$\mathrm{GL}_2^+(\mathbb{R})$-ORBIT CLOSURES VIA TOPOLOGICAL SPLITTINGS

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Introduction

For an arbitrary dynamical system, it is very hard in general to give informations on the behavior of a particular orbit. Nevertheless the situation for unipotent flows in homogeneous spaces is very well-understood. Ratner proved the striking result that the closure of *any* orbit of any group generated by unipotent elements acting on a homogenous space is also a nice homogeneous space.

While for example the unit cotangent bundle to the moduli space of Abelian varieties A_g is a homogeneous space, the unit cotangent bundle to the moduli space of curves $\mathcal{M}_g$ is not. But there is an action of $\mathrm{GL}_2^+(\mathbb{R})$ on the cotangent bundle which is natural from the point of view e.g. of translation surfaces and billiards. The cotangent bundle of $\mathcal{M}_g$ contains the bundle of holomorphic one-forms over $\mathcal{M}_g$ (i.e. the Hodge bundle). This bundle is preserved by the action of $\mathrm{GL}_2^+(\mathbb{R})$ and most phenomena of the $\mathrm{GL}_2^+(\mathbb{R})$-dynamics are already present for

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the action on the Hodge bundle. Points of this bundle will in the sequel be called translation surfaces. See Section 2 for an introduction to this terminology.

There is a strong hope to believe that the Hodge bundle behaves, with respect to the $\mathrm{GL}_2^+(\mathbb{R})$-action, as if it was a homogeneous space. More precisely, the main conjecture is that the closure of any $\mathrm{GL}_2^+(\mathbb{R})$-orbit is an algebraic suborbifold. In this case, as remarked by Kontsevich, the orbit closure admits a structure of an affine manifold, and this extra structure may then be used to classify these orbits closures.

This conjecture has been recently proven for genus $g = 2$ by McMullen ([Mc3]). One of his main techniques, splittings of translation surfaces, extends to higher genera only in special strata and only with considerable more combinatorial effort ([HLM06, HLM07]). The purpose of this survey is to explain the splitting results, how they combine with Ratner's theorem to McMullen's proof, how they generalize to higher genera and what the limits of this strategy are. For this purpose, we recall some aspects of flat surfaces. There are other surveys on this subject with different focus, for example [Es06], [Fo06], [HS06], [Ma86], [MaTa02], [Mö07], [Vi07], [Yo06], [Zo06].

Translation surfaces whose $\mathrm{GL}_2^+(\mathbb{R})$-orbit is closed are called *Veech surfaces*. There is a remarkable link between properties of $\mathrm{GL}_2^+(\mathbb{R})$-orbits and dynamical properties of the translation surface ([Ve89]). We now explain this to provide the context and terminology for the two main results stated at the end of this introduction. The precise definitions of the objects will be given with more details in the next sections of this survey. We only present here what is really needed to state the results and put things in perspective.

The stabilizer of the $\mathrm{GL}_2^+(\mathbb{R})$-orbit of a translation surface is a Fuchsian group called the *Veech group*. A translation surface is a Veech surface if and only if the Veech group is a lattice in $\mathrm{SL}_2(\mathbb{R})$ ([Ve89, Ve92, SW08]). A translation surface has optimal dynamical properties, or *satisfies the Veech dichotomy*, if and only if the flow is either uniquely ergodic or completely periodic depending only on the direction. Veech surfaces satisfy Veech dichotomy, but the converse is true only in genus two ([MaCh07],[SW06]).

It is quite difficult to construct Veech surfaces but even for candidates with relatively large stabilizer of the $\mathrm{GL}_2^+(\mathbb{R})$-orbit it is easy to falsify the property of being a Veech surface. It suffices to search for a direction for which neither of the two cases of the Veech dichotomy hold. In this case, we will say that a surface is *not a Veech surface for the most obvious reason* (see Section 2).

We will see below that large Veech groups have some impact on the $\mathrm{GL}_2^+(\mathbb{R})$-orbit closure in genus 2. For this purpose we classify elements in Veech groups. It is an easy observation that for a typical surface, the Veech group is trivial and translation surfaces with one parabolic element are abundant (see e.g. [Mö07]). Hyperbolic elements, however, correspond to pseudo-Anosov diffeomorphisms and to construct translation surfaces with a pseudo-Anosov diffeomorphism in the Veech group is not completely obvious (see [Ve82] for a very general construction). The existence of a pseudo-Anosov diffeomorphism forces many constraints on the translation surface: for instance, up to normalization, the (flat) parameters defining it belong to a number field. There are nowadays several methods to produce pseudo-Anosov diffeomorphisms. We mention explicitely the Thurston-Veech construction [Th88], since the pseudo-Anosovs that arise in this way are the product of two parabolic elements (or two multi-twists). Consequently, the Veech groups of translation surfaces arising from the Thurston-Veech construction are 'pretty large' (there are non elementary Fuchsian groups).

To compare genus two and three we finally need a way to tell which translation surfaces do not arise via covering constructions from lower genus or more generally exhibit lower genus behaviour. This is done by the *trace field* the Veech group; this is the number field generated by the traces of all elements of the Veech group. Thurston [Th88] proved that for translation surfaces the degree of the trace field over $\mathbb{Q}$ is at most g. Moreover, to determine the trace field of the Veech group, it suffices to know the trace of any single pseudo-Anosov diffeomorphism ([KS00], see also [Mc1, Mc2]).

With this terminology at hand, our aim is to illustrate the common strategy behind the proof of the following two theorems on $\mathrm{GL}_2^+(\mathbb{R})$-orbit closures.

Theorem 0.1 (McMullen, [Mc3]). *Let (X, ω) be a translation surface of genus two, obtained by the Thurston–Veech construction. Then its $GL_2^+(\mathbb{R})$-orbit closure projects to a Teichmüller curve in $\mathcal{M}_2$ or to the locus of Riemann surfaces, whose Jacobian has real multiplication. In particular, the $GL_2^+(\mathbb{R})$-orbit of such a surface is never dense in the Hodge bundle.*

This sharply contrasts to the behavior of pseudo-Anosovs and orbit closures in genus three.

Theorem 0.2 ([HLM06]). *Let (X, ω) be a translation surface of genus three obtained by the Thurston–Veech construction with cubic trace field. Suppose that X is hyperelliptic and ω has two double zeros, fixed by the hyperelliptic involution. If (X, ω) is not a Veech surface for the most obvious reason, then the $GL_2^+(\mathbb{R})$-orbit closure is the hyperelliptic locus in the corresponding stratum.*

In particular, there are infinitely many translation surfaces in genus three with a non-trivial Veech group, whose $\mathrm{GL}_2^+(\mathbb{R})$-orbit is dense in the hyperelliptic locus in its stratum.

The formulation of these results chosen to make the similarities and differences for the two genera apparent. We emphasize though, that McMullen's classification goes far beyond this statement. We explain in Section 4 the full statement and the main ingredient, a topological miracle for translation surfaces in genus two.

Remark 0.2.1. The reader may wonder why we impose the topological restrictions in the second sentence of Theorem 0.2. First, these restrictions are very natural when analysing the problem 'low genera first'. In fact, as stated at the beginning, we can address the same orbit closure question for half-translation surfaces i.e. for pairs (X, q) of a Riemann surface and a quadratic differential, i.e. for points in the cotangent bundle to $\mathcal{M}_g$. We will explain in Section 2 that there is a $\mathrm{GL}_2^+(\mathbb{R})$-equivariant isomorphism between the generic stratum of the cotangent bundle to $\mathcal{M}_2$ and the surfaces considered in Theorem 0.2. Second, the behavior may be very different in other loci as explained in Section 6.

