using this exact sequence considering a finite set of points of $\mathbb{P}^1$ instead of only one point p, we obtain that $CH_0(S) \cong CH_1(S \times U)$ for every proper non-empty subset $U \subset \mathbb{P}^1$.
Hence, by Lemma 5.9, we have that $CH_0(S \times k(\eta_C)) \cong CH_0(S)$, which is torsion free by Theorem 4.1.

□

The next example we want to present is about constant cycle curves obtained as fixed point curves of non-symplectic automorphism. For simplicity we work over an uncountable field k of characteristic zero, so that by Proposition 5.14 a curve is a constant cycle curve if and only if it is a pointwise constant cycle curve.

First consider a K3 surface $p : S \to \mathbb{P}^2$ given as a 2:1 covering of the plane $\mathbb{P}^2$ ramified over a sextic curve $C \subset \mathbb{P}^2$. We presented this example of K3 surface in Section 3.2. Let $i : S \xrightarrow{\cong} S$ be the covering involution determined by p.

Lemma 5.18. *The curve $p^{-1}(C) \subset S$ is a (pointwise) constant cycle curve.*

Proof. Since $CH_0(\mathbb{P}^2)_{hom} = 0$ we have that $i^* = -id$ on $CH_0(S)_{hom}$.
Indeed for any $p, q \in S$ we have $\pi_*(p - q) = 0 \in CH_0(\mathbb{P}^2)_{hom} = 0$ and hence $0 = \pi^*(\pi_*(p - q)) = (p - q) + i^*(p - q) \in CH_0(S)_{hom}$.
Now consider $p^{-1}(C) \subset S$ and write the class $[x]$ of a point $x \in p^{-1}(C)$ as $[x] = c_S + \alpha_x$ with $\alpha_x = [x] - c_S \in CH_0(S)_{hom}$. Since $p^{-1}(C)$ is the fixed point curve of the involution i, we have that $i^*([x]) = [x] = c_S + \alpha_x$.
On the other hand $i^*([x]) = i^*(c_S + \alpha_x) = c_S - \alpha_x$. This is because $i^* = -id$ on $CH_0(S)$ and $i^*(c_S) = c_S$ since i is an automorphism, and so in particular $i(l) \cong \mathbb{P}^1$ if l is a rational curve.
Hence for $x \in p^{-1}(C)$ we have $2 \cdot \alpha_x = 0$ and, since $CH_0(S)$ is torsion free, also $\alpha_x = [x] - c_S = 0 \in CH_0(S)_{hom}$. Hence $p^{-1}(C) \cong C$ is a (pointwise) constant cycle curve. □

Remark 5.19. Since $C \cong p^{-1}(C)$, Lemma 5.18 shows an explicit example of a smooth constant cycle curve of genus ten, and so in particular not rational. One can show that the order of C is less or equal than the order of i (see [14, Prop. 7.1]), and so it is one or two. This suggests also that in general the genus of a constant cycle curve is not determined by its order.

We can generalise the previous example in the following way.
Let S be a K3 surface and let $f : S \xrightarrow{\cong} S$ be an automorphism of finite order n. Assume that the quotient

$$\pi : S \to \overline{S} := S/\langle f \rangle,$$

which is possibly singular, satisfies $CH_0(\overline{S})_{hom} = 0$. One can show that is equivalent to have $f^* \neq id$ on $H^{2,0}(S)$ (see [14, Sec. 4]), i.e. it is equivalent to have f non-symplectic automorphism.

We have then the following.

Proposition 5.20. *If $f : S \overset{\cong}{\to} S$ is a non-symplectic automorphism of finite order n and $C \subset S$ is a curve contained in the fixed point locus $Fix(f)$, then C is a (pointwise) constant cycle curve.*

Proof. We generalise the argument of the proof of Lemma 5.18.
Write $[x] = c_S + \alpha_x \in CH_0(S)$, where $x \in C$ and $\alpha_x = [x] - c_S \in CH_0(S)_{hom}$.
We have that f is an automorphism, so that in particular $f(l) \cong \mathbb{P}^1$ for any rational curve l, and hence, since c_S is the class of the points on rational curves, we have that $\pi^*(\pi_*(c_S)) = n \cdot c_S$. Moreover $C \subset Fix(f)$ and so $\pi^*(\pi_*([x])) = n \cdot [x]$.
Thus we have that

$$n \cdot [x] = \pi^*(\pi_*([x])) = \pi^*(\pi_*(c_S)) + \pi^*(\pi_*(\alpha_x)) = \pi^*(\pi_*(c_S)) = n \cdot c_S$$

where we used $\pi^*(\pi_*(\alpha_x)) = 0$ because of the condition $CH_0(\overline{S})_{hom} = 0$.

We have then proved that $n \cdot \alpha_x = n \cdot [x] - n \cdot c_S = 0 \in CH_0(S)_{hom}$ for any $x \in C$ and so, since $CH_0(S)$ is torsion free, C is a (pointwise) constant cycle curve.
Note that the calculations suggest that the order of C should be less or equal than n and indeed one can show that the order of C divides n (see [14, Prop. 7.1]). ☐

For the last example of constant cycle curves on a K3 surface, consider an elliptic K3 surface $\pi : S \to \mathbb{P}^1$ with a zero section $C_0 \cong \mathbb{P}^1 \subset S$.

Let $S_t := \pi^{-1}(t)$ be a smooth fibre of π over $t \in \mathbb{P}^1$, so that S_t is an elliptic curve. Denote by $x_t := S_t \cap C_0 \in S_t$ the origin in S_t.

Proposition 5.21. *Let*

$$C_n \subset S$$

be the curve defined by the closure of the set of n-torsion closed points of S_t.
Then C_n is a constant cycle curve.
Moreover the union of all the curves C_n is dense and, if $k = \mathbb{C}$, even dense in the classical topology.

Proof. Recall that, since S_t is an elliptic curve, we have isomorphisms of groups $CH_0(S_t)_{hom} \cong Alb(S_t) \cong S_t$, so that the set of n-torsion closed points of S_t is equal to the set of n-torsion elements of $CH_0(S_t)_{hom}$. Then for any $x \in C_n \cap S_t$, the class $[x] - [x_t]$ is n-torsion in $CH_0(S_t)_{hom}$ by definition of C_n.
Since $CH_0(S)$ is torsion free, the class $[x] - [x_t]$ viewed in $CH_0(S)$ is zero. But by the definition of rational equivalence given by Proposition 1.8 (iii), we have that $[x_{t_1}] = [x_{t_2}] \in CH_0(S)$ for any $t_1, t_2 \in \mathbb{P}^1$ because they belong to the rational curve C_0. We call $[x_0] \in CH_0(S)$ the class representing every point of C_0. Then we have found that any $[x] \in CH_0(S)$ with $x \in C_n$, is linearly equivalent to the class $[x_0]$ and thus C_n is a pointwise constant cycle curve.
If k is uncountable, by Proposition 5.14 we have that C_n is also a constant cycle curve.
For an arbitrary field k one need a more refined definition of the curves C_n (or rather their irreducible components) in order to prove that they are constant cycle curves and to determine their order.
Let $\eta \in \mathbb{P}^1$ denote the generic point and $S_\eta := \pi^{-1}(\eta) \subset S$ the generic fibre of π.

If $x_n \in S_\eta$ is a point of order n, then one can show that $C_{x_n} := \overline{\{x_n\}} \subset S$ is an irreducible component of the curve C_n and it is a constant cycle curve of order $d|n$ (see [14, Lem. 6.1]).

The density statement is obvious because the set of torsion points of a fibre S_t, which is an elliptic curve, is dense.

□

We now want to generalise this example to covering families of elliptic curves. We need the following lemma.

Lemma 5.22. *If S is a K3 surface, then there exists a smooth elliptic surface $\pi : \mathcal{C} \to T \cong \mathbb{P}^1$ with a surjective morphism $p : \mathcal{C} \to S$.*

Proof. See [11, Thm. 4.1] for the case of characteristic zero and [14, pag. 24] for positive characteristic. □

By Theorem 3.14 we can find an ample rational curve $C_0 \subset S$. Let $\widetilde{C_0} := p^{-1}(C_0) \subset \mathcal{C}$ be its preimage in $\mathcal{C}$. Replacing $\pi : \mathcal{C} \to T$ by its base change to $\widetilde{C_0}$, we can assume that $\pi : \mathcal{C} \to T$ admits a section $\sigma_0 : T \to \mathcal{C}$ such that $p(\sigma_0(T)) = C_0$.
Let $\mathcal{C}_t := \pi^{-1}(t) \subset \mathcal{C}$ be a smooth fibre of π over $t \in T$. Denote by $x_t := \mathcal{C}_t \cap \sigma_0(T) \in \mathcal{C}_t$ the 'origin' in $\mathcal{C}_t$.

Proposition 5.23. *Let $\mathcal{C}_{t,n} \subset \mathcal{C}_t$ be the set of points $x \in \mathcal{C}_t$ such that $n \cdot ([x] - [x_t]) = 0 \in CH_0(\mathcal{C}_t)$, for $n \in \mathbb{Z}$.*

The curve

$$C_n := \overline{\bigcup_{t \in T} p(\mathcal{C}_{t,n})} \subset S$$

is a constant cycle curve.

Proof. Since $[x] - [x_t] \in CH_0(\mathcal{C}_t)$ is a torsion cycle, then $p_*([x] - [x_t]) = 0 \in CH_0(S)$ because $CH_0(S)$ is torsion free by Theorem 4.1. Hence for any $t \in T$, we have that all the points in the set $\{p(x),\ x \in \mathcal{C}_t\} \subset S$ define the same class $[p(x)] = [p(x_t)]$.
But, as in the proof of Proposition 5.21, we have that $[p(x_{t_1})] = [p(x_{t_2})] \in CH_0(S)$ for any $t_1, t_2 \in T$. Indeed $x_{t_1}, x_{t_2} \in \sigma_0(T) \cong \mathbb{P}^1$ and so they are rational equivalent, i.e. $[x_{t_1}] = [x_{t_2}] \in CH_0(\mathcal{C})$. Hence $[p(x_{t_1})] = p_*([x_{t_1}]) = p_*([x_{t_2}]) = [p(x_{t_2})] \in CH_0(S)$.
Summarising, since $[p(x)] = [p(x_t)]$ for any $x \in \mathcal{C}_t$ and since $[p(x_{t_1})] = [p(x_{t_2})]$ for any $t_1, t_2 \in T$, we have proved that $\bigcup_{t \in T} p(\mathcal{C}_{t,n}) \subset \mathcal{C}$ is a pointwise constant cycle curve, and so the same holds for its closure C_n.
Hence in the case of an uncountable field, C_n is also a constant cycle curve by Proposition 5.14. In the case of arbitrary fields, the arguments of the proof of [14, Lem. 6.1], to which we referred in the proof of Proposition 5.21, still hold and so C_n is a constant cycle curve for every algebraically closed field k. □

From this generalisation to covering families of elliptic curves, we obtain the following interesting result.

