Avoidance of $\mathfrak{Lin}_3^1$ by Teichmüller Curves in a Stratum of $\mathcal{M}_{4,2}$

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Diploma Thesis

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I hereby confirm that this document has been composed by myself, and describes my own work, unless otherwise acknowledged in the text.

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Ich erkläre hiermit die vorliegende Arbeit selbstständig verfasst zu haben und keine außer den angegebenen Quellen verwendet zu haben.

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Introduction

Mathematics is at its most fascinating when several seemingly unrelated theories coincide. In algebraic geometry, such occasions are omnipresent. The idea of studying the *geometry* of the solutions of algebraic equations is, arguably, as old as mathematics itself. Yet to what extent this is intertwined with most other areas of mathematics became apparent only fairly recently.

The close relationship between spaces obtained as the zero locus of polynomials over $\mathbb{C}$ and complex analytic spaces was already fairly well understood in the 19th century and provided the existing theory with a whole range of analytic tools and ideas. On the other hand, the revolutionary work in the 1950s and 60s, most notably by Grothendieck in [EGAI], generalised the theory by moving away from polynomial rings over a field, permitting instead algebraic equations in *any* commutative ring. This paved the way for unleashing the entire theory of commutative algebra upon geometry. At the same time, it opened the door to other branches of mathematics, strengthening for instance the ties to number theory. More recently, various attempts to extend these ideas by transporting, for example, homotopy theory into this world, have followed.

We, however, want to turn our attention to very classical objects: curves. Intuitively, we may think of a curve as the zero locus of a single polynomial in two variables, embedded in some space. In fact, we will later devote some time to introducing a theory that allows us to generalise this definition correctly and neatly. Although studying a single curve surely has its merits, we want to take it one step further: "modern" algebraic geometry allows us to consider entire families of curves and—fascinatingly—endow these with their own intrinsic geometric structures and even an algebraic one. In particular, we are interested in the space parametrising all smooth curves of genus g, which is called $\mathcal{M}_g$. Constructing these "moduli spaces" is—in general—very difficult. Some methods will be discussed in Chapter 3. Pioneering work was achieved in the analytic world by Teichmüller[1] in the 1930s and the geometric invariant theory, developed by Mumford in the 1960s (cf. [Mum65]), finally allowed an algebraic construction. In particular, it was the idea of Deligne and Mumford in [DM69] to introduce a more general concept of curves. This allowed Mumford, Knudsen and Gieseker to construct the "correct compactification" of this space which turns out to be a projective variety (cf. e.g. [Knu83]), i.e. is geometrically much better behaved than the "actual" moduli space.

Nonetheless, the geometry of these spaces is very difficult to understand. This leads us to focus our attention instead on the geometry of divisors on $\mathcal{M}_g$, i.e. certain subspaces of codi-

[1] For a discussion of the controversies surrounding the person of Teichmüller with respect to his inglorious behaviour during the 1930s, we refer to [Hau+92], in particular the facsimile of his letter to Edmund Landau [Hau+92, pp. 28–30].

mension 1 that may arise geometrically in a number of natural ways. We will be particularly interested in divisors that may be obtained by parametrising all curves with some common geometric property. Not only does this imply that the divisor itself is an interesting object, but it also allows us to use the geometry of the individual points (that are themselves curves) to understand the global structure.

On the other hand, we can consider curves *in* the moduli space, i.e. curves parametrising curves. Besides being intuitive, this approach has the advantage that general intersection theory tells us that the intersection of a curve and a divisor will be a *number*—in contrast to something of higher dimension. This allows us to use the combination of curves and divisors to study not only these curves and divisors but also the geometry of the space as a whole. These techniques were powerfully demonstrated by Harris and Mumford [HM82] and more recently by Farkas [Far09] or Logan [Log03].

In addition, we will be concerned with a special class of curves: Teichmüller curves. These have been studied intensely since the work of Veech in [Vee89] and arise quite naturally during the analytic construction of $\mathcal{M}_g$. Besides having interesting dynamical properties, these curves behave very nicely in $\overline{\mathcal{M}_g}$; in particular, plenty is known about their behaviour on the boundary $\overline{\mathcal{M}_g} \setminus \mathcal{M}_g$ and this is the property we will exploit. Additionally, Teichmüller curves can be constructed quite explicitly, providing us with a large and fairly benign class of curves to choose from when intersecting curves and divisors. This was observed by Chen and Möller: they took well-known divisors and used these to calculate invariants for families of Teichmüller curves, see [CM11]. This allowed them to verify a conjecture by Kontsevich and Zorich, stating exactly which classes of Teichmüller curves had invariant sums of Lyapunov exponents (introduced in Chapter 3) in moduli spaces of low genus.

In the following, we will carefully analyse a special case of their argument, involving some beautiful classical geometry. More concretely, we consider Teichmüller curves in the stratum $\Omega\mathcal{M}_4(3,3)^{\text{non-hyp}}$ and show in Theorem 7 that the image of any one of those in $\overline{\mathcal{M}_{4,2}}$ does not intersect the divisor $\mathfrak{Lin}_3^1$. This is the key ingredient to their proving the non-variance of the sum of Lyapunov exponents among Teichmüller curves in this stratum, as the intersection with $\mathfrak{Lin}_3^1$ depends—in essence—only on this sum (and is always 0). Following [CM11], we then turn this argument around and use the fact that if we *know* that the sum of Lyapunov exponents is non-varying, we may calculate it for any Teichmüller curve and use this to give an explicit description of the class of $\mathfrak{Lin}_3^1$. In Theorem 8 we then use the same strategy to provide examples of families of divisors avoiding all Teichmüller curves in hyperelliptic strata of any genus, as these have the advantage that the sum of Lyapunov exponents is known to be non-varying and may be calculated explicitly.

To achieve this goal, we begin by reviewing many of the fundamental tools of algebraic geometry. That is the content of Chapter 1. It is continued in Chapter 2 where we recall many results from classical algebraic geometry, translating them into a more modern language as necessary. In particular, in Theorem 1 we describe the relationship between the Picard group and the divisor class group and we sketch a proof of the Riemann-Roch theorem in Theorem 3. The chapter concludes with a brief review of intersection theory in section 2.3 and a discussion of the canonical embedding of a genus g curve into $\mathbb{P}^{g-1}$ in section 2.4.

In Chapter 3, we finally introduce moduli spaces and discuss some of the difficulties arising in and around the definition. After briefly discussing the Picard groups of the moduli spaces concerning us, we introduce Teichmüller curves and the divisor $\mathfrak{Lin}_3^1$. Chapter 4 is devoted to discussing Chen and Möller's proof. In section 4.1, the classical theory of quadrics is reviewed and we analyse the special behaviour of canonically embedded genus 4 curves in section 4.2. The last two sections assemble the gathered material to prove the above-mentioned Theorem 7 and Theorem 8.

Note that, in particular in Chapter 1, some concepts are introduced in a slightly more general version than would strictly be necessary, but as much of the literature—in particular regarding Chapter 3—requires such terminology anyway, this approach can hardly be considered harmful. To strengthen this argument, we quote Mumford [Mum65, IV]:

> It seems to me that algebraic geometry fulfills only in the language of schemes that essential requirement of all contemporary mathematics: to state its definitions and theorems in their natural abstract and formal setting in which they can be considered independently of geometric intuition. Moreover, it seems to me incorrect to assume that any geometric intuition is lost thereby.

None of this work would have been possible without the continued support of a large number of people. First and foremost, I would like to thank Gabi Weitze-Schmithüsen for supervising and guiding me, and for her patience when spending countless hours discussing my questions. I am also grateful to André Kappes for his endurance while teaching me algebraic geometry and his continued support even after leaving Karlsruhe. Martin Möller provided a number of extremely helpful comments, particularly in regard to fixing some last-minute problems. I would also like to thank Frank Herrlich and the entire Workgroup on Number Theory and Algebraic Geometry for their continued overall support and for providing an uniquely cordial working atmosphere. Special thanks also goes to Stefan Kühnlein, without whom I might never have developed an interest in algebra; Florian Nisbach, Myriam Finster and Anja Randecker, who never barred me from their office; and Fabian Januszewski, who repeatedly discussed geometric questions with me, even if they had no connection to number theory. Last but not least, I am heavily indebted to all my other "mathematical friends", most notably Michael Fütterer, Enrica Cherubini, Miriam Schwab, Felix Wellen and Tobias Columbus who were always eager to discuss mathematics, LaTeX and any other relevant parts of life.

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1 Preliminaries

We would like to begin by recalling some facts and definitions that will be indispensable for the following discussion. Even if our aim is to study the situation of curves, this will naturally lead to understanding more general objects. Therefore, it is useful to first invoke more powerful algebraic tools that may be applied to a number of geometric settings.

As this is a mix of approaches that are widely scattered about the literature, but nonetheless vital for our purposes, this introduction is rather extensive, referring at each point to the literature in which the current approach is elaborated on.

In this chapter, X shall always be a topological space and R a ring. Any ring we consider is *commutative (!)* and contains a unit element; a homomorphism of rings is thus always required to be unital.

1.1 Sheaves

We begin by defining sheaves on a topological space. For a comprehensive introduction, see e.g. [Har77, Chapter II.1], [LAGI, Chapter 3] or, of course, [EGAI, Chapitre 0, §3]. Recall that a sheaf is an object that helps us keep track of "local data" on a topological space. The precise definition is a little technical.

1.1.1 Presheaves and Sheaves

Consider the *category* $\mathfrak{Op}(X)$ *of open sets of* X, more precisely: the objects of $\mathfrak{Op}(X)$ are the open subsets of X and for open $U, U' \subseteq X$, we define

$$\operatorname{Hom}(U, U') := \begin{cases} U \hookrightarrow U' & \text{if } U \subseteq U' \text{ and} \\ \varnothing & \text{otherwise.} \end{cases}$$

Definition 1.1.1: (a) A *presheaf* of abelian groups on X is a contravariant functor

$$\mathcal{F} : \mathfrak{Op}(X) \longrightarrow \mathfrak{Ab}.$$

We denote the image of $U \hookrightarrow U'$ under $\mathcal{F}$ as $\cdot|_U : \mathcal{F}(U') \longrightarrow \mathcal{F}(U)$ and call it the *restriction morphism* from U' to U.

(b) Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves on X. Then a *morphism of presheaves* is a natural transformation $\eta : \mathcal{F} \longrightarrow \mathcal{G}$, i.e. a family of morphisms $\eta_U \in \operatorname{Hom}(\mathcal{F}(U), \mathcal{G}(U))$ for each open $U \subseteq X$ that is compatible with the restriction maps.

(c) Let $\mathcal{F}$ be a presheaf. For any open subset $U \subseteq X$ and any open cover $(U_i)_i$ of U, let $(s_i)_i$ be a family of elements with $s_i \in \mathcal{F}(U_i)$. We call $(s_i)_i$ a *consistent family on* U if $s_j|_{U_j \cap U_k} = s_k|_{U_j \cap U_k}$ for every[2] j and k.

(d) Let $\mathcal{F}$ be a presheaf and $(s_i)_i$ a consistent family on $U \subseteq X$. Then any element $s \in \mathcal{F}(U)$ is called an *amalgamate of* $(s_i)_i$ if $s|_{U_i} = s_i$ for every i.

(e) A presheaf $\mathcal{F}$ of abelian groups on X is called a *sheaf* on X if every consistent family admits a *unique* amalgamate.

Clearly, we may restrict any sheaf $\mathcal{F}$ on X to any open subset $U \subset X$ to obtain a sheaf on U. We denote this by $\mathcal{F}|_U$.

Remark 1.1.2: Let $\mathcal{F}$ and $\mathcal{G}$ be two sheaves on X. Then a *morphism of sheaves* is simply a morphism of presheaves $\eta\colon \mathcal{F} \longrightarrow \mathcal{G}$. Observe that, due to the compatibility with the restriction maps, such a morphism must map consistent families of $\mathcal{F}$ to consistent families of $\mathcal{G}$ and amalgamates to amalgamates. Hence this gives a useful notion of a morphism. We denote the *category of sheaves of abelian groups on* X by $\mathfrak{Ab}(X)$.

Remark 1.1.3: Let $\mathcal{F}$ be any sheaf on X. What is $\mathcal{F}(\varnothing)$? Well, as we have $\varnothing \cup \varnothing = \varnothing = \varnothing \cap \varnothing$, any two elements of $\mathcal{F}(\varnothing)$ form a consistent family. Hence, $\mathcal{F}(\varnothing)$ must consist of precisely one element: $\mathcal{F}(\varnothing) = 0$.

Remark 1.1.4: Fix an open subset $U \subseteq X$. Then "evaluating a sheaf at U" is functorial: we define the *global sections functor* by the association

$$\Gamma(U, -)\colon \mathfrak{Ab}(X) \longrightarrow \mathfrak{Ab}, \quad \mathcal{F} \longmapsto \mathcal{F}(U).$$

By definition, any morphism of sheaves $\mathcal{F} \longrightarrow \mathcal{G}$ is a natural transformation. Thus, it yields a morphism $\mathcal{F}(U) \longrightarrow \mathcal{G}(U)$; hence, this construction is clearly functorial.

Of course, in Definition 1.1.1, we may replace $\mathfrak{Ab}$ with any sufficiently similar category and thus define presheaves and sheaves of Rings, R-Algebras or sets, obtaining the categories $\mathfrak{Rings}(X)$, R-$\mathfrak{Alg}(X)$ and $\mathfrak{Set}(X)$, respectively. Occasionally, we will write $\mathfrak{Sh}(X)$ to denote sheaves on X (in any suitable of the above categories), and compare this to $\mathfrak{Psh}(X)$, the category of presheaves.

Example 1.1.5: (a) Consider the complex plane $\mathbb{C}$. Then the assignment

$$\mathcal{O}\colon U \longrightarrow \mathcal{O}(U) := \{f\colon U \longrightarrow \mathbb{C} \mid f \text{ is holomorphic}\} \in \mathfrak{Ob}(\mathfrak{Rings})$$

is functorial. Indeed, it is a sheaf of rings, as being holomorphic is a local property. We call $\mathcal{O}$ the *sheaf of holomorphic functions* on $\mathbb{C}$.

(b) Let G be an abelian group and p any point of X. Consider the presheaf $\mathcal{F}$ with

$$\mathcal{F}(U) := \begin{cases} G, & \text{if } p \in U; \\ 0, & \text{otherwise;} \end{cases} \quad \text{for any open } U \subseteq X.$$

The restriction maps are taken to be the identity and the trivial map, accordingly.

[2] Note that, due to the functoriality, this definition is independent of the chosen cover.

In fact, this is already a sheaf: consider any consistent family $(s_i)_i$ with $s_i \in \mathcal{F}(U_i)$ for some open covering $(U_i)_i$ of X. Let $s := s_i$ for one fixed i so that $U_i \ni p$. Observe that for *any* $U_j \ni p$ we have $s_j = s$. Indeed, $U_i \cap U_j$ is open and non-empty (it contains p) and the restriction maps are the identity, as all three sets contain p. We claim that s is an amalgamate of the whole family $(s_i)_i$. Note first that $p \in X$ so $\mathcal{F}(X) = G \ni s$. Naturally, we have $s|_{U_j} = s_j$ for any $U_j \ni p$. But if $p \notin U_j$, s_j is necessarily zero and the restriction morphism is trivial, so $s|_{U_j} = s_j$ also. Therefore, s is in fact an amalgamate. Moreover, it is also unique: let s' be another amalgamate and $U_i \ni p$. Then we must have $s'|_{U_i} = s_i = s$. But both sets contain p, hence the restriction map is the identity and $s = s'$ in $\mathcal{F}(U)$.

Of course, we are most interested in sheaves of rings on topological spaces and these allow us to construct algebras and modules in a relative setting. A space X together with a sheaf of rings $\mathcal{O}$ is sometimes called a *ringed space* $(X, \mathcal{O})$.

Definition 1.1.6: Let $\mathcal{O}$ be a sheaf of rings on X.

(a) Let $\mathcal{F}$ be a sheaf of abelian groups on X. Then $\mathcal{F}$ is an *$\mathcal{O}$-module sheaf* if $\mathcal{F}(U)$ is an $\mathcal{O}(U)$-module for every open $U \subseteq X$ and the module structure is compatible with the restriction maps.

(b) Let $\mathcal{F}$ be an $\mathcal{O}$-module sheaf. We say $\mathcal{F}$ is *locally free* if there exists an open cover $(U_i)_i$ of X so that $\mathcal{F}|_{U_i}$ is a free $\mathcal{O}|_{U_i}$-module. In this case, we call U_i a *trivialising open set* and $\mathcal{F}|_{U_i}$ a *trivialisation of* $\mathcal{F}$. If X is connected, it makes sense to speak of the *rank* of $\mathcal{F}$. Sometimes we shall refer to a locally free sheaf of rank one as an *invertible sheaf.* The reason will become apparent in Remark 2.1.12.

(c) Let $\mathcal{I}$ be a sheaf of $\mathcal{O}$-submodules. Then we call $\mathcal{I}$ an *ideal sheaf*.