1. Ratner's theorem and the special case of $\mathrm{SL}_2(\mathbb{R})^n$

In this section, we recall a classification result on the closure of unipotent orbits in quotients of Lie groups by lattices. The result due to M. Ratner and appeared in a series of papers [Ra90], [Ra90bis], [Ra91], [Ra91bis]. We only state the topological result. One can see [Mo05] for an introduction to the subject and [MaTo94] for an alternative proof.

We recall that a square matrix A is unipotent if $A - I$ is a nilpotent element. More generally, we say that an element g of a Lie group G is unipotent if its adjoint action $x \mapsto gxg^{-1}$ on the Lie algebra is unipotent.

Theorem 1.1. *(Ratner's orbit closure theorem). Let G be any finite-dimensional Lie group, Γ any lattice in G, $X = G/\Gamma$ the corresponding homogeneous space of finite volume. Let U be a connected sub-(Lie)-group of G generated by unipotent elements. Then the closure of the U-orbit $\overline{Ux}$ is itself a homogeneous space of finite volume; in particular, there exists a closed subgroup $U \leq H \leq G$ such that $\overline{Ux} = Hx$. Moreover, $(x\Gamma x^{-1}) \cap H$ is a lattice in H.*

Generalizations were proved by different authors. Shah gives a version of Ratner's theorem for cyclic unipotent groups (see [Sh98]).

We recall that $\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$ is the space of unimodular lattices in $\mathbb{R}^2$. Denote by N the unipotent subgroup of $\mathrm{SL}_2(\mathbb{R})$:

$$N = \left\{ u_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, s \in \mathbb{R} \right\}.$$

u_s is known as the *horocycle flow*. In this survey, we will only need Ratner's theorem for a very special case. We are interested in the behavior of the diagonal action of the one parameter unipotent group N on $G \cong \mathrm{SL}_2(\mathbb{R})^k \times N^n$ for (n, k) integers and

$$\Gamma = \mathrm{SL}_2(\mathbb{Z})^k \times N(\mathbb{Z})^n, \quad \text{where} \quad N(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}, z \in \mathbb{Z}. \right\}.$$

There are not that many candidates for the group H. They are listed for $k = 1, 2$ and $n = 0$ in the following lemma. Given $s \in \mathbb{R}$, we form the twisted diagonal

$$\mathrm{SL}_2(\mathbb{R})_s = \{(g, u_s g u_s^{-1}) : g \in \mathrm{SL}_2(\mathbb{R})\} \subset \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}).$$

Lemma 1.2. *If $k = 1$, $n = 0$, the N-orbit closures in $X = G/\Gamma$ have the form $X = Hx$, where $H = N$ or $H = G$.*

If $k = 2$, $n = 0$, the N-orbit closures in $X = G/\Gamma$ have the form $X = Hx$, where $H = N$ (diagonally embedded), $SL_2(\mathbb{R})_s$, $N \times N$, $N \times SL_2(\mathbb{R})$, $SL_2(\mathbb{R}) \times N$ or G.

The main point of the proof in [Mc3] Theorem 2.4 is that groups like upper triangular matrices for $k = 1$ cannot occur, since they do not intersect Γ in a lattice.

We mainly want criteria ensuring that the orbit closure is very big. The following theorem gives them for the small cases we will discuss in the following sections. Similar statements could be made for larger (k, n) but the number of conditions to be imposed grows with $k + n$.

Theorem 1.3. *Suppose $G \cong \mathrm{SL}_2(\mathbb{R})^k \times N^n$ and that $x = (\Lambda_i, i = 1, \dots, k + n) \in G/\Gamma$ has the following properties:*

i) *If $k = 1$ and $n = 1$, suppose that there does not exist a horizontal vector in Λ_1.*
ii) *If $k = 2$ and $n = 0$, suppose that neither Λ_1 nor Λ_2 contain a horizontal vector and there does not exist $u_s \in N$ such that $u_s(\Lambda_1)$ is commensurable to Λ_2.*
iii) *If $k = 2$ and $n = 1$, suppose that (Λ_1, Λ_2) satisfies the hypothesis of ii).*

Then $\overline{Nx} = Gx$.

Proof. The first claim is a classical result, in fact the converse also holds. The second claim follows from the second part of Lemma 1.2. The hypothesis of iii) implies that the projection $\mathrm{pr}_{12}(H)$ equals $\mathrm{SL}_2(\mathbb{R})^2$. Since H also contains the diagonal embedding of N, a quick calculation of the Lie algebra ([HLM06] Lemma 5.2) implies that $H = G$. $\square$

2. Translation surfaces and $\mathrm{GL}_2^+(\mathbb{R})$-action

In this section, we briefly introduce the basic notions of Teichmüller dynamics. For more on translation surfaces, see the introductory texts [Es06], [Fo06], [HS06], [Ma86], [MaTa02], [Vi07], [Yo06], [Zo06].

Translation surfaces. A surface of genus $g \geq 1$ is called a *translation surface*, if it can be obtained by edge-to-edge gluing of polygons in the plane using translations only. Examples are given in Figure 1 below. The glueing of the vertical sides is as indicated by the numbers, the glueing of the almost horizontal sides is by the unique way this can be achieved via translations.

There is a one to one correspondence between compact translation surfaces and compact Riemann surfaces equipped with a non-zero holomorphic 1–form. Let (X,ω) be a Riemann surface X with a holomorphic 1-form ω. Locally (i.e., in each coordinate patch) $\omega = f(w)dw$. Given a point $p_0 \in X$, we define new coordinates by the map $z(p) = \int_{p_0}^{p} \omega$. In these coordinates, $\omega = dz$ locally. If we change base points in some small patch, then our coordinates change by a translation:

$$c := \int_{p_0}^{p} \omega - \int_{p_1}^{p} \omega = \int_{p_0}^{p_1} \omega.$$

Since c does not depend on p, our transition maps are of the form $z \mapsto z + c$. Thus the pair (X, ω) gives a structure which is called a *translation structure*. The translation structure induces a flat metric with conical singularities. Conversely, a translation structure on a compact orientable surface minus a finite set defines a holomorphic 1-form on X, if the area (for the flat metric) is finite. Thus in the sequel, we will write (X, ω) for a translation surface where X is the underlying Riemann surface and ω the 1–form.

At a zero of ω of multiplicity k, the total angle is $2(k + 1)\pi$. The zeros are the singularities of the flat metric. The total number of zeros (counting multiplicity) of the Abelian differential ω is $2g-2$, where g is the genus of the surface X. The n-uple of the orders of the zeros $\kappa = (k_1, \dots, k_n)$ is called the *signature* of the translation surface.

If one allows glueings of the sides of the polygons by a translation composed with $-\mathrm{id}$, we obtain the notion of *half-translation surface*. A half-translation surface corresponds uniquely to a Riemann surfaces plus non-zero quadratic differential. Half-translation surfaces behave quite similarly to translation surfaces and will appear only in a few places, in the sequel, mainly for comparison.

Moduli spaces. Let $\mathcal{M}_g$ be the moduli space of curves. The set of translation surfaces is parametrized by the bundle of holomorphic one-forms (the Hodge bundle) over $\mathcal{M}_g$ minus the zero section. This space is naturally stratified by the signature of the one-form.