Corollary 5.24. *On a K3 surface S over an algebraically closed field k, the union*

$$\bigcup_{C=ccc} C \subset S$$

of all constant cycle curves of any order is dense. For $k = \mathbb{C}$ density holds in the classical topology.

Proof. Note that $\bigcup_{t\in T, n\in\mathbb{Z}} \mathcal{C}_{t,n}$ is dense in $\mathcal{C}$, since, as we noted in the proof of Proposition 5.21, the set of torsion points of the elliptic curves $\mathcal{C}_t$ is dense in $\mathcal{C}_t$.
But $p : \mathcal{C} \to S$ is a surjective morphism and so

$$p(\bigcup_{t\in T, n\in\mathbb{Z}} \mathcal{C}_{t,n}) \subset \bigcup_{t\in T, n\in\mathbb{Z}} \overline{p(\mathcal{C}_{t,n})} = \bigcup_{n\in\mathbb{Z}} C_n$$

is dense in S. □

This corollary should be compared to Conjecture 5.2.

5.6 The bitangent correspondence

In this section we go deep into a more involved example about constant cycle curves concerning a map called 'bitangent correspondence'. In this section S is a K3 quartic surface $S \subset \mathbb{P}^3_{\mathbb{C}}$ not containing a line.

First of all we explain what we mean for 'bitangent correspondence'. A line $l \subset \mathbb{P}^3$ is called a bitangent of S if at every point $x \in S \cap l$ the intersection multiplicity is at least two. If there is a unique point of intersection of degree four, we call l a quadritangent.

Lemma 5.25. *If S is a generic K3 quartic surface $S \subset \mathbb{P}^3_{\mathbb{C}}$ not containing a line, then for the generic $x \in S$ there exists exactly six bitangent lines*

$$l_{x,1}, \ldots, l_{x,6} \subset T_x S$$

passing through x and some other points $y_{x,1}, \ldots, y_{x,6} \in S$.

Proof. For the generic $x \in S$, the curve

$$C_x := T_x S \cap S$$

is of degree four and has exactly one singularity, a node at x. Let

$$\nu_x : \widetilde{C}_x \to C_x$$

be its normalisation and let

$$f_x : \widetilde{C}_x \to S$$

be the composition of ν_x with the inclusion $C_x \subset S$.

Since C_x is a degree four curve contained in $T_xS \cong \mathbb{P}^2$, its arithmetic genus can be computed with a generalisation of the Clebsch formula (Corollary 2.4) for singular curves, so that $p_a(C_x) = (4-1)(3-1)/2 = 3$. Moreover, since C_x has only one singular point that is a node, we can find the genus of its normalisation $g(\widetilde{C}_x) = p_a(C_x) - 1 = 2$ (cf. [10, Ch. V, Cor. 3.7]).

Choose now a generic line $\mathbb{P}^1 \subset T_xS \cong \mathbb{P}^2$ and consider the linear projection

$$\phi_x : C_x \to \mathbb{P}^1$$

from the node $x \in S$. As $deg(C_x) = 4$ and x is a node, ϕ_x is of degree two and clearly so is the composition

$$\widetilde{\phi}_x := \phi_x \circ \nu_x : \widetilde{C}_x \to \mathbb{P}^1.$$

Now we can use the Riemann-Hurwitz formula (see Theorem 2.3) $2-2g(\widetilde{C}_x) = deg(\widetilde{\phi}_x)(2-2g(\mathbb{P}^1)) - deg(R_{\widetilde{\phi}_x})$, where $R_{\widetilde{\phi}_x}$ is the ramification divisor, to obtain that $deg(R_{\widetilde{\phi}_x}) = 6$. Hence there are exactly six lines $l_{x,1}, \dots, l_{x,6} \subset T_xS$ passing through x such that they are bitangent to C_x, and so to S, at some points $y_{x,1}, \dots, y_{x,6} \in C_x$. □

The next lemma states that actually the $y_{x,i}$'s define the same element in $CH_0(S)$.

Lemma 5.26. *The points $y_{x,1}, \dots, y_{x,6} \in S$ defined in Lemma 5.25 are rational equivalent in $CH_0(S)$.*

Proof. Let

$$\widetilde{y}_{x,i} := \nu_x^{-1}(y_{x,i}) \in \widetilde{C}_x$$

for $i = 1, \dots, 6$, be the preimages of the y_i's in the normalisation.
Then $\widetilde{y}_{x,1}, \dots, \widetilde{y}_{x,6}$ are all linearly equivalent up to two torsion, since, by construction of $\widetilde{\phi}_x$, we have that $\widetilde{\phi}_x^* \mathcal{O}_{\mathbb{P}^1}(1) = \mathcal{O}(2 \cdot \widetilde{y}_{x,i}) \in Pic(\widetilde{C}_x)$ for all i.
Thus, considering the map $f_* : Pic(\widetilde{C}_x) \to CH_0(S)$, one finds that the elements

$$f_*(\mathcal{O}(2 \cdot \widetilde{y}_{x,1})) = 2 \cdot [y_{x,1}] \sim \dots \sim f_*(\mathcal{O}(2 \cdot \widetilde{y}_{x,6})) = 2 \cdot [y_{x,6}]$$

are rational equivalent in $CH_0(S)$. But $CH_0(S)$ is torsion free and then $[y_{x,1}] = \dots = [y_{x,6}] \in CH_0(S)$. □

Thanks to Lemma 5.26 we can now define the bitangent correspondence map.

Definition 5.27. The 1:6 correspondence $x \mapsto \{y_{x,1}, \dots, y_{x,6}\}$ induces a well defined involution, called the *bitangent correspondence*

$$\gamma : CH_0(S) \overset{\cong}{\to} CH_0(S), \quad [x] \mapsto [y_{x,1}].$$

This is clearly an involution since, if l_x is a bitangent through x and y_x, then $l_x = l_y$ is also the bitangent through $y = y_x$ and $x_y = x$.

Proposition 5.28. *The bitangent correspondence γ acts as $-id$ on $CH_0(S)_{hom}$.*

Proof. For any $x \in S$, write $[x] = c_S + \alpha_x$, where $\alpha_x := [x] - c_S \in CH_0(S)_{hom}$. To prove the Proposition we have to show that $\gamma(\alpha_x) = -\alpha_x$ for any $x \in S$.
Since any point of S is rationally equivalent to a cycle contained in a fixed non-empty open subset (see Proposition 2.15 for a more general statement), we can assume that $x \in S$ is generic. Then the construction of Lemma 5.25 still holds, so let the notation be as in the proof of Lemma 5.25.

Define the dualizing sheaf for the singular curve C_x, as in [3, pag.48], by

$$\omega_{C_x} := (\omega_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(C_x))_{|C_x}$$

(for the general definition of dualizing sheaf see [10, pag. 241]; the general definition agrees with the definition we gave by [10, Ch. III, Th. 7.11]).

Lemma 5.29. *We have the following isomorphisms:*

(i) $\omega_{\widetilde{C}_x} \cong \widetilde{\phi}_x^* \mathcal{O}_{\mathbb{P}^1}(-2) \otimes \mathcal{O}_{\widetilde{C}_x}(\sum_{i=1}^6 \widetilde{y}_{x,i})$.

(ii) $\omega_{\widetilde{C}_x} \cong \nu_x^* \omega_{C_x} \otimes \mathcal{O}_{\widetilde{C}_x}(-x_1 - x_2)$, *where* $x_1, x_2 \in \widetilde{C}_x$ *are the two points over the node* $x \in C_x$.

(iii) $\omega_{C_x} \cong \mathcal{O}_{\mathbb{P}^2}(1)_{|C_x}$.

(iv) $\widetilde{\phi}_x^* \mathcal{O}_{\mathbb{P}^1}(2) \cong \nu_x^*(\mathcal{O}_{\mathbb{P}^2}(2)_{|C_x}) \otimes \mathcal{O}_{\widetilde{C}_x}(-2x_1 - 2x_2)$.

Proof of Lemma 5.29. (i) Use the general version of Riemann-Hurwitz formula (see [10, Ch. IV, Prop. 2.3]) for the morphism $\widetilde{\phi}_x : \widetilde{C}_x \to \mathbb{P}^1$, noting that its ramification divisor is given by $\mathcal{O}_{\widetilde{C}_x}(\sum_{i=1}^6 \widetilde{y}_{x,i})$ and that $\omega_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(-2)$.

(ii) If $\nu_x : \widetilde{C}_x \to C_x$ is the normalisation, since x is node, we can think of $\widetilde{C}_x$ as the strict transform of C_x under the blow-up of the whole plane $\mathbb{P}^2$ in x. Explicitly, if we call again $\nu_x : \widetilde{\mathbb{P}^2} \to \mathbb{P}^2$ the blow-up of $\mathbb{P}^2$ in $x \in C_x \subset \mathbb{P}^2$, the normalisation $\widetilde{C}_x$ is given by $\widetilde{C}_x = \overline{\nu^{-1}(C_x - \{x\})}$. If we let $E := \nu^{-1}(x)$ be the exceptional divisor, we have that $\widetilde{C}_x \cap E = \{x_1, x_2\}$.