(d) Let $\mathcal{F}$ be a sheaf of rings on X. Then $\mathcal{F}$ is a *sheaf of $\mathcal{O}$-algebras* if $\mathcal{F}(U)$ is an $\mathcal{O}(U)$-algebra for every open $U \subseteq X$ and the structure is compatible with the restriction maps.

1.1.2 Stalks

Consider a point $p \in X$. The open neighbourhoods $p \in U \subseteq X$ of p form a directed poset via the partial ordering "$\subseteq$". Unfortunately, in most topologies the limit of this system does not exist; it is therefore senseless to speak of a "smallest open neighbourhood of p". Considering presheaves, however, we may avoid this shortcoming to some extent: Consider instead the colimit[3] of $\mathcal{F}(U)$ for all neighbourhoods $U \ni p$. That usually does exist and hence we gain an understanding of the sheaf on an "infinitesimal neighbourhood of p". This discussion should motivate the following definition.

Definition 1.1.7: Let $\mathcal{F}$ be a presheaf of abelian groups on X.

(a) For any point $p \in X$, define the *stalk* of $\mathcal{F}$ at p to be

$$\mathcal{F}_p := \operatorname*{colim}_{p \in U \text{ open}} \mathcal{F}(U).$$

[3] Recall that presheaves are *contravariant* functors.

(b) An element of $\mathcal{F}_p$ may be considered an equivalence class[4] (s, U) with U an open neighbourhood of p and $s \in \mathcal{F}(U)$. We call $(s, U) \in \mathcal{F}_p$ a *germ* of the stalk $\mathcal{F}_p$.

(c) We call the set $\operatorname{supp} \mathcal{F} := \{p \in X \mid \mathcal{F}_p \neq 1\}$, i.e. the set of points at which the stalk is not trivial, the *support of* $\mathcal{F}$.

Note that these definitions make sense as the category of abelian groups is bicomplete.

Remark 1.1.8: Given a presheaf $\mathcal{F}$ on X, there is an universal way to construct a sheaf, i.e. there exists a sheaf $\mathcal{F}^+$ so that for any sheaf morphism $\mathcal{F} \longrightarrow \mathcal{G}$ there exists an unique morphism letting the following diagram commute in the category of presheaves:[5]

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathcal{F}^+ \\ & \searrow & \downarrow \\ & & \mathcal{G} \end{array}$$

In particular, $\mathcal{F}^+$ is unique up to canonical isomorphism. Moreover, the association is functorial in $\mathcal{F}$. Luckily, this construction is very explicit. It can be shown that

$$\mathcal{F}^+(U) = \left\{ s \colon U \longrightarrow \coprod_{p \in U} \mathcal{F}_p \;\middle|\; \begin{array}{l} \forall p : s(p) \in \mathcal{F}_p \text{ and} \\ \quad \exists U_p \ni p \text{ open}, t \in \mathcal{F}(U_p) : t_q = s(q) \; \forall q \in U_p \end{array} \right\}$$

for open $U \subseteq X$ is a sheaf and fulfills the desired property [Har77, Proposition II.1.2]. We denote this process by *sheafification* and the sheaf $\mathcal{F}^+$ as the *sheaf associated to the presheaf* $\mathcal{F}$.

Observe that the stalks are unaffected, i.e. $\mathcal{F}_p^+ = \mathcal{F}_p$ for all $p \in X$.

Example 1.1.9: Let G be an abelian group. Consider the presheaf $\mathcal{F}$ that associates G to every open set of X with the identity map for restriction morphisms. This is in general not a sheaf. Indeed, if X is reducible and G is not trivial, pick two open subsets $U, U' \subseteq X$ with $U \cap U' = \emptyset$ and two distinct elements $a, b \in G$. Then $a \in \mathcal{F}(U)$ and $b \in \mathcal{F}(U')$ is a consistent family (on $U \cup U'$) that admits no amalgamate.

What is the sheafification of $\mathcal{F}$? Obviously, for every p, the stalk $\mathcal{F}_p = G$ and there exists an open neighbourhood U_p, $\mathcal{F}(U_p) = G$ also. Therefore, by Remark 1.1.8, $\mathcal{F}^+$ is the sheaf of locally constant functions on X with values in G. Nonetheless, we call $\mathcal{F}^+$ the *constant sheaf* G and denote it by G_X or $\underline{G}$.

Sheafification will be very helpful for a number of further constructions:

[4] More precisely, we have $(s, U) \sim (s', U')$ iff there exists an open $U'' \subseteq U \cap U'$ with $s|_{U''} = s'|_{U''}$.

[5] In fact, the functor $\cdot^+$ is left adjoint to the forgetful functor from $\mathfrak{Sh}$ to $\mathfrak{Psh}$ [LAGI, §3.3.2].

Remark 1.1.10: (a) Let $\eta\colon \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of sheaves, i.e. $\eta \in \mathrm{Hom}_{\mathfrak{Ab}(X)}(\mathcal{F}, \mathcal{G})$. Then the presheaf $\ker \eta(U)$ for all open $U \subseteq X$ is in fact a sheaf and we call it the *kernel*[6] of η. Note that the presheaf $\operatorname{im} \eta(U)$ fails to be a sheaf, in general; we therefore denote the sheaf associated to this presheaf as *image* of η or $\operatorname{im} \eta$.

(b) Let $\mathcal{O}$ be a sheaf of rings and $\mathcal{F}$ and $\mathcal{G}$ sheaves of $\mathcal{O}$-modules. Again, the presheaf $\mathcal{F}(U) \otimes_{\mathcal{O}(U)} \mathcal{G}(U)$ fails to be a sheaf and we therefore define the *tensor product* $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$ as the sheaf associated to this presheaf.

The above Remark 1.1.10 (a) allows us to speak of *exact sequences of sheaves*: Consider the sheaves $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ on X with morphisms

$$0 \longrightarrow \mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H} \longrightarrow 0. \tag{1.1}$$

We say this sequence is exact if $\operatorname{im} \varphi = \ker \psi$. Note additionally that we say that a map of sheaves $\varphi\colon \mathcal{F} \longrightarrow \mathcal{G}$ is *injective* if its kernel is the zero sheaf and that we say it is *surjective* if its image is $\mathcal{G}$. In fact, the injective morphisms are exactly the monomorphisms, while the surjective morphisms correspond to the epimorphisms in the category of sheaves. Moreover, $\mathfrak{Ab}(X)$ is an abelian category [LAGI, §3.5].

Remark 1.1.11: In this situation, we may form the quotient presheaf $\mathcal{G}(U)/\mathcal{F}(U)$. Again, this is not a sheaf and we call the sheafification of this presheaf the *quotient sheaf* denoted by $\mathcal{G}/\mathcal{F}$. In particular, we have $\mathcal{G}/\mathcal{F} \cong \mathcal{H}$.

What happens to φ at a point $p \in X$? Observe the following diagram:

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\varphi} & \mathcal{G} \\ \downarrow & & \downarrow \\ \mathcal{F}_p & \overset{\varphi_p}{\dashrightarrow} & \mathcal{G}_p \end{array}$$

By the universal property of the colimit, the morphism φ induces a *morphism of the stalks*[7] yielding a functor[8]

$$\cdot_p\colon \mathfrak{Ab}(X) \longrightarrow \mathfrak{Ab}.$$

It turns out that this functor is exact and this will simplify many arguments, as it is often much easier to argue on the stalks than on the sheaf as a whole.

[6] Observe that these definitions correspond to the kernel and image defined in the category $\mathfrak{Ab}(X)$ via universal mapping properties, cf. [LAGI, §3.5] or [Vak12, Proposition 3.5.1 and 3.3.I].

[7] Indeed, the natural transformation φ consists of morphisms $\varphi(U)$ for every open $U \subseteq X$ that are compatible with the restriction maps. But these morphisms on the open neighbourhoods of p induce a morphism on the stalk by the universal property of the colimit.

[8] Do we really want to call this “stalking”?

Proposition 1.1.12: A sequence (1.1) on X is exact if and only if it is exact on stalks, i.e. the sequence

$$0 \longrightarrow \mathcal{F}_p \xrightarrow{\varphi_p} \mathcal{G}_p \xrightarrow{\psi_p} \mathcal{H}_p \longrightarrow 0$$

is exact for every $p \in X$.

Consequently, a morphism $\varphi\colon \mathcal{F} \longrightarrow \mathcal{G}$ of sheaves on X is an epimorphism, monomorphism or isomorphism if and only if the corresponding morphism $\varphi_p\colon \mathcal{F}_p \longrightarrow \mathcal{G}_p$ is an epimorphism, monomorphism or isomorphism for every $p \in X$.

Proof: This is discussed in [Vak12, §3.5]. □

Note, however, that Proposition 1.1.12 *requires* a morphism of sheaves to exist. A morphism on stalks does not necessarily glue to give a morphism of sheaves!

1.1.3 Sheaves on Different Spaces

Up to this point, we have only considered sheaves on a fixed space X. In practice, however, it will frequently be necessary to compare sheaves on different spaces. Therefore, we must define how to "transport" a sheaf to another space via a continuous map.

Definition 1.1.13: Let Y be another topological space and $f: X \longrightarrow Y$ a continuous map.

(a) Let F be a sheaf on X. Then we define the *direct image sheaf* of $\mathcal{F}$ via

$$f_*\mathcal{F}(V) := \mathcal{F}(f^{-1}(V)) \text{ for every open } V \subseteq Y.$$

Note that, as f is continuous, $f^{-1}(V)$ is open in X and that $f_*(\mathcal{F})$ is in fact a sheaf on Y.

(b) Now let $\mathcal{G}$ be a sheaf on Y. For an open set $U \subseteq X$, $f(U)$ will, in general, not be open, so we must once again recede to a limit and sheafify[9]: We define the *inverse image sheaf* of $\mathcal{G}$ as the sheaf associated to the presheaf

$$f^{-1}\mathcal{G}(U) := \operatorname*{colim}_{\substack{V \supseteq f(U) \\ \text{open}}} \mathcal{G}(U).$$

(c) Both these relations are functorial, more precisely: the functors $f_*: \mathfrak{Sh}(X) \longrightarrow \mathfrak{Sh}(Y)$ and $f^{-1}: \mathfrak{Sh}(Y) \longrightarrow \mathfrak{Sh}(X)$ are in fact adjoint, i.e. there is a natural[10] bijection

$$\mathrm{Hom}_{\mathfrak{Sh}(X)}(\mathcal{F}, f^{-1}\mathcal{G}) \cong \mathrm{Hom}_{\mathfrak{Sh}(Y)}(f_*\mathcal{F}, \mathcal{G}).$$

(d) This allows us to define the notion of a *morphism of ringed spaces*: Consequently, these consist of a pair $(f, f^\sharp)\colon (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ where $f: X \longrightarrow Y$ is a continuous map and $f^\sharp\colon \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$ is a morphism of sheaves.

[9] This is necessary: consider, for example, any space X and the constant map f, taking X to a point p. A sheaf $\mathcal{G}$ on p is nothing but a group G and taking the inverse image of $\mathcal{G}$ places us in the situation of Example 1.1.9.

[10] In both $\mathcal{F}$ and $\mathcal{G}$, cf. e.g. [LAGI, §3.4.1].

(e) Consider again the case that $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are ringed spaces, f is a morphism and $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_X$ and $\mathcal{O}_Y$-modules, respectively. Then $f_*\mathcal{F}$ is an $\mathcal{O}_Y$-module[11] but $f^{-1}\mathcal{G}$, in general, fails to be an $\mathcal{O}_X$-module. Therefore we define

$$f^*\mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

and may verify that f_* and f^* are again adjoint functors (between the categories $\mathfrak{Mod}(X)$ and $\mathfrak{Mod}(Y)$), cf. e.g. [Har77, p. 110] or [Vak12, §17.3].

Example 1.1.14: Consider any point $p \in X$ together with its embedding $\iota\colon p \hookrightarrow X$ and let $\mathcal{F}$ be any sheaf on X. Then $\iota^{-1}\mathcal{F} = \mathcal{F}_p$, the stalk at p. Indeed, in both cases the limit ranges over all open neighbourhoods of p in X.

Now endow p with any sheaf $\mathcal{F}$. Note that, by Remark 1.1.3 a sheaf on a point is just a group G with $\mathcal{F}(p) = G$. Observe that $\iota_*\mathcal{F}$ is the skyscraper sheaf from Example 1.1.5 (b). Indeed, for any open $U \subseteq X$, $\iota_*\mathcal{F}(U) = G$ iff $p \in U$ and otherwise $\iota_*\mathcal{F}(U)$ is the trivial group.

1.2 Schemes

The ingenuity of algebraic geometry lies in the fact that it allows us to study geometric objects by "translating" their geometry into algebraic objects. This is best done via the construction of schemes. To define what a scheme should be, we must first recall the notion of an affine scheme. A comprehensive introduction may be found in, e.g. [Har77, §II.2] or [EGAI, Chapitre 1, §1–2][12].

1.2.1 Affine Schemes

Consider the set $\operatorname{Spec} R$ of all prime ideals[13] of the ring R, which we call the *spectrum of the ring R*. We will define a topology on it and turn it into a ringed space. We need a slightly more precise notion, though.

Definition 1.2.1: A *locally ringed space* consists of a topological space X and a sheaf of rings, $\mathcal{O}$, with the additional condition that for every $p \in X$ the stalk $\mathcal{O}_p$ is a local ring.

A *morphism of locally ringed spaces* $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ consists of a continuous map $f: X \longrightarrow Y$ and a morphism of sheaves $f^\sharp\colon \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$ satisfying the additional property that the induced morphism of local rings $f_p^\sharp\colon \mathcal{O}_{Y,f(p)} \longrightarrow f_*\mathcal{O}_{X,p}$ is a local morphism for every $p \in X$, i.e. $f_p^{\sharp-1}(\mathfrak{m}_\mathfrak{p}) = \mathfrak{m}_{f(p)}$, where $\mathfrak{m}_\mathfrak{p}$ and $\mathfrak{m}_{f(p)}$ are the corresponding unique maximal ideals.

[11] Note that $f_*\mathcal{O}_X$ is an $\mathcal{O}_Y$-algebra as f is a morphism of ringed spaces, thereby endowing $\mathcal{O}_X$ with a canonical $\mathcal{O}_Y$-module structure.

[12] Note that in the cited version, what we call a scheme today was denoted as a "prescheme". Grothendieck himself abolished that term in the 1971 reprint; what was then a scheme is now a separated scheme.

[13] Note that a prime ideal is always a proper ideal, i.e. contains no units.

Remark 1.2.2: Let X be a locally ringed space and $p \in X$ any point. Then we denote by $\kappa(p) := \mathcal{O}_{X,p}/\mathfrak{m}_p$ the *residue field* of p. Observe that the above requirement of morphisms being local implies that they descend to the residue field, i.e. for any morphism f and $q \in f^{-1}(p)$, the field $\kappa(q)$ is an extension field of $\kappa(p)$.

If $R := k[x_1, \ldots, x_n]$ is a polynomial ring and $p := (p_1, \ldots, p_n) \in k^n$ is a point, recall that f having a *zero* at p is equivalent to f lying in the maximal ideal $(x_1 - p_1, \ldots, x_n - p_n) \leq R$. We use this to motivate the following definition.

Definition 1.2.3: Let R be any (commutative) ring.

(a) We say that $f \in R$ has a *zero* at a point $\mathfrak{p} \in \operatorname{Spec} R$ iff $f \equiv 0 \mod \mathfrak{p}$, i.e. $f \in \mathfrak{p}$. Occasionally, we write $f(\mathfrak{p}) = 0$ in this case.

(b) For any ideal $I \leq R$, we set

$$\mathfrak{V}(I) := \{\mathfrak{p} \in \operatorname{Spec} R \mid \forall f \in I : f(\mathfrak{p}) = 0\} = \{\mathfrak{p} \in \operatorname{Spec} R \mid I \subseteq \mathfrak{p}\}.$$

(c) For any subset $Y \subseteq \operatorname{Spec} R$, we define

$$\mathfrak{I}(Y) := \{f \in R \mid f(\mathfrak{p}) = 0 \ \forall \mathfrak{p} \in Y\} = \bigcap_{\mathfrak{p} \in Y} \mathfrak{p} \leq R.$$

(d) For any $f \in R$, we denote the complement of $\mathfrak{V}(f)$ as $\mathfrak{D}(f) := \operatorname{Spec} R \setminus \mathfrak{V}(f)$.

Remark 1.2.4: Observe that $\mathfrak{V}(0) = \operatorname{Spec} R$, $\mathfrak{V}(R) = \mathfrak{V}(1) = \emptyset$, $\mathfrak{V}(I_1 \cdot I_2) = \mathfrak{V}(I_1) \cup \mathfrak{V}(I_2)$ and $\mathfrak{V}(\sum I_i) = \bigcap \mathfrak{V}(I_i)$, permitting us to define a topology on $\operatorname{Spec} R$ with closed sets of the form $\mathfrak{V}(I)$ for some ideal I. We call this the *Zariski topology* and, unless otherwise stated, will always endow $\operatorname{Spec} R$ with this topology.

Note that $\mathfrak{V}(I_1) = \mathfrak{V}(I_2)$ is equivalent to the equality of their radical ideals, i.e. $\sqrt{I_1} = \sqrt{I_2}$. Also, we have $\mathfrak{I}(\mathfrak{V}(I)) = \sqrt{I}$ and $\mathfrak{V}(\mathfrak{I}(Y)) = \overline{Y}$ for any ideal I and subset Y. [EGAI, Chapitre 1, Propositions 1.1.2 and 1.1.4].

