THE ARNOUX-YOCCOZ TEICHMÜLLER DISC

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Abstract. We prove that the Teichmüller disc stabilized by the Arnoux-Yoccoz pseudo-Anosov possesses two transverse hyperbolic directions. This proves that the corresponding flat surface has not a cyclic Veech group.

In addition, we prove that this Teichmüller disc is dense inside the hyperelliptic locus of the connected component $\mathcal{H}^{\mathrm{odd}}(2,2)$.

We rephrase our results in terms of quadratic differentials: We show that there exists a holomorphic quadratic differential, on a genus 2 surface, with the two following properties. (1) The Teichmüller disc is dense inside the moduli space of holomorphic quadratic differentials (which are not the global square of any Abelian differentials).
(2) The stabilizer of the $\mathrm{PSL}_2(\mathbb{R})$-action contains two non commuting pseudo-Anosov diffeomorphisms.

The proof uses Ratner's theorems.

Contents

1. Introduction

After the work of McMullen [Mc1, Mc2, Mc4, Mc5, Mc6] and Calta [Cal04] who classified Veech surfaces and completely periodic translation surfaces arising from Abelian differentials in genus 2, it is a big challenge to try to understand what happens for translation surfaces that admit pseudo-Anosov diffeomorphism in higher genus. The holonomy field of such a surfaces is a number field of degree bounded by the genus of the surface (see Kenyon-Smillie [KS00]). At the time Thurston defined pseudo-Anosov diffeomorphism no examples were known when the holonomy field is an extension of $\mathbb{Q}$ of degree more than 2.

Date: November 21, 2006.

2000 *Mathematics Subject Classification.* Primary: 32G15. Secondary: 30F30, 57R30, 37D40.

Key words and phrases. Abelian differentials, Veech group, Pseudo-Anosov diffeomorphism, Teichmüller disc, Ratner theorems.

At the beginning of the 80's, Arnoux and Yoccoz [AY81] discovered a family Φ_n, $n \geq 3$ of pseudo-Anosov diffeomorphisms with expansion factor $\lambda_n = \lambda(\Phi_n)$ the Pisot root of the irreducible polynomial $X^n - X^{n-1} - \cdots - X - 1$. The pseudo-Anosov ϕ_n acts linearly on a genus n translation surface (X_n, ω_n) (the Abelian differential ω_n having two zeroes of order $n-1$). The stable and unstable foliations of these pseudo-Anosov exhibit a lot of interesting properties studied in [Arn81]. See also [Arn88] where some strange phenomenon is discussed in the case of genus 3.

In this paper, we study properties of the Teichmüller disc stabilized by the pseudo-Anosov $\Phi = \Phi_3$. The Abelian differential $\omega = \omega_3$ has two zeroes of order 2. The stratum of Abelian differentials (see Section 2.2 for definitions) with two zeroes of order 2 is called $\mathcal{H}(2,2)$. It has two connected components (see Kontsevich-Zorich [KZ03]). The translation surface (X, ω) sits in $\mathcal{H}^{odd}(2,2)$ the non-hyperelliptic component with odd parity of the spin structure. Nevertheless, as Φ was first defined on the sphere and then lifted to (X, ω), the Riemann surface X is a hyperelliptic surface. Thus, (X, ω) belongs to the hyperelliptic locus in $\mathcal{H}^{odd}(2,2)$. We denote this locus by $\mathcal{L}$. It has complex codimension 1 in $\mathcal{H}^{odd}(2,2)$.

Hubert and Lanneau [HL06] showed that this Teichmüller disc was not stabilized by any parabolic element. Thus the Arnoux-Yoccoz surface has been considered as a good candidate for a Teichmüller disc stabilized only by a cyclic group generated by a pseudo-Anosov diffeomorphism. We show that this does not hold.

Theorem 1.1. *The surface (X, ω) has two transverse hyperbolic directions. More precisely the Teichmüller disc of (X, ω) is stabilized by Φ and by a pseudo-Anosov diffeomorphism $\tilde{\Phi}$ not commuting with Φ. The expansion factor $\tilde{\lambda} = \lambda(\tilde{\phi})$ has degree 6 over $\mathbb{Q}$.*

We recall that the stabilizer of a translation surface under the natural $\mathrm{SL}_2(\mathbb{R})$-action is a Fuchsian group called the Veech group. The Veech group is the group of the differentials of the diffeomorphisms that preserve the affine structure induced by the translation structure. Combining Theorem 1.1 with a previous result of Hubert-Lanneau ([HL06]) we get the following corollary.

Corollary 1.2. *Non-elementary Veech groups without parabolic elements do exist.*

Theorem 1.3. *The Teichmüller disc stabilized by the Arnoux-Yoccoz pseudo-Anosov $\phi = \phi_3$ is dense inside the hyperelliptic locus $\mathcal{L}$.*

Theorem 1.1 and Theorem 1.3 show that in genus 3 the situation is very different from what we know in genus 2. McMullen proved that, as soon as a genus 2 translation surface is stabilized by a pseudo-Anosov diffeomorphism with orientable stable and unstable foliation, the image of its Teichmüller disc in the moduli space of curves is contained in a Hilbert modular surface, in particular not dense in its stratum. Here, the Veech group is non elementary, nevertheless, the closure of the Teichmüller disc is as big as it can be.

It is also interesting to rephrase Theorems 1.1 and 1.3 using quadratic differentials. In particular we show that the orientability assumption in genus 2 is necessary.

Corollary 1.4. *Let $(\mathbb{P}^1, q)$ be the quotient of (X, ω) by the hyperelliptic involution. Then the $SL_2(\mathbb{R})$-orbit of $(\mathbb{P}^1, q)$ is dense in the stratum of meromorphic quadratic differentials* *having two simple zeores and six simple poles. The stabilizer of the $SL_2(\mathbb{R})$-action contains two (non commuting) pseudo-Anosov diffeomorphisms.*

One can see the last result in terms of holomorphic quadratic differentials on a genus 2 surface into the following way. Let $\mathcal{Q}_2$ be the moduli space of holomorphic quadratic differentials which are *not* the global square of any Abelian differentials. Let $\pi : Y \to \mathbb{P}^1$ be a double covering of $\mathbb{P}^1$ ramified precisely over the six poles of q. Let $(Y, \tilde{q})$ be the lift of q. Obviously $(Y, \tilde{q}) \in \mathcal{Q}_2$.

Corollary 1.5. *The $SL_2(\mathbb{R})$-orbit of $(Y, \tilde{q})$ is dense inside the whole moduli space $\mathcal{Q}_2$. Moreover the stabilizer of the $SL_2(\mathbb{R})$-action of $(Y, \tilde{q})$ contains two non-commuting pseudo-Anosov diffeomorphisms.*

Proof. As we have seen, $(\mathbb{P}^1, q)$ belongs to the stratum $\mathcal{Q}(1, 1, -1^6)$ of meromorphic quadratic differentials having two simple zeores and six simple poles. The stratum $\mathcal{Q}(1, 1, -1^6)$ is isomorphic to the stratum $\mathcal{Q}(1, 1, 1, 1)$, the principal stratum of holomorpic quadratic differentials on genus 2 surfaces (see Lanneau [Lan1]). Moreover the $\mathrm{SL}_2(\mathbb{R})$-action is equivariant with respect to this isomorphism. Therefore corollary 1.4 shows that the closure of the Teichmüller disc $\mathrm{SL}_2(\mathbb{R}) \cdot (Y, \tilde{q})$ contains $\mathcal{Q}(1, 1, 1, 1)$. This last stratum is dense inside $\mathcal{Q}_2$ which gives the corollary. □

2. Background

In order to establish notations and preparatory material, we review basic notions concerning translation surfaces, affine diffeomorphisms, and moduli spaces.

2.1. **Pseudo-Anosov diffeomorphisms and Veech groups.** A *translation surface* is a (real) genus g surface with a translation structure (i.e. an atlas such that all transition functions are translations). As usual, we consider maximal atlases. These surfaces are precisely those given by a Riemann surface X and a holomorphic (non-null) one form $\omega \in \Omega(X)$; see say [MT02] for a general reference on translation surfaces and holomorphic one forms.

Let (X, ω) be a translation surface. The stabilizer of (X, ω) under the $\mathrm{SL}(2, \mathbb{R})$-action is called the *Veech group* of (X, ω) and is denoted by $\mathrm{SL}(X, \omega)$. A more intrinsic definition is the following. An *affine diffeomorphism* is an orientation preserving homeomorphism of X which is affine in the charts of ω and permutes the zeroes of ω. The derivative (in the charts of ω) of an affine diffeomorphism defines an element of $\mathrm{SL}(X, \omega)$. Vice versa any such element is the derivative of an affine diffeomorphism.

A diffeomorphism f is *pseudo-Anosov* if and only if the linear map Df is hyperbolic; that is $|\mathrm{trace}(Df)| > 2$. In this case, Df has two real eigenvalues $\lambda^{-1} < 1 < \lambda$. The number λ is called the expansion factor of the pseudo-Anosov diffeomorphism f.

2.2. **Connected components of the strata.** The moduli space of Abelian differentials is stratified by the combinatorics of the zeroes. We denote by $\mathcal{H}(k_1, \dots, k_n)$ the stratum of Abelian differentials consisting of holomorphic one form with n zeroes of multiplicity $(k_1, \dots, k_n)$. These strata are non-connected in general but each stratum has at most three connected components (see [KZ03] for a complete classification). In particular the stratum with two zeroes of multiplicity 2, $\mathcal{H}(2,2)$, has two connected components. The *hyperelliptic component* $\mathcal{H}^{hyp}(2,2)$ contains precisely pairs (X,ω) of a hyperelliptic surface X and a one-form whose zeros are interchanged by the hyperelliptic involution. The other (*non-hyperelliptic*) component $\mathcal{H}^{odd}(2,2)$ is distinguished by an odd parity of the spin structure. There are two ways to compute the parity of the spin structure of a translation surface X. The first way is to use the Arf formula on a symplectic basis (see [KZ03]). The second possibility applies if X comes from a quadratic differential, i.e. if X possesses an involution such that the quotient produces a half-translation surface (see [Lan2]).

Let $\mathcal{Q}(1,1,-1^6)$ be the stratum of meromorphic quadratic differentials on the projective line with two simple zeroes and six simple poles. It is easy to see that this stratum is connected (see [KZ03, Lan1]). Taking the orientating double covering, one gets a local embedding

$$\mathcal{Q}(1,1,-1^6) \to \mathcal{H}(2,2).$$

We will denote by $\mathcal{L}$ the image in $\mathcal{H}(2,2)$ of the previous map. The construction of the Arnoux-Yoccoz surface (X,ω) is given below. Here we record for later use:

Lemma 2.1. *The Arnoux-Yoccoz surface (X,ω) lies in $\mathcal{L} \subset \mathcal{H}^{odd}(2,2)$.*

Proof. Thanks to the decomposition of the Arnoux-Yoccoz surface (X,ω) into cylinders (see Section 4.2) it is easy to define an affine diffeomorphism of (X,ω), e.g. by a rotation of 180 around the center of C_1' (see Figure 6). This diffeomorphism fixes 8 points, but not the two zeros of ω. Hence X is hyperelliptic and lies in $\mathcal{H}^{odd}(2,2)$.