By the adjunction formula (see [9, pag. 147]) we have that

$$\omega_{\widetilde{C}_x} = \omega_{\widetilde{\mathbb{P}^2}|\widetilde{C}_x} \otimes \mathcal{O}_{\widetilde{\mathbb{P}^2}}(\widetilde{C}_x)_{|\widetilde{C}_x}.$$

If now we use the canonical bundle formula for the blow-up (see [10, Ch. II, ex. 8.5, (b)] or [17, Prop. 2.5.3]) applied to $\nu_x : \widetilde{\mathbb{P}^2} \to \mathbb{P}^2$ we obtain

$$\omega_{\widetilde{\mathbb{P}^2}|\widetilde{C}_x} = (\nu_x^* \omega_{\mathbb{P}^2})_{|\widetilde{C}_x} \otimes \mathcal{O}_{\widetilde{\mathbb{P}^2}}(E)_{|\widetilde{C}_x} = (\nu_x^* \omega_{\mathbb{P}^2})_{|\widetilde{C}_x} \otimes \mathcal{O}_{\widetilde{C}_x}(x_1 + x_2).$$

Hence (ii) is equivalent to $(\nu_x^* \omega_{\mathbb{P}^2})_{|\widetilde{C}_x} \otimes \mathcal{O}_{\widetilde{C}_x}(x_1 + x_2) \otimes \mathcal{O}_{\widetilde{\mathbb{P}^2}}(\widetilde{C}_x)_{|\widetilde{C}_x} \cong \nu_x^* \omega_{C_x} \otimes \mathcal{O}_{\widetilde{C}_x}(-x_1 - x_2)$. To conclude we then need to prove that $\nu_x^* \omega_{C_x} \cong (\nu_x^* \omega_{\mathbb{P}^2})_{|\widetilde{C}_x} \otimes \mathcal{O}_{\widetilde{C}_x}(2x_1 + 2x_2) \otimes \mathcal{O}_{\widetilde{\mathbb{P}^2}}(\widetilde{C}_x)_{|\widetilde{C}_x}$.

But by definition of ω_{C_x}, we have that

$$\nu_x^* \omega_{C_x} = \nu_x^*((\omega_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(C_x))_{|C_x}) = (\nu_x^* \omega_{\mathbb{P}^2})_{|\widetilde{C}_x} \otimes (\nu_x^* \mathcal{O}_{\mathbb{P}^2}(C_x))_{|\widetilde{C}_x}.$$

But since $x \in C_x$ is a point of multiplicity two, by [10, Ch. V, Prop. 3.6] we have that $\nu_x^*\mathcal{O}_{\mathbb{P}^2}(C_x) = \mathcal{O}_{\widetilde{\mathbb{P}^2}}(\widetilde{C}_x) + [E]^{\otimes 2}$ and so

$$(\nu_x^*\mathcal{O}_{\mathbb{P}^2}(C_x))_{|\widetilde{C}_x} = \mathcal{O}_{\widetilde{\mathbb{P}^2}}(\widetilde{C}_x)_{|\widetilde{C}_x} \otimes [E]^{\otimes 2}_{|\widetilde{C}_x} = \mathcal{O}_{\widetilde{\mathbb{P}^2}}(\widetilde{C}_x)_{|\widetilde{C}_x} \otimes \mathcal{O}_{\widetilde{C}_x}(2x_1 + 2x_2)$$

where we have used that $[E]_{|\widetilde{C}_x} = \mathcal{O}_{\widetilde{C}_x}(x_1 + x_2)$ since $\widetilde{C}_x \cap E = \{x_1, x_2\}$.

(iii) Since C_x is of degree four, by definition of the dualizing sheaf ω_{C_x} we obtain $\omega_{C_x} \cong (\omega_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(C_x))_{|C_x} \cong (\mathcal{O}_{\mathbb{P}^2}(-3) \otimes \mathcal{O}_{\mathbb{P}^2}(4))_{|C_x} \cong \mathcal{O}_{\mathbb{P}^2}(1)_{|C_x}$.

(iv) Using respectively (iii), (ii) and (i) we obtain that $\nu_x^*(\mathcal{O}_{\mathbb{P}^2}(1)_{|C_x}) \otimes \mathcal{O}_{\widetilde{C}_x}(-x_1 - x_2) \cong \widetilde{\phi}_x^*\mathcal{O}_{\mathbb{P}^1}(-2) \otimes \mathcal{O}_{\widetilde{C}_x}(\sum_{i=1}^6 \widetilde{y}_{x,i})$. Then iv) is equivalent to the isomorphism $\widetilde{\phi}_x^*\mathcal{O}_{\mathbb{P}^1}(2) \cong \widetilde{\phi}_x^*\mathcal{O}_{\mathbb{P}^1}(-4) \otimes \mathcal{O}_{\widetilde{C}_x}(2\sum_{i=1}^6 \widetilde{y}_{x,i})$, that is in turn equivalent to $\widetilde{\phi}_x^*\mathcal{O}_{\mathbb{P}^1}(6) \cong \mathcal{O}_{\widetilde{C}_x}(2\sum_{i=1}^6 \widetilde{y}_{x,i})$.

We have seen in the proof of Lemma 5.26 that, since $\widetilde{\phi}_x^*\mathcal{O}_{\mathbb{P}^1}(1) = \mathcal{O}_{\widetilde{C}_x}(2 \cdot \widetilde{y}_{x,i}) \in Pic(\widetilde{C}_x)$ for all i, the $\widetilde{y}_{x,i}$'s are linearly equivalent up to two torsion in $Pic(\widetilde{C}_x)$, i.e.

$$\mathcal{O}_{\widetilde{C}_x}(2 \cdot \widetilde{y}_{x,1}) = \cdots = \mathcal{O}_{\widetilde{C}_x}(2 \cdot \widetilde{y}_{x,6}) \in CH_0(S).$$

Hence we have that $\mathcal{O}_{\widetilde{C}_x}(2\sum_{i=1}^6 \widetilde{y}_{x,i}) = \mathcal{O}_{\widetilde{C}_x}(\sum_{i=1}^6 2\widetilde{y}_{x,i}) = \mathcal{O}_{\widetilde{C}_x}(12 \cdot \widetilde{y}_{x,1}) \in Pic(\widetilde{C}_x)$. We have then that (iv) is equivalent to $\widetilde{\phi}_x^*\mathcal{O}_{\mathbb{P}^1}(6) \cong \mathcal{O}_{\widetilde{C}_x}(12 \cdot \widetilde{y}_{x,1})$.

But we observed in the proof of Lemma 5.26 that $\widetilde{\phi}_x^*\mathcal{O}_{\mathbb{P}^1}(1) = \mathcal{O}_{\widetilde{C}_x}(2 \cdot \widetilde{y}_{x,i}) \in Pic(\widetilde{C}_x)$ and so, by multiplying by six, we are done. □

Using respectively (i), (ii) and (iii) of Lemma 5.29, we obtain $\mathcal{O}_{\widetilde{C}_x}(\sum_{i=1}^6 \widetilde{y}_{x,i}) \cong (\nu_x^*(\mathcal{O}_{\mathbb{P}^2}(1)_{|C_x}) \otimes \mathcal{O}_{\widetilde{C}_x}(-x_1 - x_2) \otimes \widetilde{\phi}_x^*\mathcal{O}_{\mathbb{P}^1}(2)$.

But by (iv) of Lemma 5.29 we have also that $\widetilde{\phi}_x^*\mathcal{O}_{\mathbb{P}^1}(2) \cong \nu_x^*(\mathcal{O}_{\mathbb{P}^2}(2)_{|C_x}) \otimes \mathcal{O}_{\widetilde{C}_x}(-2 \cdot x_1 - 2 \cdot x_2)$, so that we finally have

$$\mathcal{O}_{\widetilde{C}_x}(\sum_{i=1}^6 \widetilde{y}_{x,i}) \cong \nu_x^*(\mathcal{O}_{\mathbb{P}^2}(3)_{|C_x}) \otimes \mathcal{O}_{\widetilde{C}_x}(-2 \cdot x_1 - 2 \cdot x_2).$$

Recall now that $f_x : \widetilde{C}_x \to S$ is the composition of the normalisation and the inclusion $j : C_x \hookrightarrow S$. Then, since $C_x \subset \mathbb{P}^2$ is of degree four, $\mathcal{O}_{\mathbb{P}^2}(1)_{|C_x} \cong \mathcal{O}_{C_x}(\sum_{i=1}^4 \overline{x}_i)$, for $\{\overline{x}_1, \ldots, \overline{x}_4\} = C_x \cap t$ where $t \cong \mathbb{P}^1$ is a generic line in $\mathbb{P}^2$. By definition of c_S, we have then that $j_*(\overline{x}_1) = c_S \in CH_0(S)$ and so $f_{x*}(\nu_x^*(\mathcal{O}_{\mathbb{P}^2}(1)_{|C_x})) = 4 \cdot c_S$.

Hence we have

$$f_{x*}(\mathcal{O}_{\widetilde{C}_x}(\sum_{i=1}^6 \widetilde{y}_{x,i})) = f_{x*}(\nu_x^*(\mathcal{O}_{\mathbb{P}^2}(3)_{|C_x}) + f_{x*}(\mathcal{O}_{\widetilde{C}_x}(-2 \cdot x_1 - 2 \cdot x_2)) = 3 \cdot (4 \cdot c_S) - 6 \cdot [x] = 6 \cdot c_S - 6 \cdot \alpha_x.$$

But by Lemma 5.26 we have $[y_{x,1}] = \cdots = [y_{x,6}] \in CH_0(S)$ and so we have also $f_{x*}(\mathcal{O}_{\widetilde{C}_x}(\sum_{i=1}^6 \widetilde{y}_{x,i})) = \sum_{i=1}^6 [y_{x,i}] = 6 \cdot [y_{x,1}] \in CH_0(S)$.
Since by definition $\gamma([x]) = [y_{x,1}] \in CH_0(S)$, we obtain

$$(6 \cdot \gamma)([x]) = f_*(\sum_{i=1}^6 \widetilde{y}_{x,i}) = 6 \cdot c_S - 6 \cdot \alpha_x.$$

Thus for $[x] \in CH_0(S)$ with $\alpha_x = 0$ this shows that $(6 \cdot \gamma)([x]) = 6 \cdot c_S$ and so for an arbitrary $[x] = c_S + \alpha_x$ we have that $6 \cdot c_S - 6 \cdot \alpha_x = (6 \cdot \gamma)([x]) = (6 \cdot \gamma)(c_S) + (6 \cdot \gamma)(\alpha_x) = 6 \cdot c_S + (6 \cdot \gamma)(\alpha_x)$ so that

$$6 \cdot (\gamma(\alpha_x)) = -6 \cdot \alpha_x.$$

But since $CH_0(S)$ is torsion free, we have that $\gamma(\alpha_x) = -\alpha_x \in CH_0(S)_{hom}$ and then Proposition 5.28 is proved. □

We now give a more geometric way of defining the bitangent correspondence that will lead us to the construction of some constant cycle curves.