For Noetherian R, this construction turns $\operatorname{Spec} R$ into a Noetherian topological space [EGAI, Chapitre 1, Corollaire 1.1.6]. Meanwhile, for any ring R the sets $\mathfrak{D}(f)$ for $f \in R$ form a basis of the topology and $\operatorname{Spec} R$ is always a quasi-compact[14] space [EGAI, Chapitre 1, Proposition 1.1.10].

Example 1.2.5: (a) Let k be an algebraically closed field. Then we define *affine space over* k, $\mathbb{A}^n_k$, as $\operatorname{Spec} k[x_1, \ldots, x_n]$. Note that the closed points are in one-to-one correspondence with the points of k^n. Additionally, the zero ideal corresponds to one non-closed point, which is dense. In analogy to this, we define *affine space over* R, $\mathbb{A}^n_R$, as $\operatorname{Spec} R[x_1, \ldots, x_n]$, for any (commutative) ring R.

(b) Consider any ideal $I \leq R$. Then $\operatorname{Spec} R/I$ can be canonically identified with the closed set $\mathfrak{V}(I) \subseteq \operatorname{Spec} R$. Indeed, the prime ideals of R/I bijectively correspond to those containing I in R. In fact, this is a homeomorphism.

[14] Note that the Zariski topology is Hausdorff only in trivial cases: for example, any prime ideal that is not maximal corresponds to a non-closed point in the spectrum.

Proposition 1.2.6: Spec R is irreducible as a topological space if and only if R is an integral domain.

Proof: [EGAI, Chapitre 1, Proposition 1.1.13]. □

Consider the following presheaf of rings on $X = \operatorname{Spec} R$: to each set[15] $\mathfrak{D}(f)$, we assign the localisation of R by[16] f. This is in fact a sheaf, which we call the *sheaf of regular functions, $\mathcal{O}$*, cf. [EGAI, Chapitre 1, Théorème 1.3.7]. More specifically we write $\mathcal{O}_X$ if more than one space is involved and—by abuse of notation—often also just $\mathcal{O}_R$.

This construction turns $(\operatorname{Spec} R, \mathcal{O}_R)$ into a locally ringed space. By abuse of notation, we will continue speaking of Spec R only, even when considering the locally ringed space.

Definition 1.2.7: Let $(X, \mathcal{O}_X)$ be a locally ringed space. Then we say that $(X, \mathcal{O}_X)$ is an *affine scheme* if it is isomorphic—as a locally ringed space—to $(\operatorname{Spec} R, \mathcal{O}_R)$ for some ring R.

Remark 1.2.8: By abuse of notation, we will occasionally only speak of the affine scheme R, of course referring to the affine scheme $(\operatorname{Spec} R, \mathcal{O}_R)$.

More generally, consider any R-module M. We then obtain a sheaf of $\mathcal{O}$-modules in the following fashion: to each $\mathfrak{D}(f)$, we assign the R_f-module[17] M_f; correspondingly the stalk at $\mathfrak{p}$ is $M_\mathfrak{p}$, cf. [EGAI, Chapitre 1, Théorème 1.3.7]. We call this the *sheaf associated to M, $\widetilde{M}$*. Observe that $\widetilde{R} = \mathcal{O}_R$ and that the association $M \longmapsto \widetilde{M}$ is functorial.

Remark 1.2.9: Observe that for R-modules M and N, $\widetilde{M \otimes N} \cong \widetilde{M} \otimes \widetilde{N}$ [EGAI, Chapitre 1, Corollaire 1.3.12].

Now consider any ring homomorphism $\varphi\colon R \longrightarrow S$. For a prime ideal $\mathfrak{p} \leq S$, $\varphi^{-1}(\mathfrak{p}) \leq R$ is also prime. This allows us to define a map

$$\operatorname{Spec}\varphi\colon \operatorname{Spec} S \longrightarrow \operatorname{Spec} R, \quad \mathfrak{p} \longmapsto \varphi^{-1}(\mathfrak{p}).$$

It turns out that this map is in fact continuous and gives rise to a morphism of locally ringed spaces. Hence, we may consider Spec as a functor from $\mathfrak{Rings}$ to affine schemes, $\mathfrak{AffSch}$. In fact, it induces an equivalence of categories, which we capture in the following proposition.

Proposition 1.2.10: The association $R \longmapsto (\operatorname{Spec} R, \mathcal{O}_R)$ is functorial, associating $\operatorname{Spec}\varphi$ to any ring homomorphism φ. This induces an anti-equivalence of the categories $\mathfrak{Rings}$ and $\mathfrak{AffSch}$.

Proof: [LAGII, Theorem 6.1.20]. □

[15] Note that it suffices to define a presheaf on the basis of a topology, as any open set may be considered as a limit of open sets of the basis, cf. [EGAI, Chapitre 0, §3.2].

[16] I.e. we localise by the multiplicative system $\{1, f, f^2, \dots\}$. This is not to be confused with the localisation $R_\mathfrak{p}$ of R by a prime ideal $\mathfrak{p}$, where the multiplicative system is $R \setminus \mathfrak{p}$.

[17] Recall that localisation of a module is defined via the tensor product: $M_f := M \otimes_R R_f$ and accordingly for any prime ideal $\mathfrak{p}$.

1.2.2 Schemes

Finally, we are able to define a scheme.

Definition 1.2.11: A locally ringed space $(X, \mathcal{O}_X)$ is a *scheme* if there exists a covering $(U_i)_i$ of X so that $(U_i, \mathcal{O}_X|_{U_i})$ is an affine scheme, for every i. We call $\mathcal{O}_X$ the *structure sheaf of* X. By abuse of notation, we tend to write X for the scheme, implicitly assuming the existence of the structure sheaf $\mathcal{O}_X$.

A *morphism of schemes* is a morphism of ringed spaces, giving rise to the category of schemes, $\mathfrak{Sch}$.

In the case of affine schemes, the entire geometric structure was contained in a single ring. For general schemes, this is no longer the case. Indeed, many interesting schemes will have only constant global sections of the structure sheaf. However, when studying morphisms between schemes and affine schemes, it is enough to consider the ring morphisms between the corresponding global sections. More precisely:

Proposition 1.2.12: The Spec functor and the global sections functor form an adjoint pair, i.e. for any scheme X and any ring R there is a natural bijection

$$\mathrm{Hom}_{\mathfrak{Sch}}(X, \operatorname{Spec} R) \cong \mathrm{Hom}_{\mathfrak{Rings}}(R, \Gamma(X, \mathcal{O}_X)).$$

Proof: [EGAI, Chapitre 1, Proposition 2.2.4]. □

Definition 1.2.13: Let X, Y be schemes. An *open subscheme* of X consists of an open subspace $U \subseteq X$ endowed with the structure sheaf $\mathcal{O}_U = \mathcal{O}_X|_U$. An *open immersion* is a morphism $f : X \longrightarrow Y$ inducing an isomorphism of X with an open subscheme of Y.

For closed subschemes we are motivated by Example 1.2.5 (b): a *closed subscheme* is a scheme $\iota\colon Y \longrightarrow X$ with ι the inclusion map and Y a closed subspace of X and the additional requirement that $\iota^\sharp : \mathcal{O}_X \longrightarrow \iota_*\mathcal{O}_Y$ is surjective. Consequently, a *closed immersion* is an isomorphism onto a closed subscheme of X.

We say that $\iota\colon X \longrightarrow Y$ is an *immersion* if it gives an isomorphism of X with an open subscheme of a closed subscheme of Y.

Often, it will be useful to look at schemes in a relative setting. This is formalised as follows.

Definition 1.2.14: Let S be a scheme. Then we define a *scheme over* S or S-scheme to be a scheme X with a fixed morphism $f : X \longrightarrow S$. We sometimes refer to f as the *structure morphism* of X.

Let X and Y be S-schemes. Then a morphism of S-schemes is a morphism $f : X \longrightarrow Y$ letting the following diagram commute in $\mathfrak{Sch}$:

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & Y \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

Example 1.2.15: (a) Let R and S be k-algebras for some ring k. Then the corresponding affine schemes are k-schemes and the morphisms of k-schemes correspond to the k-algebra morphisms.

(b) Note that for any scheme X, there is a unique morphism $\mathbb{Z} \longrightarrow \Gamma(\mathcal{O}_X, X)$ and hence, by Proposition 1.2.12, a unique morphism of schemes $X \longrightarrow \operatorname{Spec}\mathbb{Z}$. Therefore, we may consider any scheme as a scheme over $\mathbb{Z}$ in a canonical way.

1.2.3 Products

Recall that whenever two categories are anti-equivalent, products in one category correspond to coproducts in the other category [Mac98, §§IV.4 and V.5]. By Proposition 1.2.10, it will thus be sufficient to look at (fibred) coproducts[18] in $\mathfrak{Rings}$ to find (fibred) products of affine schemes. We have thereby proven the following result.

Proposition 1.2.16: Let $X := \operatorname{Spec} S$ and $Y := \operatorname{Spec} S'$ be affine schemes over R. Then

$$X \times_R Y = \operatorname{Spec}(S \otimes_R S').$$

In particular, we have $X \times Y = \operatorname{Spec}(S \otimes_{\mathbb{Z}} S')$.

For general schemes, this construction must be applied locally and these affines must then be glued together to build the fibred product.

Proposition 1.2.17: Let X and Y be schemes over a scheme S. Then the *fibred product* $X \times_S Y$ exists.

Proof: Take an affine cover $\operatorname{Spec} A_i$ of S and affine covers of the preimages in X and Y. These affine fibre products can be glued [Har77, Theorem II.3.3]. □

Example 1.2.18: Note that, in general, the fibred product of schemes will not be homeomorphic to the fibred product of the underlying topological spaces. Consider a Galois extension L/k of degree n. Then the inclusion $k \hookrightarrow L$ makes L into a k-scheme and we have

$$L \times_k L = \operatorname{Spec}(L \otimes_k L) = \operatorname{Spec} L^n = \coprod_n \operatorname{Spec} L$$

by Galois theory and the fact that products correspond to coproducts. Thus, this fibred product consists of n distinct points. But the topological fibre product is simply the product of one point with itself over another point and thus consists of only a single point.

However, if we limit ourselves to schemes over algebraically closed fields, these problems do not occur, i.e. the points of the fibred products of schemes correspond to the points of the fibred products of the topological spaces, cf. [Vak12, 10.1.3]. This allows us to generalise many geometric notions via fibre products.

Definition 1.2.19: Let $f : X \longrightarrow Y$ be a morphism. For any subscheme $Y' \subseteq Y$, we define the *preimage* of Y' under f as $X \times_Y Y'$. In particular, we thus obtain the *fibre* $X_p := X \times_Y \kappa(p)$ for any point $p \in Y$.

[18] Recall that the fibred coproduct in $\mathfrak{Rings}$ is the tensor product, which in turn is the coproduct in the category of R-algebras, where R is the base ring of the tensor product [Lan02, Proposition XVI.6.1].

REMARK 1.2.20: In particular, for any $p \in Y$, the scheme $f^{-1}(p)$ corresponds to the topological fibre product, i.e. the fibre as a scheme is homeomorphic to the fibre of the continuous map f.

For an affine morphism $f\colon \operatorname{Spec} A \longrightarrow \operatorname{Spec} B$ and any closed subscheme $\mathfrak{V}(I) \subseteq B$, the preimage is $\operatorname{Spec}(A \otimes_B B/I) \cong A/(f^\sharp(I))$, which again corresponds to the preimage of the *map* f. See also the discussion in [Vak12, §10.3], in particular [Vak12, §10.3.2].

Recall that, in the category $\mathfrak{Set}$, given subsets $X, X' \subseteq Y$, the corresponding fibre product is the intersection: $X \cap X' = X \times_Y X'$. This motivates the following definition.

DEFINITION 1.2.21: Let X, X' be subschemes of Y. Then we define the *intersection scheme* $X \cap X'$ as the fibre product $X \times_Y X'$, obtained via the inclusion morphisms.

1.2.4 Projective Schemes

An important class of schemes is formed by projective schemes. As they will provide an essential environment for our later observations, this is a good time to recall some of their useful properties. To do this, we must first consider graded algebras.

DEFINITION 1.2.22: Let R be a ring and S an R-algebra. Then we say that S is a *($\mathbb{Z}$-)graded R-algebra* if there exists a decomposition

$$S = \bigoplus_{d \in \mathbb{Z}} S_i$$

so that S_0 is an R-algebra and the multiplication satisfies $S_i \cdot S_j \subseteq S_{i+j}$.

An element f of S_d is called a *homogeneous element* and d is called the *degree* of f.

Let S and S' be graded R-algebras. A *morphism of graded algebras* is an R-algebra homomorphism $\varphi\colon S \longrightarrow S'$ that respects the grading, i.e. there exists some $n \in \mathbb{N}$ so that $\varphi(S_d) \subseteq S'_{nd}$ for all d.

We call S a $\mathbb{Z}^{\geq 0}$*-graded ring* if $S_d = 0$ for $d < 0$. From now on, we assume *all* graded rings to be $\mathbb{Z}^{\geq 0}$-graded and finitely generated by elements of S_1 as an S_0-algebra unless otherwise stated.

Note that S is an S_0-algebra and that $S_+ := \bigoplus_{d>0} S_d$ is an ideal, which we call the *irrelevant ideal.* Consequently, an ideal I is called a *relevant ideal* if $S_+ \not\subseteq I$.

An ideal in S is called a *homogeneous ideal* if it is generated by homogeneous elements.

EXAMPLE 1.2.23: Consider the polynomial ring $S := R[x_1, \ldots, x_n]$ over any ring R. Then S is a graded R algebra with $S_0 = R$ and the standard grading. Note especially that $S_d = 0$ for $d < 0$.

REMARK 1.2.24: Note that an ideal I is homogeneous iff for any $I \ni f = \oplus f_i$, $f_i \in I$ for all i. This induces a natural grading on the quotient S/I [Bou73, p. 368].

Further, observe that $\mathfrak{p}$ is a homogeneous prime ideal if and only if for any *homogeneous* $a, b \in S$, $ab \in \mathfrak{p}$ implies $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$ [Bou72, §III.1.4, Proposition 4].

Given any multiplicative set T of homogeneous elements, $T^{-1}S$ may be endowed with a canonical structure of a $\mathbb{Z}$-graded ring. In particular, for any homogeneous $f \in S$, we obtain a grading on S_f by setting $\deg f^{-1} = -\deg f$ [Bou72, §II.2.9, Proposition 29 and the following remarks].

Now we will define a "projective variant" of the Spec construction. For a graded ring S, consider the set

$$\operatorname{Proj} S := \{\mathfrak{p} \leq S \mid \mathfrak{p} \text{ is a homogeneous relevant prime ideal}\}.$$

REMARK 1.2.25: In analogy to Definition 1.2.3, we may define $\mathfrak{V}(I)$ for any homogeneous ideal $I \subset S_+$, again giving rise to the *Zariski topology* on $\operatorname{Proj} S$. As before, we set $\mathfrak{D}(f) := \operatorname{Proj} S \setminus \mathfrak{V}(f)$ for any homogeneous $f \in S_+$, yielding a basis of the topology.

By observing $\mathfrak{D}(f) = \operatorname{Spec}(S_f)_0$, i.e. the spectrum of the subring of degree zero elements[19] of the homogeneous localisation at f, we may assign a sheaf of rings to $\operatorname{Proj} S$ and see immediately that this endows $\operatorname{Proj} S$ with a scheme structure. Details may be found, e.g., in [EGAII, Chapitre II, §2.3] and [EGAII, Chapitre II, 2.4.1].

DEFINITION 1.2.26: A scheme of the form $\operatorname{Proj} S$ for some finitely generated graded R-algebra S is called a *projective scheme over R*; a *quasi-projective scheme over R* is a quasi-compact open subscheme of some projective scheme over R.

In particular, we call $\mathbb{P}^n_R := \operatorname{Proj} R[x_0, \dots, x_n]$ *projective R-space.* Often we omit the R if no confusion can arise and in most cases concerning us, R will be a field.

Given any scheme Y, we define *projective space over Y* as $Y \times_{\mathbb{Z}} \mathbb{P}^n_{\mathbb{Z}}$. Note that for $Y = \operatorname{Spec} R$, we have $\mathbb{P}^n_Y = \mathbb{P}^n_R$ as expected.

EXAMPLE 1.2.27: Projective space $\mathbb{P}^n_R$ comes with a natural cover by affines: observe that

$$\mathfrak{D}(x_i) = \operatorname{Spec}(R[x_0, \dots, x_n]_{x_i})_0 = \operatorname{Spec} R\left[\tfrac{x_0}{x_i}, \dots, \tfrac{x_n}{x_i}\right] = \mathbb{A}^n_R.$$

In the case that $R = k$, an algebraically closed field, this allows us to assign coordinates to the closed points of $\mathbb{P}^n_k$. Clearly, these correspond to elements of $(k^{n+1} \setminus \{0\})/k^\times$. We denote such a point by $(x_0 : \dots : x_n)$ where two points are equal if there exists $\lambda \in k^\times$ so that

$$(x'_0 : \dots : x'_n) = (\lambda x_0 : \dots : \lambda x_n).$$

Hence the points of $\mathfrak{D}(x_i)$ correspond to points $(p_0 : \dots : p_n)$ with $p_i \neq 0$. In this case, we may of course fix $p_i = 1$.