Let us recall the formula in Theorem 1.2. of [Lan2] (p. 516) in order to calculate the parity of the spin structure. If X is a half translation surface belonging to the stratum $\mathcal{Q}(k_1,\dots,k_l)$ then the parity of the spin structure of $\hat{X}$, the orientating surface, is

$$\left[\frac{|n_{+1}-n_{-1}|}{4}\right] \mod 2$$

where $n_{\pm 1}$ is the number of zeros of degrees $k_j = \pm 1 \mod 4$, and where all the remaining zeros satisfy $k_r = 0 \mod 4$ (the square brackets denote the integer part). This shows that the parity of the spin structure of (X_3,ω_3) is then $\frac{1}{4}(6-2) = 1 \mod 2$. □

Therefore the hyperelliptic locus $\mathcal{L}$ belongs to the odd part $\mathcal{H}^{odd}(2,2)$ of $\mathcal{H}(2,2)$. We recall that the complex dimension of $\mathcal{Q}(1,1,-1^6)$ is 6 and the complex dimension of $\mathcal{H}(2,2)$ is 7.

Lemma 2.2. *There exists a linear isomorphism between the stratum $\mathcal{Q}(1,1,1,1)$ and the stratum $\mathcal{Q}(1,1,-1^6)$.*

Here linear implies in particular that the $\mathrm{SL}_2(\mathbb{R})$-action commutes with this isomorphism.

Proof. We recall here the proof presented in [Lan1]. Let us consider a meromorphic quadratic differential q on $\mathbb{P}^1$ having the singularity pattern $(1,1,-1^6)$. Consider a ramified double covering π over $\mathbb{P}^1$ having ramification points over the simple poles of q, and no other ramification points. We obtain a genus 2 hyperelliptic Riemann surface X with a quadratic differential $\tilde{q} = \pi^* q$ on it. It is easy to see that the induced quadratic differential has the singularity pattern $(1, 1, 1, 1)$. Hence we get locally an $\mathrm{SL}_2(\mathbb{R})$-equivariant mapping

$$\mathcal{Q}(1, 1, -1^6) \to \mathcal{Q}(1, 1, 1, 1).$$

Since the dimensions coincide, e.g.

$$\dim_{\mathbb{C}} \mathcal{Q}(1, 1, -1^6) = 2 \cdot 0 + 8 - 2 = 6 = 2 \cdot 2 + 4 - 2 = \dim_{\mathbb{C}} \mathcal{Q}(1, 1, 1, 1)$$

and since the geodesic flow acts ergodically on the strata the image of the above map equals $\mathcal{Q}(1, 1, 1, 1)$. □

3. Completely periodic directions

Let (X, ω) be a translation surface. A *cylinder* is a topological cylinder embedded in X, isometric to a flat cylinder $\mathbb{R}/w\mathbb{Z} \times]0, h[$. The boundary of a maximal cylinder is a union of a finite number of saddle connections.

A direction θ is *completely periodic* on X if all the regular geodesics in the direction θ are closed. This means that X is the closure of a finite number of maximal cylinders in the direction θ. In a periodic direction, all the geodesics emanating from singularities are saddle connections.

Let θ be a completely periodic direction on a translation surface (X, ω). A translation surface comes with a horizontal and vertical direction, and we will henceforth assume that θ is different from the vertical direction. The saddle connections in the direction θ are labeled by $\gamma_1, \ldots, \gamma_k$. The cylinders are labeled by $\mathcal{C}_1, \ldots, \mathcal{C}_p$ and $w_1, \ldots, w_p$ will stand for the widths (or perimeters) of the cylinders.

For each cylinder $\mathcal{C}_i$ one can encode the sequence of saddle connections contained in the bottom of $\mathcal{C}_i$ and ordered in the cyclic ordering of the boundary of $\mathcal{C}_i$ by a cyclic permutation σ_i^b. We get an analogous definition if we replace bottom by top. Therefore one gets two $p-$tuples of cyclic permutations $(\sigma_1^b \ldots \sigma_p^b)$ and $(\sigma_1^t \ldots \sigma_p^t)$. Note that these data define two permutations on k elements $\pi_b = \sigma_1^b \circ \cdots \circ \sigma_p^b$ and $\pi_t = \sigma_1^t \circ \cdots \circ \sigma_p^t$.

These data form the *combinatorics* $\mathcal{G}$ of the direction θ on the surface X. This notion is very close to the one of separatrix diagram introduced by Kontsevich and Zorich (see [KZ03]) but will be more convenient for our purposes.

To give a complete description of the surface (X, ω) in the direction θ we also need continuous parameters:

- the lengths of the saddle connections,
- the *heights* of the cylinders with respect to the vertical direction,
- the *twists* of the cylinders.

The only parameters which are non trivial to define are the twists. For that, one has first to fix a *marking* on the combinatorics $\mathcal{G}$, i.e. on each cycle of σ_i^b and σ_i^t, we mark an arbitrary element, denoted by $m_b(i)$ (resp. $m_t(i)$). When representing the combinatorics by a table, we will underline the marked elements.

On a translation surface with marked combinatorics in some direction θ we will normalize the first twist to zero, but define a twist vector in the cylinder $\mathcal{C}_1$. This is a saddle connection contained in $\mathcal{C}_1$ joining the origin of the saddle connection $\gamma_{m_b(1)}$ to the origin of the saddle connection $\gamma_{m_t(1)}$. The corresponding vector may be decomposed into its vertical component (equal to h_1, the height of $\mathcal{C}_1$) and its component in the direction θ, denoted by v_1 (see Figure 1). The vector v_1 is well defined up to an additive constant nw_1 where $n \in \mathbb{Z}$. We normalize v_1 by the requirement $|v_1| < |w_1|$.

To have enough flexibility we will define twists for cylinders $\mathcal{C}_i$ $(i = 2, \dots, p)$ *with respect to the direction* $\theta^\perp = \theta^\perp(n_0)$ given by $h_1 + v_1 + n_0 w_1$ for $n_0 \in \mathbb{Z}$. This is done in the following way.

Let $\mathcal{C}_i$ be a cylinder. Let $(h_i)_y$ be its height. The endpoint P of the vertical vector $h_i = \left(\begin{smallmatrix} 0 \\ (h_i)_y \end{smallmatrix}\right)$ based at the origin of the saddle connection $\gamma_{m_b(i)}$ is located on the top of $\mathcal{C}_i$. Let v_i be the vector joining P to the origin of the saddle connection $\gamma_{m_t(i)}$ in the direction θ. The vector v_i is well-defined up an additive constant nw_i where $n \in \mathbb{Z}$. The twist t_i of $\mathcal{C}_i$ is defined to be the difference

$$t_i = v_i - \frac{(h_i)_y}{(h_1)_y}(v_1 + n_0 w_1)$$

(see Figure 1). The affine invariant will be the normalized twist, namely

$$\frac{|t_i|}{|w_i|} \in [0, 1[$$

Therefore for each completely periodic direction θ, each marking m on $\mathcal{G}(\theta)$ and each $n_0 \in \mathbb{Z}$, one gets the following quantities:

- $\overrightarrow{L}(\theta) \in \mathbb{R}^k$
- $\overrightarrow{H}(\theta) \in \mathbb{R}^p$
- $\mathcal{G}(\theta)$
- $\overrightarrow{T}(\theta, m, n_0) \in [0, 1[^{p-1}$ (normalized twists)

Remark 3.1. In Figure 1 one has, with previous notations, $\sigma_1^t = (1\ 2), \sigma_2^t = (3\ 5\ 4)$ and $\sigma_1^b = (3\ 4\ 1)$, $\sigma_2^b = (2\ 5)$. Or equivalently: $\mathcal{G} = \left(\left(\begin{smallmatrix} 1 & 2 & \\ 3 & 4 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 3 & 5 & 4 \\ 2 & 5 & \end{smallmatrix}\right)\right)$ and a marking is $(\mathcal{G}, m) = \left(\left(\begin{smallmatrix} 1 & \underline{2} & \\ \underline{3} & 4 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 3 & 5 & \underline{4} \\ \underline{2} & 5 & \end{smallmatrix}\right)\right)$.

For a vector $u \in \mathbb{R}^n$ and a permutation π on n symbols, we will use the following convention $\pi\left((u_1, \dots, u_n)\right) = (u_{\pi(1)}, \dots, u_{\pi(n)})$.

We then have the obvious

Lemma 3.1. *Let* $f : X \longrightarrow X$ *be an affine diffeomorphism. Let* $\theta \in \mathbb{S}^1$ *be a completely periodic direction and* $\theta' := f(\theta)$. *Then* f *induces a bijection of the saddle connections* $\{\gamma_1, \dots, \gamma_k\}$ *(and thus a permutation* π_f^{sc} *on* k *elements) and a bijection of the cylinders* $\{\mathcal{C}_1, \dots, \mathcal{C}_p\}$ *(and thus a permutation* π_f^{cy} *on p elements). Moreover one has:*

$$\overrightarrow{L}(\theta) = \pi_f^{sc}\left(\overrightarrow{L}(\theta')\right) \in \mathbb{RP}(k)$$

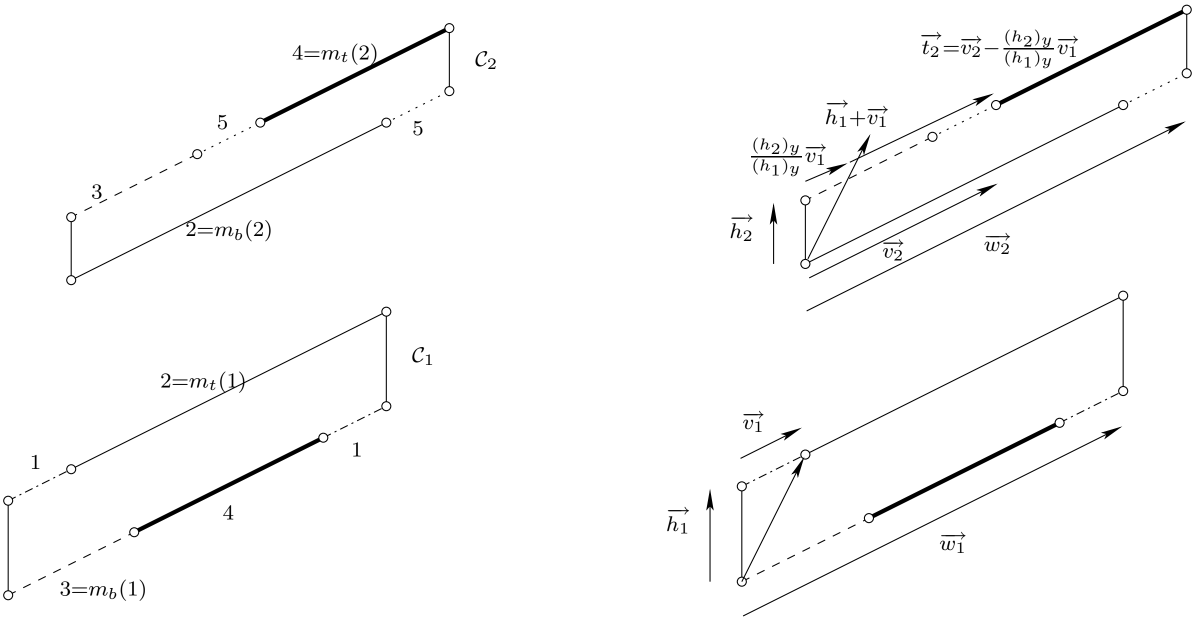

FIGURE 1. Combinatorics and marking of a translation surface.