Fixing a signature κ, we call the associated subset of translation surfaces a stratum $\mathcal{H}(\kappa)$. More precisely, the stratum $\mathcal{H}(\kappa)$ is obtained as the quotient of the set of translation surfaces with a given signature by the action of the diffeomorphisms (diffeomorphisms act by precomposition). Such a moduli space possesses a complex structure given by the *period coordinates*: given a basis of the relative homology (a symplectic basis of the absolute homology and cycles joining a zero to the other ones), one gets complex coordinates by integrating ω along this basis. The complex dimension of the orbifold $\mathcal{H}(\kappa)$ is $2g + n - 1$, where g is the genus and n the number of zeros. The period coordinates also define a *Liouville measure*

ν, the Lebesgue measure in the period coordinates normalized so that the lattice $(\mathbb{Z} + i\mathbb{Z})^{2g+n-1} \subset \mathbb{C}^{2g+n-1}$ has area 1. This measure is globally well defined because the Jacobian of every transition function is equal to 1.

Half-translation surfaces similarly are parametrized by a vector bundle over $\mathcal{M}_g$ minus the zero section. It is also stratified according to the signature and we denote its strata by $\mathcal{Q}(\kappa)$.

$GL_2^+(\mathbb{R})$-action. The group $GL_2^+(\mathbb{R})$ acts on the set of translation surfaces by its natural action on planar polygons. In the charts of the translation structure, it acts by postcomposition. The subgroup $SL_2(\mathbb{R})$ preserves the area of a translation surface. The following one-parameter subgroups will play an important role in the sequel: the horocycle flow introduced in the previous section, the *geodesic flow* generated by the action of

$$\left\{ g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, t \in \mathbb{R} \right\},$$

and the *rotational flow* generated by the action of

$$\left\{ R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}, s \in \mathbb{R} \right\}.$$

We denote by $\mathcal{H}^{(1)}(\kappa)$ the translation surfaces of $\mathcal{H}(\kappa)$ of area 1. The measure ν on $\mathcal{H}(\kappa)$ induces a smooth measure $\nu^{(1)}$ on $\mathcal{H}^{(1)}(\kappa)$ defined in the following way. If E is a measurable subset of $\mathcal{H}^{(1)}(\kappa)$ then we define

$$\nu^{(1)}(E) = \nu\left(\{\lambda(X,\omega);\ (X,\omega) \in E \text{ and } 0 < \lambda \leq 1\}\right).$$

By construction, the measure $\nu^{(1)}$ is $SL_2(\mathbb{R})$ invariant. We now have all the material to state the following an important result.

Theorem 2.1 (Masur [Ma82], Veech [Ve82, Ve86])**.** *The $\nu^{(1)}$-volume of the stratum $\mathcal{H}^{(1)}(\kappa)$ is finite. Moreover, the geodesic flow acts ergodically on each of the connected components of $\mathcal{H}^{(1)}(\kappa)$.*

The connected components were classified by Kontsevich and Zorich ([KZ03]). They found two invariants that are complete if the genus is at least 4. One of them is hyperellipticity, the other one is a parity of a spin structure, that plays no role in the sequel. The *hyperelliptic locus* of a stratum is the set of translation surfaces (X, ω) that have an holomorphic involution i such that $X/i \cong \mathbb{P}^1$ A hyperelliptic locus is closed and $GL_2(\mathbb{R})$ invariant. *Hyperelliptic connected components* are components of strata consisting of hyperelliptic surfaces. They only exist in the strata $\mathcal{H}(2g)$ and $\mathcal{H}(g-1, g-1)$.

The Masur-Veech theorem implies that almost every $GL_2^+(\mathbb{R})$-orbit is dense in the connected component containing this orbit. This results does not say anything for a specific orbit closure. Kontsevich conjectures that an avatar of Ratner's theorem holds in this non homogeneous situation. The hope is that *every* $\mathrm{GL}_2^+(\mathbb{R})$-orbit closure is a linear orbifold in period coordinates. This is the main conjecture in Teichmüller dynamics. McMullen solved this question in genus 2 (see [Mc3]) as we will explain in the sequel. In this survey, we give a general method to compute orbit closures in some strata. For a list of candidates of orbit closures under the assumption that Kontsevich's linearity conjecture holds, see [Mö08].

Veech groups and Veech surfaces, closed $\mathbf{GL}_2^+(\mathbb{R})$ orbits. We denote the stabilizer of $\mathrm{SO}_2(\mathbb{R})\cdot(X,\omega)$ under the action of $\mathrm{GL}_2^+(\mathbb{R})$ by $\mathrm{SL}(X,\omega)$. It is called the *Veech group* of (X,ω). This is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$.

A more intrinsic definition of the Veech group is as follows. An *affine diffeomorphism* is an orientation preserving homeomorphism of X that permutes the singularities of the flat metric and acts affinely on the polygons defining X. The group of affine diffeomorphisms is denoted by $\mathrm{Aff}(X,\omega)$. For a translation surface (X,ω), the image of the derivation

$$d:\begin{cases}\mathrm{Aff}(X,\omega) & \to & \mathrm{SL}_2(\mathbb{R})\\ f & \mapsto & df\end{cases}$$

is the Veech group $\mathrm{SL}(X,\omega)$. The kernel of d is finite (if the genus $g>1$) because the group of holomorphic automorphisms of a compact Riemann surface of genus $g>1$ is finite by Hurwitz' theorem. The Veech group of a generic surface is trivial. However, Veech groups can be complicated objects (see [HS06], [Mc2] for instance). Closed $\mathrm{GL}_2^+(\mathbb{R})$-orbits can be characterized in terms of Veech groups.

Theorem 2.2 (Smillie)**.** *The $GL_2^+(\mathbb{R})$ orbit of (X,ω) is closed if and only if $SL(X,\omega)$ is a lattice in* $\mathrm{SL}_2(\mathbb{R})$.

A proof of Smillie's result can be found in [MW02]. See also the paper of Veech [Ve92] for hyperelliptic surfaces. If $SL(X,\omega)$ is a lattice then (X,ω) is called a *Veech surface* because the lattice condition was introduced in [Ve89]. The classification of Veech surfaces is a difficult problem solved by McMullen in genus two ([Mc4, Mc5, Mc6], see also [Ca04] and [Mö06]).

Flat geometry. Since the gluings are performed by translations, the linear flow in any given direction are well defined on a translation surface. We will denote by $\mathcal{F}_\theta$ the linear flow of slope (direction) θ. A *saddle connection* is a geodesic segment for the flat metric starting and ending at a zero, not containing any zero in its interior. A *cylinder* on (X,ω) is a maximal connected set of homotopic simple closed geodesics. If the genus of X is greater than one then every cylinder is bounded by saddle connections. A cylinder is *simple* if every boundary consists of only one saddle connection. A cylinder has a width (or circumference) c and a height h. The *modulus* of a cylinder is $\mu = h/c$.

A direction θ on a translation surface is called *periodic*, if the translation surface is the union of the closures of cylinders in this direction. θ is *parabolic*, if moreover the moduli of all the cylinders are commensurable. The Veech dichotomy states that for Veech surface, each direction θ is either uniquely ergodic or parabolic.

Definition 2.3. We will say that a translation surface is *not Veech for the most obvious reason*, if there exists a direction θ that is completely periodic but not parabolic.

A *splitting* of a translation surface (X, ω) is a 'partition' of X into translation surfaces of lower genus with boundary, such that (X, ω) can be restored by glueing together the boundary components and such that all the boundaries are geodesic segments in one fixed direction θ. In the sequel, we will only consider splittings of translation surfaces into two kinds of pieces: tori and cylinders. A *(splitting) cylinder* is a cylinder of the translation surface such that the boundary segments of the splitting are closed loops homotopic to the core curve of the cylinder. A *(splitting) torus* is a splitting piece, isomorphic to a flat torus with a slit which is isomorphic to an interval, not to a loop. Such a torus may or may not be swept out by closed geodesics. In the latter case, it is of course a (metric) cylinder of the surface. This is a source of confusion, but the reader may keep in mind that our emphasis when distinguishing tori and cylinders is on the topology of the boundary segments.