One can show that the bitangents of S can be parametrized by a variety F_S, so that every bitangent $l \subset S$ corresponds to a point $[l] \in F_S$. Moreover one can construct a variety B_S whose points are (l, x), with l bitangent of S and $x \in S \cap l$ a point of contact. It is known that F_S and B_S are smooth irreducible surfaces.

We have then the following situation

$$\begin{array}{ccc} B_S & \xrightarrow{p} & S \\ \downarrow{\scriptstyle q} & & \\ F_S & & \end{array}$$

where $p(l, x) = x$ and $q(l, x) = l$ are the projections. Since every bitangent l has two points of contact with S, the map q is of degree two with a ramification divisor $D_{hf} \subset F_S$ given by points of F_S corresponding to quadritangents (note that, since F_S is a surface, then D_{hf} is a curve). Moreover we know that, since by Lemma 5.25 there are exactly six bitagents through a generic $x \in S$, the map p is of degree six.

Consider now the covering involution $i : B_S \to B_S$ with respect to the double covering $q : B_S \to F_S$, so that, for the generic bitangent l, we have $i(l, x) = (l, y)$, where $S \cap l = \{x, y\}$.

Lemma 5.30. *In this context the map $6 \cdot \gamma : CH_0(S) \to CH_0(S)$ can be written as*

$$6 \cdot \gamma = p_* \circ i^* \circ p^* : CH_0(S) \to CH_0(B_S) \to CH_0(B_S) \to CH_0(S).$$

Proof. Recall that $6 \cdot \gamma([x]) = [6 \cdot y_{x,1}] = [\sum_{i=1}^{6} y_{x,i}] \in CH_0(S)$. Then it is enough to write explicitly the action of $p_* \circ i^* \circ p^*$. We have indeed, for $x \in S$ generic,

$$[x] \xmapsto{p^*} \sum_{i=1}^{6} (l_{x,i}, x) \xmapsto{i^*} \sum_{i=1}^{6} (l_{x,i}, y_{x,i}) \xmapsto{p_*} \sum_{i=1}^{6} y_{x,i}.$$

□

Consider now the curve of quadritangents $D_{hf} \subset F_S$ and let

$$C_{hf} := p(q^{-1}(D_{hf})) \subset X$$

be the curve of all points $x \in S$ such that there exists a quadritangent at x.

Remark 5.31. The curve $C_{hf} \subset S$ should be seen as the analogue of the degree six branch curve $C \subset S \to \mathbb{P}^2$ of Lemma 5.18.

We have now finally arrived to the example of constant cycle curve we aimed for.

Proposition 5.32. *For a quartic $S \subset \mathbb{P}^3_{\mathbb{C}}$ not containing a line, the curve $C_{hf} \subset S$ of contact points of quadritangents is a constant cycle curve.*

Proof. For a rigorous proof of the fact that C_{hf} is a constant cycle curve, in which it is shown also that the order of C_{hf} divides four, see [14, Prop 8.6]. We now give a pointwise geometric argument. First of all note that any quadritangent $l \subset T_xS$ at a point $x \in C_{hf}$ is the limit of proper bitangents $l_t := \overline{x_t, y_t} \to l$, where $x_t \to x$ and $y_t \to x$. As in the proof of Proposition 5.28, write $[x_t] = c_S + ([x_t] - c_S)$ and $[y_t] = c_S + ([y_t] - c_S)$. In that proof we showed that $\gamma(c_S) = c_S$ and $\gamma_{|CH_0(S)_{hom}} = -id$, so that we obtain

$$\gamma([x_t]+[y_t]) = \gamma(2c_S)+\gamma([x_t]-c_S)+\gamma([y_t]-c_S) = 2c_S+(-[x_t]+c_S)+(-[y_t]+c_S) = 4c_S-[x_t]-[y_t].$$

But by definition $\gamma([x_t]) = [y_t]$ and $\gamma([y_t]) = [x_t]$, so that we have also $\gamma([x_t] + [y_t]) = [x_t] + [y_t]$. Then we obtain $4c_S - [x_t] - [y_t] = [x_t] + [y_t]$ and so we have

$$[x_t] + [y_t] = 2 \cdot c_S \in CH_0(S).$$

But $[x_t], [y_t] \to [x]$, so that, to the limit, we have $2 \cdot [x] = 2 \cdot c_S \in CH_0(S)$. Since $CH_0(S)$ is torsion free we have then proved that, for every $x \in C_{hf}$, we have $[x] = c_S \in CH_0(S)$ and so C_{hf} is a pointwise constant cycle curve. Since we are working over $\mathbb{C}$, by Proposition 5.14 we obtain that C_{hf} is indeed also a constant cycle curve. □

6. Symplectic involutions of K3 surfaces act trivially on CH_0

6.1 Introduction

In this chapter we will present the results of the paper [25] by Voisin. Unlike the previous chapter, we now work only on complex projective K3 surfaces.

Recall the result established at the end of Chapter 1 in Remark 1.41, which states that if a correspondence $\Gamma \in CH_k(X \times Y)$ between two smooth complex projective varieties induces the 0-map $\Gamma_* : CH_0(X)_{hom} \to CH_0(Y)_{hom}$, then the maps $[\Gamma]^* : H^i(Y, \mathbb{C}) \to H^i(X, \mathbb{C})$ vanish on $H^{i,0}(Y) \subset H^i(Y, \mathbb{C})$ for any $i > 0$.

We recall now Conjecture 1.45 which is a special case of the famous Bloch conjecture.

Conjecture (Bloch Conjecture)**.** *Let S be a smooth projective complex surface and let X be a smooth projective complex variety of dimension d. If $\Gamma \in CH_d(X \times S)$ is a correspondence such that the maps $[\Gamma]^* : H^{i,0}(S) \to H^{i,0}(X)$ vanish for $i > 0$, then*

$$\Gamma_* : CH_0(X)_{alb} \to CH_0(S)$$

vanishes, where $CH_0(X)_{alb} := Ker(alb_X : CH_0(X)_{hom} \to Alb(X))$.

One can study a particular case of this conjecture considering the action of a symplectic automorphism on a K3 surface. Since for a K3 surface S by definition we have that $H^0(S, \Omega^1_S) = 0$ and the sheaf of holomorphic 2-forms is trivial, a finite order automorphism g on S is symplectic if and only if g acts trivially on $H^{2,0}(S)$, i.e. $g^*(\omega) = \omega$, where ω is the holomorphic 2-form on S.

From now on let S be a complex projective K3 surface provided with a symplectic automorphism $g : S \to S$ of finite order. We can take a correspondence $\Gamma \in CH_2(S \times S)$ given by

$$\Gamma := \Gamma_g - \Delta$$

where Γ_g is the graph of a finite order symplectic automorphism g and Δ is the diagonal of $S \times S$. Since $H^{1,0}(S) = 0$ and g is symplectic, we have that $[\Gamma]^* = g^* - id : H^{i,0}(S) \to H^{i,0}(S)$ is the zero map for $i = 0, 1, 2$.
Recall now that $Alb(S) = 0$ (cf. 1.29), so that $CH_0(S)_{alb} = CH_0(S)_{hom}$. Hence in this case, if the Bloch conjecture were true, we would have that

$$\Gamma_* = g_* - id : CH_0(S)_{hom} \to CH_0(S)$$

is the zero map, so that g_* would act as the identity on $CH_0(S)_{hom}$.

In her paper, Voisin proved this particular case of the Bloch conjecture for a symplectic involution $g = i$, i.e. a symplectic automorphism of order two. In this chapter we will present her proof of the fact that a symplectic involution on a K3 surface S acts trivially on $CH_0(S)$.

The proof is based on two propositions. The first result (Proposition A) states that the group of i-anti-invariant 0-cycles on S is finite dimensional in Roitman's sense. This, combined with the other result (Proposition B) which is about the factorisation of the homomorphism induced by a correspondence through the Albanese morphism, allows us to conclude that the group of i-anti-invariant 0-cycles on S is in fact trivial.

Proposition B is a general result not concerning K3 surfaces or symplectic involutions, so that the fundamental ingredient specific for the case of involutions on K3 surfaces lies in Proposition A. We will see (cf. Key Lemma in Section 6.3) that it is the fact that the Prym varieties of étale double covers of curves of genus g are of dimension $g-1$. This observation will be applied to the étale double covers of generic smooth ample curves $C \subset S/i$.

Note that in order to define the Prym variety we need specifically that i is an involution and so this proof cannot be generalised to automorphisms of higher order.

However this result by Voisin on symplectic involutions, together we other advanced results about a derived category of coherent sheaves on S, has been used by Huybrechts in [16] to prove that any symplectic automorphism of finite order, not only of order two, on a K3 surface S acts trivially on $CH_0(S)$. We will not however present the proof of this last result.

6.2 Proof of the triviality on CH_0 of symplectic involutions

In this section we want to prove the following theorem, that is the central result of the chapter.

Theorem 6.1 (Voisin, cf. [25]). *Let S be an algebraic K3 surface, and let $i : S \to S$ be a symplectic involution. Then i_* acts as the identity on $CH_0(S)$.*

The proof of Theorem 6.1 is essentially based on two propositions, whose proofs by Voisin (cf. [25]) will be shown in Sections 6.3 and 6.4.

Before stating the two propositions, we need to generalise one of the equivalent definitions of finite dimensionality in Roitman's sense (cf. Proposition 1.38 (iv)).

Definition. Let X be a smooth connected projective variety over $\mathbb{C}$ and let $P \subset CH_0(X)$ be a subgroup.

We say that P is *finite dimensional in Roitman's sense* if there exists a smooth projective variety W and a correspondence $\Gamma \in CH_k(W \times X)$, with $k = dim(W)$, such that P is contained in the set $\{\Gamma_*(w),\ w \in W\}$.

Remark 6.2. If $P \subset CH_0(X)$ is a finite dimensional subgroup in Roitman's sense, then $P \subset CH_0(X)_{hom}$.