and

$$\overrightarrow{H}(\theta) = \pi_f^{cy}\left(\overrightarrow{H}(\theta')\right) \in \mathbb{RP}(p).$$

Lemma 3.2. *With the same assumptions as in lemma 3.1, let us choose a marking m on $\mathcal{G} = \mathcal{G}(\theta)$. Then π_f^{sc} induces a marking m' on $\mathcal{G}' = \mathcal{G}(\theta')$. Moreover there exists $n_0' \in \mathbb{Z}$ such that the normalized twists $\overrightarrow{T}(\theta, m, 0)$ and $\pi_f^{sc}\left(\overrightarrow{T}(\theta', m', n_0')\right)$ are the same.*

In fact, given a combinatorics in the direction θ, the lengths of the saddle connections, the width of the cylinders (with respect to the vertical direction) and the twists characterize the surface (X, ω) in the moduli space. Namely one has:

Theorem 3.3. *Let X be a flat surface. Let $\theta, \theta' \in \mathbb{S}^1$ be two completely periodic directions. Let us choose a marking m on $\mathcal{G}$. Let us also assume there exist two permutations π_1 on k elements and π_2 on p elements such that*

(1) $\overrightarrow{L} = \pi_1\left(\overrightarrow{L'}\right) \in \mathbb{RP}(k)$ *and* $\overrightarrow{H} = \pi_2\left(\overrightarrow{H'}\right) \in \mathbb{RP}(p)$,

(2) *$\mathcal{G}$ and $\mathcal{G}'$ are isomorphic via (π_1, π_2) and there exists $n_0' \in \mathbb{Z}$ such that the normalized twists $\overrightarrow{T}(\theta, m, 0)$ and $\pi_1\left(\overrightarrow{T'}(\theta', m', n_0')\right)$ are the same.*

Then there exists an affine diffeomorphism $f \in Aff(X, \omega)$. Moreover $f(\theta) = \theta'$, $f(\theta'^{\perp}) = \theta'^{\perp}$ and $\pi_f^{sc} = \pi_1$, $\pi_f^{cy} = \pi_2$.

4. Application to the Arnoux-Yoccoz flat surface

4.1. **Construction.** (after P. Arnoux)

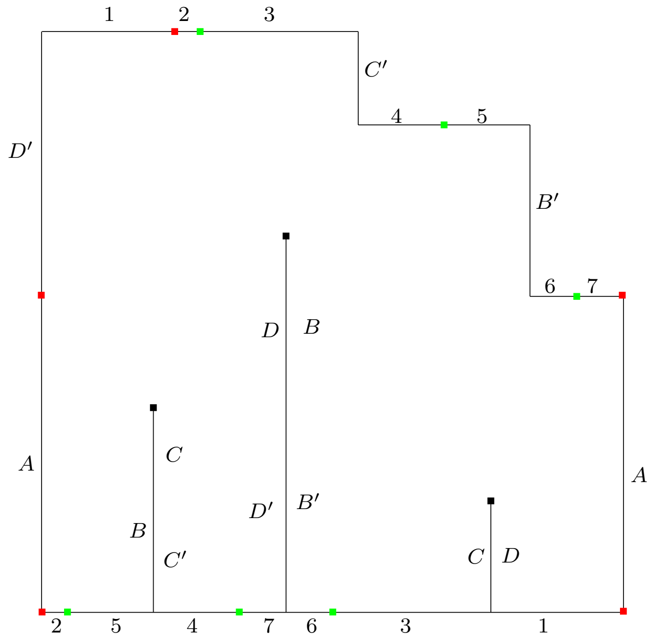

FIGURE 2. The Arnoux-Yoccoz flat surface. Identifications of the boundaries are made with respect to the labels.

Here we present the idea of the construction of the Arnoux-Yoccoz flat surface. What appears to be ad hoc here was constructed originally by a self-similar interval exchange transformation whose canonical suspension is a quadratic differential on $\mathbb{P}^1$. One can find all details in the original paper [AY81]. See also [Arn88], p. 496–498.

Let us consider three rectangles in $\mathbb{R}^2$ glued on the horizontal segment $[0,2]$ with the following parameters. The bases have length 2α, $2\alpha^2$ and $2\alpha^3$ and the heights are respectively 2, $2(\alpha+\alpha^2)$ and 2α (here α is the positive real root of polynomial X^3+X^2+X-1). In each of theses rectangles, we make a small vertical cut starting on the base with a specified length. The cuts start at the points $\alpha-\alpha^3$, $\alpha+\alpha^2$, $1+\alpha$ respectively on the base and they have heights $\alpha+\alpha^3$, $1+\alpha^2$, $\alpha^2+\alpha^4$ respectively (see Figure 2).

We then identify horizontal boundaries with the following rules: we glue the point (x,y) on the top boundary with $(f(x),0)$ for any $x\in[0,2]$, where $f:[0,2[\to[0,2[$ is an interval exchange transformation. Following [Arn88] page 489, this interval exchange transformation has permutation $\pi=\left(\begin{smallmatrix}1&2&3&4&5&6&7\\2&5&4&7&6&3&1\end{smallmatrix}\right)$ and the interval lengths are indicated in Figure 2.

In this manner we obtain a flat surface with vertical boundary components. To get a closed flat surface we have to identify the vertical boundaries. This is done according to the rules presented in Figure 2. The parameters are chosen in such a way that the gluing are isometric.

The flat surface (X,ω) obtained in this way has two conical singularities of total angle 6π, so $(X,\omega)\in\mathcal{H}(2,2)$.

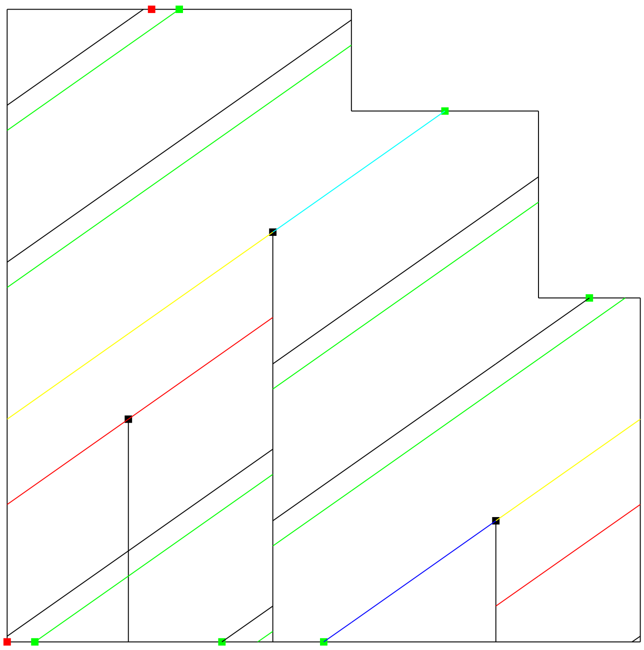

FIGURE 3. Singular geodesics on the Arnoux-Yoccoz flat surface in the direction $\theta = 1 - \alpha^2$: a 3C-direction.

We review the construction of the pseudo-Anosov diffeomorphism due to Arnoux-Yoccoz. Let us define a new parametrization of X. We will do that in two steps. First in the above chart of X, let us take the two small rectangles and glue them above the big rectangle by an isometry (in this operation, we have to cut the medium rectangle into two parts). Then we cut the figure following the vertical line $x = \alpha + \alpha^4$ and permute the two obtained pieces. In this way we get a new parametrization of our surface which is exactly the same except that the horizontal coordinates are multiplied by α and the vertical coordinates are multiplied by α^{-1}. Therefore we can define an affine diffeomorphism Φ on (X, ω):

$$\Phi(x, y) \text{ in the first parametrization} = (\alpha x, \alpha^{-1} y) \text{ in the second parametrization.}$$

The derivative of Φ is

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}, \ \frac{1}{\alpha} = \lambda = \lambda(\Phi) > 1$$

Notation. According to Figure 2 we denote the white points by $P_1,\ P_2,\ P_3$ and the black points by $P_4,\ P_5,\ P_6$ in the order presented. The reason for this order will become clear in the sequel. Note that the points $\{P_i\}_{i=1,2,3}$ and $\{P_i\}_{i=4,5,6}$ respectively define a singularity in X.

Let us assume that the foliation in direction θ is completely periodic. For each of the two singularities there are three emanating saddle connections. We label the saddle connection emanating from P_i by γ_i.

With these conventions the labels of the singularities appear in the counterclockwise around a singularity. The length of γ_i is denoted by $|\gamma_i| = |\int_{\gamma_i}\omega|$. We will frequently abuse notation and write γ_i for the vector $\begin{pmatrix} \int_{\gamma_i} \Re(\omega) \\ \int_{\gamma_i} \Im(\omega) \end{pmatrix}$

For a vector $u = (u_1, \dots, u_n)$ we will use the notation u^2 for $(u_1^2, \dots, u_n^2)$.