[19] Or put differently: the elements $\frac{a}{b} \in S_f$ with $\deg a = \deg b$.

Remark 1.2.28: Given an affine scheme $\operatorname{Spec} R$, all relevant information about the sheaf of regular functions was contained in the global sections, the ring R. In contrast, $\mathcal{O}(\mathbb{P}^n_k) = k$ for any field k[20], i.e. the global sections contain no real information about the regular functions. In particular, this shows that $\mathbb{P}^n_k$ is not affine for $n > 0$. Similarly, $\mathcal{O}(\operatorname{Proj} S) = S_0$ for any (non-trivially) graded ring S.

Unfortunately, in contrast to Proposition 1.2.10, the Proj construction is *not* functorial. The major obstruction is that given a morphism $\varphi\colon S' \longrightarrow S$ of graded rings and a homogeneous prime $\mathfrak{p} \leq S$, the irrelevant ideal of S' may be contained in $\varphi^{-1}(\mathfrak{p})$, i.e. this does not necessarily yield a point of $\operatorname{Proj} S'$.

Remark 1.2.29: Let $\varphi\colon S' \longrightarrow S$ be a morphism of graded rings. We set $G(\varphi) := \operatorname{Proj} S \setminus \mathfrak{V}(\varphi(S'_+))$. Then—as in the affine case—φ induces a continuous map $\operatorname{Spec}\varphi\colon G(\varphi) \longrightarrow \operatorname{Proj} S'$. Thus, if $G(\varphi) = \operatorname{Proj} S$, we obtain a morphism $\operatorname{Spec}\varphi\colon \operatorname{Proj} S \longrightarrow \operatorname{Proj} S'$ [EGAII, Chapitre II, Proposition 2.8.2]. Note in particular, however, that different morphisms of rings may induce the same morphism of schemes, cf. also the discussion in [Vak12, §7.4].

Example 1.2.30: Again, consider any graded ring S. Then, by Proposition 1.2.12, the canonical morphism $S_0 \longrightarrow \Gamma(\operatorname{Proj} S, \mathcal{O})$ induces a morphism $\operatorname{Proj} S \longrightarrow \operatorname{Spec} S_0$, making any projective scheme $\operatorname{Proj} S$ a $\operatorname{Spec} S_0$-scheme.

Despite these limitations, we observe some similarities to Proposition 1.2.16.

Remark 1.2.31: Let S and T be finitely generated graded R-algebras. Then we may describe the coproduct in the category of graded R-algebras by

$$S \otimes_R T = \bigoplus (S \otimes_R T)_i.$$

Note that the grading is given by $(S \otimes T)_i = \bigoplus S_\lambda \otimes T_\mu$ where the sum is taken over all λ, μ satisfying $\mu + \lambda = i$ [Bou73, §II.11.5] and [Bou73, §III.4.7, Proposition 10].

This helps us formulate a practical analogy to the affine situation.

Proposition 1.2.32: Let S, T be finitely generated graded R-algebras. Then

$$\operatorname{Proj} S \times_R \operatorname{Proj} T \cong \operatorname{Proj}(S \otimes_R T).$$

Proof: The morphism is given in, e.g., [Stacks, Lemma 24.43.6 (01WD)] and it is clear from the proof that the image is $\operatorname{Proj}(S \otimes_R T)$. Alternatively, as Proj is a special case of a projective bundle, this is a special case of the discussion in [EGAII, Chapitre II, §4.3]. A detailed construction can also be found in [Vak12, §10.6]. □

[20] Indeed, consider any $f \in \mathcal{O}(\mathbb{P}^n_k)$. Then we may restrict f to, say, $f|_{\mathfrak{D}(x_i)} =: g$ and $f|_{\mathfrak{D}(x_j)} =: h$ where g is a polynomial in $\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}$ and h a polynomial in $\frac{x_0}{x_j}, \dots, \frac{x_n}{x_j}$ and these must coincide on $\mathfrak{D}(x_i) \cap \mathfrak{D}(x_j) = \mathfrak{D}(x_i x_j)$. But x_i appears in g with only non-positive and in h with only non-negative exponents (the same holds for x_j, respectively), hence it may appear only with exponent 0. Therefore g must already be constant on $\mathfrak{D}(x_i)$ and h on $\mathfrak{D}(x_j)$. As this works for any i and j, f must be globally constant, cf. also [Vak12, 5.4.E] or [GW10, Proposition 1.61].

Example 1.2.33: In particular, this provides a closed embedding of $\mathbb{P}^n_R \times \mathbb{P}^m_R$ into $\mathbb{P}^{mn+m+n}_R$ and it is known as the *Segre embedding*. This morphism is given locally, i.e. as a map $\mathfrak{D}(x_i) \times \mathfrak{D}(y_j) \longrightarrow \mathfrak{D}(z_{ij})$ via

$$R\Big[\tfrac{z_{00}}{z_{ij}}, \tfrac{z_{01}}{z_{ij}} \dots, \tfrac{z_{nm}}{z_{ij}}\Big] \longrightarrow R\Big[\tfrac{x_0}{x_i}, \dots, \tfrac{x_n}{x_i}\Big] \otimes R\Big[\tfrac{y_0}{y_j}, \dots, \tfrac{y_m}{y_j}\Big]$$
$$\tfrac{z_{ab}}{z_{ij}} \longmapsto \tfrac{x_a}{x_i} \otimes \tfrac{y_b}{y_j}.$$

Choosing coordinates, this amounts to

$$\big((x_0 : \dots : x_n), (y_0 : \dots : y_m)\big) \longmapsto (x_0 y_0 : x_0 y_1 : \dots : x_n y_m).$$

See also [Vak12, §10.6].

1.3 Geometric Aspects

We now introduce a series of definitions that will allow us to speak precisely of several geometric aspects of a scheme. We loosely follow [Har77, §II.3 and §II.4].

Definition 1.3.1: Let X be a scheme, then we call X *irreducible* if the underlying topological space is irreducible. We call X *reduced* if $\mathcal{O}_X(U)$ is reduced for every open $U \subseteq X$.

While being irreducible seems independent of the structure sheaf, we can provide a sufficient algebraical condition.

Remark 1.3.2: X is irreducible and reduced if and only if $\mathcal{O}_X(U)$ is an integral domain for every open $U \subseteq X$. In this case, we call X *integral.*

Proof: Let X be integral. Any integral domain admits no nilpotent elements, i.e. X is reduced. Let $U \subseteq X$ be a reducible set. Then we find disjoint open non-empty subsets $U_1, U_2 \subset U$ and $\mathcal{O}(U_1 \sqcup U_2) = \mathcal{O}(U_1) \times \mathcal{O}(U_2)$ is no integral domain.

Conversely, let X be irreducible and reduced and assume, for some open U, we find $r, s \in \mathcal{O}(U)$ with $rs = 0$. As $\mathcal{O}(U)$ is reduced, this implies $r \neq s$. Hence, we have open sets[21] $\mathfrak{D}(r) \cap \mathfrak{D}(s) = \mathfrak{D}(rs) = \mathfrak{D}(0) = \varnothing$ and as U is irreducible, one of these must be empty, i.e. r or s must be nilpotent. But as the ring is reduced, there are no non-zero nilpotent elements; therefore there exist no zero divisors. □

Example 1.3.3: For any integral domain R, $\mathbb{A}^n_R$ and $\mathbb{P}^n_R$ are integral.

We now have the terminology to speak of dimension.

Definition 1.3.4: Let X be a scheme. The *dimension of* X, $\dim X$, is the dimension of X as a topological space, i.e. the supremum of lengths of ascending chains of irreducible subspaces. In particular, $\dim \operatorname{Spec} R = \dim R$, i.e. the Krull dimension.

To obtain the correct notion of codimension, we should restrict ourselves to irreducible subschemes. Let $Y \subset X$ be irreducible. Then the *codimension* of Y in X is the supremum of lengths of chains starting with Y. In particular, in $\operatorname{Spec} R$, the codimension of a prime $\mathfrak{p}$ is the height of $\mathfrak{p}$.

[21] For any $r \in \mathcal{O}(U)$ we may define $\mathfrak{D}(r)$ by taking an affine cover $U_i := \operatorname{Spec} R_i$ of U and setting $\mathfrak{D}(r)$ to the union of $\mathfrak{D}(r|_{U_i})$, an open set in U. In other words, $\mathfrak{D}(r)$ consists of those points where the corresponding germ is invertible in the stalk.

Example 1.3.5: For any field k, the dimension of $\mathbb{A}_k^n$ and $\mathbb{P}_k^n$ are both n.

Remark 1.3.6: Observe that the preimage of an affine set is in general not affine. We call a morphism $f\colon X \longrightarrow Y$ an *affine morphism* if $f^{-1}(V)$ is affine for every affine $V \subseteq Y$. Note that it suffices to check this condition for *one fixed* open affine cover of Y [GW10, Proposition and Definition 12.1].

Now consider the morphism $\operatorname{Spec}\overline{\mathbb{Q}} \longrightarrow \operatorname{Spec}\mathbb{Q}$. Geometrically, this fibre consists only of one point, but algebraically, it essentially contains the absolute Galois group of $\mathbb{Q}$. In a geometric setting, it is therefore often useful to require additional finiteness conditions for morphisms to minimise unexpected behaviour on the fibres.

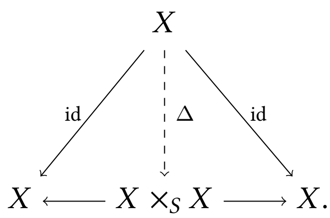
Definition 1.3.7: Let $f\colon X \longrightarrow Y$ be a morphism. For any affine cover $\operatorname{Spec} A_i$ of Y and any affine cover $\operatorname{Spec} B_{ij}$ of $f^{-1}(\operatorname{Spec} A_i)$, f induces an A_i-algebra structure on each B_{ij}.

We call f *locally of finite type* if every B_{ij} may be finitely generated as an A_i-algebra and *of finite type* if for every i finitely many j's suffice, i.e. every $f^{-1}(\operatorname{Spec} A_i)$ possesses a finite cover by affines. We call f *finite* if every $f^{-1}(\operatorname{Spec} A_i)$ is affine and may be finitely generated as an A_i-module. Note that this implies that any fibre over a point is finite.

Remark 1.3.8: Fortunately, in the above definitions it is again equivalent to require that the conditions hold for one particular or any cover, cf. [GW10, Proposition and Definition 10.5 and 12.9].

Note that in footnote 14 we already observed that the Zariski topology is hardly ever Hausdorff. The correct analogy of this criterion is checking if the "diagonal embedding" is closed, cf. also the discussion in [EH00, §III.1.2].

Definition 1.3.9: Let $f\colon X \longrightarrow S$ be a morphism of schemes and consider the associated fibred product $X \times_S X$. Then we obtain a morphism $\Delta\colon X \longrightarrow X \times_S X$ induced by the universal mapping property via



We call f *separated* if Δ is a closed immersion and we call the scheme X *separated* if $f\colon X \longrightarrow \operatorname{Spec}\mathbb{Z}$ is separated.

Another topological notion we would like to have is that of compactness. However, as discussed, the Zariski topology is far from Hausdorff but almost always[22] quasicompact. Intuitively, compactness should be something $\mathbb{A}^n$ does not fulfill but $\mathbb{P}^n$ does. It turns out that properness is the property of compact spaces that offers a useful generalisation into this setting.

[22] Indeed, even for a non-Noetherian ring R the space $\operatorname{Spec} R$ is quasicompact: consider any open cover $\operatorname{Spec} R = \bigcup U_i$. As the $\mathfrak{D}(r)$ form a basis of the topology, we may assume $U_i = \mathfrak{D}(f_i)$ for $f_i \in R$. Hence, we have $R = \bigoplus(f_i) \ni 1$, i.e. 1 may be expressed as a *finite* linear combination of the f_i. In other words, these finitely many $\mathfrak{D}(f_i)$ suffice for covering $\operatorname{Spec} R$.

Definition 1.3.10: Let $f: X \longrightarrow Y$ be a morphism of schemes. We call f *closed* if closed sets are mapped to closed sets. We call f *proper* if it is separated and *universally closed*, i.e. for any base change $Y' \longrightarrow Y$ the pullback $f': X \times_Y Y' \longrightarrow Y'$ of f is also closed.

We say that f is *projective* if it factors via $\mathbb{P}^n_Y$ by a closed immersion and the canonical projection; f is *quasi-projective* if the same holds for an open immersion into $\mathbb{P}^n_Y$ followed by a projective morphism.

Remark 1.3.11: A morphism of affine schemes is always separated; in particular, $\mathbb{A}^n$ is always separated. Open and closed immersions are separated and closed immersions are proper; both are stable under composition and base change. As promised, $\mathbb{A}^n_k$ is not proper over $\operatorname{Spec} k$ but for any graded ring S, $\operatorname{Proj} S$ is proper and projective over $\operatorname{Spec} S_0$. Furthermore, any projective morphism of Noetherian schemes is proper and any quasi-projective morphism of Noetherian schemes is of finite type and separated. See [Har77, §II.4].

Two more topological notions will be important in the next section. The definitions might seem slightly peculiar but they will turn out to be just the right conditions for a morphism to behave "nicely" when transfering quasi-coherent sheaves along it.

Recall that a scheme is *quasi-compact* if every open cover contains a finite subcover. A scheme is *quasi-separated* if the intersection of two quasi-compact open sets is again quasi-compact. We now state relative versions of these conditions.

Definition 1.3.12: Let $f: X \longrightarrow Y$ be a morphism of schemes. We call f *quasi-compact* if the preimage of every open affine subset of Y is again quasi-compact. Similarly, f is *quasi-separated* if the pre-image of every open affine set is a quasi-separated scheme. In particular, the absolute conditions again amount to the relative conditions over $Y = \operatorname{Spec} \mathbb{Z}$.

Remark 1.3.13: As above, it suffices to check both conditions on a fixed open affine cover of Y [Vak12, 8.3.C]. Note that if X is Noetherian (i.e. all cases of concern to us), any such f will be quasi-separated and quasi-compact [Vak12, 8.3.B].

1.4 Projective Subschemes and Coherent Sheaves

We briefly collect some important results about the classification of closed subschemes and several important classes of sheaves.

1.4.1 Closed Subschemes

Recall the notion of a closed $\iota: Y \hookrightarrow X$ subscheme (Definition 1.2.13). Denote by $\mathcal{I}$ the kernel of the sheaf morphism $\iota^\sharp$. By Remark 1.1.10 (a), this gives rise to an exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \longrightarrow \iota_* \mathcal{O}_Y \longrightarrow 0 \tag{1.2}$$

of sheaves on X. We call $\mathcal{I}$ the *ideal sheaf of* Y. Observe that for any open $U \subseteq X$, $\mathcal{I}(U)$ is an ideal in $\mathcal{O}(U)$. In particular, if X and Y are affine, we have $Y = \mathfrak{V}(I)$ for some ideal and in this case $\mathcal{I} = \widetilde{I}$. We will now generalise these concepts to obtain a similar result for general schemes.

Definition 1.4.1: (a) Let X be a Noetherian[23] scheme, $\mathcal{F}$ an $\mathcal{O}_X$-module. Then $\mathcal{F}$ is called *quasi-coherent* if we can cover X by open affine $U_i = \operatorname{Spec} A_i$ so that $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$ for some A_i-module M_i.

(b) Let M be an R-module. We call M *coherent* if M is finitely generated and for any map $\bigoplus_n R \longrightarrow M$, the kernel is finitely generated. Consequently, we call $\mathcal{F}$ *coherent* if the M_i are coherent modules.

Remark 1.4.2: Note that $\mathcal{F}$ is (quasi-)coherent if the above conditions hold for *any* cover of X by affine open sets. See, for example, the discussion in [Vak12, §14.2], in particular [Vak12, Theorem 14.2.1 and 14.6.3].

Example 1.4.3: Not every $\mathcal{O}_X$-module is quasi-coherent. Indeed, quasi-coherence implies that all local information is contained in the global sections over an affine set. This can fail dramatically. Consider, for example, a discrete valuation ring R with maximal ideal $\mathfrak{m}$ and quotient field K. Then the open sets of $X := \operatorname{Spec} R$ are (0) and X, while $\mathfrak{m}$ is the only closed point. Define $\mathcal{F}$ via $\mathcal{F}(0) = K$ and $\mathcal{F}(X) = 0$. As the only covers are by the points themselves, $\mathcal{F}$ is trivially a sheaf and an $\mathcal{O}_X$-module (note that $\mathcal{O}_X(0) = R_{(0)} = K$). But $\mathcal{F}$ is not (quasi-)coherent, as this would imply $\mathcal{F} = \widetilde{0}$ but $\mathcal{F}_{(0)} = K$ while $0 = \widetilde{0}_0$.

Note that quasi-coherent and coherent sheaves form abelian categories $\mathfrak{QCoh}(X)$ and $\mathfrak{Coh}(X)$ [Vak12, §14.4 and 14.6.3].