Indeed P is a subgroup of $CH_0(X)$, so, if $p \in P$, then $\{np,\ n \in \mathbb{Z}\} \subset P$. Hence, if $deg(p) \neq 0$, then the degrees of cycles in $\{np,\ n \in \mathbb{Z}\} \subset P$ would have infinitely many possible values.

This cannot happen if $P \subseteq \{\Gamma_*(w),\ w \in W\}$ because the degrees of the cycles in the set $\{\Gamma_*(w),\ w \in W\}$ can only have finitely many possible values.

Let now

$$CH_0(S)^- := \{z \in CH_0(S),\ i_*(z) = -z\} \subseteq CH_0(S)$$

be the subgroup of $CH_0(S)$ given by the anti-invariant part with respect to i_*. Clearly $CH_0(S)^- \subseteq CH_0(S)_{hom}$.

Lemma 6.3. *$CH_0(S)^-$ is a divisible group.*

Proof. Theorem 4.2 states that $CH_0(S)$ is a divisible group, i.e. we have that, for any $n \in \mathbb{Z}$ and $z \in CH_0(S)^-$, there exists a $z' \in CH_0(S)$ such that $nz' = z$. But, if $z' \in CH_0(S)$ is such that $nz' = z$ with $z \in CH_0(S)^-$, then $z' \in CH_0(S)^-$, so that indeed also $CH_0(S)^-$ is a divisible group . But $ni_*(z') = i_*(nz') = i_*(z) = -z = -nz'$ and we know that $CH_0(S)$ is torsion free by Theorem 4.1, so we obtain $i_*(z') = -z'$. □

Consider the correspondence $Z \subset S \times S$ given by $Z = \Delta_S - \Gamma_i$, where Δ_S is the diagonal of S and Γ_i is the graph of i. Then $Z_* : CH_0(S) \to CH_0(S)$ is the map $Z_*(z) = z - i_*(z),\ z \in CH_0(S)$ and it is related to the subgroup $CH_0(S)^-$ by the following lemma.

Lemma 6.4. *$Z_{*|CH_0(S)^-}$ is the multiplication by 2 on $CH_0(S)^-$ and $Im(Z_*) = CH_0(S)^-$.*

Proof. It is easy to see that $Z_{*|CH_0(S)^-} = 2\cdot$ is the multiplication by 2 on $CH_0(S)^- \subseteq CH_0(S)_{hom}$: if $z \in CH_0(S)^-$ then $Z_*(z) = z - i_*(z) = z - (-z) = 2z$.
We have also that $Im(Z_*) \subseteq CH_0(S)^-$: if $z \in Im(Z_*)$, then $z = Z_*(z') = z' - i_*(z')$ for some $z' \in CH_0(M)$, and hence $i_*(z) = i_*(z' - i_*(z')) = i_*(z') - z' = -z$. Therefore $z \in CH_0(S)^-$.

To conclude we observe that, by Lemma 6.3, $CH_0(S)^-$ is a divisible group, so the map $Z_{*|CH_0(S)^-} = 2\cdot : CH_0(S)^- \to CH_0(S)^-$ must be surjective. □

We can now state the first proposition that will be used to prove Theorem 6.1.

Proposition A. *Let S be an algebraic K3 surface, and let $i : S \to S$ be a symplectic involution. Then the anti-invariant part $CH_0(S)^-$ is finite dimensional in Roitman's sense.*

The proof of Proposition A will be given in Section 6.3.

The other key proposition that will be used to prove Theorem 6.1 is a general proposition about the factorisation of the homomorphism induced by a correspondence through the Albanese morphism.

Proposition B. *Let M and X be smooth projective varieties, $dimX = d$. Let $Z \in CH_d(M \times X)$ be a correspondence and let $Z_* : CH_0(M) \to CH_0(X)$ be the induced map. Assume that $Z_*(CH_0(M))$ is finite dimensional in Roitman's sense.*
Then the map $Z_* : CH_0(M)_{hom} \to CH_0(X)$ factors through the Albanese morphism $alb_M : CH_0(M)_{hom} \to Alb\ M$.

The proof of Proposition B will be given in Section 6.4.

Now we are ready to prove Theorem 6.1.

Proof of Theorem 6.1. We will apply Proposition B to the case of $X = M = S$, where S is our algebraic K3 surface and the correspondence $Z = \Delta_S - \Gamma_i \subset S \times S$ is given by $Z = \Delta_S - \Gamma_i$ as above. Since $Z_* : CH_0(S) \to CH_0(S)$ is the map $Z_*(z) = z - i_*(z),\ z \in CH_0(S)$, if we show that $Z_* \equiv 0$ on $CH_0(S)$ then Theorem 6.1 is proved.

Combining Lemma 6.4 and Proposition A, we obtain that $Im(Z_*)$ is finite dimensional in Roitman's sense. So we can apply Proposition B.
Then $Z_* : CH_0(S)_{hom} \to CH_0(S)$ factors through *Alb* S. But, since S is a K3 surface, *Alb* $S = 0$ and so we have that $Z_{*|CH_0(S)_{hom}} = 0$.
By Lemma 6.4, we know also that $Z_{*|CH_0(S)^-}$ is the multiplication by 2 on $CH_0(S)^- \subseteq CH_0(S)_{hom}$.
So we have proven that $Z_{*|CH_0(S)^-} = 2\cdot = 0$.
Hence, if $z \in CH_0(S)^-$, then $Z_*(z) = 2z = 0$, i.e. $CH_0(S)^-$ is a 2-torsion subgroup of $CH_0(S)$.
But we know that $CH_0(S)$ is torsion free (Theorem 4.1) and so we must have $CH_0(S)^- = 0$.
Again by Lemma 6.4, we have that $CH_0(S)^- = Im(Z_*)$ and so we have proven that $Im(Z_*) = 0$, that is to say that Z_* is the zero map.
Therefore $Z_*(z) = z - i_*(z) = 0$ for all $z \in CH_0(S)$ and then $i_*(z) = z$ for all $z \in CH_0(S)$.
Hence i_* acts as the identity on $CH_0(S)$ and Theorem 6.1 is proven. □

In the next two sections we will show Voisin's proofs of Propositions A and B presented in [25].

6.3 Proof of Proposition A

In this section we will show the proof of Proposition A.

Proposition A. *Let S be an algebraic K3 surface and let $i : S \to S$ be a symplectic involution. We want to prove that*

$$CH_0(S)^- = \{z \in CH_0(S),\ i_*(z) = -z\}$$

is finite dimensional in Roitman's sense.

6.3.1. Setting.

We now need to fix some notation and establish basic facts that we will use in 6.3.2.

Let

$$\pi : S \to \Sigma = S/i \text{ be the canonical projection}$$

and let

$$b : \widetilde{\Sigma} \to S/i \text{ be the blow up of the singular points of } \Sigma.$$

In Section 3.3 we proved that $\widetilde{\Sigma}$ is a K3 surface.

Let now

$$L \in Pic(\Sigma) \text{ be a very ample line bundle}$$

and define $g \in \mathbb{Z}$ such that

$$c_1(L)^2 = 2g - 2.$$

Lemma 6.5. *We have $g+1 = h^0(\Sigma, L)$ and $g = g(C)$ is the genus of any smooth irreducible curve $C \in |L|$.*

Proof of Lemma 6.5. Note that, L being very ample, we can move $C \in |L|$ to avoid the singular points of Σ. But then we have that C is isomorphic to a curve $b^*(C)$ in the K3 surface $\widetilde{\Sigma}$. Then Lemma 6.5 follows from the Riemann-Roch Theorem for K3 surfaces with L ample (see Corollary 3.5 (i)), $h^0(\Sigma, L) = c_1(L)^2/2+2$, and the genus formula (see Corollary 3.5 (iii)) $g(C) = 1+\frac{1}{2}(C.C) \in \mathbb{Z}$, where $C \in |L|$ is a smooth irreducible curve. □

Lemma 6.6. *For a smooth ample curve $C \subset \Sigma$ not containing any singular point of Σ, the inverse image $\widetilde{C} := \pi^{-1}(C) \subset S$ is smooth, connected and it is an étale double covering of C.*

Proof of Lemma 6.6. As C is smooth, we have that C does not contain the singular points of σ and outside of these points, π is an étale double covering. So we have that $\widetilde{C}$ is smooth and an étale double covering of C.
We must only check the connectedness.
If $\widetilde{C}$ is not connected then

$$\widetilde{C} = \widetilde{C}_1 + \widetilde{C}_2, \quad \widetilde{C}_1.\widetilde{C}_2 = 0$$

where $\widetilde{C}_1$, $\widetilde{C}_2$ are the connected components of $\widetilde{C}$ and $. : Pic(S) \otimes Pic(S) \to \mathbb{Z}$ is the intersection product defined in Section 3.1 given by $D_1.D_2 := deg(D_1 \cdot D_2) \in \mathbb{Z}$ for any $D_1, D_2 \in Pic(S)$.
First note that

$$\widetilde{C}_1.\widetilde{C}_1 = i^*(\widetilde{C}_2).i^*(\widetilde{C}_2) = deg(i^*(\widetilde{C}_2 \cdot \widetilde{C}_2)) = deg(\widetilde{C}_2 \cdot \widetilde{C}_2) = \widetilde{C}_2.\widetilde{C}_2.$$

Since C is ample we have $C.C > 0$ (cf. Lemma 3.3) and so $\widetilde{C}.\widetilde{C} = \pi^*(C).\pi^*(C) = deg(\pi^*(C{\cdot}C)) > 0$.
But then we have $\widetilde{C}_1.\widetilde{C}_1 + \widetilde{C}_2.\widetilde{C}_2 = \widetilde{C}.\widetilde{C} > 0$ and so

$$\widetilde{C}_1.\widetilde{C}_1 = \widetilde{C}_2.\widetilde{C}_2 > 0.$$

We have thus a contradiction by the Hodge index Theorem (cf. Theorem 3.4.4), which states that, if $L_1 \in Pic(S)$ is big, i.e. $L_1.L_1 > 0$, and $L_2 \neq 0 \in Pic(S)$, then $(L_1.L_2) < 0$. Indeed we have found that $\widetilde{C}_1.\widetilde{C}_1 > 0$ and $\widetilde{C}_1.\widetilde{C}_2 = 0$.