The Thurston–Veech construction The derivative d of an affine diffeomorphism is a hyperbolic matrix if and only if the diffeomorphism is a pseudo-Anosov diffeomorphism. In our setting, we have good charts for the pseudo-Anosov diffeomorphism. Up to conjugacy, the matrix is $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ with $\lambda > 1$, the diffeomorphism expands the horizontal foliation by a factor λ and contracts the vertical foliation by λ^{-1}. We recall a construction due to Thurston and Veech that produces a lot of pseudo-Anosov diffeomorphisms. An affine diffeomorphism is *parabolic* if the absolute value of the trace of its derivative is equal to 2. There is a canonical way to construct parabolic elements in the affine group.

Lemma 2.4. *If (X, ω) has a decomposition (splitting) into metric cylinders for the horizontal direction, with commensurable moduli, then the Veech group $SL(X, \omega)$ contains*

$$Df = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$$

where c is the least common multiple of the moduli.

Assume that we have 2 parabolic elements in transverse directions, then the group generated by these elements contains infinitely many pseudo-Anosov diffeomorphisms ([Th88], [Ve89]). In fact, the Thurston–Veech construction goes the other way around ([Th88]). Given two multicurves on a topological surface, from topological data (the intersection numbers of the curves), Thurston defines the flat metric (a quadratic differential) possessing parabolic elements in the horizontal and vertical directions.

3. Ingredients of the proof: Topological splittings, Ratner's theorem and change of direction

A first observation is that $\mathrm{SL}_2(\mathbb{R})$ contains a lot of unipotent subgroups. Given a suitable topological splitting of a translation surfaces in some direction, we may use the unipotent subgroup in this direction and apply Ratner's theorem. In this way, given some irrationality hypothesis the orbit closure can be shown to be pretty large. We start with the precise definition of a suitable splitting.

Definition 3.1. A *configuration* on a translation surface (X, ω) is a collection $\beta_1, \ldots, \beta_k$ of homologous saddle connections. A *unipotent-admissible configuration* is a configuration, such that the complement $X \setminus \{\cup_{i=1}^k \beta_i\}$ is a union of cylinders C_i and tori (with a slit) T_j.

Let $\mathcal{L}$ be a connected component of a stratum or the hyperelliptic locus inside such a connected component. We are heading for an abstract proposition that encodes the strategy to show that the closure of $\mathrm{GL}_2^+(\mathbb{R}) \cdot (X, \omega)$ equals all of $\mathcal{L}$. Usually one application of Ratner's theorem is not enough. We specify the notions that we need in order to apply the argument sketched above in *several* directions.

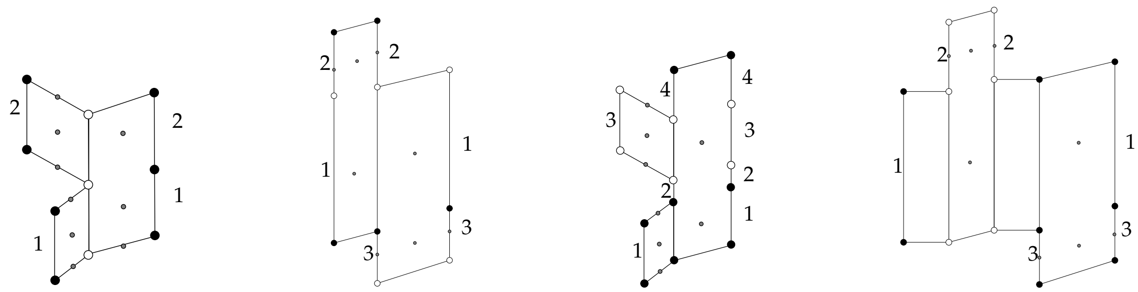
Definition 3.2. A direction v on a translation surface (X, ω) is called *resplitting-admissible direction* if X decomposes in the direction v completely into cylinders and if all but one of these cylinders are simple.

A translation surface (X, ω) with a unipotent-admissible configuration $\{\beta_i\}$ is called *irrational*, if it is not completely periodic.

If $g \leq 3$, this configuration is called *strongly incommensurable* if the splitting pieces C_i and T_j satisfy the following condition. Let C_i' and $T_j' = \mathbb{C}/\Lambda_i'$ denote the splitting pieces normalized to area one. There does not exist a unipotent element $u_t \in \mathrm{SL}_2(\mathbb{R})$, fixing β_1, such that $u_t(\Lambda_i') = \Lambda_j'$ for some $i \neq j$.

If $g > 3$ we call a unipotent-admissible configuration *strongly incommensurable*, if the conclusion of Theorem 1.3 holds for the splitting pieces normalized to area one.

Note that the existence of a unipotent-admissible configuration is a property that persists in a small neighborhood, while the existence of a resplitting-admissible direction does not. Examples of such directions are given in Figure 1 in genus $g = 2$ and in the locus $\mathcal{H}(2,2)^{\mathrm{odd,hyp}}$ in $g = 3$. The definition of *strongly incommensurable* for $g \leq 3$ is made such that the hypothesis of Theorem 1.3 are met. Of course, the definition of strong incommensurability for $g > 3$ is lazy, but we have at present no use of a more concrete one , which would involve a list of excluded cases whose length growths with g.



(a) (b) (c) (d)

Figure 1. The vertical directions are resplitting-admissible ((a) and (c)) respectively contain unipotent-admissible configurations ((b) and (d)) for genus two respectively for genus three.

Proposition 3.3. *Suppose $(X, \omega) \in \mathcal{L}$ has a strongly irrational and strongly non-commensurable unipotent-admissible configuration and a resplitting-admissible direction v. Then*

$$\overline{GL_2^+(\mathbb{R}) \cdot (X, \omega)} = \mathcal{L}.$$

Proof. Let us denote by Z the closure of (X, ω) under $\mathrm{SL}_2(\mathbb{R})$ inside $\mathcal{L}^{(1)}$, the real hypersurface of translation surfaces of area one. One has to show that $Z = \mathcal{L}^{(1)}$.

Let U be the subgroup of $\mathrm{SL}_2(\mathbb{R})$ of unipotent elements u having the β_i has eigenvectors. Then the action of U on (X, ω) is very simple: it stabilizes globally the direction of β and it acts on each component of the splitting. Thus U acts on the parameter space of the splitting pieces, which is isomorphic to $G = (\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z}))^k \times U^n$ for some (k, n). Thanks to Ratner's theorem (Theorem 1.1), the closure of $U \cdot (X, \omega)$ is algebraic, i. e. $H \cdot (X, \omega)$ where H is a closed subgroup of $\mathrm{SL}_2(\mathbb{R})^2 \times U$ containing U diagonally embedded.

We can be more precise. The hypothesis on the unipotent-admissible direction and Theorem 1.3 imply that $H = G$. In other words, the closure of the unipotent group orbit in that direction contains only surfaces with same ratios of the areas of the splitting pieces. Since $H = G$, this is the only constraint, by the following dimension count. The dimension of H is $3k + n$ and using the action of $\mathrm{SL}_2(\mathbb{R})$ we may move β_1 arbitrarily. Thus the closure is of dimension at least $d = 3k + 2$. On the other hand, the saddle connection β_1 and each of the cylinders contribute one complex period, while the tori contribute two. Taking the global area-one constraint into account, we obtain

$$\dim_{\mathbb{R}} \mathcal{L}^{(1)} = 2(1 + n + 2k) - 1 = 4k + 2n - 1 = d + (k + n - 1).$$

Thus we need to find surfaces in the orbit closure where these ratios are different. Let us consider the configuration of saddle connections $\tilde{\beta}_i$ obtained by applying a simple Dehn twist around the vertical non-simple cylinder of the resplitting-admissible direction. Those saddle connections are shown in Figure 2 for $k = 2$, $n = 2$. The configuration $\tilde{\beta}_i$ is obviously again unipotent-admissible. Since the $\tilde{\beta}_i$

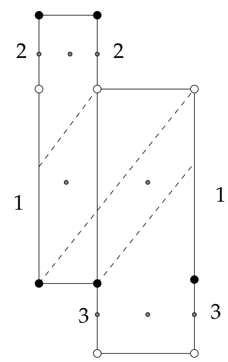


FIGURE 2. A Dehn twisted unipotent-admissible configuration)

are homologous, this configuration still exists in a neighborhood of (X, ω).