Remark 1.4.4: Let $X = \operatorname{Spec} R$ be an affine scheme. Then the functor $\widetilde{\cdot}$ from the category of R-modules to $\mathfrak{QCoh}(X)$ is exact and even gives rise to an equivalence of categories. The inverse is given by the global sections functor. In particular, note that in this case the global sections functor is exact.

For Noetherian X, the same functors induce an equivalence of the category of finitely generated R-modules and $\mathfrak{Coh}(X)$. See [Har77, Corollary II.5.5 and Proposition II.5.6].

In Example 1.2.5 (b) we observed that ideals of a ring give rise to closed subschemes of the corresponding affine scheme; at the beginning of this section, we noted that closed subschemes give rise to ideal sheaves. We now have the language to make this relationship more precise.

Proposition 1.4.5: Let X be a scheme. Then the quasi-coherent ideal sheaves are in one-to-one correspondence to the closed subschemes of X. Concretely, the support[24] of the sheaf $\mathcal{O}_X/\mathcal{I}$ is a closed subscheme determined uniquely by an ideal sheaf $\mathcal{I}$.

If X is Noetherian, we may replace "quasi-coherent" by "coherent".

Proof: [Har77, Proposition II.5.9]. □

Remark 1.4.6: Observe that combining this with Remark 1.4.4 yields a one-to-one correspondence of closed subschemes of $\operatorname{Spec} R$ to ideals of R. In particular, note that any closed subscheme of an affine scheme is affine.

[23] Strange things may happen if X is not Noetherian, cf. [Vak12, §14.8].

[24] That is the points of X where the stalk is not zero.

1.4.2 Twisted Sheaves

To obtain a similar result for projective schemes, we want an analogy of Remark 1.4.4 for coherent sheaves on projective space. To achieve this, we must replace R-modules by graded S-modules for some graded ring S.

DEFINITION 1.4.7: Let S be a graded ring and $M := \bigoplus_{i\in\mathbb{Z}} M_i$ a graded abelian group with an additional S-module structure, satisfying $S_i \cdot M_j \subseteq M_{i+j}$ for any i, j. Then we call M a *graded S-module.*

For some integer l, we define the *twist of M by l* to be the graded module $M(l)$ with grading $M(l)_d := M_{d+l}$. Observe in particular that $M(l)_0 = M_l$.

Next we define the *sheaf associated to M* as $\widetilde{M}|_{\mathfrak{D}(f)} := \widetilde{(M_f)_0}$, i.e. sheaf associated to the module of degree 0 elements in M_f, for any homogeneous $f \in S_+$. Note that $(M_f)_0$ is in fact a $\mathcal{O}_{\operatorname{Proj} S}(\mathfrak{D}(f)) = (S_f)_0$-module and that $\mathcal{O}_{\operatorname{Proj} S} = \widetilde{S}$. In particular, $\widetilde{M}_{\mathfrak{p}} = M_{\mathfrak{p}}$ for any $\mathfrak{p} \in \operatorname{Proj} S$. Details may be found in, e.g., [Vak12, §16.1].

Note that $\widetilde{\cdot}$ is again an exact functor. However, in contrast to Remark 1.4.4, it does not (yet) give rise to an equivalence of categories. Achieving this is quite a bit more subtle.

DEFINITION 1.4.8: Let n be some integer. Then we define $\mathcal{O}_X(n) := \widetilde{S(n)}$ for $X := \operatorname{Proj} S$. Now, given any $\mathcal{O}_X$-module sheaf $\mathcal{F}$, we may define the *twisted sheaf* $\mathcal{F}(n) := \mathcal{F} \otimes_{\mathcal{O}} \mathcal{O}(n)$.

REMARK 1.4.9: Observe that, for any graded S-module M and any $n \in \mathbb{Z}$, we have $\widetilde{M(n)} \cong \widetilde{M}(n)$. In particular, this implies $\mathcal{O}(m) \otimes \mathcal{O}(n) \cong \mathcal{O}(m+n)$ for any m, n. Note, in addition, that any $\mathcal{O}(n)$ is locally free of rank 1 [Har77, Proposition II.5.12].

EXAMPLE 1.4.10: Recall that in Remark 1.1.10 (b) we defined the tensor sheaf as the sheaf *associated* to the tensor presheaf. Without any effort, we can now show that this is in fact necessary: consider $\mathcal{O}(-1)$ and $\mathcal{O}(1)$ on $\mathbb{P}^n_k$. Then $\Gamma(\mathbb{P}^n, \mathcal{O}(1)) = k[x_0, \ldots, x_n]_1$, i.e. it consists of all degree 1 polynomials, while—by the same argument—$\Gamma(\mathbb{P}^n, \mathcal{O}(-1)) = 0$. But by Remark 1.4.9, $\mathcal{O}(-1) \otimes_{\mathcal{O}} \mathcal{O}(1) = \mathcal{O}(0) = \mathcal{O}$ and $\Gamma(\mathbb{P}^n, \mathcal{O}) = k$ by Remark 1.2.28 while $\Gamma(\mathbb{P}^n, \mathcal{O}(1)) \otimes_k \Gamma(\mathbb{P}^n, \mathcal{O}(-1)) = 0$.

Via twisting, we may now assign a graded module to any sheaf. We follow [Har77, p. 118].

DEFINITION 1.4.11: Let $X = \operatorname{Proj} S$ for some graded ring S and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We define the *graded S-module associated to $\mathcal{F}$* to be

$$\Gamma_*(\mathcal{F}) := \bigoplus_{n\in\mathbb{Z}} \Gamma(X, \mathcal{F}(n)).$$

Note that $\Gamma_*(\mathcal{F})$ is in fact a graded S-module: consider any $s \in S_n$ and $t \in \Gamma(X, \mathcal{F}(m))$. Then we may consider s as an element of $\Gamma(X, \mathcal{O}(n)) \cong S_n$ and define $s \cdot t := s \otimes t \in \mathcal{O}(n) \otimes \mathcal{F}(m)$. But, by definition, this yields a global section of $\mathcal{F}(m + n)$, i.e. our multiplication respects the grading.

EXAMPLE 1.4.12: Let $S = k[x_0, \ldots, x_n]$. Then $\Gamma_*(\operatorname{Proj} S, \mathcal{O}) \cong S$.

It was a fundamental observation by Serre [Ser55, §59 and §65] that this construction "retrieves" the module that induced a quasi-coherent sheaf.

Proposition 1.4.13: Let $X = \operatorname{Proj} S$ for some graded ring S and let $\mathcal{F}$ be some quasi-coherent sheaf on X. Then $\widetilde{\Gamma_*(\mathcal{F})}$ is naturally isomorphic to $\mathcal{F}$.

Proof: For a detailed discussion, see [Vak12, §16.4] or [Har77, Proposition II.5.15]. □

Finally, this lets us state the correspondence between projective subschemes and ideals.

Proposition 1.4.14: (a) Any closed subscheme $Y \subseteq \mathbb{P}^n_R$ is of the form $\operatorname{Proj}\big(R[x_0, \dots, x_n]/I\big)$ for some homogeneous ideal I.

(b) A scheme over $\operatorname{Spec} R$ is projective if and only if it is isomorphic to $\operatorname{Proj} S$ for some graded ring S with $S_0 = R$.

Proof: [Har77, Corollary II.5.16]. □

1.4.3 Pulling Back and Pushing Forward

Recall that in Definition 1.1.13 we observed the relationship between sheaves on different spaces. We now want to briefly examine what we can say in the special case of coherent and quasi-coherent sheaves. We begin by noting an obvious obstruction.

Example 1.4.15: Consider $f\colon \mathbb{A}^1_k \longrightarrow \operatorname{Spec} k$ induced by the inclusion $k \hookrightarrow k[t]$. Then $\mathcal{O}_{\mathbb{A}^1}$ is a coherent sheaf but as $k[t]$ is no finitely generated k-module, $f_*\mathcal{O}_{\mathbb{A}^1}$ is not coherent. Clearly, being finite is a necessary requirement for a morphism to preserve coherence.

In most cases, however, quasi-coherent sheaves behave well.

Proposition 1.4.16: Let $f\colon X \longrightarrow Y$ be a morphism of schemes.

(a) If f is quasi-compact and quasi-separated, $f_*\colon \mathfrak{QCoh}(X) \longrightarrow \mathfrak{QCoh}(Y)$ is functorial.

(b) $f^*\colon \mathfrak{QCoh}(Y) \longrightarrow \mathfrak{QCoh}(X)$ is functorial.

(c) If f is quasi-compact and quasi-separated, f^* and f_* are an adjoint pair.

(d) $f^*\mathcal{O}_Y \cong \mathcal{O}_X$.

(e) If f is a finite morphism of Noetherian schemes, $f_*\colon \mathfrak{Coh}(X) \longrightarrow \mathfrak{Coh}(Y)$ is functorial.

(f) If X and Y are Noetherian, $f^*\colon \mathfrak{Coh}(Y) \longrightarrow \mathfrak{Coh}(X)$ is functorial.

(g) Let $\mathcal{G} \in \mathfrak{QCoh}(Y)$. If $\mathcal{G}$ is locally free of rank r then so is $f^*\mathcal{G}$. In particular, pullbacks of invertible sheaves are invertible and preimages of trivialising neighbourhoods trivialise the pullback.

(h) $(f^*\mathcal{G})_p \cong \mathcal{G}_p \otimes_{\mathcal{O}_{Y,f(p)}} \mathcal{O}_{X,p}$ for any $\mathcal{G} \in \mathfrak{QCoh}(Y)$ and $p \in Y$.

(i) $f^*(\mathcal{G} \otimes_{\mathcal{O}_Y} \mathcal{G}') \cong f^*\mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{G}'$ for any $\mathcal{G}, \mathcal{G}' \in \mathfrak{QCoh}(Y)$.

Proof: See, e.g. [Vak12, §§17.2–3] or [Har77, Proposition II.5.8]. □

Observe that the adjoint property yields a natural map $\mathcal{G} \longrightarrow f_*f^*\mathcal{G}$ [Mac98, p. 81]. In other words, the sections of $\mathcal{G}(U)$ give rise to sections of $f^*\mathcal{G}(f^{-1}(U))$. In this context, we speak of *pulling back sections* and, by abuse of notation, given a section $s \in \mathcal{G}(U)$ we write $f^*(s)$ for the corresponding section of $f^*\mathcal{G}(f^{-1}(U))$.

Example 1.4.17: In particular, if $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, we can be very concrete. Any quasi-coherent sheaf $\mathcal{F}$ on X is of the form $\widetilde{M}$ for some A-module M. The induced morphism $B \longrightarrow A$ allows us to consider M as a B-module which we denote by ${}_BM$. We then have

$$f_*\widetilde{M} = \widetilde{{}_BM}.$$

Similarly, let N be any B-module. Then we have

$$f^*\widetilde{N} \cong \widetilde{N \otimes_B A}.$$

See, e.g. [Har77, Proposition II.5.2] or [Vak12, 17.2A and 17.3.2].

Note that the pulling back of coherent sheaves is geometrically a very natural notion. For example, given a map $f\colon X \longrightarrow \mathbb{P}^n$, by Proposition 1.4.16, $f^*\mathcal{O}_{\mathbb{P}^n}(1)$ is an invertible sheaf on X. This will be studied in section 2.1.3.

Example 1.4.18: Consider first the embedding $\iota\colon p =: Y \hookrightarrow X$ for some $p \in X$. Recall that $\iota^{-1}\mathcal{F} = \mathcal{F}_p$ for any sheaf $\mathcal{F}$ on X. Now let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then

$$\iota^*\mathcal{F} = \iota^{-1}\mathcal{F} \otimes_{\iota^{-1}\mathcal{O}_X} \mathcal{O}_Y = \mathcal{F}_p \otimes_{\mathcal{O}_{X,p}} \kappa(p) =: \mathcal{F}|_p,$$

as the structure sheaf on p is the constant sheaf $\kappa(p)$. We call this the *fibre of $\mathcal{F}$ over p*.

Now we replace p by some open subset $\iota\colon U \hookrightarrow X$. Then

$$\iota^*\mathcal{F} = \iota^{-1}\mathcal{F} \otimes_{\iota^{-1}\mathcal{O}_X} \mathcal{O}_U = \mathcal{F}|_U \otimes_{\mathcal{O}_U} \mathcal{O}_U = \mathcal{F}|_U$$

as ι is an open map. See also [Vak12, §17.3].

1.4.4 Global Constructions

If S is an R-algebra, $\operatorname{Spec} S$ is a $\operatorname{Spec} R$-scheme and Proposition 1.2.12 tells us that all *affine* $\operatorname{Spec} R$-schemes arise in this way. We want to "globalise" this to describe any affine *morphism*. The idea is to replace all rings by sheaves in the Spec and Proj constructions.

Given a quasi-coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-algebras, we may glue the $\operatorname{Spec} \mathcal{F}(U)$ for affine $U \subseteq X$ to obtain an X-scheme $\operatorname{Spec} \mathcal{F}$. This is sometimes called *global* Spec. Note that it comes with a canonical *affine* morphism $\operatorname{Spec} \mathcal{F} \longrightarrow X$ (as every $\mathcal{F}(U)$ is an $\mathcal{O}(U)$-algebra). Conversely, given any affine[25] morphism $f\colon Y \longrightarrow X$, we obtain a quasi-coherent sheaf $f_*\mathcal{O}_Y$ of $\mathcal{O}_X$-algebras and $Y \cong \operatorname{Spec} f_*\mathcal{O}_Y$. In other words, the functor $\mathcal{F} \longmapsto \operatorname{Spec} \mathcal{F}$ induces an anti-equivalence of the category of quasi-coherent $\mathcal{O}_X$-algebras and the category of affine X-schemes. The quasi-inverse is given by associating $f_*\mathcal{O}_Y$ to any affine morphism $f\colon Y \longrightarrow X$. See [GW10, Proposition 11.1 and Corollary 12.2] or [Vak12, §18.1].

In particular, locally, $\operatorname{Spec} \mathcal{F} \longrightarrow X$ is of the form $\operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ for some R-algebra S.

A similar construction works with Proj: given a graded quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{F} := \bigoplus \mathcal{F}_i$, we may glue the schemes $\operatorname{Proj} \mathcal{F}(U)$ (for all affine $U \subseteq X$) to obtain an X-scheme

[25] Note that affine implies quasi-compact and quasi-separated.

$\operatorname{Proj}\mathcal{F}$ that we call *global* Proj. The structure morphism $\pi\colon \operatorname{Proj}\mathcal{F} \longrightarrow X$ is always separated and under certain benign conditions also projective [Har77, Proposition II.7.10]. Unfortunately, in this case we are unable to state such nice universal property as we had for global Spec. See also [Vak12, Remark 18.2.6].

EXAMPLE 1.4.19: Observe that, by definition, for any affine scheme X, $\operatorname{Spec}\mathcal{O}_X = X$ while for any projective scheme X, $\operatorname{Proj}\mathcal{O}_X = X$.

Note that, in analogy to Definition 1.4.7, we obtain an exact $\widetilde{\cdot}$-functor from the category of graded quasi-coherent $\mathcal{F}$-modules into the category of quasi-coherent $\mathcal{O}_{\operatorname{Proj}\mathcal{F}}$-modules and may define the twisting operation in the same manner. In particular, we have

$$\mathcal{O}_{\operatorname{Proj}\mathcal{F}}(n) := \widetilde{\mathcal{F}(n)}.$$

Note that the results of section 1.4.2 may be globalised with little effort [GW10, Theorem 13.29]. In particular, $\mathcal{O}_{\operatorname{Proj}\mathcal{F}}(n)$ is again invertible for all n and we obtain $\mathcal{F}_0$-module homomorphisms

$$\alpha_n\colon \mathcal{F}_n \longrightarrow \pi_*\Big(\mathcal{O}_{\operatorname{Proj}\mathcal{F}}(n)\Big). \tag{1.3}$$

An important application of these global constructions is that they allow us to define the "correct" analogue of topological vector bundles in the world of schemes, cf. [GW10, §11.4].

Recall that there is a universal construction to turn an arbitrary module into a ring: for any R-module M, we define the *tensor algebra* $T(M)$ by setting

$$T_0(M) := R, \quad T_n(M) := M^{\otimes n}, \quad \textit{and} \quad T(M) := \bigoplus T_n(M).$$

$T(M)$ comes with a natural structure of a graded (non-commutative!) R-algebra by defining the multiplication

$$T_n(M) \times T_m(M) \ni (t_n, t_m) \longmapsto t_n \otimes t_m \in T_{n+m}(M).$$

We define the *symmetric algebra* $\operatorname{Sym}_R M$ as the quotient of $T(M)$ by the ideal generated by (homogeneous) elements of the form $a \otimes b - b \otimes a$ ($a \in T_n(M)$, $b \in T_m(M)$). Observe that this yields a (commutative) graded R-algebra and that any R-morphism from M into some commutative R-algebra factors uniquely through $\operatorname{Sym}_R M$. Conversely, $\operatorname{Sym}_R M$ is uniquely determined by this universal mapping property.

EXAMPLE 1.4.20: In particular, if $M = R^d$, $\operatorname{Sym}_R M$ is the polynomial ring in d variables.