□

6.3.2. Proof of Proposition A

Let M, X be smooth projective varieties with X of dimension d. Let $Z \in CH_d(M \times X)$ be a correspondence and let $Z_* : CH_0(M) \to CH_0(X)$ be the induced map. Each point $(m_1, \ldots, m_k) \in M^k$ determines an element $\sum_i m_i \in CH_0(M)$ and so we have a map

$$Z_*^k : M^k \to CH_0(X), \quad (m_1, \ldots, m_k) \mapsto Z_*(\sum_i m_i).$$

Lemma 6.7. *Assume there is a point $m \in M$ such that $Z_*(m) = 0 \in CH_0(X)$ and that for some integer $g > 0$, one has $Z_*^{g-1}(M^{g-1}) = Z_*^g(M^g)$ in $CH_0(X)$. Then $Im(Z_*)$ is finite dimensional in Roitman's sense.*

Proof. Since $Z_*^{g-1}(M^{g-1}) = Z_*^g(M^g)$, by induction we have that $Z_*^{g-1}(M^{g-1}) = Z_*^k(M^k)$, $\forall k \geq g-1$.
Any cycle $z \in CH_0(M)$ can be written as $z^+ - z^-$, with z^+ and z^- effective cycles of degree k^+, k^-. Up to adding multiples of m to z^+ and z^- (which does not change $Z_*(z)$) we may assume $k = k^+ = k^- \geq g$. Then $Z_*(z) = Z_*(z^+) - Z_*(z^-)$ with $Z_*(z^+),\ Z_*(z^-) \in Z_*^k(M^k) = Z_*^{g-1}(M^{g-1})$.

So we have proved that the correspondence $Z' \in CH_d(M^{2g-2} \times X)$ given by

$$Z' := \sum_{i \leq g-1} (pr_i, p_X)^*(Z) - \sum_{g \leq i \leq 2g-2} (pr_i, p_X)^*(Z)$$

satisfies $Im(Z_*) = \{Z'_*(m),\ m \in M^{2g-2}\}$.

Hence $Im(Z_*)$ is finite dimensional in Roitman's sense. □

Let now S be a K3 surface endowed with a symplectic involution i.

Let $Z \in CH_0(S \times S)$ be the correspondence

$$Z = \Delta_S - \Gamma_i, \quad Z_* = id - i^* : CH_0(S) \to CH_0(S).$$

We call c_S the effective 0-cycle of degree 1 defined in Theorem 4.3. This is the class of any point on any rational curve contained in S.

The following Lemma is the key ingredient of this section.

Lemma (Key Lemma)**.** *We have $Z_*(c_S) = 0$ and $Z_*^g(S^g) = Z_*^{g-1}(S^{g-1})$.*

If we admit the Key Lemma, we can finally easily prove Proposition A.

Proof of Proposition A. By the Key Lemma the correspondence $Z = \Delta_S - \Gamma_i$ satisfies the hypothesis of Lemma 6.7 and hence $Im(Z_*)$ is finite dimensional. But by Lemma 6.4 we have $Im(Z_*) = CH_0(S)^-$ and so we have finally proved that $CH_0(S)^-$ is finite dimensional. □

So the only thing left is to prove the Key Lemma.

Proof of Key Lemma. We have clearly that $Z_*(c_S) = i_*(c_S) - c_S = 0$ because if $c_S = [x],\ x \in D$, a rational curve, then $i(x) \in i(D)$ is again a rational curve as i is an automorphism.

Let $L \in Pic(\Sigma)$ be a very ample line bundle with $c_1(L)^2 = 2g - 2$ as in 6.3.1.

Let now $s = (s_1, \ldots, s_g)$ be a generic point of S^g. Then if we denote $\sigma_i := \pi(s_i) \in \Sigma$, then also $(\sigma_1, \ldots, \sigma_g)$ is generic in Σ^g.

Remark 6.8. There exists a *unique* curve $C_s \in |L|$ containing all the $\sigma_i' s$.
This is because L is very ample, so $|L|$ can be seen as the linear system of hyperplane sections given by the embedding $\phi_{|L|}$ and, by Lemma 6.5, we have $g + 1 = h^0(\Sigma, L)$ so the curve passing through g points is unique (the condition of passing through a point is a linear condition so decreases by 1 the dimension of a linear system).

Since the σ_i's are generic, we have that C_s is generic in $|L|$ and so, by Bertini's theorem (cf. Theorem 2.1), it is smooth outside the base locus of $|L|$ that is zero (L is very ample). Denote by

$$U \subset S^g$$

the open subset such that, for every $s = (s_1, \dots, s_g) \in U$, C_s is smooth.
By Lemma 6.6, for any $s = (s_1, \dots, s_g) \in U$, we have that $\widetilde{C_s} := \pi^{-1}(C_s)$ is smooth, connected and it is an étale double covering of C_s. Moreover $s_i \in \widetilde{C_s}$ for any $i = 1, \dots, g$.

Now consider the 0-cycle

$$z_s = \sum_l s_l - i(\sum_l s_l) = Z_*(\sum_l s_l) = Z_*^g(s) \in CH_0(S)_{hom}.$$

Since $s_i \in \widetilde{C_s}$, we have $z_s \in CH_0(\widetilde{C_s})_{hom}$. Let now $alb_{\widetilde{C_s}}$ be the albanese map

$$alb_{\widetilde{C_s}} : CH_0(\widetilde{C_s})_{hom} = Pic^0(\widetilde{C_s}) \to \mathcal{J}(\widetilde{C_s}) = Alb(\widetilde{C_s})$$

which is an isomorphism by Theorem 2.10 of Abel.

Since $i(z_s) = -z_s$ we have that, up to 2-torsion, $alb_{\widetilde{C_s}}(z_s) \in \mathcal{P}(\widetilde{C_s}/C_s)$ where $\mathcal{P}(\widetilde{C_s}/C_s)$ is the Prym variety (see Section 2.3).

Moreover by Section 2.3 we know that $\mathcal{P}(\widetilde{C_s}/C_s)$ is a $g-1$ dimensional Abelian variety .

Remark 6.9. The fact that the construction of the Prym variety is specific for the case of involutions is the reason why this proof doesn't fit for symplectic automorphisms of order bigger than 2.

If $|L|_0 \subset |L|$ is the open set parametrizing the smooth curves, denote by $\mathcal{P}(\widetilde{C}/C) \to |L|_0$ the Prym fibration. This is the fibration whose fibre over a point $c_s \in |L|_0$ representing the curve C_s is the Prym variety $\mathcal{P}(\widetilde{C_s}/C_s)$.

We have so far proved that, for any $s \in U$, $alb_{\widetilde{C_s}}(Z_*^g(s)) = alb_{\widetilde{C_s}}(z_s) \in \mathcal{P}(\widetilde{C_s}/C_s)$. Then we have a factorisation of the map Z_*:

$$\begin{array}{ccc} U & \xrightarrow{Z_*^g} & CH_0(S) \\ & \overset{f}{\searrow} & \downarrow alb \\ & & \mathcal{P}(\widetilde{C}/C) \end{array}$$

In order to compute the dimension of the fibre of Z_*^g we need the following result.

Lemma 6.10 (Dimension of the fibres of a morphism)**.** *If $f : Y' \to Y$ is a dominant morphism of irreducible projective varieties then $dim(Y') \geq dim(Y)$ and, for any $y \in f(Y')$, $dim(f^{-1}(y)) \geq dim(Y') - dim(Y)$.*

Proof. See [10, Ch. II, ex.3.22] or [24, II, Ch. 2, Th. 4.4]. □

Since $\mathcal{P}(\widetilde{C_s}/C_s)$ is a $g-1$ dimensional Abelian variety, the fibration $\mathcal{P}(\widetilde{C}/C)$ has dimension $2g-1$. Indeed $dim(\mathcal{P}(\widetilde{C}/C) = dim(\text{base}) + dim(\text{fibre}) = dim(|L|_0) + dim(\mathcal{P}(\widetilde{C_s}/C_s)) = g + g - 1 = 2g - 1$. Therefore, by Lemma 6.10, the fibres of the morphism f are of dimension at least one, since $dim(U) - dim(\mathcal{P}(\widetilde{C}/C)) = 2g - (2g - 1) = 1$.
But $alb_{\widetilde{C}_s}$ is an isomorphism for any $s \in U$, and so the dimension of the fibres of f is the same as the dimension of the fibres of Z^g_*.
It follows that, for $s \in U$, there exists a curve $F_s \subset (Z^g_*)^{-1}(z_s) \subset S^g$ such that

$$Z^g_*(t) = \sum_l t_l - i(\sum_l t_l) = z_t \sim z_s \text{ for any } \; t = (t_1, \ldots, t_g) \in F_s.$$

By Corollary 3.16, we can choose a very ample divisor $D \subset S$ whose irreducible components are rational curves. If $pr_l : S^g \to S$ is the l-th projection, we have that F_s meets the ample divisor $\sum_l pr_l^{-1}(D) = (D \times S^{g-1}) \cup (S \times D \times S^{g-2}) \cup \cdots \cup (S^{g-1} \times D)$ (this is clearly ample because D is ample in S) and hence there exists a point $\bar{t} = (\bar{t}_1, \ldots, \bar{t}_g) \in F_s \bigcap \sum_l pr_l^{-1} D$.
This means that there exists a point $\bar{t} = (\bar{t}_1, \ldots, \bar{t}_g) \in F_s$ such that

$$\bar{t}_{l_0} \in D \text{ for some } l_0 \in \{1, \ldots, g\}$$

with D divisor whose irreducible components are rational curves.
But then $[\bar{t}_{l_0}] = c_S$ ($\bar{t}_{l_0}$ belongs to a rational curve, a component of D) and so $\bar{t}_{l_0} - i(\bar{t}_{l_0}) \sim 0$ in $CH_0(S)$ because at the beginning of the proof of the Key Lemma we noted that $i_*(c_S) = c_S$.
Hence we have $z_{\bar{t}} = \sum_l \bar{t}_l - i(\sum_l \bar{t}_l) \sim \sum_{l \neq l_0} \bar{t}_l - i(\sum_{l \neq l_0} \bar{t}_l) \in Z^{g-1}_*(S^{g-1})$.
But $z_s \sim z_t$ for any $(t_1, \ldots, t_g) \in F_s$ and so in particular $z_s \sim z_{\bar{t}} \in Z^{g-1}_*(S^{g-1})$.
So we have proven that, if $z_s \subset Z^g_*(U)$ with $U \subset S^g$ open subset, then $z_s \in Z^{g-1}_*(S^{g-1})$.
To conclude the proof we need that the above result is true for any $s \in S^g$.
But now we apply Proposition 2.15 which implies that any cycle in S^g is rationally equivalent to a cycle in $U \subset S^g$. Then the cycles $z_s \in CH_0(S)$ with $s \in U$, fill up the entire $Z^g_*(S^g)$.
We have then shown that $Z^g_*(S^g) = Z^{g-1}_*(S^{g-1})$ and the Key Lemma is proved.

