Buildings and Berkovich Spaces

Annette Werner

Goethe-Universität, Frankfurt am Main

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This talk reports on joint work with Amaury Thuillier and Bertrand Rémy (Lyon). Our results generalize results of Vladimir Berkovich who investigated the case of split groups.
Non-archimedean fields

$K$ non-Archimedean field, i.e. $K$ is complete with respect to a non-trivial absolute value $| \cdot |_K$ satisfying

$$|a + b|_K \leq \max\{|a|_K, |b|_K\}.$$ 

$K$ is called discrete if the value group $|K^*| \subset \mathbb{R}^*$ is discrete.

Non-archimedean analysis has special charms:

$$\sum_{n=1}^{\infty} a_n$$ converges if and only if $a_n \to 0.$
Non-archimedean fields

Examples:

- $K = k((T))$ formal Laurent series over any ground field $k$, with $|\sum_{n \geq n_0} a_n T^n| = e^{-n_0}$ if $a_{n_0} \neq 0$
- $K = \mathbb{C}\{\{T\}\}$ Puiseux series
- $K = \mathbb{Q}_p$, the completion of $\mathbb{Q}$ with respect to $|x| = p^{-\nu_p(x)}$
- algebraic extensions of $\mathbb{Q}_p$
- $K = \mathbb{C}_p$, the completion of the algebraic closure of $\mathbb{Q}_p$
$G$ semisimple group over $K$, i.e.

$G \hookrightarrow \mathrm{GL}_{n,K}$ closed algebraic subgroup such that

$\text{rad}(G)(= \text{biggest connected solvable normal subgroup}) = 1$

Examples: $\mathrm{SL}_n, \mathrm{PGL}_n, \mathrm{Sp}_{2n}, \mathrm{SO}_n$ over $K$

$\mathrm{SL}_n(D)$ $D$ central division algebra over $K$
Main result

Goal: Embed the Bruhat-Tits building $\mathcal{B}(G, K)$ associated to $G$ in the Berkovich analytic space $G^{an}$ associated to $G$.

Hope: Investigate the building with the help of the amiant Berkovich space $G^{an}$. 
Archimedean Example:

\[ G = \text{SL}(2, \mathbb{R}) \]
\[ H = \text{SO}(2, \mathbb{R}) \text{ maximal compact subgroup} \]
\[ G/H = \mathbb{H} = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \} \]
upper half-plane

Non-Archimedean analog:

\[ p \text{ prime number} \]
\[ G = \text{SL}(2, \mathbb{Q}_p) \]
\[ H = \text{SL}(2, \mathbb{Z}_p) \text{ maximal compact subgroup.} \]
\[ G/H \text{ is a totally disconnected topological space.} \]
Note: \( \mathbb{H} = \{ \text{norms on } \mathbb{R}^2 \}/\text{scaling}. \)

Goldman-Iwahori:

\[ \mathcal{B}(SL_2, \mathbb{Q}_p) = \{ \text{Non-archimedean norms on } \mathbb{Q}_p^2 \}/\text{scaling} \]

- Topology of pointwise convergence
- \( SL(2, \mathbb{Q}_p) \)-action
- Stabilizer of the norm \( \begin{pmatrix} a \\ b \end{pmatrix} \mapsto \max\{|a|, |b|\} \) is the maximal compact subgroup \( SL(2, \mathbb{Z}_p) \)
\[ \mathcal{B}(SL_2, \mathbb{Q}_p) \text{ is an infinite } (p + 1)-\text{valent tree:} \]

\[ (p = 2). \]
In general: The building $\mathcal{B}(G, K)$ is obtained by glueing real vector spaces (apartments). Every maximal split torus $T \subset G$, i.e. $T \simeq \mathbb{G}_m^r$, induces an apartment $A(T)$, which is defined as the real cocharacter space $A(T) = \text{Hom}_K(\mathbb{G}_m, T) \otimes_{\mathbb{Z}} \mathbb{R}$.

The glueing process is defined with deep (and quite technical) results by Bruhat and Tits.

$\mathcal{B}(G, K)$ is a complete metric space with a continuous $G(K)$–action.

If $K$ is discrete, $\mathcal{B}(G, K)$ carries a (poly-)simplicial structure.
Apartment for $Sp_4$
Apartment for $PGL_3$
Some part of $B(PGL_3, \mathbb{Q}_p)$
Why are Bruhat-Tits buildings useful?

- \( \mathcal{B}(G, K) \) is a nice space on which \( G(K) \) acts
- Cohomology of arithmetic groups (Borel-Serre)
- \( \mathcal{B}(G, K) \) encodes information about the compact subgroups of \( G(K) \)
- Representation theory of \( G(K) \) (Schneider-Stuhler)
- Bruhat-Tits buildings are non-Archimedean analogs of Riemann symmetric spaces of non-compact type
- Buildings can be used to prove results for symmetric spaces (e.g. Kleiner-Leeb)
A Berkovich space is a non-Archimedean analytic space with good topological properties.