The universal property implies that Sym commutes with tensor products, i.e. for any morphism $A \longrightarrow B$ there is an isomorphism

$$\operatorname{Sym}_A(M) \otimes_A B \xrightarrow{\sim} \operatorname{Sym}_B(M \otimes_A B). \tag{1.4}$$

Now we imitate this construction for sheaves. Let $\mathcal{F}$ be some sheaf of $\mathcal{O}_X$-modules. Then we define $\operatorname{Sym}(\mathcal{F})$ as the sheaf associated to the presheaf $U \longmapsto \operatorname{Sym}_{\mathcal{O}(U)}\mathcal{F}(U)$. This is a sheaf of graded $\mathcal{O}_X$-algebras and (1.4) shows that any quasi-coherent $\mathcal{F}$ gives rise to a quasi-coherent $\mathcal{O}_X$-algebra $\operatorname{Sym}(\mathcal{F})$ and for any $f\colon Y \longrightarrow X$ we have $f^*\operatorname{Sym}(\mathcal{F}) \cong \operatorname{Sym}(f^*\mathcal{F})$.

Moreover, if $f : X \longrightarrow Y$ and X is quasi-separated and quasi-compact (e.g. if X is Noetherian), $f_* \operatorname{Sym}(\mathcal{F})$ is a quasi-coherent $\mathcal{O}_Y$-algebra. See also [GW10, §11.1].

Given a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, we call $\mathbb{V}(\mathcal{F}) := \operatorname{Spec}\operatorname{Sym}(\mathcal{F})$ the *quasi-coherent bundle* defined by $\mathcal{F}$. Note that this association is functorial [GW10, §11.3]. As $\operatorname{Sym}(\mathcal{F})$ is a graded sheaf of $\mathcal{O}_X$-algebras, we may define $\mathbb{P}(\mathcal{F}) := \operatorname{Proj}\operatorname{Sym}(\mathcal{F})$, the *projective bundle* associated to $\mathcal{F}$.

Note that for any invertible sheaf $\mathcal{L}$ on X, $\mathbb{P}(\mathcal{F}) \cong \mathbb{P}(\mathcal{F} \otimes \mathcal{L})$ [GW10, Lemma 13.31]. In particular, we have $\mathbb{P}(\mathcal{L}) \cong \mathbb{P}(\mathcal{O}_X) = S$.

Example 1.4.21: Consider $X = \operatorname{Spec} k$ and let $\mathcal{F}$ be the sheaf associated to k^n. Then $\mathbb{V}(\mathcal{F}) = \mathbb{A}^n_k$ and $\mathbb{P}(\mathcal{F}) = \mathbb{P}^n_k$ by Example 1.4.20.

Observe that in this case, as $\operatorname{Sym}_1 \mathcal{F} = \mathcal{F}$, (1.3) specialises to

$$\alpha_1 : \mathcal{F} \longrightarrow \pi_*\Big(\mathcal{O}_{\mathbb{P}\mathcal{F}}(1)\Big)$$

and the adjoint property of $\cdot^*$ then gives us a natural morphism of $\mathcal{O}_{\mathbb{P}\mathcal{F}}$-modules

$$\alpha_1^\sharp : \pi^*\mathcal{F} \longrightarrow \mathcal{O}_{\mathbb{P}\mathcal{F}}(1).$$

1.4.5 Ample and Very Ample Sheaves

We end this section by introducing some special classes of invertible sheaves that will be very important to us later on. We follow [Har77, §II.5].

Recall that if S is a graded ring, we required S to be generated by S_1 as an S_0-algebra. Given a scheme $\operatorname{Proj} S$, this implies that we are able to construct the entire sheaf $\mathcal{O}$ given only the twisted sheaf $\mathcal{O}(1)$ (in fact, knowing the global sections of $\mathcal{O}(1)$ will suffice). This motivates the following definition.

Definition 1.4.22: Given any scheme Y, define the twisting sheaf $\mathcal{O}(1)$ on $\mathbb{P}^n_Y$ as $g^*(\mathcal{O}(1))$, where $g : \mathbb{P}^n_Y \longrightarrow \mathbb{P}^n_{\mathbb{Z}}$.

Now let X be any scheme over Y and $\mathcal{L}$ an invertible sheaf on X. We say that $\mathcal{L}$ is *very ample* relative to Y if there is an immersion $\iota : X \longrightarrow \mathbb{P}^n_Y$ for some n so that $\iota^*(\mathcal{O}(1)) \cong \mathcal{L}$.

Very ample sheaves will be an important tool in embedding varieties into projective spaces. For many purposes, it will be convenient, however, to replace this notion by an absolute one. Before we may do so, we must introduce some finiteness conditions.

Definition 1.4.23: Let X be a scheme, $\mathcal{F}$ an $\mathcal{O}$-module. Then we say that $\mathcal{F}$ is *globally generated* if a surjection $\bigoplus \mathcal{O} \twoheadrightarrow \mathcal{F}$ exists. $\mathcal{F}$ is *finitely globally generated* if a finite number of summands suffice.

Example 1.4.24: Let $\mathcal{F}$ be a quasi-coherent scheme on some affine scheme $\operatorname{Spec} R$. Then $\mathcal{F}$ is globally generated. Indeed, $\mathcal{F} = \widetilde{M}$ for some R-module M. Picking generators of M, we obtain a surjection $\bigoplus R \twoheadrightarrow M$ and this induces a morphism of sheaves.

The following important result is again due to Serre:

Remark 1.4.25 ([Har77, Theorem II.5.17]): Let X be a projective scheme over a Noetherian ring R, let $\mathcal{O}(1)$ be a very ample invertible sheaf on X and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then there exists an integer n_0 so that for all $n \geq n_0$, the sheaf $\mathcal{F}(n)$ can be generated by a finite number of global sections.

This yields a useful generalisation of "very ample".

Definition 1.4.26: Let X be a Noetherian scheme. We call an invertible sheaf $\mathcal{L}$ on X *ample* if, for every coherent $\mathcal{F}$ on X, we find some integer n_0 so that for every $n \geq n_0$ the sheaf $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by its global sections.

In particular, we see that any very ample sheaf is ample. The relationship, however, is more intricate.

Proposition 1.4.27 ([Har77, Proposition II.7.5]): Let $\mathcal{L}$ be an invertible sheaf on a Noetherian scheme X. Then the following conditions are equivalent:

(a) $\mathcal{L}$ is ample;

(b) $\mathcal{L}^{\otimes m}$ is ample for all $m > 0$; and

(c) $\mathcal{L}^{\otimes m}$ is ample for some $m > 0$.

1.5 Cohomology of Sheaves

Recall the global sections functor from Remark 1.1.4. Γ is additive and, by Remark 1.1.10 (a), preserves kernels, i.e. is left exact but in general not right exact.

Example 1.5.1: Let $X = \mathbb{P}^1_k$ for some algebraically closed k and let Y be the disjoint union of two closed points p, q of X. Then Y is a closed subscheme of X via the canonical embedding ι and—in analogy to (1.2)—we obtain an exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow \iota_* \mathcal{O}_Y \longrightarrow 0$$

for the ideal sheaf $\mathcal{I}$ of Y. Now, taking global sections, we observe that $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = k$ by Remark 1.2.28. But $\Gamma(\mathbb{P}^1, \iota_* \mathcal{O}_Y) = \Gamma(Y, \mathcal{O}_Y) = k^2$ as the sheaf $\iota_* \mathcal{O}_Y$ is zero outside of Y and $\mathcal{O}_Y(Y) = \mathcal{O}_Y(p) \times \mathcal{O}_Y(q) = k^2$. Hence the k-linear map

$$\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \longrightarrow \Gamma(Y, \mathcal{O}_Y)$$

is not surjective, cf. [Har77, Exercise II.1.21].

The idea of sheaf cohomology is to now take the tools offered by homological algebra to carefully examine to what extent a functor deviates from being exact.

Concretely, consider any exact sequence of sheaves

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0.$$

Then $\mathcal{F}/\mathcal{F}' \cong \mathcal{F}''$ as sheaves. Recall, however, that the quotient sheaf was the sheafification of the quotient presheaf. In a sense, the cohomology will measure to what extent the quotient presheaf deviates from the quotient sheaf.

This turns out to be an indispensable element in the study of the geometry of schemes.

1.5.1 Derived Functors

The approach we choose may be traced back to Alexander Grothendieck's famous "Tôhoku" paper, [Tôhoku, Chapitre II, §1–3]. A quick overview of the facts and results most important for algebraic geometry may be found in [Har77, §III.1–2]. A greater emphasis is placed on the relationship to "classical" topological (co)homology theories in [Wei95], while a rigorous discussion of the relevant algebraic results may be found in, e.g. [Lan02, Chapter XX].

First, we must introduce some terminology and recall some general results from homological algebra. In the following, let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories[26].

Definition 1.5.2: We call a sequence

$$\cdots \longrightarrow F^i \xrightarrow{d^i} F^{i+1} \xrightarrow{d^{i+1}} F^{i+2} \longrightarrow \cdots$$

a *complex* if $d^{i+1} \circ d^i = 0$ for every i. A complex is said to be *exact* if $\operatorname{im} d^i = \ker d^{i+1}$ for every i. We define the i-th *cohomology object* of the complex C as $H^i(C) := \ker d^i / \operatorname{im} d^{i+1}$.

Hence, we may picture the cohomology as measuring the "exactness" of a complex.

Remark 1.5.3: Note that the H^i are functorial and that any short exact sequence of complexes gives rise to a long exact sequence of cohomology groups. See for example [Lan02, §XX.2].

In order to carry these notions into the world of sheaves, it will be helpful to generalise the described situation.

Definition 1.5.4: We call F a (covariant cohomological) δ*-functor* from $\mathcal{A}$ to $\mathcal{B}$ if F consists of a family $\{F^n\}_{n\geq 0}$ of covariant additive[27] functors and, for every short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0 \tag{1.5}$$

in $\mathcal{A}$, a $\delta^n\colon F^n(M'') \longrightarrow F^{n+1}(M')$, naturally transforming (1.5) into a long exact sequence. More precisely: F must satisfy

($\Delta 1$) the long sequence

$$0 \longrightarrow F^0(M') \longrightarrow F^0(M) \longrightarrow F^0(M'') \xrightarrow{\delta^1} F^1(M') \longrightarrow \cdots \tag{1.6}$$

is exact; and

($\Delta 2$) for any two short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M'' & \longrightarrow & M & \longrightarrow & M' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N'' & \longrightarrow & N & \longrightarrow & N' & \longrightarrow & 0 \end{array}$$

[26] An introduction to abelian categories may be found, e.g., in [Tôhoku, Chapitre I] or [Vak12, §2.6].

[27] Recall that a functor $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ is *additive* if the induced $\operatorname{Hom}(A, B) \longrightarrow \operatorname{Hom}(F(A), F(B))$ is a group homomorphism for any objects A, B.

and every n, we require that the following diagram commutes:

$$\begin{array}{ccc} F^n(M'') & \xrightarrow{\delta^n} & F^{n+1}(M') \\ \downarrow & & \downarrow \\ F^n(N'') & \xrightarrow{\delta^n} & F^{n+1}(N'). \end{array}$$

To summarise, we may imagine F as being a functor from the category of short exact sequences in $\mathcal{A}$ into the category of long exact sequences in $\mathcal{B}$.

In complete analogy, we may of course define contravariant δ-functors. Note that in the contravariant case, all arrows in (1.6) must be reversed.

Remark 1.5.5: Observe that F^0 is always left exact.

Definition 1.5.6: We call a δ-functor *universal* if for any other δ-functor G and any natural transformation $f_0\colon F^0 \longrightarrow G^0$ there exist unique natural transformations $f_n\colon F^n \longrightarrow G^n$ extending f_0 that are compatible with the δ^n.

Remark 1.5.7: Given any two universal δ-functors F and G with $F^0 \cong G^0$, the universality yields that $F \cong G$.

We now introduce an important criterion for a δ-functor to be universal.

Definition 1.5.8: Let F be any additive functor. Then we call F *erasable*[28] if, given any object A in $\mathcal{A}$, there exists some monomorphism $u\colon A \hookrightarrow M$ satisfying $F(u) = 0$.

Proposition 1.5.9: Let F be any covariant δ-functor. If the F^n are erasable for $n > 0$ then F is universal.

Proof: [Lan02, Theorem XX.7.1]. □

Next, we study an example that forms the prime reason of our interest in δ-functors.

First, recall that an object A is called *injective* if any morphism $B \longrightarrow A$ may be extended via any monomorphism $B \hookrightarrow C$, i.e. for any B, C, the following diagram commutes:

$$\begin{array}{ccccc} 0 & \longrightarrow & B & \hookrightarrow & C \\ & & \downarrow & \swarrow_{\exists} & \\ & & A. & & \end{array}$$

The dual notion is that of a *projective object*. See, e.g., [Lan02, §XX.4] for an introduction to injective and projective objects.

In particular, if F is an erasable functor for which the M from Definition 1.5.8 can be chosen as being injective or projective, we say that F is *erasable by injectives or projecties*, respectively.

We say that a category has *enough injectives* (respectively, *enough projectives*), if any object may be embedded into an injective (respectively, surjects into some projective) object. The categories $\mathfrak{Ab}$, R-$\mathfrak{Mod}$, $\mathfrak{Ab}(X)$ and $\mathfrak{Mod}(X)$ all have enough injectives, cf. e.g. [Lan02, Theorem XX.4.1], [Vak12, §24.2.6] and [Har77, §III.2].

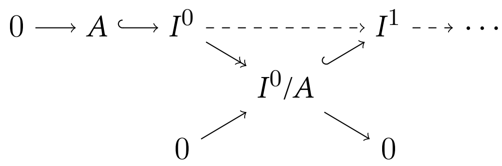
[28] Often this is also referred to as *effaceable*, but see the "linguistic note" in [Lan02, p. 800].

Definition 1.5.10: Let A be any object of $\mathcal{A}$. Then an *injective resolution* of A is an exact sequence

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

where all I^i are injective objects in $\mathcal{A}$.

Remark 1.5.11: Observe that we can construct an injective resolution for any object of a category that has enough injectives. Indeed, by assumption, given any A, we find an injective object $A \hookrightarrow I_0$. The same holds for the object I_0/A: we find an injective object $I_0/A \hookrightarrow I_1$. Inductively, this yields an injective resolution as may be observed in the following diagram.



Let C be a complex in $\mathcal{A}$. Given any functor $F\colon \mathcal{A} \longrightarrow \mathcal{B}$, $F(C)$ yields a complex in $\mathcal{B}$ as any functor of abelian categories respects the zero morphism. However, if C is an exact complex, $F(C)$ will, in general, not be exact. This motivates the following definition.

Definition 1.5.12: Let $\mathcal{A}$ have enough injectives and let $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ be a covariant left exact functor. For each object of A, we fix an injective resolution. Then we define the *right derived functors of F* via $R^iF(A) := H^i(F(I))$ where I is the chosen resolution of A.

Example 1.5.13: F is an exact functor if and only if $R^iF(A) = 0$ for every A and every $i > 0$, as the image of the injective resolutions remain exact in this case.

Note that the derived functors are independent of the choice of resolution: given a second injective resolution I', we have $H^i(F(I)) \cong H^i(F(I'))$ for every i and the isomorphism is in fact unique [Lan02, p. 791].

Proposition 1.5.14: Let $\mathcal{A}$ be a category with enough injectives and let F be any covariant left exact functor. Then the derived functors R^iF form a universal δ-functor with $F \cong R^0F$. Conversely, if G^i is a universal δ-functor then G^0 is left exact and $G^i \cong R^iG^0$ for all i.

Proof: The fact that the derived functors are a δ-functor may be found in, e.g., [Lan02, Theorem XX.6.1]. The main ingredient is showing that $R^nF(I) = 0$ for any F and any injective I and any $n > 0$. Then Proposition 1.5.9 implies that the derived functor is a universal δ-functor. The second part follows immediately from the definition of universal δ-functors. □

In particular, any derivation of a left exact functor transforms short exact sequences into long exact sequences.

Finally, we are able to define sheaf cohomology.

Definition 1.5.15: Consider the global sections functor $\Gamma(X, -)\colon \mathfrak{Ab}(X) \longrightarrow \mathfrak{Ab}$. We set $\mathrm{H}^i(X, -) := R^i\Gamma(X, -)$ and call these the *cohomology functors*. For any sheaf $\mathcal{F} \in \mathfrak{Ab}(X)$, we call $\mathrm{H}^i(X, \mathcal{F})$ the i-th *cohomology group* of $\mathcal{F}$.

In practice, however, injective resolutions are no fun to calculate. Hence, we finish this excursion into homological algebra by briefly describing sheaves that are sometimes easier to calculate than injective sheaves but work just as well for describing cohomology.

Definition 1.5.16: Let $\mathcal{F}$ be a sheaf. Then we call $\mathcal{F}$ *flasque* if all restriction morphisms are surjective.

In other words, if $\mathcal{F}$ is flasque any section of $\mathcal{F}(U)$ may be extended to *any* larger $U' \supseteq U$.