We write $(X, \omega) = \#_{i=1}^{k} T_i \ \#_{i=1}^{n} C_i$ to denote that (X, ω) is obtained as the connected sum of the T_i and C_i. They are glued along the β_i, which is suppressed in the notation. By the preceding discussion, the orbit closure of (X, ω) contains

$$(Y_{\underline{u}}, \eta_{\underline{u}}) = \#_{i=1}^{k} u_i T_i \ \ \#_{i=1}^{n} u_{k+i} C_i,$$

where $\underline{u} = (u_1, \dots, u_{k+n})$. For any $k + n$-tuple $\underline{u}$ close enough to zero, the Dehn-twisted unipotent-admissible configuration $\tilde{\beta}_i$ still exists. We denote this decomposition of the modified surface in the new direction by

$$(Y_{\underline{u}}, \eta_{\underline{u}}) = \#_{i=1}^{k} \widetilde{T}_{i\underline{u}} \ \ \#_{i=1}^{n} \widetilde{C}_{i\underline{u}}$$

If the u_i are chosen such that this unipotent-admissible configuration is irrational and strongly non-commensurable, too, then we may apply Theorem 1.3 again with the conclusion $H = G$.

Two things now need to be checked by direct calculation. First, the set of $\underline{u}$ where the configuration $\tilde{\beta}_i$ is not irrational or not strongly non-commensurable is a countable union of subvarieties of real codimension at least one. Second, the u_i-twisting can indeed be used to adjust the ratios of areas. More precisely, we fix $u_{k+n} = 1$. Then the map

$$\begin{aligned}\varphi : (u_1, \ldots, u_{k+n-1}) \mapsto & (\text{area}(\widetilde{T_{1\underline{u}}})/\text{area}(\widetilde{C_{n\underline{u}}}), \ldots, \text{area}(\widetilde{T_{k\underline{u}}})/\text{area}(\widetilde{C_{n\underline{u}}}) \\ & \text{area}(\widetilde{C_{1\underline{u}}})/\text{area}(\widetilde{C_{n\underline{u}}}), \ldots, \text{area}(\widetilde{C_{n-1\underline{u}}})/\text{area}(\widetilde{C_{n\underline{u}}}))\end{aligned}$$

is an invertible function in a neighborhood of zero. This is checked in [HLM06] Lemma 5.9.

In conclusion, the orbit closure of (X, ω) contains points with all ratios of splitting pieces close to the original ratios and for almost all these ratios (with respect to the Lebesgue measure) we can apply Ratner's theorem in the new splitting direction. Thus the orbit closure of (X, ω) contains a subset of $\mathcal{L}$ of positive measure. Recall that the geodesic flow is ergodic on $\mathcal{L}$ (see Theorem 2.1). Therefore Z has full measure in $\mathcal{L}$. Since Z is closed, this completes the proof. □

4. Genus two: McMullen's complete classification

It is an easy topological exercise to see that the configuration given in Figure 1 (b) is the only unipotent-admissible configuration in $\mathcal{H}(1, 1)$. Shrinking the height of the saddle connection with label 2 to zero, one obtains the corresponding picture for $\mathcal{H}(2)$. The surprising fact and first key step to the classification is that such a configuration always exists in $g = 2$.

Theorem 4.1 ([Mc3]). *Any translation surface (X, ω) in $\mathcal{H}(2)$ or $\mathcal{H}(1, 1)$ admits a unipotent-admissible configuration. More precisely, the set of directions of those configurations is dense in the unit circle for any given (X, ω).*

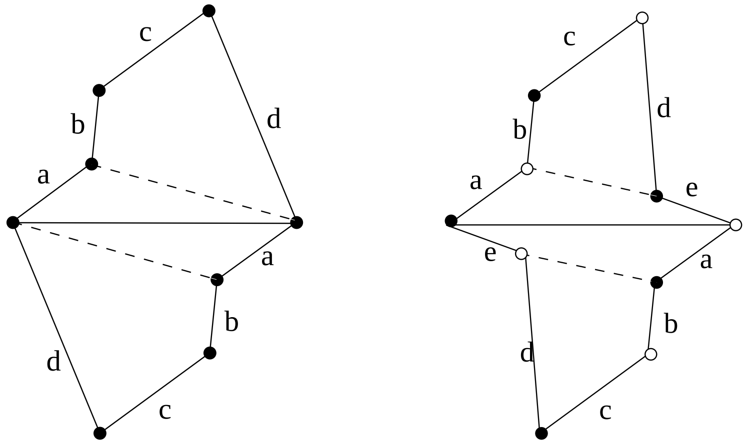
Proof. We present an "elementary" proof different from the one in [Mc3]. It only enables us to prove the existence of the unipotent-admissible configuration but not the density of the set of directions. Let (X, ω) be a genus two translation surface and let $\tau : X \to X$ be the hyperelliptic involution. Let us assume that we have constructed a saddle connection β such that $\beta \neq \tau(\beta)$. Observe that if ω has a single zero then X splits along $\beta \cup \tau(\beta)$ as a connected sum of two tori (see below) and if ω has two zeroes connected by β then X also splits along $\beta \cup \tau(\beta)$ as a connected sum of two tori. Hence (X, ω) admits a unipotent-admissible configuration with the set of homologous saddle connections $\{\beta,\ \tau(\beta)\}$. A second observation is that $\beta = \tau(\beta)$ if and only if β contains a Weierstrass point in its interior.

Thus we only need to proof that there always exists a saddle connection (connecting the two different singularities if they are two) that does not pass through a Weierstrass point. In order to prove this, we will use the representation of (X, ω) by a centrally symmetric polygon (see [Ve86]). Let us recall the construction here.

Consider a collection of vectors $v_1, \dots, v_n$ in the complex plane $\mathbb{C}$ with $n = 4$ or $n = 5$. Let us construct from these vectors a broken line in a natural way: a $i - th$ edge of this broken line is represented by the vector v_i. Construct another broken line starting at the same point as the initial one by taking the same vectors but in the reversing order: $v_n, \dots, v_1$. We label the points of this polygon by $P_1, \dots, P_{2n}$. By construction these two broken lines have the same endpoints (namely P_1 and P_{2n}). Suppose that they define a polygon without self-intersections of the boundary (other than P_1 and P_{2n}). Then by identifying the opposite sides v_i by a translation we get a translation surface (P, dz) in the stratum $\mathcal{H}(2)$ (respectively $\mathcal{H}(1, 1)$) if $n = 4$ (respectively $n = 5$) (see Figure 3). The Weierstrass points are the middle of the vectors v_i, the center of the polygon P and (if $n = 4$) the singularity (i.e. the vertices of v_i). Using the $\mathrm{GL}_2^+(\mathbb{R})$)-action we normalize so that $\sum_{i=1}^n v_i$ is horizontal. Veech's result [Ve86] says that any hyperelliptic translation surface admits such a representation.