4.2. **A first direction: $\theta = 1 - \alpha^2$.** In this direction the surface X decomposes into three metric cylinders. See Figure 3 and also Figure 6 (in the vertical direction) for the way the 6 saddle connection bound the cylinders. We call such a direction a *3C-direction*. A straightforward computation gives the lengths of the γ_i (in the direction θ) summarized in the following table.

saddle connection γ_i	$(\lvert\gamma_i\rvert^2)_{i=1,\dots,6} \in \mathbb{R}^6$	$L(\theta)^2 \in \mathbb{RP}(6)$
$i=1$	$16+18\alpha+10\alpha^2$	$40+34\alpha+22\alpha^2$
$i=2$	$4-6\alpha+\alpha^2$	1
$i=3$	$16+18\alpha+10\alpha^2$	$40+34\alpha+22\alpha^2$
$i=4$	$4-6\alpha+6\alpha^2$	$4+2\alpha+2\alpha^2$
$i=5$	$4-6\alpha+\alpha^2$	1
$i=6$	$4-6\alpha+6\alpha^2$	$4+2\alpha+2\alpha^2$

Let us now describe the combinatorics of 3C-direction θ. We will denote these cylinders by $\mathcal{C}_b, \mathcal{C}_m, \mathcal{C}_s$ for big, medium and small according to their widths. For the top boundaries of the cylinders, the saddle connections appear in the following order

$$\sigma^t_{big} = (2\ 4\ 5\ 1),\ \sigma^t_{med} = (6),\ \sigma^t_{small} = (3).$$

And for the bottom boundaries one has

$$\sigma^b_{big} = (3\ 2\ 6\ 5),\ \sigma^b_{med} = (4),\ \sigma^b_{small} = (1).$$

Thus we get the following combinatorics, on which we have chosen arbitrarily a marking.

$$(\mathcal{G}, m) = \left(\begin{pmatrix} \underline{2}\ 4\ 5\ 1 \\ \underline{3}\ 2\ 6\ 5 \end{pmatrix},\ \begin{pmatrix} \underline{6} \\ \underline{4} \end{pmatrix},\ \begin{pmatrix} \underline{3} \\ \underline{1} \end{pmatrix} \right)$$

The heights of the cylinders are summarized below.

cylinder	$(\ (h_b)_y, (h_m)_y, (h_s)_y\ ) \in \mathbb{R}^3$	$[\ (h_b)_y, (h_m)_y, (h_s)_y\ ] \in \mathbb{RP}(3)$
$\mathcal{C}_b$	$2-4\alpha+2\alpha^2$	$3+3\alpha+2\alpha^2$
$\mathcal{C}_m$	$-2+2\alpha+4\alpha^2$	$2+2\alpha+\alpha^2$
$\mathcal{C}_s$	$-2+6\alpha-4\alpha^2$	1

Equipped with this marking one can calculate the twists in the medium and small cylinder (recall that we normalize the twist in the first, i.e. the big cylinder to zero). We first calculate the direction $\theta^\perp$.

The twist vector along the cylinder $\mathcal{C}_b$ is a vector (contained in $\mathcal{C}_b$) joining the origin of the saddle connection γ_3 to the origin of the saddle connection γ_2. This vector can be calculated into the following way. Let P be the endpoint of the vector $h_b = \begin{pmatrix} 0 \\ (h_b)_y \end{pmatrix}$ based at the origin of γ_3, namely P_3. The point P is located on the top of $\mathcal{C}_b$. Let v_b be the vector joining P to the origin of the saddle connection γ_2 (namely P_2) in the direction θ.

The twist vector of $\mathcal{C}_b$ is then $h_b + v_b$. A simple computation shows that $v_b = \begin{pmatrix} 1-\alpha^2 \\ 2\alpha-2\alpha^2 \end{pmatrix}$. Hence the twist vector is $h_b + v_b = \begin{pmatrix} 1-\alpha^2 \\ 2-2\alpha \end{pmatrix}$. Note that here we require that $|v_b| < |w_b|$. Finally with our marking, one obtains the direction $\theta^\perp := \theta^\perp(0)$ to be $h_b + v_b$, namely $\theta^\perp = \frac{2-2\alpha}{1-\alpha^2} = 1 + \alpha^2$.

Let us compute the twists of cylinders $\mathcal{C}_m$ and $\mathcal{C}_s$ with respect to the direction $\theta^\perp$. We first begin with cylinder $\mathcal{C}_m$.

As above let P be the endpoint of the vector $h_m = \begin{pmatrix} 0 \\ (h_m)_y \end{pmatrix}$ based at the origin of γ_4, namely P_4. The point P is located on the top of $\mathcal{C}_m$. Let v_m be the vector joining P to the origin of the saddle connection γ_6 (namely P_6) in the direction θ. A simple computation shows $v_m = \begin{pmatrix} 1-\alpha \\ 2-2\alpha-2\alpha^2 \end{pmatrix}$. The vector v_m is well defined up to an additive constant nw_m where $n \in \mathbb{Z}$. The twist t_m of $\mathcal{C}_m$ will be the difference

$$t_m = v_m - \frac{(h_m)_y}{(h_b)_y} v_b.$$

Thus

$$\frac{(h_m)_y}{(h_b)_y} v_b = \frac{1}{2}(1+\alpha^2) \begin{pmatrix} 1-\alpha^2 \\ 2\alpha-2\alpha^2 \end{pmatrix} = \begin{pmatrix} 1-\alpha \\ 2-2\alpha-2\alpha^2 \end{pmatrix} = v_m.$$

Therefore the (normalized) twist $\frac{|t_m|}{|w_m|} \in [0,1[$ of $\mathcal{C}_m$ is null.

Now let us finish the calculation of the twist of $\mathcal{C}_s$. As above let P be the endpoint of the vector $h_s = \begin{pmatrix} 0 \\ (h_s)_y \end{pmatrix}$ based at the origin of γ_1, namely P_1. The point P is located on the top of $\mathcal{C}_s$. Let v_s be the vector joining P to the origin of the saddle connection γ_3 (namely P_3) in the direction θ. A simple computation shows $v_s = \begin{pmatrix} 1+\alpha+2\alpha^2 \\ 2-2\alpha+2\alpha^2 \end{pmatrix}$. The vector v_s is well defined up to an additive constant nw_s where $n \in \mathbb{Z}$. The twist t_s of $\mathcal{C}_s$ will be the difference $t_s = v_s - \frac{(h_s)_y}{(h_b)_y} v_b$. But

$$\frac{(h_s)_y}{(h_b)_y} v_b = \frac{1}{2}(-1+2\alpha+\alpha^2) \begin{pmatrix} 1-\alpha^2 \\ 2\alpha-2\alpha^2 \end{pmatrix} = \begin{pmatrix} -1+\alpha+2\alpha^2 \\ -2\alpha+4\alpha^2 \end{pmatrix}.$$

Therefore the twist vector in the small cylinder is $t_s = \begin{pmatrix} 2 \\ 2-2\alpha^2 \end{pmatrix}$. Using $w_s = \gamma_3 = \gamma_1$ we calculate the normalized twist of $\mathcal{C}_s$ to be

$$\frac{|t_s|}{|w_s|} = \left(\frac{4+8\alpha-8\alpha^2}{16+18\alpha+10\alpha^2} \right)^{1/2} = \left(1-2\alpha+\alpha^2\right)^{1/2} \in [0,1[$$

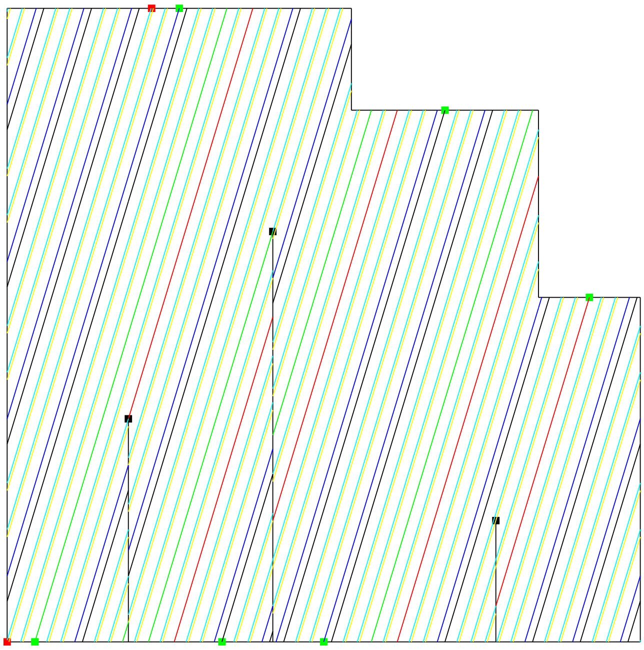

FIGURE 4. Singular geodesics on the Arnoux-Yoccoz flat surface in the direction $\theta' = 3 + \alpha^2$: again a 3C-direction.

We summarize the above computations for the direction $\theta = 1 - \alpha^2$.

$$\overrightarrow{L}^2(\theta) = \begin{bmatrix} 40 + 34\alpha + 22\alpha^2 \\ 1 \\ 40 + 34\alpha + 22\alpha^2 \\ 4 + 2\alpha + 2\alpha^2 \\ 1 \\ 4 + 2\alpha + 2\alpha^2 \end{bmatrix} \qquad \overrightarrow{W}(\theta) = \begin{bmatrix} 3 + 3\alpha + 2\alpha^2 \\ 2 + 2\alpha + \alpha^2 \\ 1 \end{bmatrix}$$

$$(\mathcal{G}, m) = \left(\begin{pmatrix} \underline{2} & 4 & 5 & 1 \\ \underline{3} & 2 & 6 & 5 \end{pmatrix}, \begin{pmatrix} \underline{6} \\ \underline{4} \end{pmatrix}, \begin{pmatrix} \underline{3} \\ \underline{1} \end{pmatrix} \right) \qquad \overrightarrow{T}^2(\theta, m, 0) = (0, 1 - 2\alpha + \alpha^2)$$

normalized twists

4.3. **A second direction:** $\theta' = 3+\alpha^2$**.** Using Figure 4 one checks that θ' is also a 3C-direction with the following invariants.

saddle connection γ'_i	$(\lvert\gamma'_i\rvert^2)_{i=1,\dots,6} \in \mathbb{R}^6$	$L(\theta')^2 \in \mathbb{RP}(6)$
$i=1$	$26+20\alpha+11\alpha^2$	1
$i=2$	$144+114\alpha+74\alpha^2$	$4+2\alpha+2\alpha^2$
$i=3$	$144+114\alpha+74\alpha^2$	$4+2\alpha+2\alpha^2$
$i=4$	$26+20\alpha+11\alpha^2$	1
$i=5$	$1612+1354\alpha+878\alpha^2$	$40+34\alpha+22\alpha^2$
$i=6$	$1612+1354\alpha+878\alpha^2$	$40+34\alpha+22\alpha^2$

As previously we label the cylinders by $\mathcal{C}'_b, \mathcal{C}'_m, \mathcal{C}'_s$ according to their widths. The combinatorics with some choice of marking is given by

$$(\mathcal{G}', m') = \left(\left(\begin{smallmatrix} \underline{4} & 3 & 1 & 6 \\ \underline{5} & 4 & 2 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} \underline{2} \\ \underline{3} \end{smallmatrix}\right), \left(\begin{smallmatrix} \underline{5} \\ \underline{6} \end{smallmatrix}\right) \right).$$

A straightforward computation shows the expected results for the heights of the cylinders in the direction θ'.

cylinder	$(\ (h'_b)_y, (h'_m)_y, (h'_s)_y\ ) \in \mathbb{R}^3$	$[\ (h'_b)_y, (h'_m)_y, (h'_s)_y\ ] \in \mathbb{RP}(3)$
$\mathcal{C}'_b$	$-6+8\alpha+6\alpha^2$	$3+3\alpha+2\alpha^2$
$\mathcal{C}'_m$	$-2+6\alpha-4\alpha^2$	$2+2\alpha+\alpha^2$
$\mathcal{C}'_s$	$10-14\alpha-8\alpha^2$	1

To calculate the twists of $\mathcal{C}'_m$ and $\mathcal{C}'_s$ we first calculate the direction $\theta'^{\perp}$. The twist vector of $\mathcal{C}'_b$ is a vector (contained in this cylinder) joining the origin of the saddle γ'_5 to the origin of the saddle connection γ'_4. We will use the previous way for the computation of the twist vector of $\mathcal{C}'_b$.