Flasque sheaves have a series of convenient properties. For example, if $\mathcal{F}$ is any flasque sheaf, $\mathrm{H}^i(X, \mathcal{F}) = 0$ for $i > 0$ [Har77, Proposition III.2.5]. In general, given any covariant left exact functor F, we call an object A *(F-)acyclic* if $R^iF(A) = 0$ for all $i > 0$.

Proposition 1.5.17: Let F be a left exact covariant functor and A an object with an *acyclic resolution* B, i.e. there exist objects B^i so that

$$0 \longrightarrow A \longrightarrow B^1 \longrightarrow B^2 \longrightarrow \cdots$$

is exact and the B^i are acyclic. Then there exist unique isomorphisms $H^i(F(B)) \cong R^iF(A)$.

Proof: Again, this is an immediate consequence of Proposition 1.5.9. See, e.g., [Wei95, p. 50] or [Lan02, Theorem XX.6.2]. □

Remark 1.5.18: Note that if X is a Noetherian scheme, any quasi-coherent sheaf on X may be embedded into a flasque sheaf [Har77, Corollary III.3.6]. Just as in Remark 1.5.11, we may construct a resolution of flasque sheaves for any sheaf and, by Proposition 1.5.17, we thus obtain cohomology groups of quasi-coherent sheaves.

We conclude by briefly mentioning that one may analogously define *left derivatives* of contravariant functors using projective resolutions. This approach is taken, e.g. by [Wei95, §2.4].

1.5.2 Invariants

Having spent the previous section tediously applying the tools of homological algebra to the world of sheaves, it is now time to reap the harvest of our expedition. Cohomology groups will allow us to define a series of invariants that will be indispensable in the following discussion.

To begin, we must briefly collect some fundamental results about the finiteness and vanishing of certain cohomology groups. In general, we should assume that our schemes are Noetherian for these results to hold, but as the geometric situations we shall be concerned with satisfy this property anyway, this turns out to be no real restriction for us.

Proposition 1.5.19: Let $X = \operatorname{Spec} R$ Noetherian. Then $\mathrm{H}^i(X, \mathcal{F}) = 0$ for any $i > 0$ and any quasi-coherent sheaf $\mathcal{F}$ on X.

Proof: Essentially, once we have Remark 1.5.18, this is a consequence of Remark 1.4.4, as the global sections functor is exact in this setting. □

Remark 1.5.20: In fact, these notions are equivalent. It can be shown ([Har77, Theorem III.3.7]) that for any Noetherian scheme X, the following statements are equivalent.

(a) X is affine;

(b) $\mathrm{H}^i(X, \mathcal{F}) = 0$ for all quasi-coherent $\mathcal{F}$ and all $i > 0$; and

(c) $\mathrm{H}^1(X, \mathcal{I}) = 0$ for all coherent sheaves of ideals $\mathcal{I}$.

Remark 1.5.21: In the projective case, the situation is also well understood. Let $X = \mathbb{P}^n_R$ for some Noetherian ring R and $d \geq 1$. By Proposition 1.4.13, we have $\mathrm{H}^0(X, \mathcal{O}_X(m)) = R[x_0, \ldots, x_n]_m$ for $m \geq 0$. By counting generators, we see that this is a free R-module of rank $\binom{n+m}{n}$ (we choose m of n variables without caring for the order). Also, $\mathrm{H}^n(X, \mathcal{O}_X(m))$ is a free R-module of rank $\binom{-m-1}{-n-m-1}$ if $m \leq -n-1$ and $\mathrm{H}^i(X, \mathcal{O}_X(m)) = 0$ in all other cases [Vak12, Theorem 19.1.2].

Remark 1.5.22: Note that if $\mathcal{F}$ is a coherent sheaf on X and $\iota\colon X \hookrightarrow Y$ is a closed immersion of separable Noetherian schemes, we have $\mathrm{H}^i(X, \mathcal{F}) \cong \mathrm{H}^i(Y, \iota_*\mathcal{F})$. See [Vak12, 19.2.E] or [Stacks, Lemma 25.2.4 (089W)] in combination with Remark 1.4.6.

Finally, there are two fundamental vanishing theorems, which we cite here.

Proposition 1.5.23: Let $X = \operatorname{Proj} S$ for a Noetherian graded ring S. Then:

(a) for all coherent sheaves $\mathcal{F}$, all $\mathrm{H}^i(X, \mathcal{F})$ are finitely generated S_0-modules; and

(b) for every coherent $\mathcal{F}$ there exists an n_0 so that for $n \geq n_0$, $\mathrm{H}^i(X, \mathcal{F}(n)) = 0$ for all $i > 0$.

Proof: Originally, this was proven in [Ser55, §66]. See also [Har77, Theorem III.5.2]. □

Proposition 1.5.24 (Grothendieck's Vanishing Theorem): Let X be a Noetherian topological space of dimension n. Then $\mathrm{H}^i(X, \mathcal{F}) = 0$ for any $\mathcal{F} \in \mathfrak{Ab}(X)$ and all $i > n$.

Proof: Originally, this was proven in [Tôhoku, Théorème 3.6.5]. See also [Har77, Theorem III.2.7]. □

This allows us to define several invariants well-known from topology (compare this *definition* to the *theorem* [Hat01, Theorem 2.44]).

Definition 1.5.25: Let X be a Noetherian projective scheme over a field k and $\mathcal{F}$ a coherent sheaf on X. Then we define the *Euler characteristic* $\chi(\mathcal{F})$ as

$$\chi(\mathcal{F}) \coloneqq \sum_{i \geq 0} (-1)^i \dim_k \mathrm{H}^i(X, \mathcal{F}).$$

Note that this sum is finite by Proposition 1.5.24 and Proposition 1.5.23. Additionally, we define the *arithmetic genus* $g_a(X)$ via

$$g_a(X) \coloneqq (-1)^{\dim X}(\chi(\mathcal{O}_X) - 1).$$

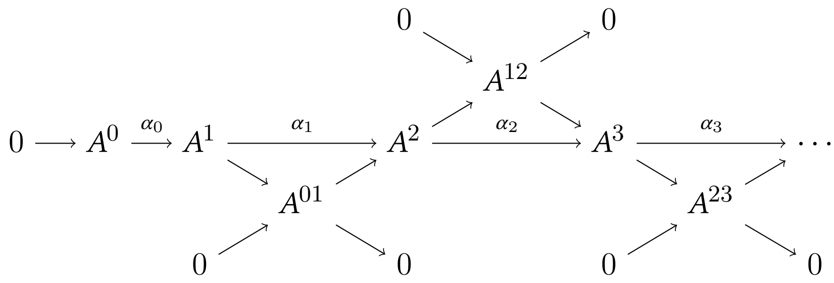
Remark 1.5.26: Note that the Euler characteristic is additive on short exact sequences: Let

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0 \tag{1.7}$$

be an exact sequence of coherent sheaves. Then we have $\chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'')$. Indeed, given any short exact sequence of vector spaces,

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

the dimension is clearly additive, i.e. $\dim A - \dim B + \dim C = 0$. If we have a longer sequence, we may factor it into short exact sequences:



where $A^{ij} := \operatorname{im} \alpha_i = \ker \alpha_j$ as the sequence is exact. This shows that

$$\dim A^0 - \dim A^1 + \dim A^2 - \dim A^3 + \dim A^{23} = 0$$

and, inductively, in particular that for any *finite* exact sequence, the dimension is again additive, i.e. $\sum(-1)^i \dim A_i = 0$. Applying this to the cohomology sequence of (1.7),

$$0 \longrightarrow \mathrm{H}^0(X, \mathcal{F}') \longrightarrow \mathrm{H}^0(X, \mathcal{F}) \longrightarrow \cdots \longrightarrow \mathrm{H}^n(X, \mathcal{F}'') \longrightarrow 0,$$

we see that the Euler characteristic is additive:

$$\begin{aligned} 0 &= \sum(-1)^i \dim \mathrm{H}^i(X, \mathcal{F}) + \sum(-1)^i \dim \mathrm{H}^i(X, \mathcal{F}') + \sum(-1)^i \dim \mathrm{H}^i(X, \mathcal{F}'') \\ &= \sum(-1)^i \Big(\dim \mathrm{H}^i(X, \mathcal{F}') - \dim \mathrm{H}^i(X, \mathcal{F}) + \dim \mathrm{H}^i(X, \mathcal{F}'')\Big) = 0. \end{aligned}$$

2 Algebraic Curves

Having dutifully reviewed the basic concepts of Algebraic Geometry, we will now focus our attention on the case of curves. In particular, we will mostly limit this discussion to a "purely geometric" setting, i.e. everything should happen over an *algebraically closed* field k of characteristic 0.

However, as our ultimate aim is to study families of curves and how they vary, it will often be worthwhile to take a slightly more general view on things. This will permit us to eventually apply some of these techniques to moduli spaces.

Definition 2.0.1: A *variety* is a Noetherian integral separated scheme of finite type over an algebraically closed field k. We say a variety is *complete* if it is also proper over k.

A *curve* is a complete variety of dimension one. We call a curve *regular* or *non-singular* if all local rings are regular. In particular, this implies that a regular curve is always projective (cf. e.g. [Har77, Proposition II.6.7]).

We shall denote the structure sheaf by $\mathcal{O} := \mathcal{O}_X$ and the (constant sheaf of the) field of rational functions by $k(X)$.

Remark 2.0.2: Recall that the category of non-singular curves with *surjective* k-morphisms is anti-equivalent to the category of finitely generated fields of transcendence degree 1 over k, see [Vak12, Theorem 18.4.3] or [Har77, Corollary I.6.12]. Note also that for any morphism $\varphi\colon X \longrightarrow Y$ of curves, $\varphi(X)$ is either a point or surjective. In the second case, the induced $k(X) \subseteq k(Y)$ is a finite *algebraic* field extension and φ is a finite morphism [Har77, Proposition II.6.8].

This allows us to define the *degree of* φ as $\deg\varphi := [k(X) : k(Y)]$, the degree of the field extension.

In particular, this implies that if X is any curve and $f \in k(X)$ is a rational function, f induces a morphism $\varphi\colon X \longrightarrow \mathbb{P}^1$. If f is constant, this is clear. If $f \notin k$, $k[f]$ is of transcendence degree 1 over k as f cannot be algebraic but $k[f] \subseteq k(X)$. Therefore, $k(f) \cong k(\mathbb{P}^1)$ and the inclusion into $k(X)$ induces a morphism $X \longrightarrow \mathbb{P}^1$, cf. e.g. [LAGII, §9.3].

2.1 Linear Systems and Projective Morphisms

Our first aim is to study maps of a curve X into a projective space $\mathbb{P}^n$. Embeddings are of course of special interest to us, as they allow us to regard our curve as a projective subscheme, paving the way for the application of many different techniques. We will see that in many cases, the canonical sheaf ω_X will give rise to such an embedding, and—incidentally—this will be the most important for our purposes.

2.1.1 Divisors and Invertible Sheaves

Let us review some facts about divisors and their relationship to invertible sheaves. A reference for this is, e.g., [Har77, §II.6], [LAGII, §9.4] or [Vak12, Chapter 15]. We start with the simplest case:

Definition 2.1.1: Let X be a variety that is *regular in codimension one*, i.e. every local ring $\mathcal{O}_x$ of dimension one is regular. Then we call a closed integral subscheme of codimension one a *prime divisor on* X. A *Weil Divisor* is then defined to be an element of the free abelian group generated by the prime divisors. We will denote this group by $\mathfrak{Div}(X)$.

For $\mathfrak{Div}(X) \ni D =: \sum n_Y Y$ we call the set of closed subschemes Y so that $n_Y \neq 0$ the *support of* D, in symbols: $\operatorname{supp} D$.

This yields a natural group homomorphism $\deg\colon \mathfrak{Div}(X) \longrightarrow \mathbb{Z}$ via $\sum n_Y Y \longmapsto \sum n_Y$ we call the *degree*.

Example 2.1.2: Let X be a curve. Then X is regular in codimension one iff it is regular and in this case the prime divisors are exactly the closed points $P \in X$. A (Weil) divisor may thus be considered as a (finite) linear combination of points.

If $X = \operatorname{Spec} R$ is any affine variety that is regular in codimension 1, the prime divisors correspond to the prime ideals $\mathfrak{p}$ of height 1 in R. Note, in particular, that $R_\mathfrak{p}$ is required to be a regular local ring and that $\mathfrak{p}$ is the (unique) generic point of the closed subscheme $Y := \overline{\mathfrak{p}}$ of codimension 1.

Remark 2.1.3: Note that we obtain a partial ordering on $\mathfrak{Div}(X)$: let $D := \sum n_Y Y$ and $D' := \sum m_Y Y$ be divisors. Then we say that $D \geq D'$ iff $n_Y \geq m_Y$ for every prime divisor Y of X. In the case that D' is the *zero divisor* (i.e. $m_Y = 0$ for all Y), we call D an *effective divisor*.

Definition 2.1.4: For any prime divisor Y on X, let $\eta \in Y$ be its generic point. As Y is a prime divisor, $\mathcal{O}_\eta$ will be a discrete valuation ring and as X is integral, its quotient field is equal to $k(X)$, the function field of X. Hence we obtain a discrete valuation v_Y on $k(X)$ which we call the *valuation of* Y.

Let f be in $k(X)^\times$. Then, in accordance with Definition 1.2.3 (a), we say that f has a *zero at* Y if $v_Y(f) > 0$ (i.e. f lies in the maximal ideal $\mathfrak{m}_\eta$) and that it has a *pole at* Y if $v_Y < 0$. If $v_Y = 0$, we say that f is *invertible at* Y.

We call $v_Y(f)$ the *order of* f *at* Y and write $\operatorname{ord}_Y f := v_Y(f)$.

For any variety X that is regular in codimension one, any non-zero regular function $f \in k(X)^\times$ is invertible at all but finitely many prime divisors of X [Har77, Lemma II.6.1], giving rise to the following definition.

Definition 2.1.5: Let f be a non-zero rational function on X. Then we define the *divisor of f* by

$$\operatorname{div} f := \sum v_Y(f) \cdot Y,$$

where the sum is taken over all prime divisors Y of X.

In fact, by the property of valuations, it is evident that the map

$$\operatorname{div}\colon k(X)^\times \longrightarrow \mathfrak{Div}(X)$$

is actually a homomorphism of groups. The image of this map forms the (normal) subgroup $\mathfrak{Prin}(X)$ of *principal divisors* and any divisor $D \in \mathfrak{Div}(X)$ that lies in this subgroup may be referred to as a *principal divisor.* The quotient is called the *divisor class group of X*, in symbols:

$$\mathfrak{Cl}(X) := \mathfrak{Div}(X)\big/\mathfrak{Prin}(X)\,.$$

Consequently, if $D - D' \in \mathfrak{Prin}(X)$ for $D, D' \in \mathfrak{Div}(X)$, we write $D \sim D'$ and say that D and D' are *linearly equivalent.*

Example 2.1.6: Consider $X = \mathbb{P}^1_k$. Then $\mathfrak{Cl}(X) \cong \mathbb{Z}$. Indeed, let $\mathbb{P}^1_k = \operatorname{Proj} k[x_0, x_1]$ and $P_0 := (x_0)$ and let Q be any (closed) point $(ax_0 + bx_1)$. Then

$$f := \frac{x_0}{ax_0 + bx_1}$$

is a rational function with $\operatorname{div} f = P - Q$, i.e. $P \sim Q$ in $\mathfrak{Div}\,\mathbb{P}^1_k$.

In fact, the same argument works for arbitrary dimensions, i.e. $\mathfrak{Cl}(\mathbb{P}^n_k) \cong \mathbb{Z}$ for all n. Moreover, this isomorphism is given by the degree map, see [Har77, Proposition II.6.4].

Remark 2.1.7: Consider the case of non-singular curves, $f\colon X \longrightarrow Y$. In the case that f is a finite morphism, it gives rise to a morphism $f^*\colon \mathfrak{Div}(Y) \longrightarrow \mathfrak{Div}(X)$. Indeed, as Y is a non-singular curve, for any closed point $Q \in Y$, the corresponding maximal ideal $\mathfrak{m}_Q =: (t_Q)$ is a principal ideal. This allows us to define

$$f^*Q := \sum_{P \in f^{-1}(Q)} v_p(t_Q \circ f)P,$$

a divisor on X (note that as f is finite, the sum is finite). As any other generator t'_Q differs only by a unit from t_Q, this is well-defined. By linear extension, we obtain a morphism f^* as claimed. Note that the morphism of the function fields induced by f ensures that f^* respects linear equivalence. This allows us to consider the morphism of class groups:

$$f^*\colon\ \mathfrak{Cl}(Y) \longrightarrow \mathfrak{Cl}(X).$$

What happens to the degree of a divisor under the map f^*? Counting preimages weighted by the valuation of $t_Q \circ f$ (cf. e.g. [Har77, Proposition II.6.9]) yields

$$\deg f^*D = \deg f \cdot \deg D.$$

In particular, as any non-constant rational function on a curve is a finite morphism onto $\mathbb{P}^1$, *any principal divisor on a non-singular curve is of degree* 0 by Example 2.1.6 (cf. [Har77, Corollary II.6.10]). As a consequence, for any non-singular curve the degree function descends onto the class groups.