Indeed, let p be a Weierstrass point. The set of directions of geodesic segments which emanate from p to a singularity is dense in the circle (see say Proposition 3.1 in [Vor96] or Lemma 1 in [HS06]). Thus let I be a geodesic emmanating from p to a zero of ω. Then necessarily I is fixed by τ and therefore I is closed. Observe that I is a loop if ω has one zero. Now let us consider the first return map of a minimal transverse foliation to I. By a straightforward computation, one checks that the surface is decomposed in terms of a centrally symmetric polygon.

Therefore we have to show that there always exists a saddle connection in any flat centrally symmetric polygon that does not pass neither through the middle of the vectors v_i nor the center of the polygon. The proof is straightforward by the following algorithm.



(a) $\mathcal{H}(2)$ (b) $\mathcal{H}(1,1)$

Figure 3. Representation of any translation genus two surface in terms of flat centrally symmetric polygon. The last case of the algorithm is indicated by dotted lines in both cases.

$$\text{If } n = 4 \text{ i.e. } (X,\omega) \in \mathcal{H}(2), \quad \begin{cases} \text{let } \beta = \overrightarrow{P_1P_2} & \text{if } (P_3)_y \le (P_2)_y, \\ \text{let } \beta = \overrightarrow{P_5P_2} & \text{if } (P_3)_y \le (P_4)_y, \\ \text{let } \beta = \overrightarrow{P_2P_4} & \text{otherwise.} \end{cases}$$

$$\text{If } n = 5 \text{ i.e. } (X,\omega) \in \mathcal{H}(1,1), \quad \begin{cases} \text{let } \beta = \overrightarrow{P_1P_4} & \text{if } (P_4)_y \le \min\{(P_2)_y, (P_3)_y\}, \\ \text{let } \beta = \overrightarrow{P_6P_3} & \text{if } (P_3)_y \le \min\{(P_4)_y, (P_5)_y\}, \\ \text{let } \beta = \overrightarrow{P_2P_5} & \text{otherwise.} \end{cases}$$

Then β satisfies the required condition. This ends the proof of the existence of a unipotent-admissible configuration. □

As a consequence, we state the orbit closure result in genus $g = 2$ in its full strength.

Corollary 4.2 ([Mc3])**.** *The $GL_2^+(\mathbb{R})$-orbit closure of a translation surface (X,ω) with $g(X) = 2$ is one of the following possibilities.*

i) *a stratum $\mathcal{H}(2)$ or $\mathcal{H}(1,1)$, or*

ii) *the locus of eigenforms on Riemann surfaces whose Jacobian admits real multiplication, or*

iii) *the orbit is closed and projects to a Teichmüller curve in $\mathcal{M}_2$.*

If (X, ω) arises from the Thurston-Veech construction, case ii) or case iii) holds.

Proof. Pick one of the unipotent-admissible configurations $\beta := \{\beta_1, \beta_2\}$ on (X, ω) given by Theorem 4.1 and let U be the unipotent subgroup of $\mathrm{SL}_2(\mathbb{R})$ stabilizing the β_i. If the configuration β is irrational (in the sense of Definition 3.2, then we claim that there is a surface in the U-orbit closure of (X, ω) that satisfies the hypothesis of Proposition 3.3. The proof is the same as Lemma 5.2, just with a fewer number of splitting pieces.

Consequently, we are in case i) or all the unipotent-admissible configurations give periodic directions. Suppose for one of the configurations $\{\beta_1, \beta_2\}$ the two periodic tori have incommensurable moduli, i.e. the direction is not periodic. Then, writing $(X, \omega) = T_1 \# T_2$, we know that the U-orbit closure contains all the surfaces

$$(Y_u, \eta_u) = uT_1 \# T_2.$$

There is a u_0 such that (Y_{u_0}, η_{u_0}) contains a resplitting-admissible direction. To illustrate this, we may apply a vertical shear to the left cylinder in Figure 1 (b) such all the almost-horizontal saddle connection on the boundary of the figure have the same slope. Then this direction is resplitting-admissible. The surface (Y_{u_0}, η_{u_0}) has a new unipotent-admissible configuration, obtained by Dehn-twist around the long cylinder as in Figure 2. Since unipotent-admissible configurations are stable under deformation, this configuration also exists on (Y_u, η_u) for all u close enough to u_0. One easily calculates that this new configuration is irrational for all u outside a countable set. For a u in the complement we now apply the initial argument.

Consequently, we are in case i) or all the unipotent-admissible configurations give parabolic directions. In the latter case, the orbit closure has to be strictly smaller than the whole stratum by the following theorem. If $(X, \omega) \in \mathcal{H}(2)$, then the orbit has to be closed since the intersection or the eigenform locus and $\mathcal{H}(2)$ has real dimension 4, the same as $\mathrm{GL}_2^+(\mathbb{R})$.

It remains to show that if the orbit of $(X, \omega) \in \mathcal{H}(1, 1)$, is not closed, its closure Z is the whole eigenlocus and we are in case ii). The argument given in [Mc3] Theorem 12.1 is again a combination of splitting techniques and a Ratner type theorem. The main step consists of showing that if the orbit of (X, ω) ist not closed, its closure Z has non-empty interior. Roughly, one can use non-closedness of the orbit to approximate (X, ω) by a sequence of surfaces

$$(X_n, \omega_n) = (T_1, \omega_1) \#_{I_n} (T_2, \omega_2),$$

where the two tori $T_i = \mathbb{C}/\Lambda_i$ are commensurable by the preceding discussion and where the glueing is along intervals I_n. By resplitting, one may moreover suppose that the I_n have an infinite number of different slopes. Thus Z contains

$$(T_1, \omega_1) \#_{\gamma I_n} (T_2, \omega_2),$$

for all γ in the lattice Γ that stabilizes both Λ_i and for all n. It remains to show ([Mc3] Theorem 2.10) that $\bigcup_n \Gamma \cdot I_n$ is dense in $\mathbb{R}^2$. □

The last step of the proof can also be achieved differently. For the eigenform locus, the closures of the horocycle orbits are known due to [CW07] and completed by Smillie and Weiss [SW]. In the short list of possible closures only the above-mentioned cases are $\mathrm{GL}_2^+(\mathbb{R})$-invariant.

The proof of the preceding corollary is completed by the following characterization of real multiplication. It is the second ingredient that is special to genus two.

Theorem 4.3 ([Mc3] Theorem 6.1)**.** *Suppose that (X, ω) is a translation surfaces of genus $g = 2$ such that for two unipotent-admissible configurations the splitting pieces are isogenous tori. Then (X, ω) is in the eigenlocus for real multiplication.*

5. Genus three: The locus $\mathcal{L} = (\mathcal{H}(2,2)^{\mathrm{odd}})^{\mathrm{hyp}}$ and a similar case

In this section we study $\mathrm{GL}_2^+(\mathbb{R})$-orbits closures in the locus $\mathcal{L} = (\mathcal{H}(2,2)^{\mathrm{odd}})^{\mathrm{hyp}}$. One can describe these surfaces without defining odd spin structures as those translation surfaces with two double zeros and a hyperelliptic involution that fixes the two zeros. There are two reasons to study $\mathcal{L}$. First, the hyperelliptic quotient maps define a $\mathrm{GL}_2^+(\mathbb{R})$-equivariant isomorphism from $\mathcal{L}$ to $\mathcal{Q}(1,1,1,1)$. Consequently, the orbit closures in $\mathcal{L}$ are the same are those in the generec stratum of half-translation surfaces in genus two. Second, $\mathcal{L}$ is one the smallest GL-invariant loci besides $g = 2$. It thus exhibts dynamics different from $g = 2$, without much annoyance from too many parameters.