□

6.4 Proof of Proposition B

In this section we give the proof of Proposition B.

Proposition B. *Let M and X be smooth projective varieties, with $dim X = d$. Let $Z \in CH_d(M \times X)$ be a correspondence and let $Z_* : CH_0(M) \to CH_0(X)$ be the induced map. Assume that $Z_*(CH_0(M))$ is finite dimensional in Roitman's sense.*
We want to prove that the map $Z_* : CH_0(M)_{hom} \to CH_0(X)$ factors through the Albanese morphism $alb_M : CH_0(M)_{hom} \to Alb\ M$, that is to say that $Z_{*|ker Alb\ M} = 0$.

6.4.1. Setting and preliminary results.

First of all, we want to construct a smooth curve $C \subset M$ with genus as high as we will need.

Let $M \subset \mathbb{P}^N$, $m := dim(M)$ and $deg(M)$ the degree of M.

Lemma 6.11. *For fixed $g \in \mathbb{Z} > 0$, there are ample hypersurfaces $H_i \in M$, for $i = 1, \ldots, m-1$, such that $C := \bigcap_{i=1}^{m-1} H_i$ is a smooth curve of genus $g(C) > g$.*

Proof of Lemma 6.11. By Bertini's Theorem (cf. Theorem 2.1), the generic element of a linear system is smooth away the base locus of the system.

Consider then the linear system on M cut by hypersurfaces of $\mathbb{P}^N$ of sufficiently high degree. Since this linear system is very ample, the base locus is empty and then the generic element of the system is smooth.
Then, if $V_i \in |\mathcal{O}_{\mathbb{P}^N}(n_i)|$, call $H_i := V_i \cap M$ a smooth generic element of the linear system on M cut by hypersurfaces of $\mathbb{P}^N$ of degree n_i, for $i = 1, \ldots, m-1$.
We want to prove that, taking the n_i's large as we want, we find a curve $C := \bigcap_{i=1}^{m-1} H_i$ of genus large as we want.
First of all we note that, again by Bertini's Theorem, C is smooth.
Indeed we have that $H_1 \cap H_2 = M \cap V_1 \cap V_2 = H_1 \cap V_2$ so that we can view $H_1 \cap H_2$ as a generic element of the linear system on H_1 cut by hypersurfaces of $\mathbb{P}^N$ of degree n_2. Then by Bertini's Theorem $H_1 \cap H_2$ is smooth and, by induction, we obtain that C is smooth.
Now, if we fix $n_2, \ldots, n_{m-1}$, we can view C as a curve on M_2, where M_2 is the smooth surface

$$M_2 := \bigcap_{i=2}^{m-1} H_i$$

of degree $e := deg(M_2) = deg(M) \prod_{i=2}^{m-1} n_i$.
Then $C = M_2 \cap V_1$ and, by the genus formula (cf. Theorem 3.4 (iii)),

$$g(C) = 1 + \frac{1}{2}(C^2 + C \cdot K_{M_2}).$$

Since C is in the linear divisor of M_2 cut by hyperplanes of $\mathbb{P}^N$ of degree n_1 we have that $C^2 = deg(C)^2 = n_1^2 e^2$. Moreover $C \cdot K_{M_2} = n_1 \cdot l$, where $l = deg((\mathcal{O}_{\mathbb{P}^N}(1))_{|M_2} \cdot K_{M_2}) \in \mathbb{Z}$.
So for $C \in |\mathcal{O}_{\mathbb{P}^N}(n_1)_{|M_2}|$, we have that $g(C) = 1 + \frac{1}{2}(n_1^2 e^2 + n_1 l)$. Hence $g(C) \to \infty$ for $n_1 \to \infty$.
Thus for any $g \in \mathbb{Z} > 0$, by varying n_1, we can find a curve $C = \bigcap_{i=1}^{m-1} H_i$ that is smooth of genus $g(C) > g$. Lemma 6.11 is then proved.

□

Now fix a smooth curve C obtained as in Lemma 6.11 and call $j : C \hookrightarrow M$ the inclusion. Let $\mathcal{J}(C) \cong Alb(C)$ be the Jacobian of C and let

$$f := Alb(j) : \mathcal{J}(C) \to Alb(M), \quad K(C) := Ker(f) \subset \mathcal{J}(C).$$

Lemma 6.12. *The map f is surjective and $K(C)$ is an Abelian variety.*

Proof. By the Lefschetz theorem on hyperplane sections (cf. [9, pag. 156]), for a hyperplane section H of M, the map $H^p(M, \Omega^q_M) \to H^p(H, \Omega^q_H)$ induced by the inclusion of H in M is an isomorphism for $p + q \leq dim(M) - 2$ and injective for $p + q = dim(M) - 1$.
We apply this theorem inductively to obtain

$$H^0(M, \Omega^1_M) \cong H^0(H_0, \Omega^1_{H_0}) \cong \ldots \hookrightarrow H^0(C, \Omega^1_C), \quad C := \bigcap_i H_i.$$

So we obtain $f = alb(j) : H^0(C, \Omega^1_C)^* \twoheadrightarrow H^0(M, \Omega^1_M)^*$ is surjective ($Hom(\ ,\mathbb{C})$ is exact for $\mathbb{C}$-vector finite dimensional spaces). One can show that, if we consider the inclusion of the lattices $H_1(C, \mathbb{Z}) \subseteq H^0(C, \Omega^1_C)^*$ and $H_1(M, \mathbb{Z}) \subseteq H^0(M, \Omega^1_M)^*$ as in Remark 1.28, the map $f_{|H_1(C,\mathbb{Z})}$: $H_1(C, \mathbb{Z}) \to H_1(M, \mathbb{Z})$ is equal to the push-forward map j_* on homology defined by the inclusion $j : C \hookrightarrow M$.
So we obtain a surjective map

$$f : \mathcal{J}(C) = Alb(C) = \frac{H^0(C, \Omega^1_C)^*}{H_1(C, \mathbb{Z})} \twoheadrightarrow Alb(M) = \frac{H^0(M, \Omega^1_M)^*}{H_1(M, \mathbb{Z})}.$$

We now want to prove that $K(C) = Ker(f)$ is an Abelian variety.
Since $K(C)$ is contained in the abelian variety $\mathcal{J}(C)$, we need only to prove that $K(C)$ is a complex torus.
The key ingredient that we need now is the fact that, again by the Lefschetz theorem on hyperplane sections, we have that also

$$f_{|H_1(C,\mathbb{Z})} : H_1(C, \mathbb{Z}) \twoheadrightarrow H_1(M, \mathbb{Z})$$

is surjective.

Let then in general $f : \mathbb{C}^g/\Lambda \to \mathbb{C}^h/\Gamma$ be a map of complex torus with $g \geq h$ and with $f(\Lambda) = \Gamma$ as in our case.
We can choose the $\mathbb{Z}$-basis of Λ and Γ such that $f_{|\Lambda} : \Lambda \cong \mathbb{Z}^g \to \Gamma \cong \mathbb{Z}^h$ is of the form

$$f_{|\Lambda} = \begin{bmatrix} \alpha_1 & 0 & 0 & 0 & \ldots \\ 0 & \alpha_2 & 0 & 0 & \ldots \\ & & \ddots & & \\ & & & \alpha_h & \ldots \end{bmatrix}$$

for $\alpha_i \in \mathbb{Z} \geq 0$. This is the Smith normal form of $f_{|\Lambda}$.

But since $f(\Lambda) = \Gamma$ then the α_i must be all equal to 1 and so we have the direct sum decomposition $\Lambda \cong \Lambda' \oplus \Gamma$ induced by f, where $\Lambda' = Ker(f) \cap \Lambda$.

But then $\mathbb{C}^g/\Lambda = (\Lambda \otimes_{\mathbb{Z}} \mathbb{R})/\Lambda$ with $\Lambda = \Lambda' \oplus \Gamma$. Hence f can be written as

$$f = (1, 0) : \frac{\Gamma \otimes_{\mathbb{Z}} \mathbb{R}}{\Gamma} \times \frac{\Lambda' \otimes_{\mathbb{Z}} \mathbb{R}}{\Lambda'} \to \frac{\Gamma \otimes_{\mathbb{Z}} \mathbb{R}}{\Gamma}$$

and so $Ker(f) = \frac{\Lambda' \otimes_{\mathbb{Z}} \mathbb{R}}{\Lambda'}$ is a complex torus.
Returning to our specific case, we have then proved that $K(C) = Ker(f)$ is a complex torus. □

We now state a stronger result about $K(C)$ that we will need later. We say that a certain property holds for the very general element of a linear system if it holds for every element of the system outside a countable union of Zariski closed subsets.

Proposition 6.13. *For a very general C as in Lemma 6.11 we have that $K(C)$ is a simple Abelian variety.*

Proof. See [25, Prop 1.4]. □

If $j : C \hookrightarrow M$ is the inclusion of the curve in M, let $j_* : CH_0(C)_{hom} \to CH_0(M)$ induced map. Since by Theorem 2.10 of Abel we can identify $CH_0(C)_{hom} \cong \mathcal{J}(C)$, we can view $K(C) \subset \mathcal{J}(C)$ as a subset of $CH_0(C)_{hom}$ and consider the map

$$j_{*|K(C)} : K(C) \subset CH_0(C)_{hom} \to CH_0(M).$$

By hypothesis we have a correspondence $Z \in CH_d(M \times X)$, with $Z_* : CH_0(M) \to CH_0(X)$, such that $Z_*(CH_0(M))$ is finite dimensional in Roitman's sense, i.e. there exists a smooth projective variety W and a correspondence $\Gamma \subset W \times X$ such that $Im(Z_*) \subset \{\Gamma_*(w),\ w \in W\}$.
Fix such a W and Γ.