Let P be the endpoint of the vector $h'_b = \left(\begin{smallmatrix} 0 \\ (h'_b)_y \end{smallmatrix}\right)$ based at the origin of γ'_5, namely P_5. The point P is located on the top of $\mathcal{C}'_b$. Let v'_b be the vector joining P to the origin of the saddle connection γ'_4 (namely P_4) in the direction θ'. A simple computation using the requirement $|v'_b| < |w'_b|$ shows $v'_b = \left(\begin{smallmatrix} 5+2\alpha-\alpha^2 \\ 18+2\alpha \end{smallmatrix}\right)$. Given $n'_0 \in \mathbb{Z}$, the twist vector of $\mathcal{C}'_b$ is then $h'_b+v'_b+n'_0w'_b$. The width of the big cylinder is given by the vector $w'_b = \gamma'_5+\gamma'_4+\gamma'_2+\gamma'_1 = \left(\begin{smallmatrix} 14+12\alpha+8\alpha^2 \\ 46+40\alpha+26\alpha^2 \end{smallmatrix}\right)$. Finally with our marking, one obtains the direction $\theta'^{\perp} := \theta'^{\perp}(n_0) = h'_b + v'_b + n'_0 w'_b$.

Proceeding as in the previous section we obtain $v'_m = \left(\begin{smallmatrix} 4-3\alpha-\alpha^2 \\ 10-8\alpha+4\alpha^2 \end{smallmatrix}\right)$ and

$$\frac{(h'_m)_y}{(h'_b)_y}(v'_b + n'_0 w'_b) = \frac{1}{2}(1+\alpha^2)\left(\left(\begin{smallmatrix} 5+2\alpha-\alpha^2 \\ 18+2\alpha \end{smallmatrix}\right) + n'_0\left(\begin{smallmatrix} 14+12\alpha+8\alpha^2 \\ 46+40\alpha+26\alpha^2 \end{smallmatrix}\right)\right) =$$

$$= \left(\begin{smallmatrix} 4-\alpha+\alpha^2 \\ 10+8\alpha^2 \end{smallmatrix}\right) + n'_0\left(\begin{smallmatrix} 9+8\alpha+5\alpha^2 \\ 30+26\alpha+16\alpha^2 \end{smallmatrix}\right).$$

This yields

$$t'_m = v'_m - \frac{(h'_m)_y}{(h'_b)_y}(v'_b + n'_0 w'_b) = \left(\begin{smallmatrix} -2\alpha-2\alpha^2 \\ -8\alpha-4\alpha^2 \end{smallmatrix}\right) - n'_0\left(\begin{smallmatrix} 9+8\alpha+5\alpha^2 \\ 30+26\alpha+16\alpha^2 \end{smallmatrix}\right).$$

The width of $\mathcal{C}'_m$ is the norm of the vector $w'_m = \gamma_2 = \gamma_3 = \left(\begin{smallmatrix} 3+2\alpha+\alpha^2 \\ 10+6\alpha+4\alpha^2 \end{smallmatrix}\right)$. Therefore the normalized twist of $\mathcal{C}'_m$ is

$$\text{(4.1)}\quad \frac{|t'_m|}{|w'_m|} \;=\; \left(4+2n'_0+7{n'_0}^2+(-6+6{n'_0}^2)\alpha+(-2+2n'_0+4{n'_0}^2)\alpha^2\right)^{1/2} \quad\in\quad \mathbb{R}/\mathbb{Z}$$

Remark 4.1. If $n'_0 = -1$ then $\frac{|t'_m|}{|w'_m|}$ equals 3. Thus in case $n'_0 = -1$, the normalized twist of $\mathcal{C}'_m$ is null, more precisely $t'_m = 3w'_m$.

Now let us finish our computation of the twists invariants by the one of $\mathcal{C}'_s$. We obtain $v'_s = \left(\begin{smallmatrix} -1+5\alpha+6\alpha^2 \\ -4+22\alpha+12\alpha^2 \end{smallmatrix}\right)$ and

$$\frac{(h'_s)_y}{(h'_b)_y}(v'_b+n'_0w'_b) = \frac{1}{2}(-1+2\alpha+\alpha^2)\left(\left(\begin{smallmatrix} 5+2\alpha-\alpha^2 \\ 18+2\alpha \end{smallmatrix}\right)+n'_0\left(\begin{smallmatrix} 14+12\alpha+8\alpha^2 \\ 46+40\alpha+26\alpha^2 \end{smallmatrix}\right)\right) = \\ = \left(\begin{smallmatrix} -2+3\alpha+5\alpha^2 \\ -8+16\alpha+10\alpha^2 \end{smallmatrix}\right)+n'_0\left(\begin{smallmatrix} 3+2\alpha+\alpha^2 \\ 10+6\alpha+4\alpha^2 \end{smallmatrix}\right)$$

Together this gives

$$t'_s = v'_s - \frac{(h'_s)_y}{(h'_b)_y}(v'_b+n'_0w'_b) = \left(\begin{smallmatrix} 1+2\alpha+\alpha^2 \\ 4+6\alpha+2\alpha^2 \end{smallmatrix}\right) - n'_0\left(\begin{smallmatrix} 3+2\alpha+\alpha^2 \\ 10+6\alpha+4\alpha^2 \end{smallmatrix}\right).$$

The width w'_s of the small cylinder is given by $w_s = \gamma'_5 = \gamma'_6$, hence

$$\text{(4.2)}\quad \frac{|t'_s|}{|w'_s|} \;=\; \left(2-{n'_0}^2+(-2+2n'_0+2{n'_0}^2)\alpha+(-3-4n'_0)\alpha^2\right)^{1/2} \quad\in\quad \mathbb{R}/\mathbb{Z}$$

Remark 4.2. If $n'_0 = -1$ then $\frac{|t'_s|}{|w'_s|}$ equals $(1-2\alpha+\alpha^2)^{1/2} \in [0,1[$.

One can summarize the above computations (for the direction $\theta' = 3+\alpha^2$)

$$\overrightarrow{L}^2(\theta') = \begin{bmatrix} 1 \\ 4+2\alpha+2\alpha^2 \\ 4+2\alpha+2\alpha^2 \\ 1 \\ 40+34\alpha+22\alpha^2 \\ 40+34\alpha+22\alpha^2 \end{bmatrix} \qquad \overrightarrow{W}(\theta') = \begin{bmatrix} 3+3\alpha+2\alpha^2 \\ 2+2\alpha+\alpha^2 \\ 1 \end{bmatrix}$$

$$(\mathcal{G}', m') = \left(\left(\begin{smallmatrix} \underline{4} & 3 & 1 & 6 \\ \underline{5} & 4 & 2 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} \underline{2} \\ \underline{3} \end{smallmatrix}\right), \left(\begin{smallmatrix} \underline{5} \\ \underline{6} \end{smallmatrix}\right) \right) \qquad \overrightarrow{T}^2(\theta', m', n'_0) = (\text{ see (4.1), see (4.2) })$$

normalized twists

4.4. An other pseudo-Anosov diffeomorphism.

Theorem 4.1. *The stabilizer of the Arnoux-Yoccoz Teichmüller disc is not cyclic. More precisely, there exists a pseudo-Anosov diffeomorphism $\tilde{\Phi}$ such that $\tilde{\Phi}$ and the Arnoux-Yoccoz diffeomorphism $\Phi = \Phi_3$ do not possess a common power.*

Proof. If we take $n'_0 = -1$ then Theorem 3.3 applies with above θ, θ', and permutations π_1 on $k = 6$ elements and π_2 on $p = 3$ elements given by

$$\pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 5 & 3 & 1 & 2 \end{pmatrix} \qquad \text{and} \qquad \pi_2 = \mathrm{id}$$

We claim that the diffeomorphism $\tilde{\Phi}$ thus obtained has the required property. For this purpose we calculate its derivative in the coordinates (x, y) given by the Arnoux-Yoccoz construction at the beginning of Section 4. By construction, one knows the image by $\tilde{\Phi}$ of saddle connections in direction θ. In particular, $\tilde{\Phi}(\gamma_i) = \gamma'_{\pi_1(i)}$. For $i = 3$ we obtain

$$\tilde{\Phi}(\gamma_3) = \gamma'_5 \qquad \text{i.e.} \qquad D\tilde{\Phi}\begin{pmatrix} 3+2\alpha+\alpha^2 \\ 2+2\alpha \end{pmatrix} = \begin{pmatrix} 9+8\alpha+5\alpha^2 \\ 30+26\alpha+16\alpha^2 \end{pmatrix}$$

Moreover, with our previous normalization, the image of the vector $h_b + v_b$ is the vector $h'_b + v'_b + n'_0 w'_b$ (with $n'_0 = -1$). Therefore

$$D\tilde{\Phi}\begin{pmatrix} 1-\alpha^2 \\ 2-2\alpha \end{pmatrix} = \begin{pmatrix} -9-10\alpha-9\alpha^2 \\ -34-30\alpha-20\alpha^2 \end{pmatrix}.$$

Applying a change of basis, we get the matrix $D\tilde{\Phi}$ in the coordinate (x, y):

$$D\tilde{\Phi} = \begin{pmatrix} 9+8\alpha+5\alpha^2 & -9-10\alpha-9\alpha^2 \\ 30+26\alpha+16\alpha^2 & -34-30\alpha-20\alpha^2 \end{pmatrix}\begin{pmatrix} 3+2\alpha+\alpha^2 & 1-\alpha^2 \\ 2+2\alpha & 2-2\alpha \end{pmatrix}^{-1} = \begin{pmatrix} 23+18\alpha+12\alpha^2 & -29-24\alpha-16\alpha^2 \\ 74+62\alpha+40\alpha^2 & -95-80\alpha-52\alpha^2 \end{pmatrix}.$$

Since the trace of this matrix is greater than 2, the diffeomorphism $\tilde{\Phi}$ is pseudo-Anosov. Moreover neither the horizontal direction nor the vertical direction is an eigenvector of $D\tilde{\Phi}$. The eigenvectors of the derivate correspond to the stable and unstable foliation. Since the stable and unstable foliation of Φ are by construction the horizontal and vertical one we conclude that both the stable and unstable foliation of Φ and $\tilde{\Phi}$ are different. Since two pseudo-Anosov diffeomorphism have a power in common if and only if their stable and unstable foliation coincide we conclude the proof of the theorem.