The purpose of this section is to explain the reduction steps that allow to say that the $\mathrm{GL}_2^+(\mathbb{R})$-orbit closure of a translation surface in $\mathcal{L}$ with a suitable direction and 'some irrationality' is the same as the $\mathrm{GL}_2^+(\mathbb{R})$-orbit closure of another translation surface that meets the hypothesis of Proposition 3.3. This yields a proof of Theorem 0.2 at the end of this section. Finally, we add some remarks about another stratum where similar techniques apply.

Proposition 5.1. *Almost every surface in $\mathcal{L}$ has a unipotent-admissible configuration. There exist translation surfaces in $\mathcal{L}$ that do not have a unipotent-admissible configuration*

Proof. A unipotent-admissible configuration is depicted in Figure 1 (d). Since the possession of a unipotent-admissible configuration is an open condition, invariant under the action of $\mathrm{GL}_2^+(\mathbb{R})$ and since the geodesic flow is ergodic, the first statement is clear. An example for the second claim, a square-tiled surface with 6 squares, is given in [HLM07] Figure 14. □

We start with a strengthening of Proposition 3.3, which however relies on topological properties of $\mathcal{L} = (\mathcal{H}(2,2)^{\mathrm{odd}})^{\mathrm{hyp}}$

Lemma 5.2. *If $(X,\omega) \in \mathcal{L}$ has an irrational unipotent-admissible configuration, then*

$$\overline{GL_2^+(\mathbb{R}) \cdot (X,\omega)} = \mathcal{L}.$$

Proof. The list of all possible completely periodic directions has been compiled in [HLM07] Figure 1. It shows that the only unipotent-admissible configuration is given by the vertical direction of Figure 1 (d) above.

Suppose first, that moreover this configuration is strongly irrational and strongly incommensurable. Let U the unipotent subgroup of $\mathrm{SL}_2(\mathbb{R})$ generated by unipotent elements u having the β_i has eigenvectors. Then

$$\overline{U \cdot (X,\omega)} \supset C_1 \# UT_1 \# C_2 \# UT_2.$$

Consequently, we may arrange another direction which is almost a resplitting-admissible direction, like the horizontal one in Figure 1 (d), to be resplitting-admissible. That is, there is a surface (Y,η) in the U-orbit closure of (X,ω), such that a direction given by a saddle connection in C_1 is a resplitting-admissible direction and such that the direction of the β_i is untouched. Hence (Y,η) satisfies the hypothesis of Proposition 3.3.

The idea to reduce next to the above situation, is to use not the given unipotent-admissible direction but a Dehn-twisted one. Consider again Figure 1 (d). There, the horizontal direction does not even contain a cylinder, but it is nevertheless close to a resplitting-admissible direction in the following sense. Consider the saddle connection from the lower left to the upper right corner. If the tori are not too far from horizontal, this saddle connections comes with 3 other homologous saddle connections, topologically like the Dehn twist around the middle almost cylinders. Hence this direction is again a unipotent-admissible configuration.

To make this idea work, one has to check two things ([HLM06] Lemma 4.5 and Lemma 4.6). First, using merely irrationality, one can find a surface (Y,η) in the U-orbit closure that possesses another ('Dehn-twisted') unipotent-admissible configuration. Since the existence of such a configuration is open, there is an open real interval I of surfaces (Y_u, η_u), $u \in I$ with this second unipotent-admissible configuration. Second, one has to check that the property of this second unipotent-admissible configuration not being strongly irrational or not being strongly incommensurable is a condition that holds only for countably many surfaces. On the complement, we are in the situation we discussed at the beginning of the proof. □

Proof of Theorem 0.2. The proof contains two steps: first derive a condition on any kind of configuration that admits a reduction to the favorable situation of Proposition 3.3. Second, show that this condition holds for surfaces.

Consider the vertical direction in Figure 4. Note that the glueings are different

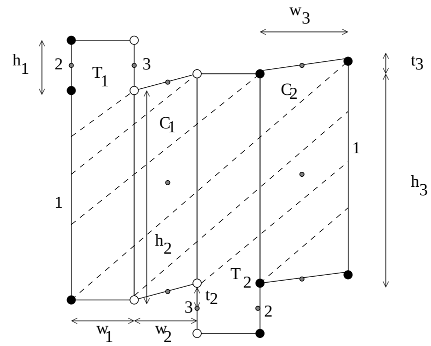


FIGURE 4. A 'Dehn-twisted' unipotent-admissible configuration

than in Figure 1 (d). Consequently, the long vertical saddle connections are not homologous, h_2 may or may not be equal to h_3. If the moduli of C_1, C_2 and T_1 are incommensurable, one might hope that there is a surface in the U-orbit closure of (X, ω), such that the 'Dehn-twisted' direction with dotted lines exists (i.e. that for example t_2 is not too big) and that it is moreover irrational. Then one could apply Lemma 5.2.

This does not quite work. Unramified double coverings of genus two surfaces provide examples, where the irrationality holds and yet the $GL_2^+(\mathbb{R})$-orbit closure is small ([HLM07] Section 7). But if the moduli of C_1, C_2 and T_1 are *pairwise* incommensurable, this strategy works. To complete the first step, one has to check this not only for the direction in Figure 4, but also for four other cases ([HLM07] Section 4.)

In the second step, we deduce pairwise incommensurability from the existence of a pseudo-Anosov φ with trace field K and $[K : \mathbb{Q}] = 3$. Suppose, for simplicity, that $[K : \mathbb{Q}]$ is Galois and write one and two primes for the Galois conjugates. Then $\psi := \varphi^* + (\varphi^*)^{-1} \in \mathrm{End}(H^1(X, \mathbb{R}))$ preserves the symplectic form and we have a decomposition into ψ-eigenspaces,

$$H^1(X,\mathbb{R}) = S \oplus S' \oplus S'' \quad \text{where} \quad S = \langle \mathrm{Re}\omega, \mathrm{Im}\omega \rangle,$$

orthogonal with respect to the cup product. Write c_i and h_i for the circumference and height C_i ($i = 1, 2$) and $c_3/2$ and h_3 for the circumference and height of T_1 respectively. Let $m_i = h_i/c_i$ be the modulus. Then

$$\sum_{i=1}^{3} m_i c_i c_i' = \sum_{i=1}^{3} h_i c_i' = \int_X \mathrm{Re}(\omega) \wedge \mathrm{Im}(\omega') = \frac{i}{4\pi} \int_X (\omega + \overline{\omega}) \wedge (\omega' - \overline{\omega'}) = 0.$$

The same calculation with the other Galois conjugate gives

$$\sum_{i=1}^{s} m_i c_i c_i'' = 0, \quad \text{hence} \quad \sum_{i=1}^{3} (m_i - m_i'')\delta_i c_i c_i'' = 0.$$

Moduli and circumferences of the cylinders exchanged by the hyperelliptic involution are the same.

Suppose that the moduli in the given direction are not pairwise incommensurable, i.e. there is a relation $a_1 m_1 + a_2 m_2 = 0$. Applying a matrix in $\mathrm{SL}_2(K)$ to (X, ω), we may suppose that m_3 is rational without changing the ratios of the m_i. We deduce from the above equations $m_i = m_i''$ for $i = 1, 2$. Hence, in fact all $m_i \in \mathbb{Q}$ and we are in case i). □

Remark 5.2.1. The stratum $\mathcal{H}(4)^{\mathrm{odd}}$ has two connected components (see [KZ03]). Although we have not carried out the details, it is very likely that statements similar to Lemma 5.2 and Theorem 0.2 hold for both components of this stratum, too. A unipotent-admissible configuration to start with is given by the "necklace" in Figure 5 with $P = 1$ and $M = 2$. It is easy to compile the list of completely periodic directions with three cylinders in this stratum. Following the above strategy, one has to check that under a suitable irrationality condition each of them reduces to the situation of Proposition 3.3.