Since $K(C)$ is the kernel of the surjective map $f : \mathcal{J}(C) \to Alb(M)$ defined in Lemma 6.12, we have
$mathcalJ(C)/K(C) \cong Alb(M)$ as complex vector spaces. But $Alb(M)$ has fixed dimension and, by Lemma 6.11, we can choose C with $g(C) = dim(\mathcal{J}(C))$ high as we want.
Hence we can take $dim(K(C))$ high as we want. We may then assume that $dim(K(C)) > dim(W)$.

Summarizing, we have the following situation:

$$\begin{array}{ccccc} K(C) \subset CH_0(C)_{hom} & \xrightarrow{j_{*|K(C)}} & CH_0(M) & \xrightarrow{Z_*} & CH_0(X) \\ & & & & \uparrow{\scriptstyle \Gamma_*} \\ & & & & CH_0(W) \end{array}$$

with $Z_*(CH_0(M)) \subset \{\Gamma_*(w),\ w \in W\}$ and $dim(K(C)) > dim(W)$.

Proof of Proposition B

Let $R \subset K(C) \times W$ be

$$R = \{(k, w) \in K(C) \times W \ s.t.\ Z_*(j_*(k)) = \Gamma_*(w) \in CH_0(X)\}.$$

Call $p_1 : R \to K(C)$ and $p_2 : R \to W$ the two projections and let

$$F_w := p_1(pr_2^{-1}(w)) = \{k \in K(C) \ : \ Z_*(j_*(k)) = \Gamma_*(w)\} \subset K(C)$$

be the fibre of p_2 over $w \in W$ viewed in $K(C)$.

Lemma 6.14. *For every $w \in W$, we have that $dim(F_w) > 0$.*

Proof of Lemma 6.14. First of all note that $R = \bigcup_{i\in\mathbb{N}} R_i$ is a countable union of closed algebraic subset $R_i \subset K(C) \times W$.
Indeed we can view R as

$$R = (Z_* \circ j_* - \Gamma_*)^{-1}(0), \text{ where } Z_* \circ j_* - \Gamma_* : R \subset K(C) \times W \to CH_0(X).$$

Then the proof of Lemma 10.7 in [27] can be adapted to this case to find that R consists of a countable union of closed subset R_i of $K(C) \times W$.
As $Im(Z_*) \subset \{\Gamma_*(w),\ w \in W\}$, we have that $p_1 : R \to K(C)$ is surjective and then $p_1(R) = \bigcup_i p_{1|R_i}(R_i) = K(C)$. Moreover, since R_i are closed in $K(C)\times W$, we have that $p_{1|R_i}$ is a projective morphism and so is closed.
Then we have that $K(C) = \bigcup_{i\in I} p_{1|R_i}(R_i)$ is a countable union of closed subsets.
Using a Baire category argument (see Lemma 2.13), we obtain that there exists a $j \in I$ such that $p_{1|R_j}(R_j)$ is dense in $K(C)$. Hence for such j we have that $p_{1|R_j} : R_j \to K(C)$ is a dominant map and thus $dim(R_j) \geq dim(K(C))$ (cf. Lemma 6.10).
But since in 6.4.1 we have chosen C such that $dim(K(C)) > dim(W)$, we have that

$$dim(R_j) > dim(W).$$

By the same argument used to prove that p_1 is dominant, we find that also $pr_{2|R_j} : R_j \to p_2(R_j)$ is a dominant morphism. Then we have that $dim(pr_2^{-1}(w)) \geq dim(R_j) - dim(pr_2(R_j))$ for any $w \in W$ (cf. Lemma 6.10).
But $dim(R_j) > dim(W)$ and so we have

$$dim(pr_2^{-1}(w)) > 0 \text{ for any } w \in W.$$

Recall now that $F_w := p_{1|R_j}(pr_2^{-1}(w)) \subset K(C)$ is the fibre of w viewed in $K(C)$. Then using again that $p_{1|R_j}$ is dominant, we obtain that $dim(F_w) > 0$ for any $w \in W$.

□

We now want to prove that, for any $w \in W$, the group generated by F_w is all of $K(C)$.

Lemma 6.15. *For any $w \in W$, we have that $\langle F_w \rangle = K(C)$, where $\langle F_w \rangle$ is the group generated by $F_w \subset K(C)$.*

Proof. Since by Lemma 6.14 we have that the fibre $F_w \subset K(C)$ is non-empty of dimension bigger than zero, we can consider a connected component of F_w, call it $N \subseteq F_w$, with $dim(N) > 0$.
Since N is a connected subset of $K(C)$ of dimension > 0, we have that the subgroup $\langle N \rangle \subset K(C)$ generated by N is closed and connected in $K(C)$.
Indeed consider the map

$$\phi_k : N^k \times N^k \to K(C), \quad ((p_1, \ldots, p_k), (p_{k+1}, \ldots, p_{2k})) \mapsto \sum_{i=1}^{k} p_i - \sum_{i=k+1}^{2k} p_i$$

where the sum is taken considering the group structure of the abelian variety $K(C)$.
Since N is connected then also $N^k \times N^k$ is connected and so $Im(\phi_k)$ is a closed connected subset of $K(C)$.
Moreover, since $K(C)$ is a finite dimensional Abelian variety, the chain $Im(\phi_k) \subseteq Im(\phi_{k+1}) \subseteq \dots$ must become stationary at some point, so that there exists $K \in \mathbb{N}$ such that $Im(\phi_K) = Im(\phi_{K+n})$ for any $n \in \mathbb{N}$. But then we have that $\langle N \rangle = Im(\phi_K)$ and so we have finally proved that $\langle N \rangle$ is closed and connected in $K(C)$.
Since the only closed connected subgroups of an abelian variety are abelian varieties, we have found that $\langle N \rangle$ is an Abelian subvariety of $K(C)$.
But by Proposition 6.13, we have that $K(C)$ is simple, i.e. $K(C)$ cannot have proper Abelian subvarieties, and so $\langle N \rangle = K(C)$. Then, since $N \subseteq F_w$, we have also that the group $\langle F_w \rangle$ generated by F_w is equal to all of $K(C)$, for any $w \in W$. □

We are finally able to prove Proposition B.

Proof of Proposition B. As by hypothesis $Im(Z_*) \subset \{\Gamma_*(w),\ w \in W\}$, there exists a point $w_0 \in W$ such that $\Gamma_*(w_0) = 0 \in CH_0(X)$. By Lemma 6.15 we have that $\langle F_{w_0} \rangle = K(C)$ and so every element $k \in K(C)$ is of the form $k = \sum_i n_i z_i$, with $n_i \in \mathbb{Z}$ and $z_i \in F_{w_0}$.
But by definition of F_{w_0}, we have $Z_* \circ j_*(z_i) = \Gamma_*(w_0) = 0$ for any $z_i \in F_{w_0}$.
Hence for any $k = \sum_i n_i z_i \in K(C)$, we have that $Z_*(j_*(k)) = \sum_i n_i Z_*(j_*(z_i)) = 0$.
We have then proved that

$$Z_{*|j_*(K(C))} = 0.$$

To prove Proposition B we want $Z_{*|Ker(alb_M)} \equiv 0$ on all of $Ker(alb_M)$ and not only on $j_*(K(C)) \subset Ker(alb_M)$.
We now use the following:

Fact 6.16 (See [25]). *For k large enough, there is a connected subvariety M' of $M^k \times M^k$ such that $Ker(alb_M) \subset CH_0(M)_{hom}$ is generated by the cycles $z_m = z_m^+ - z_m^-$ with $z_m^+ = \sum_{l \leq k} m_l$, $z_m^- = \sum_{k+1<l<2k} m_l$, where $m = (m_1, \dots, m_{2k}) \in M'$.*
Furthermore, if the H_i's of Lemma 6.11 are taken ample enough, a very general point $m \in M'$ is supported on a very general curve C constructed as in Lemma 6.11.

The proof of the second part of Fact 6.16 is trivial because we only need to show that, if the H_i's of Lemma 6.11 are taken ample enough, a very general point $m = (m_1, \dots, m_{2k}) \in M'$ is supported on a general curve C constructed as in Lemma 6.11.
But this is clear since, if $H_i = M \cap V_i$, where $V_i \in |(O)_{P^N}(n_i)|$, for n_i large enough we can impose that every H_i contains all the points $m_1, \dots, m_{2k}$ which are general in M.

Even if we will not prove the first part of Fact 6.16 in general, for our purposes it is sufficient to prove it for a K3 surface M. Indeed in the proof of Theorem 6.1 we have used Proposition B only in the case $M = S$, where S is a K3 surface.

In this case the first part of Fact 6.16 is trivial because, since $alb_S = 0$, then $Ker(alb_S) = CH_0(S)_{hom}$ is generated by the cycles of the form $z_m = m_1 - m_2$ for $(m_1, m_2) \in S \times S$. So by taking $M' = S \times S$ we are done.

Return now to the general case of a projective variety M.
By Fact 6.16, for a very general $m = (m_1, \dots, m_{2k}) \in M' \subset M^k \times M^k$, i.e. for m contained in the complement of a countable union of closed algebraic subsets of M', the 0-cycles $z_m = z_m^+ - z_m^- = \sum_{l \leq k} m_l - \sum_{k+1 < l < 2k} m_l$ are supported on C and annihilated by alb_M. Then, since j_* is the homomorphism induced by the inclusion of C in M and $K(C) = Ker(alb_M)$,we have that the 0-cycles z_m belong to $j_*(K(C))$, for any $m \in M'$ very general.
Then we have proved that

$$Z_*(z_m) = 0 \text{ for } m \text{ very general in } M'.$$

If we prove that $Z_*(z_m) = 0$ for any $m \in M'$, then we are done because, by Fact 6.16, $Ker(alb_M)$ is generated by these cycles z_m.
But now we can use Proposition 2.15, which states that if Y is a connected complex projective variety and $U \subset Y$ is the complement of a countable union of proper closed algebraic subsets Z_i, then any 0-cycle of Y is rationally equivalent to a 0-cycle supported on U.
This tells us that for any $m \in M'$, the cycle z_m is rationally equivalent to a cycle z'_m given by a very general point $m' \in M'$.
Hence indeed Z_* vanishes on all of M' and so on all of $Ker(alb_M)$ and Proposition B is then proven.

□

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