□

5. Closure of the disc

The set of translation surfaces $(X, \omega) \in \mathcal{L}$ with fixed area α (with respect to ω) form a real hypersurface $\mathcal{L}_\alpha$ in $\mathcal{L}$.

This section is devoted to prove

Theorem 5.1. *The Teichmüller disc stabilized by the Arnoux-Yoccoz pseudo-Anosov is dense inside $\mathcal{L}_\alpha$.*

Equivalently, we claim that the $\mathrm{GL}_2^+(\mathbb{R})$-orbit of the Arnoux-Yoccoz surface is dense inside $\mathcal{L}$. We will switch between both statements in the proof: Using $\mathrm{GL}_2^+(\mathbb{R})$ is more convenient for normalization while we need an ergodicity argument that holds for the $\mathrm{SL}_2(\mathbb{R})$-action on $\mathcal{L}_\alpha$.

We sketch the strategy of proof. First, by topological considerations we find a direction on the Arnoux-Yoccoz surface with a set of homologous saddle connections that allows to decompose the surface into tori and cylinders. The parabolic subgroup of $\mathrm{SL}_2(\mathbb{R})$ that fixes these saddle connections acts on the set of pairs of lattices that define the tori. Second we apply Ratner's theorem and the explicit geometry of the surface to see that orbit closure in the homogeneous space of pairs of lattices is as big as one could hope for. Hence the closure of the disc contains at least the submanifold of $\mathcal{L}_\alpha$ defined by fixing the areas of the decomposition pieces. We will give details of this using period coordinates on $\mathcal{L}$. The third step consists of splitting Arnoux-Yoccoz surface in another direction to remove this area constraint.

Let (X, ω) be the Arnoux-Yoccoz surface. Besides the completely periodic directions (for example the one with slope $\theta = 1 - \alpha^2$ used in the previous section) there are directions Ξ on (X, ω) with the following property. There are 4 homologous saddle connections β_i such that β_2 and β_3 (resp. β_4 and β_1) bound a cylinder C_1 (resp. C_2). The hyperelliptic involution interchanges C_1 and C_2. The complement of $X \setminus \{C_1 \cup C_2\}$ consists of two components T_1 and T_2. If we identify the cuts lines β_1 and β_2 on T_1 and β_3 and β_4 on T_2 we obtain two tori, which we also denote by the same letter. In fact T_1 and T_2 are irrationally foliated as we will see below.

Such a direction is given by $\Xi = \alpha + \alpha^2$ (see Figure 5).

We apply a matrix in $\mathrm{GL}_2^+(\mathbb{R})$ such that Ξ becomes horizontal and θ vertical and moreover such that γ_2 becomes $\gamma_2' = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ and β_i becomes $\beta_i' = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. We can draw this surface in the $\mathrm{GL}_2^+(\mathbb{R})$-orbit of (X, ω) as indicated in Figure 6. Note that the primes in this section do not the same meaning as the primes in Section 4.3. Note moreover that the figure does not display scales in the vertical and horizontal direction correctly.

We we refer to this surface as the adjusted Arnoux-Yoccoz surface (X', ω'). Moreover we call a splitting of a translation surface in $\mathcal{L}$ with the same topology and dynamics as the horizontal one a *$2T2C$-splitting.*

5.1. **Applying Ratner's theorem.** Let G be a lie group, Γ a lattice in G and $U \subset G$ a 1-parameter subgroup generated by unipotent elements. U acts on the left on $\mathcal{X} = G/\Gamma$. Ratner's theorem ([Rat91]) states that for any $x \in \mathcal{X}$ the closure $\overline{U \cdot x}$ is an orbit $H \cdot x \in \mathcal{X}$, where H is a unimodular subgroup depending on x with the property that $x\Gamma x^{-1} \cap H$ is a lattice an H.

The arguments we will use here are somehow similar to the ones in [Mc3] §2. Let Λ_i, $i = 1, 2, 3$ be lattices in $\mathbb{R}^2$ normalized such that

$$\mathrm{area}(\mathbb{R}^2/\Lambda_i) = 1.$$

Triples of normalized lattices are parameterized by the homogeneous space $\mathcal{X} = G^3/\Gamma^3$ where $G = \mathrm{SL}_2(\mathbb{R})$ and $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. We denote the projections onto the factors resp. pairs

FIGURE 5. Decomposition of the Arnoux-Yoccoz surface: shaded regions represent a 2T2C-direction, black lines the 3C-direction studied in Section 4.2.

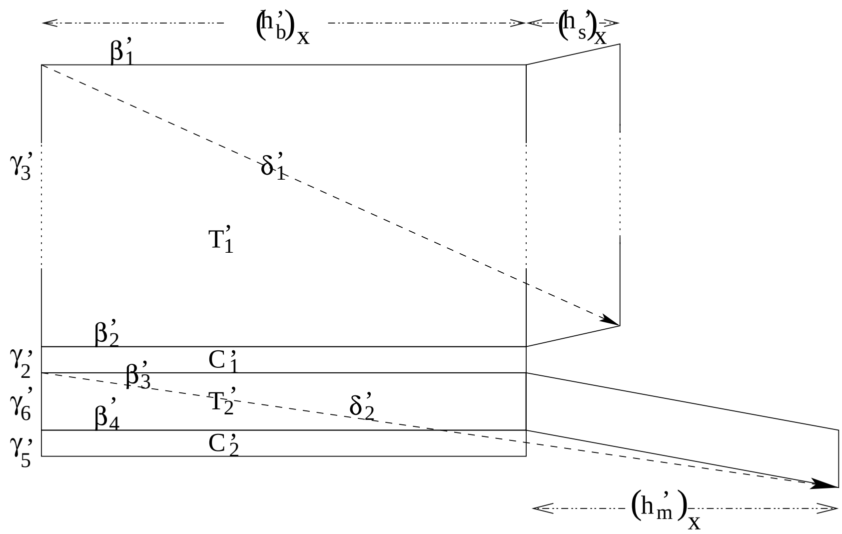

FIGURE 6. The adjusted Arnoux-Yoccoz flat surface.

of factors by pr_i resp. pr_{ij}. Some more notation: Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{n}$ (resp. $\mathfrak{u}$, resp. $\mathfrak{a}$) be the Lie algebra of the unipotent upper triangular matrices N (resp. unipotent lower triangular matrices U, resp. diagonal matrices A). Choose standard generators

$$n = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{n}, \quad a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{a}, \quad u = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathfrak{u}$$

such that $[n, a] = -2n$, $[n, u] = a$ and $[a, u] = -2u$.

Suppose $U \subset G^3$ is a connected group generated by unipotent elements. Then Ratner's theorem ([Rat91]) states that for any $x \in \mathcal{X}$ the closure $\overline{U \cdot x}$ is an orbit $H \cdot x \in \mathcal{X}$, where H is a unimodular subgroup depending on x with the property that $x\Gamma^3 x^{-1} \cap H$ is a lattice an H.

We will apply this to the case where U is the group diagonal embedding N_Δ of the unipotent upper triangular matrices N. For comparison with [Mc3] we remark that McMullen has to deal in genus 2 with the case of one lattice and the action of $N \cap \mathrm{SL}_2(\mathbb{Z})$ or the case of two lattices and the action of N_Δ (Theorems 2.3 and 2.6 of loc. cit.). We remark that we could also proceed Shah's version of Ratner's theorem for cyclic groups ([Sha98]) and the action of $\mathbb{N}_\Delta \cap \mathrm{SL}_2(\mathbb{Z})$ on pairs of lattices, fixing C_i''. We will not list all the possible closures of N_Δ-actions but restrict to what we actually need.

Lemma 5.2. *Suppose that Λ_3 is the standard lattice and $C := \overline{N_\Delta \cdot (\Lambda_1, \Lambda_2, \Lambda_3)}$ projects to the whole space G^2/Γ^2 via pr_{12}. Then $C = (G \times G \times N) \cdot (\Lambda_1, \Lambda_2, \Lambda_3)$.*

Proof. By Ratner's theorem $C = H \cdot (\Lambda_1, \Lambda_2, \Lambda_3)$ for some H. By the hypothesis on Λ_3 the Lie algebra $\mathfrak{h}$ of H is contained in

$$(\mathfrak{n}_1 \oplus \mathfrak{a}_1 \oplus \mathfrak{u}_1) \oplus (\mathfrak{n}_2 \oplus \mathfrak{a}_2 \oplus \mathfrak{u}_2) \oplus \mathfrak{n}_3.$$

Either this is an equality and we are done or $\mathfrak{h}$ is given by one equation $\sum_{i=1}^{7} \alpha_i e_i = 0$, where e_i are the standard generators of the summands defined above. For $i \neq 7$ let $b_i = e_i$ if $\alpha_i = 0$ and $b_i = \alpha_7/\alpha_i e_i - e_7$ otherwise. We have $b_i \in \mathfrak{h}$ in both cases. One checks that $[b_1, b_3]$ is a non-zero multiple of e_2, that $[b_1, b_2]$ is a non-zero multiple of e_1 and that $[b_3, b_2]$ is a non-zero multiple of e_3. Continuing like this $e_i \in \mathfrak{h}$ for $i \neq 7$. Since $\mathfrak{h}$ contains the diagonal we also have $e_7 \in \mathfrak{h}$ and we are done. □

Corollary 5.3. *Let Λ_3 be the standard lattice. Suppose that neither of the lattices Λ_1 and Λ_2 contains a horizontal vector and suppose there does not exist an element $M_t \in N$ such that Λ_1 and $M_t \cdot \Lambda_2$ are commensurable. Then*

$$C := \overline{N_\Delta \cdot (\Lambda_1, \Lambda_2, \Lambda_3)} = (G \times G \times N) \cdot (\Lambda_1, \Lambda_2, \Lambda_3)$$

Proof. The hypothesis on Λ_1 and Λ_2 are just the one needed to apply Theorem 2.6 in [Mc3]. Its conclusion is the condition on pr_{12} needed to apply Lemma 5.2. □

Let C_1' and T_i', $i = 1, 2$, be the components obtained by splitting the adjusted Arnoux-Yoccoz surface (X', ω'). We denote by the same symbols the tori obtained by gluing the slits. Let Λ_i', $i = 1, 2$ and Λ_C' be defined by $\mathbb{R}^2/\Lambda_i' \cong T_i'$ and $\Lambda_3' := \Lambda_C'$ be defined by $\mathbb{R}^2/\Lambda_C' \cong C_1'$. Finally we apply homotheties to Λ_i' in order to obtain lattices $\widetilde{\Lambda}_i$ with area one. We will generally denote with a tilde lattices that are *area-normalized.*

Lemma 5.4. *The area-normalized lattices $\widetilde{\Lambda}_i$ obtained by splitting the adjusted Arnoux-Yoccoz surface (X', ω') in the horizontal direction, i.e. along the saddle connections β_i', satisfy the conditions of Corollary 5.3.*
More generally, let Λ_1, Λ_2 be the lattices

$$\Lambda_i = \langle \left(\begin{smallmatrix} 0 \\ a_i \end{smallmatrix}\right), \left(\begin{smallmatrix} b_i \\ c_i \end{smallmatrix}\right) \rangle.$$

Let $A_i = \sqrt{\operatorname{area}(\mathbb{R}^2/\Lambda_i)}$ with $i = 1, 2$. Assume that a_i, b_i, c_i belong to $\mathbb{Q}(\alpha)$ for $i = 1, 2$. If $\dfrac{c_i}{a_i} \notin \mathbb{Q}$ and $\mathbb{Q}(A_1) \neq \mathbb{Q}(A_2)$ then the area-normalized lattices $\widetilde{\Lambda_1}$ and $\widetilde{\Lambda_2}$ satisfy the conditions of Corollary 5.3.