6. General case: limits of the strategy

In last sections we have seen that the strategy of finding an irrational unipotent-admissible configuration is very useful to obtain informations on the orbit closure of the surface. In this section we reveal the limits of this strategy by showing that only few strata possess surfaces with a unipotent-admissible splitting. We will prove that such surfaces are precisely those obtained from tori connected in a "necklace" by a chain of cylinders. We want to determine the strata these surfaces belong to.

Theorem 6.1. *The genus g surface $(X,\omega) \in \mathcal{H}(k_1,\dots,k_n)$ has an irrational unipotent-admissible configuration if and only if X is constructed from tori and cylinders cyclically glued to a "necklace" where two neighboring tori might be glued directly or by a cylinder. The waist curves of all the cylinders and all saddle connections representing the boundaries of the tori are homologous (see Figure 5). Moreover the configuration has either:*

1. *g tori and no cylinders; $\mathcal{H}(k_1,\dots,k_n) = \mathcal{H}^{hyp}(g-1,g-1)$.*
2. *$g-1$ tori and $0 < n \leq g-1$ cylinders; $\mathcal{H}(k_1,\dots,k_n) = \mathcal{H}(2l_1,\dots,2l_n)$.*

In case (2) one can check, if $n = 1$ or $n = 2$, then the surface of genus g belongs to the connected component having the parity of the spin structure equal to g mod 2; The component is not the hyperelliptic component unless $g = 2$.

As for $g = 3$ and contrary to genus two, those configurations do not exist everywhere.

Proposition 6.2. *For each $g \geq 3$, there exist translation surfaces in $\mathcal{H}^{hyp}(g-1,g-1)$ and that do not have a unipotent-admissible configuration into a "necklace" of tori and cylinders (as in the preceding theorem).*

Proof of Theorem 6.1. We will use a result of Naveh [Na08] in order to bound the number of cylinders and tori in terms of the genus of the surface. Let us recall this result here. For the unipotent-admissible configuration, let M denote the number of minimal components (i.e. tori) and P the number of cylinders. We can summarize Theorem 1 and Theorem 2 of [Na08] for our purpose in the following way. Let B be the number of indexes $i \in \{1,\dots,n\}$ such that k_i is odd (B is odd). Then $M \leq g$ and

$$\begin{cases} M+P \;\leq\; 2(g-1)+n-M-B/2 & \text{if} \quad g-1-B/2 \;\leq M \leq g-1, \\ M+P \;\leq\; g-1+n & \text{if} \quad g-1-B/2 \;\geq M, \\ P \;=\; 0 & \text{if} \quad g \;= M. \end{cases}$$

If (X,ω) has an irrational unipotent-admissible configuration into M tori and P cylinders then $2M+P+1$ is equal to the complex dimension of the stratum, that is $2g+n-1$. Indeed each torus contributes two complex dimensions, each cylinder one complex dimension and the homologous waist curves also one complex dimension. Hence

$$2g+n-1 \;=\; 2M+P+1 = (M+1)+(M+P)$$

Let us examine the three cases following Naveh's theorem.

(1) If $g-1-B/2 \leq M \leq g-1$ then

$$2g+n-1 \;\leq\; M+1+2(g-1)+n-M-B/2$$

which leads to $B = 0$ and therefore $M = g - 1$. Substituting in the previous equation this gives

$$2g + n - 1 = 2(g - 1) + P + 1 \qquad \text{or} \qquad P = n$$

(2) If $M \leq g - 1 - B/2$ then

$$2g + n - 1 \quad \leq \quad M + 1 + g - 1 + n.$$

Thus $g \leq M + 1$ and $M = g - 1$; this corresponds to the previous case.

(3) If $M = g$, then one has $P = 0$. We get $2g + 1 = 2g + n - 1$ so that $n = 2$.

Hence we have proven that there are either $g - 1$ tori and n cylinders or g tori and no cylinders. The only possible configurations so that the waist curves are homologous is given by Figure 5. One then checks the type of the singularities. The proposition is proven. □

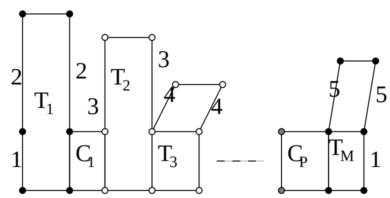


FIGURE 5. The tori and cylinders are cyclically glued to a "necklace" where two 'neighboring' tori might be glued directly or by a cylinder in between them.

Proof of Proposition 6.2. Let us consider the Veech surface given by a centrally symmetric polygon $4g+2$-gon (see Figure 6). The Veech group has two cusps ([Ve89]), namely the horizontal and the vertical direction. For these two directions, one checks that the surface is not decomposed into tori and cylinders as in Theorem 6.1.

□

7. OPEN QUESTIONS

This survey on flat surfaces is focussed on the closures of $\mathrm{GL}_2^+(\mathbb{R})$-orbits. Without doubt, the classification of these closures is the main open question. We list here some open problems in this direction, more specific and thus probably easier

Question 7.1 (Unipotent orbit closures)**.** *Can one classify the closures of the orbit of the unipotent subgroup? This has been achieved for branched coverings of Veech surfaces (*[EMM06]*) and for the locus of eigenforms (*[CW07]*), but it is open even for $\mathcal{H}(2)$. A*

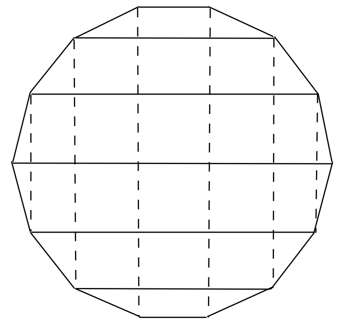


FIGURE 6. A surface in $\mathcal{H}^{hyp}(g-1, g-1)$ (drawn for $g = 3$) without unipotent-admissible configuration. Idenfication of the sides are by parallel translation.

solution to this question enables to treat $GL_2^+(\mathbb{R})$-orbit closures by splitting techniques where the splitting pieces might now be more complicated than just tori and cylinders.

Question 7.2 (Athreya's question: give a generic flat surface)**.** *The closure of almost all $GL_2^+(\mathbb{R})$-orbits in a stratum is the whole stratum by ergodicity of the Teichmüller geodesic flow. Write down explicitly a surface that is generic in this sense! Can one specify such a surface (of genus g) with all saddle connections in a field K, $[K : \mathbb{Q}] \leq g$?*

Question 7.3 (pseudo-Anosov diffeomorphisms and Veech groups)**.** *What can be said about the Veech groups containing the derivative of a pseudo-Anosov diffeomorphism? Right now, we do not have a single example of a cyclic Veech group generated by an hyperbolic element. In genus 2, if a Veech group contains a hyperbolic element then its boundary (as a Fuchsian group) is equal to $P^1(\mathbb{R})$ (see* [Mc2] *). In genus $g \geq 3$, even for pseudo-Anosov obtained by Thurston-Veech construction, we do not know any interesting property of the associated Veech group.*

Question 7.4 (The role of pseudo-Anosov diffeomorphisms)**.** *What is the size of the $GL_2^+(\mathbb{R})$-orbit closure of (X, ω), if (X, ω) has a pseudo-Anosov in its Veech group? The main theorems in the introduction answer this for $g = 2$ and a special locus in genus* 3*. Does the behaviour for $g = 3$ generalize to all strata in $g \geq 3$?*

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