**Archimedean case:**
$X$ smooth projective variety over $\mathbb{C}$. Then $X(\mathbb{C})$ is a complex projective manifold.

**Non-archimedean case:**
$X$ smooth projective variety over $K$. Then $X(K)$ inherits a non-Archimedean topology from $K$ with bad topological properties, e.g. it is totally disconnected.
Berkovich Spaces

Tate, Raynaud... Define non-Archimedean analytic functions by a suitable Grothendieck topology

Berkovich Enlarge $X(K)$ to a topological space $X^{an}$ with good properties.
Example: The Berkovich unit disc

Assume for simplicity that $K$ is algebraically closed.

$A = K \{ z \} = \{ \text{formal series } f(z) = \sum_{n \geq 0} a_n z^n \text{ with } a_n \to 0 \}$

$\| f \| = \max_n |a_n|_K$  Gauss norm on $A$
$\mathcal{M}(A) = \{ \text{bounded multiplicative seminorms on } A \text{ extending } | |_K \}$ is the Berkovich unit disc.

Hence every $\gamma \in \mathcal{M}(A)$ is a function $\gamma : A \rightarrow \mathbb{R}_{\geq 0}$ satisfying

- $\gamma |_K = | |_K$
- $\gamma(fg) = \gamma(f) \gamma(g)$
- $\gamma(f + g) \leq \gamma(f) + \gamma(g)$
- $\gamma \leq c |||$

Every $a \in K$ with $|a|_K \leq 1$ induces a point $|f|_a = |f(a)|_K$ in $\mathcal{M}(A)$.

The Gauss norm is multiplicative, i.e. a point in $\mathcal{M}(A)$. 
Non-archimedean balls

The other seminorms in $\mathcal{M}(A)$ can be described with closed non-Archimedean discs $D(a, r) = \{x \in K : |x - a| \leq r\}$.

Note: Two non-Archimedean closed discs are either disjoint or nested.
Berkovich unit disc

Basic fact: The Gauss norm is the supremum norm on $D(0, 1)$.

The Berkovich unit disc consists of the following points:

- **Points of type 1:** $|f|_a = |f(a)|_K$ for $a \in D(0, 1)$.

- **Points of type 2:**
  
  $|f|_{a,r} = \sup_{x \in D(a,r)} |f(x)|_K$ for $D(a, r) \subset D(0, 1)$
  
  and $r \in |K^*|$

- **Points of type 3:**
  
  $|f|_{a,r} = \sup_{x \in D(a,r)} |f(x)|$ for $D(a, r) \subset D(0, 1)$
  
  and $r \notin |K^*|$

- **Points of type 4:**
  
  $|f|_{a,r} = \lim_{n \to \infty} |f|_{a_n,r_n}$ for a nested sequence $D(a_1, r_1) \supset D(a_2, r_2) \ldots$ of closed discs in $D(0, 1)$
Berkovich unit disc

(from J.H. Silverman: The arithmetic of dynamical systems)
Endow the Berkovich unit disc $\mathcal{M}(A)$ with the topology of pointwise convergence of seminorms evaluated on $A$.

Then $\mathcal{M}(A)$ is a compact, uniquely path-connected Hausdorff space containing $\{x \in K : |x|_K \leq 1\}$ as a dense subspace.
Similary one can define Berkovich discs of any radius $r > 0$.

**Berkovich affine line:**

$$(\mathbb{A}^1_K)^{an} = \text{union of all Berkovich discs of positive radius}$$

$$= \{\text{multiplicative seminorms on } K[z]\}.$$

**Berkovich projective line:**

$$(\mathbb{P}^1_K)^{an} \text{ can be constructed by glueing two Berkovich unit discs.}$$
In general:

\[ X = \text{Spec } A \text{ for } A = K[x_1, \ldots, x_n]/\mathfrak{a} \]

Berkovich space \( X^{an} \) corresponding to \( X \):

\[ X^{an} = \{ \text{multiplicative seminorms on } A \text{ extending } | |_K \} \]

An analogous definition over the complex numbers yields \( X(\mathbb{C}) \) by a theorem of Gelfand-Mazur.
Berkovich spaces have found a variety of applications, e.g.