Proof. We proof the general statement first. Nonexistence of horizontal vectors in $\widetilde{\Lambda}_i$ (and thus in Λ_i) is equivalent to the first condition on the lattices Λ_i. Now let us prove that the area-normalized lattices $\widetilde{\Lambda_1}$ and $M_t \cdot \widetilde{\Lambda_2}$ are not commensurable for any $t \in \mathbb{R}$. By contradiction let us assume there exist integers n, m and p such that

$$n \begin{pmatrix} 0 \\ \frac{a_1}{A_1} \end{pmatrix} = m M_t \begin{pmatrix} 0 \\ \frac{a_2}{A_2} \end{pmatrix} + p M_t \begin{pmatrix} \frac{b_2}{A_2} \\ \frac{c_2}{A_2} \end{pmatrix}.$$

This yields to

$$n \begin{pmatrix} 0 \\ \frac{a_1}{A_1} \end{pmatrix} = m \begin{pmatrix} \frac{ta_2}{A_2} \\ \frac{a_2}{A_2} \end{pmatrix} + p \begin{pmatrix} \frac{b_2 + tc_2}{A_2} \\ \frac{c_2}{A_2} \end{pmatrix}.$$

Taking the second coordinate one gets:

$$n\frac{a_1}{A_1} = m\frac{a_2}{A_2} + p\frac{c_2}{A_2}$$

implying $A_1 \in \mathbb{Q}(A_2)$ which is a contradiction.

Now let us prove that the area-normalized lattices $\widetilde{\Lambda}_i$ obtained by splitting the adjusted Arnoux-Yoccoz surface satisfy the conditions of Corollary 5.3. For that we will use the general statement we have just proved.

Claim 5.5. *The lattices Λ_1' and Λ_2' (and thus $\widetilde{\Lambda_1}$ and $\widetilde{\Lambda_2}$) have no horizontal vectors.*

Proof of the claim. This exactly means that the horizontal flow on T_1' is irrational (resp on T_2'). We explain the method for T_1'. Flowing from P the intersection of γ_3' and β_1' (see Figure 6) in the horizontal direction, we cross again γ_3' at Q. The flow in the horizontal direction is irrational if and only if the first return map on the circle γ_3' is an irrational rotation. We prove that the normalized twist $\frac{PQ}{\gamma_3'} \in \mathbb{Q}(\alpha)$ is irrational. A direct computation shows that the value of the normalized twist for T_1' is $\frac{1}{2} - \frac{\alpha^2}{2}$ and for T_2' is $\frac{10}{11} - \frac{5}{11\alpha} + \frac{3}{11\alpha^2}$. Both numbers are irrational which proves the claim. □

Now let us verify the second hypothesis. We have to prove that the numbers $\sqrt{\mathrm{area}(T_1')}$ and $\sqrt{\mathrm{area}(T_2')}$ generate two different quadratic extensions of $\mathbb{Q}(\alpha)$. According to Figure 6 one has

$$\mathrm{area}(T_1') = \gamma_3' \wedge \beta_1' + \gamma_3' \wedge h_s' \quad \text{and} \quad \mathrm{area}(T_2') = \gamma_6' \wedge \beta_4' + \gamma_6' \wedge h_m', \tag{5.1}$$

where $\wedge$ is the cross product. The adjusted Arnoux-Yoccoz surface is obtained from the original one by the linear map which sends the vectors γ_2 to $\binom{0}{1}$ and β_2 to $\binom{1}{0}$. These calculations are done in $\mathbb{Q}(\alpha)$. Since

$$\frac{\mathrm{area}(T_1')}{\mathrm{area}(T_1)} = \frac{\mathrm{area}(T_2')}{\mathrm{area}(T_2)} \in \mathbb{Q}(\alpha),$$

we may as well prove the same assertion for $\sqrt{\mathrm{area}(T_1)}$ and $\sqrt{\mathrm{area}(T_2)}$ using Formula 5.1 without the primes and the values from table in Section 4.2. This leads to:

$$\mathrm{area}(T_1) = 4\alpha \qquad \text{and} \qquad \mathrm{area}(T_2) = -8 + 12\alpha + 8\alpha^2.$$

and the proof of Lemma 5.4 is completed by the following claim. □

Claim 5.6. *The numbers* $\sqrt{\mathrm{area}(T_1)}$ *and* $\sqrt{\mathrm{area}(T_2)}$ *generate two different quadratic extensions of* $\mathbb{Q}[\alpha]$.

Proof of the Claim. We first prove that $x := \sqrt{\mathrm{area}(T_1)}$ does not belong to $\mathbb{Q}(\alpha)$. The polynomial $Q(X) = X^6 + X^4 + X^2 - 1$ annihilates x by definition of α and it is irreducible over $\mathbb{Q}$. Indeed $Q(X+1) = 2+12X+22X^2+24X^3+16X^4+6X^5+X^6$ satisfies Eisenstein's criterion with respect to the prime number 2.

Second, assume by contradiction, that $y := \sqrt{-2 + 3\alpha + 2\alpha^2}$ belongs to $\mathbb{Q}(x)$. Then, there exist $a, b \in \mathbb{Q}(\alpha)$ such that $y = a + bx$. Taking the square of this equation we get $a^2 + b^2\alpha + 2abx = -2 + 3\alpha + 2\alpha^2$. In the basis $\{1, x\}$ this leads to the two following equations.

$$\begin{cases} ab &= 0 \\ a^2 + b^2\alpha &= -2 + 3\alpha + 2\alpha^2 \end{cases}$$

If $b = 0$, we get $a^2 = -2 + 3\alpha + 2\alpha^2$. We have already proved that $-2 + 3\alpha + 2\alpha^2$ is not a square in $\mathbb{Q}(\alpha)$ which is a contradiction with $a \in \mathbb{Q}(\alpha)$.
If $a = 0$, we get $b^2 = 1 - 2\alpha^2$. A straightforward computation shows that b is a root of the polynomial $S(X) = X^6 - 5X^4 + 19X^2 - 7$. This polynomial is irreducible over $\mathbb{Q}$. Consequently $\mathbb{Q}(b)$ has degree 6 over $\mathbb{Q}$ which is a contradiction to $b \in \mathbb{Q}(\alpha)$. The claim is proven. □

5.2. **Splitting in different directions.** We want to conclude the proof of Theorem 5.1. By the following observation it suffices to show that the closure of the Arnoux-Yoccoz-disc contains a set of positive measure of $\mathcal{L}_\alpha$ or equivalently that the $\mathrm{GL}_2^+(\mathbb{R})$-orbit of (X', ω') contains a set of positive measure of $\mathcal{L}$.

Lemma 5.7. *The action of* $\mathrm{SL}_2(\mathbb{R})$ *on the hypersurface* $\mathcal{L}_\alpha$ *of fixed area* α *in hyperelliptic locus* $\mathcal{L} \subset \mathcal{H}(2,2)$ *is ergodic.*

Proof. Let $\mathcal{Q}(1,1,-1^6)$ be the stratum of quadratic differentials on the projective line with two simple zeros and 6 simple poles. Taking a double cover, ramified at each of the poles and zeroes of the quadratic differential, yields a $\mathrm{SL}_2(\mathbb{R})$-equivariant local diffeomorphism ([KZ03], [Lan1])

$$\mathcal{Q}(1,1,-1^6) \to \mathcal{H}^{\mathrm{odd}}(2,2)$$

The action of $\mathrm{SL}_2(\mathbb{R})$ on the hypersurface of fixed area in any stratum of quadratic differentials is ergodic by [Vee86], see also [MS91]. These two statements imply the Lemma. □

Given a basis of the relative homology of a surface in $\mathcal{H}^{\mathrm{odd}}(2,2)$ integration of the one-form defines a map to $\mathbb{C}^7$. This map is a local biholomorphism ([DH75], [MS91], [Vee90]), the coordinates are called *period coordinates*. In case of the adjusted Arnoux-Yoccoz surface a basis of the relative homology is given by $\{\gamma_2', \gamma_3', \gamma_6', \gamma_5', \beta_1', \delta_1', \delta_2'\}$. If $\gamma_2' = \gamma_5'$ then the 180 degree rotation around the center of C_1' defines an affine diffeomorphism with 8 fixed points. Hence this equation singles out the hyperelliptic locus $\mathcal{L}$ inside $\mathcal{H}^{\mathrm{odd}}(2,2)$.

To motivate the resplitting below consider another, neither complex nor linear, coordinate system for $\mathcal{L}$ around (X', ω')

$$\{\mathrm{area}(T_1'), \mathrm{area}(T_2'), \mathrm{area}(C_1'), |\gamma_2'|/\sqrt{\mathrm{area}(C_1')}, |\beta_1|/\sqrt{\mathrm{area}(C_1')},$$
$$|\gamma_3'|/\sqrt{\mathrm{area}(T_1')}, |\delta_1'|/\sqrt{\mathrm{area}(T_1')}, |\gamma_6|/\sqrt{\mathrm{area}(T_2')}, |\delta_2'|/\sqrt{\mathrm{area}(T_2')}\}.$$

We deduce from Lemma 5.4 that the orbit closure of the Arnoux-Yoccoz disc contains a full neighborhood of the initial value for $|\gamma_3'|/\sqrt{\mathrm{area}(T_1')}$, $|\delta_1'|/\sqrt{\mathrm{area}(T_1')}$, $|\gamma_6'|/\sqrt{\mathrm{area}(T_2')}$ and $|\delta_2'|/\sqrt{\mathrm{area}(T_2')}$ while the other parameters are kept fixed. Using the $\mathrm{GL}_2^+(\mathbb{R})$-action we may vary $|\gamma_2'|/\sqrt{\mathrm{area}(C_1')}$, $|\beta_1'|/\sqrt{\mathrm{area}(C_1')}$ and $\mathrm{area}(C_1')$ arbitrarily while the ratios of areas of the splitting pieces is kept fixed.

Hence till now we know that in a neighborhood of (X', ω')

$$\overline{\mathrm{GL}_2^+(\mathbb{R}) \cdot (X', \omega')} \supset \{\mathrm{area}(T_i')/\mathrm{area}(C_1') = \kappa_i, i = 1, 2\}$$

for some constants κ_1 and κ_2. We have to vary these constants next using resplittings. The final argument below will use yet another coordinate system since we have to control when the hypothesis of Lemma 5.4 are satisfied.

Apply a Dehn twist to the vertical cylinder of height $(h_b)_x := (\beta_1)_x = 1$ in Figure 6. The corresponding direction is also a $2T2C$-direction. We denote objects in this direction by double-primes , i.e. we have a *twisted splitting*

$$(X', \omega') = T_1'' \# C_1'' \# T_2'' \# C_2''.$$

By Lemma 5.4, the N–orbit closure of $(T_1' \# C_1' \# T_2' \# C_2')$ contains $N \times N \times \{id\} \cdot (T_1' \# C_1' \# T_2' \# C_2')$. In particular for (u_1, u_2) in a neighborhood of $(0,0)$

$$(X'(u_1, u_2), \omega'(u_1, u_2)) := M_{u_1}(T_1') \# M_{u_2} C_1' \# T_2' \# M_{u_2} C_2',$$

lies in the N–orbit closure, where $M_{u_i} = \begin{pmatrix} 1 & u_i \\ 0 & 1 \end{pmatrix}$.

Lemma 5.8. *For (u_1, u_2) in a neighborhood of $(0,0)$ the twisted decomposition persists.*

The statement is obvious from the construction via Dehn twists. We denote this decomposition as follows:

$$(X'(u_1,u_2),\omega'(u_1,u_2)) := T_1''(u_1,u_2)\,\#\,C_1''(u_1,u_2)\,\#\,T_2''(u_1,u_2))\#\,C_2''(u_1,u_2).$$

We fix some more notation: $T_i''(u_1,u_2) = \mathbb{R}^2/\Lambda_i''(u_1,u_2)$ and $\Lambda_3''(u_1,u_2) := \Lambda_C''(u_1,u_2)$ is defined by $C_1''(u_1,u_2) = \mathbb{R}^2/\Lambda_1''(u_1,u_2)$. By definition we have

$$\begin{aligned}
\Lambda_1''(u_1,u_2) &= \left\langle \begin{pmatrix} u_1|\gamma_3'| \\ |\gamma_3'| \end{pmatrix}, V(u_1,u_2) + \begin{pmatrix} (h_s')_x + u_1(h_s')_y \\ (h_s')_y \end{pmatrix} \right\rangle \\
\Lambda_2''(u_1,u_2) &= \left\langle \begin{pmatrix} 0 \\ |\gamma_6'| \end{pmatrix},\ V(u_1,u_2) + \begin{pmatrix} (h_m')_x \\ (h_m')_y \end{pmatrix} \right\rangle \\
\Lambda_3''(u_1,u_2) &= \left\langle \begin{pmatrix} u_2|\gamma_2'| \\ |\gamma_2'| \end{pmatrix}, V(u_1,u_2) \right\rangle
\end{aligned}$$

where

$$V(u_1,u_2) = \begin{pmatrix} 2u_2\ |\gamma_2'| \\ 2|\gamma_2'| \end{pmatrix} + \begin{pmatrix} 0 \\ |\gamma_6'| \end{pmatrix} + \begin{pmatrix} u_1\ |\gamma_3'| \\ |\gamma_3'| \end{pmatrix} + \begin{pmatrix} \beta_1' \\ 0 \end{pmatrix}.$$

The next Lemma shows that the (u_1,u_2)-twisting can indeed be used to adjust the areas.

Lemma 5.9. *The map*

$$\varphi : (u_1,u_2) \mapsto (\mathrm{area}(T_1''(u_1,u_2))/\mathrm{area}(C_1''(u_1,u_2)), \mathrm{area}(T_2''(u_1,u_2))/\mathrm{area}(C_1''(u_1,u_2)))$$

is an invertible function in a neighborhood of $(u_1,u_2) = (0,0)$.

Proof. We remark that φ is the composition of

$$\psi : (u_1,u_2) \mapsto (\mathrm{area}(T_1''(u_1,u_2)), \mathrm{area}(T_2''(u_1,u_2)))$$

and

$$\eta : (x,y) \mapsto (\frac{2x}{1-x-y}, \frac{2y}{1-x-y}).$$

A direct computation shows that the Jacobian of η is $\frac{4}{(1-x-y)^3}$. This number is non zero when $x+y$ is far from 1, a condition which is satisfied for (u_1,u_2) in a neighborhood of $(0,0)$.

The rest of the proof of the lemma consists in computing the Jacobian of ψ.

$$\mathrm{area}(T_1'') = u_1|\gamma_3'|V(u_1,u_2)_y - |\gamma_3'|(2u_2|\gamma_2'| + u_1|\gamma_3'| + |\beta_1'| + (h_s')_x)$$

and

$$\mathrm{area}(T_2'') = |\gamma_6'|(2u_2|\gamma_2'| + u_1|\gamma_3'| + |\beta_1'| + (h_m')_x).$$

A direct computation leads to

$$\mathrm{Jacobian}(\psi) = 2|\gamma_3'||\gamma_2'||\gamma_6'|V_y.$$

This number is non zero therefore, ψ is locally one to one. □

We want to apply the result of Corollary 5.3 to the decomposition $(T_1''(u_1,u_2), C_1''(u_1,u_2), T_2''(u_1,u_2)))$. The "horizontal direction" is the direction of the vector $V(u_1,u_2)$. We denote by N_V the conjugate of N fixing $V(u_1,u_2)$.

Lemma 5.10. *For almost all (u_1, u_2) in a neighborhood of $(0,0)$ with respect to the Lebesgue measure the N_V–orbit of $(T_1''(u_1,u_2), T_2''(u_1,u_2), C_1''(u_1,u_2))$ contains*

$$(G \times G \times N_V) \cdot (T_1''(u_1,u_2), T_2''(u_1,u_2), C_1''(u_1,u_2)).$$

Proof. We have to check the hypothesis of Corollary 5.3 for almost all (u_1, u_2) in a neighborhood of $(0,0)$ replacing the horizontal direction by the direction of $V(u_1,u_2)$.

If $\Lambda_1''(u_1,u_2)$ has a vector parallel to $V(u_1,u_2)$, there exist $t \in \mathbb{R}$, $n, p \in \mathbb{Z}$ such that

$$tV(u_1,u_2) = n \begin{pmatrix} u_1|\gamma_3'| \\ |\gamma_3'| \end{pmatrix} + p \begin{pmatrix} (h_s')_x + u_1 (h_s')_y \\ (h_s')_y \end{pmatrix}$$

The second coordinate equation implies $t = \frac{p(h_s')_y + n|\gamma_3'|}{2|\gamma_2'|+|\gamma_6'|+|\gamma_3'|}$. Thus, if n and p are fixed, the first coordinate yields a linear equation in (u_1,u_2). Almost all parameters (u_1,u_2), do not belong to this countable union of lines which gives the first claim. The same reasoning holds for the lattice $\Lambda_2''(u_1,u_2)$.

Let $\hat{M}_t$ for $t \in \mathbb{R}$ denote the elements of N_V, define $A_i'' := \text{area}(\Lambda_i''(u_1,u_2))$ and abbreviate $V := V(u_1,u_2)$. If the normalized lattice $\hat{M}_t(\widetilde{\Lambda_2''}(u_1,u_2))$ is commensurable to $\widetilde{\Lambda_1''}(u_1,u_2)$ for some t, there are integers $m \neq 0$, n, p such that:

$$\frac{m}{\sqrt{A_1}} \left(tV + \begin{pmatrix} 0 \\ |\gamma_6'| \end{pmatrix} \right) = \frac{1}{\sqrt{A_2}} \left(n \begin{pmatrix} u_1|\gamma_3'| \\ |\gamma_3'| \end{pmatrix} + p \left(V + \begin{pmatrix} (h_s')_x + u_1(h_s')_y \\ (h_s')_y \end{pmatrix} \right) \right)$$

We want to show that for each fixed $(m,n,p) \in \mathbb{Z}^3$ but arbitrary t the set of solutions (u_1,u_2) for this equation is of measure zero. Let $|\gamma'| := 2|\gamma_2'| + |\gamma_3'| + |\gamma_6'|$ be the height of the big vertical cylinder. From the second coordinate we obtain

$$t = \frac{\sqrt{A_1}(n|\gamma_3'| + p(|\gamma'| + (h_s')_y))}{\sqrt{A_2} m |\gamma'|} - m|\gamma_6'|.$$

Plugging this into the first coordinate and using the area expressions from the proof of Lemma 5.9 one obtains an algebraic equation for (u_1,u_2), whose non-triviality we have to decide. Suppose this equation is trivial. Then $(u_1,u_2) = (0,0)$ is a solution. But for $(u_1,u_2) = (0,0)$ we have by the Dehn twist construction obviously $\text{area}(\Lambda_i''(0,0)) = \text{area}(\Lambda_i')$ for $i = 1,2$. The field extension argument from Lemma 5.4 and Claim 5.6 applies and yields a contradiction. □

Now, we have all the material to complete the proof of Theorem 1.3. The surfaces $(X'(u_1,u_2))$ belong to the closure of the $\text{SL}_2(\mathbb{R})$-orbit of (X', ω') if (u_1,u_2) belongs to a small neighborhood of $(0,0)$. Take (u_1,u_2) in the set of full measure which satisfies Lemma 5.10. The N_V–orbit closure of $(\Lambda_1''(u_1,u_2), \Lambda_2''(u_1,u_2), \Lambda_3''(u_1,u_2))$ equals

$$(G \times G \times N) \cdot (\Lambda_1''(u_1,u_2), \Lambda_2''(u_1,u_2), \Lambda_3''(u_1,u_2)).$$

Applying an element of $\text{SL}_2(\mathbb{R})$, we can map $V(u_1,u_2)$ to any vector in $\mathbb{R}^2$. The previous analysis is $\text{SL}_2(\mathbb{R})$ equivariant. Thus, given $A \in \text{SL}_2(\mathbb{R})$, the orbit closure under the action of the unipotent subgroup of the decomposition of $A \cdot (X'(u_1,u_2))$ in the direction $A \cdot V$ equals the $G \times G \times N_{A.V}$-orbit. As in the motivating discussion after Lemma 5.7 we see, using period coordinates and still for fixed (u_1, u_2), that the ratios of the splitting pieces are unchanged by this process. But within the real codimension 2 neighborhood of (X', ω') in $\mathcal{L}_\alpha$ determined by the ratios of the splitting pieces the $\mathrm{SL}_2(\mathbb{R})$-orbit contains an open set. On the other hand, by Lemma 5.9 the data of (u_1, u_2) is equivalent to the areas of the splitting pieces. Thus, the $\mathrm{SL}(2,\mathbb{R})$-orbit closure of (X', ω') contains a set of positive measure in $\mathcal{L}_\alpha$. Applying Lemma 5.7, we conclude that the $\mathrm{SL}_2(\mathbb{R})$-orbit closure of (X, ω) is equal to $\mathcal{L}_\alpha$.

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