- to prove a conjecture of Deligne on vanishing cycles (Berkovich)
- in local Langlands theory (Harris-Taylor)
- to develop a $p$–adic avatar of Grothendieck’s “dessins d’enfants” (André)
- to develop a $p$–adic integration theory over genuine paths (Berkovich)
- for $p$–adic harmonic analysis and $p$–adic dynamics with applications in Arakelov Theory (Baker, Chambert-Loir, Rumely, Thuillier,...)
Embedding Theorem

$G$ semisimple algebraic group over $K$

$G^{an}$ Berkovich space associated to $G$

We define a continuous, $G(K)$–equivariant embedding

$$\nu : \mathcal{B}(G, K) \longrightarrow G^{an}$$

using the following theorem:
Embedding Theorem

Theorem

i) For all $x \in \mathcal{B}(G, K)$ there exists an (affinoid) subgroup $G_x = M(A_x) \subset G^{an}$ such that

$$G_x(L) = \text{Stab}_{G(L)}(x)$$

for all non-Archimedean fields $L \supset K$.

ii) $G_x$ has a unique maximal point in $G^{an}$ (Shilov boundary point), i.e. there exists a unique $\nu(x) \in G^{an}$ such that all $f \in A_x$ achieve their maximum on $\nu(x)$.
Embedding Theorem

Tools: Bruhat-Tits theory, Berkovich’s characterization of Shilov boundary points, descent theory for affinoids

New idea: Any point $x$ becomes special after base extension with a suitable $L/K$. 
Example

\[ G = SL_2 \]

\[ T \subset G \text{ torus of diagonal matrices} \]

\[ A(T) = \text{Hom}(\mathbb{G}_m, T) \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}\mu \text{ for } \mu: a \mapsto \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}. \]

\[ U_- = \left\{ \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} : u \in K \right\} \quad U_+ = \left\{ \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} : v \in K \right\} \]

\[ \Omega = U_- TU_+ \subset SL_2 \quad \text{big cell} \]

Note that \( \Omega = \text{Spec } K[a, a^{-1}, u, v] \).
Embedding Theorem: Example

The embedding $\nu$ is constructed apartment-wise. It maps $A(T)$ to the analytified big cell $\Omega^{an} \subset SL_2^{an}$.

**Explicit description:** Let $x_\mu \in A(T)$.

Then $\nu(x_\mu) \in \Omega^{an}$ is the following multiplicative seminorm on $K[a, a^{-1}, u, v]$:

$$| \sum_{k \in \mathbb{Z}, m, n \in \mathbb{N}_0} c_{kmn} a^k u^m v^n |_{\nu(x_\mu)} = \max_{k,m,n} |c_{kmn}|_K |e^{x(m-n)}|.$$ 

In particular, for $0 \in A(T)$ we get

$$| \sum_{k,m,n} c_{kmn} a^k u^m v^n |_{\nu(0)} = \max_{k,m,n} |c_{kmn}|.$$
**Application:** Compactifications of Bruhat-Tits buildings

$G$ semisimple algebraic group over $K$

$P \subset G$ parabolic subgroup

$G/P$ proper $K$–variety

**Example:**

$G = SL_n$ over $K$

$F = (V_0 \subset \ldots \subset V_k)$ flag of linear subspaces of $K^n$

$P = \text{Stab}(F) \subset SL_n$

$G/P$ flag variety.
Compactifications

**Definition**
\[ \nu_P : \mathcal{B}(G, K) \xrightarrow{\nu} G^{an} \rightarrow (G/P)^{an} \]

The closure of the image of \( \mathcal{B}(G, K) \) under \( \nu_P \) is a compactification \( \overline{\mathcal{B}_P(G, K)} \) of \( \mathcal{B}(G, K) \) (or of some almost simple factors).

**Theorem**
\[ \overline{\mathcal{B}_P(G, K)} = \bigcup_{Q \text{ "good" parabolic}} \mathcal{B}(Q_{ss}, K) \]

**Theorem**
Any two points \( x, y \) in \( \overline{\mathcal{B}_P(G, K)} \) are contained in one compactified apartment.

**Theorem**
(Mixed Bruhat decomposition)
Let \( x, y \in \overline{\mathcal{B}_P(G, K)} \) with stabilizers \( P_x, P_y \subset G(K) \).

Then \( G(K) = P_x N(K) P_y \).
Example:

\[ G = SL_n \text{ over } K. \]

\[ P = \begin{cases} \begin{pmatrix} * & \cdots & * \\ \vdots & \vdots & \vdots \\ * & \cdots & * \\ 0 & \cdots & 0 \end{pmatrix} \end{cases} \text{ the stabilizer of a hyperplane} \]

\[ \mathcal{B}(G, K) = \{ \text{non-Archimedean norms on } K^n \}/\text{scaling} \]

\[ \cap \]

\[ \overline{\mathcal{B}}_P(G, K) = \{ \text{non-Archimedean seminorms on } K^n \}/\text{scaling} \]

\[ \cap \]

\[ (G/P)^{an} = (\mathbb{P}^{n-1})^{an} \]