To deal with more general schemes, we must generalise our concept of divisor:

Definition 2.1.8: Let X be a Noetherian integral[29] scheme. Then we define the *group of Cartier Divisors* to be

$$\mathfrak{CaDiv}(X) := \mathrm{H}^0\left(X, {k(X)^\times}/{\mathcal{O}_X^\times}\right)$$

where $k(X)^\times$ denotes the constant sheaf of invertible rational functions. Hence we may consider a Cartier divisor to be a family (f_i, U_i) where U_i is an open cover of X and $f_i \in k(X)^\times$ subject to the condition that $\frac{f_i}{f_j} \in \mathcal{O}_X(U_i \cap U_j)^\times$.

Correspondingly, we call $D \in \mathfrak{CaDiv}(X)$ a *principal divisor*, if it is already an element of $\mathrm{H}^0(X, k(X)^\times)$, i.e. all the f_i come from a single global $f \in k(X)^\times$.

The factor group is called the *Cartier class group* and denoted by

$$\mathfrak{CaCl}(X) := {\mathfrak{CaDiv}(X)}/{\mathrm{H}^0(X, k(X)^\times)} .$$

Remark 2.1.9: In a sense, the group $\mathfrak{CaCl}(X)$ describes the effect of sheafification on the quotient sheaf $k(X)^\times/\mathcal{O}_X^\times$, i.e. the difference between the global sections of the quotient sheaf and the quotient of the global sections of the sheaves. We can make this precise using cohomology groups: Consider the short exact sequence of sheaves on X

$$1 \longrightarrow \mathcal{O}^\times \longrightarrow k(X)^\times \longrightarrow {k(X)^\times}/{\mathcal{O}^\times} \longrightarrow 1.$$

Taking cohomology gives rise to a long exact sequence

$$0 \longrightarrow \mathrm{H}^0(X, \mathcal{O}^\times) \longrightarrow \mathrm{H}^0(X, k(X)^\times) \xrightarrow{\gamma} \mathrm{H}^0(X, {k(X)^\times}/{\mathcal{O}^\times}) \xrightarrow{\delta}$$
$$\xrightarrow{\delta} \mathrm{H}^1(X, \mathcal{O}^\times) \longrightarrow \mathrm{H}^1(X, k(X)^\times) \longrightarrow \cdots$$

where we have $H^0(X, k(X)^\times/\mathcal{O}^\times) = \mathfrak{CaDiv}(X)$. Consider the group $\mathrm{H}^1(X, \mathcal{O}^\times)$. As our sequence is exact, we have $\ker\delta = \operatorname{im}\gamma$. But this is just the group of principal divisors. Therefore, we obtain an *injective* map $\mathfrak{CaCl}(X) \hookrightarrow \mathrm{H}^1(X, \mathcal{O}_X^\times)$.

When is this map also surjective? If X is an integral scheme, surely[30] any constant sheaf is a flasque sheaf and hence $\mathrm{H}^1(X, k(X)^\times) = 0$. Consequently, in this case we have

$$\mathfrak{CaCl}(X) \cong \mathrm{H}^1(X, \mathcal{O}_X^\times).$$

Closely related to these is another group:

Definition 2.1.10: Let X be a Noetherian scheme. Recall that by Definition 2.1.10, we call an $\mathcal{O}_X$-module sheaf $\mathcal{L}$ on X *invertible* if it is locally free of rank one, i.e. there exists an open covering U_i of X so that $\mathcal{L}|_{U_i} \cong \mathcal{O}_{U_i}$ for all i. We denote the set of isomorphism classes of invertible sheaves on X by $\mathfrak{Pic}(X)$.

[29] It is also possible to define Cartier Divisors on more general schemes, however, then we may no longer talk of the function field of X. Instead, we must work with the sheaf of total quotient rings.

[30] Indeed, in this case X—and therefore any open subset—is irreducible and thus connected. But that means that $k(X)^\times(U) = k(X)^\times$ for any open $U \subseteq X$ and all restriction maps are in fact the identity map.

Definition 2.1.11: Let X be a Noetherian scheme and $\mathcal{L}$ an $\mathcal{O}_X$-module sheaf. Then we call any section of $\mathcal{L}$ a *regular section* or *regular function* of $\mathcal{L}$, while we refer to elements of $\mathcal{L} \otimes k(X)$ as *rational sections* or *rational functions* of $\mathcal{L}$. In other words, for any irreducible component Y of X, the rational sections are the elements of $\mathcal{L}_\eta$ where η is the generic point of Y.

Now let $\mathcal{L}$ be invertible and choose a trivialising open cover $(U_i)_i$ of X, i.e. $\mathcal{L}|_{U_i}$ is generated by some x_i. Note that, as X is Noetherian, it is quasi-compact and hence we may choose our cover to be finite. Therefore, any regular (respectively rational) section of $\mathcal{L}$ is locally of the form $f_i x_i$ for some regular (respectively rational) f_i. This allows us to apply the construction of div from Definition 2.1.5 to this situation, as the finiteness condition assures that the sum in the divisor will be finite. Observe that this is well-defined as different local generators of $\mathcal{L}$ differ by some unit $s \in \mathcal{O}_X(U_i)^\times$ and $\operatorname{div} s = 0$ for any such s.

Now consider a non-zero rational section f of $\mathcal{L}$ and the corresponding divisor $\operatorname{div} f$. Then $\operatorname{supp} \operatorname{div} f$ is a finite (and therefore closed) union of irreducible subschemes of codimension one. Call the complement U. Then $f|_U$ is a regular section and U is dense in X. Indeed, f is invertible at every point outside of $\operatorname{supp} \operatorname{div} f$, in particular it is therefore regular on U. As U lies dense in every irreducible component, it is dense in all of X. Therefore, we may consider any rational section as consisting of a dense open subset U and a section f that is regular on U.

Before we move on, we note a technical detail:

Remark 2.1.12: Recall that for any two sheaves of $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$ we may form the tensor product $\mathcal{F} \otimes \mathcal{G} := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$, cf. Remark 1.1.10 (b). In particular, this implies that we may obtain the stalks of the tensor sheaf just by tensoring the corresponding modules over the corresponding local ring[31]. Thereby it is clear that the tensor operation gives a monoid structure to $\mathfrak{Pic}(X)$ (the neutral element being of course the sheaf $\mathcal{O}_X$ itself). We claim that $(\mathfrak{Pic}(X), \otimes)$ is actually a group.

To see this, recall that the sheaf $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ was defined as the sheafification of the presheaf $U \longmapsto \operatorname{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ and consider the sheaf $\mathcal{L}^{-1} := \mathcal{H}om(\mathcal{L}, \mathcal{O}_X)$, the *dual sheaf* to $\mathcal{L}$. Then $\mathcal{L} \otimes \mathcal{L}^{-1} \cong \mathcal{O}_X$, i.e. for any $\mathcal{L}$ we find an inverse element $\mathcal{L}^{-1}$. Indeed, on the stalks this is little more than linear algebra; especially, it is clear that $\mathcal{L}^{-1}$ is again an invertible sheaf. The interesting part is that these isomorphisms patch together to give a "global" isomorphism of sheaves. Locally we may assume that $\mathcal{O}(U) = R$ for some ring R and some open subset U and $\mathcal{L}(U) = l \cdot R$ for some generator l. Then any R-morphism of $l \cdot R$ into R is determined solely by the image of $l \cdot 1$ and hence[32] we see that $\operatorname{Hom}_{\mathcal{O}|_U}(\mathcal{L}|_U, \mathcal{O}|_U) \cong \operatorname{Hom}_R(l \cdot R, R)$ via the tilde functor. Now consider the bilinear map

$$\begin{aligned} \varphi_U \colon \mathcal{L}(U) \otimes \operatorname{Hom}_R(\mathcal{L}(U), \mathcal{O}(U)) &\longrightarrow \mathcal{O}(U) \\ s \otimes \psi &\longmapsto \psi(s). \end{aligned}$$

[31] Indeed, tensor products commute with limits, as the tensor functor is a left adjoint [Lan02, §XVI.2] and left adjoint functors preserve colimits [Mac98, §V.5].

[32] Note that $\mathcal{L}|_U \cong \widetilde{l \cdot R}$ and $\mathcal{O}|_U \cong \widetilde{R}$ and that the tilde functor is compatible with the tensor operation.

In fact, φ_U is an isomorphism. Indeed, φ_U is surjective: given an arbitrary $r \in \mathcal{O}(U)$, we need only to consider $s = l$ and we will find a linear $\psi\colon \mathcal{L}(U) \longrightarrow \mathcal{O}(U)$ sending l to r. To see that φ_U is injective consider[33] $s \otimes \psi$ so that $\varphi_U(s \otimes \psi) = 0$. Then we must have $\psi(s) = 0$. But $s = r \cdot l$ for some $r \in \mathcal{O}(U)$, yielding $\psi(rl) = r\psi(l) = 0$ and—as l generates $\mathcal{L}(U)$—this shows that $r\psi$ is the zero map. Now the bilinearity implies that

$$s \otimes \psi = rl \otimes \psi = l \otimes r\psi = l \otimes 0 = 0$$

and hence φ_U is injective.

But as φ_U does nothing more than evaluating ψ, it is actually a map of sheaves $\mathcal{O} \otimes \mathcal{H}om(\mathcal{L}, \mathcal{O}) \longrightarrow \mathcal{O}$ and by what we have just seen, it is in fact an isomorphism. Therefore $\mathcal{L}^{-1}$ is indeed the inverse to $\mathcal{L}$ (up to isomorphism) and $\mathfrak{Pic}(X)$ is in fact a group.

In the "classical case", it turns out that these three groups are isomorphic and we may therefore switch freely between the various points of view. This result, while being well-documented in the literature, e.g. [Har77, §II.6] or [LAGII, §9.4], is usually obtained rather implicitly. However, as we will later want to explicitly transform sheaves into divisors and vice-versa, we must go into some more detail.

Theorem 1: *Let X be a variety that is* locally factorial*, i.e. all local rings are UFDs, and denote by η its generic point. Then*[34] *we obtain isomorphisms*

$$\mathfrak{Cl}(X) \cong \mathfrak{CaCl}(X) \cong \mathfrak{Pic}(X).$$

More concretely, these maps are given by

$$\begin{aligned} \mathfrak{Cl}(X) &\xrightarrow{\sim} \mathfrak{CaCl}(X) \\ D &\longmapsto (U_x, f_x)_{x \in X} \\ \sum v_x(f_i)x &\longleftarrow\!\shortmid (U_i, f_i)_i \end{aligned}$$

where, for every prime divisor[35] *$x \in X$, $\operatorname{div} f_x$ is the principal*[36] *divisor corresponding to the restriction of D to $\operatorname{Spec} \mathcal{O}_x$ and U_x is an open neighbourhood of x on which D coincides with $\operatorname{div} f_x$ and, for the inverse map, choose*[37] *some f_i so that $U_i \cap \overline{x} \neq \emptyset$;*

$$\begin{aligned} \mathfrak{CaCl}(X) &\xrightarrow{\sim} \mathfrak{Pic}(X) \\ (U_i, f_i)_i &\longmapsto (f_i^{-1} \cdot \mathcal{O}|_{U_i})_i \\ (U_i, l_i^{-1})_i &\longleftarrow\!\shortmid (l_i \cdot \mathcal{O}|_{U_i})_i \end{aligned}$$

[33] Note that this is in fact an arbitrary element, as both modules are of rank 1.

[34] As a matter of fact, the condition locally factorial is only required for the first isomorphism. In particular, the groups $\mathfrak{CaCl}(X)$ and $\mathfrak{Pic}(X)$ are isomorphic for any integral scheme X.

[35] Any prime divisor is irreducible, hence contains a unique generic point, x.

[36] Recall that as $\mathcal{O}_x$ is a UFD, any prime ideal of height 1 is a principal ideal. But every prime divisor corresponds to a prime ideal of height 1, the restriction to $\mathcal{O}_x$ is consequently the principal divisor belonging to the generator. Hence every prime—and thereby any—divisor is principal on some open neighbourhood of x.

[37] Note that this is well-defined as $\overline{x}$ is irreducible and choosing some other f_j implies that $f_i f_j^{-1}$ is invertible, as $(U_i, f_i)_i$ is a Cartier divisor.

where we assume both the Cartier divisor as well as the invertible sheaf to consist of an open covering $(U_i)_i$ *of* X *and elements of* $k(X)/\mathcal{O}(U_i)^\times$ *and local generators of* $\mathcal{O}|_{U_i}$*, respectively; and*

$$\begin{aligned} \mathfrak{Cl}(X) &\xrightarrow{\sim} \mathfrak{Pic}(X) \\ D &\longmapsto \mathcal{O}(D) \\ \operatorname{div} s &\longleftarrow\!\shortmid \mathcal{L} \end{aligned}$$

where $\mathcal{O}(D)(U) := \{f \in k(X) \mid \operatorname{div} f + D|_U \geq 0\} \cup \{0\}$ *for any open subset* $U \subseteq X$ *and* s *is an* $\mathcal{O}_\eta$*-generator of* $\mathcal{L}_\eta$*.*

Proof: The first isomorphism is described in detail in [Har77, Proposition II.6.11]. At this point it should be clear that the two maps are inverse group homomorphisms; a number of checks are, however, still required to see that these maps are in fact well-defined.

Consider the second isomorphism. Note that for any Cartier Divisor $(U_i, f_i)_i$ the $\mathcal{O}_X$-module $(f_i^{-1} \cdot \mathcal{O}_{U_i})_i$ is well-defined and invertible. Indeed, it is obviously locally free of rank one and as—by definition of Cartier Divisor—$f_i f_j^{-1} \in \mathcal{O}(U_i \cap U_j)^\times$, the modules $f_i^{-1} \cdot \mathcal{O}_{U_i}$ and $f_j^{-1} \cdot \mathcal{O}_{U_j}$ coincide on the intersection and hence glue together to form a sheaf.

Now, as X is integral, we may consider any invertible sheaf as a subsheaf of the constant sheaf $k(X)$ (this is done simply by tensoring with $k(X)$, cf. [Har77, Proposition II.6.15]). Therefore, we may consider any invertible sheaf as an open cover $(U_i)_i$ of X together with elements $f_i \in k(X)$ that generate the (free) $\mathcal{O}_{U_i}$-module. But this yields a Cartier Divisor as both f_i^{-1} and f_j^{-1} generate the $\mathcal{O}_{U_i \cap U_j}$-module and hence $f_i f_j^{-1}$ is invertible on the intersection. This shows that we have a bijection of Cartier Divisors and invertible sheaves. Indeed, it also respects the group structures and linear equivalence[38] and is hence an isomorphism of groups.

The third isomorphism is obtained simply by composition of the first two. Let Y be any prime (Weil) divisor. What will the image of its class in $\mathfrak{Pic}(X)$ be? Well, for any $x \in X$ the restriction of Y to x is a principal ideal, as X is assumed locally factorial. More explicitly, if $x \notin Y$, the restriction is the zero divisor, i.e. locally $\operatorname{div} 1$; if $x \in Y$, Y restricts to a minimal, i.e. height 1, prime ideal (f) in $\mathcal{O}_x$, which is principal by factoriality. Hence, in this case, Y is locally $\operatorname{div} f$. This gives rise to an invertible sheaf that is generated by f^{-1} on an open neighbourhood of Y and by a unit on every neighbourhood not containing Y. Indeed, as f is invertible outside of Y, this is well-defined. But by the properties of valuations, this is $\mathcal{O}(Y)$.

Conversely, let $\mathcal{L}$ be any invertible sheaf. Observe that given an $\mathcal{O}_\eta$-generator s of $\mathcal{L}_\eta$, we may construct a divisor $\operatorname{div} s$: choose a trivialising open cover U_i of X. Then $s|_{U_i} = f_i t_i$ for some $f_i \in \mathcal{O}(U_i)$ and $t_i \in \mathcal{L}(U_i)$. As they come from a global s, the divisors patch together, i.e.

$$\operatorname{div} s := \sum \operatorname{div} f_i$$

[38] The tensor product of two free modules is just the module generated by the products of the original generators. Therefore the product of two Cartier Divisors is simply the tensor product of the two sheaves. Now, respecting linear equivalence means nothing more than $D - D'$ is principal iff $\mathcal{O}(D - D') \cong \mathcal{O}$. But both are equivalent to the local generators gluing together to give a global generator, cf. e.g. [Har77, Proposition II.6.13].

is well-defined. Note that this is well-defined as choosing instead some generators t_i' implies that $t_i' t_i^{-1} \in \mathcal{O}(U_i)^\times$ and therefore the associated divisor is 0. Choosing a different $\mathcal{O}_\eta$-generator s' of $\mathcal{L}_\eta$ changes the resulting divisor only by a rational function, hence yields a linear equivalent divisor and the same arguments hold when replacing $\mathcal{L}$ by an isomorphic $\mathcal{L}'$.

Now, given any $r \in \mathcal{L}(U)$, we may consider it as an element of $\mathcal{L}_\eta$ and hence find some $r' \in \mathcal{O}_\eta = k(X)$ so that $r = r's$. Hence, $\operatorname{div} r = \operatorname{div} r' + \operatorname{div} s \geq 0$ as r is regular, i.e. $r' \in \mathcal{O}(\operatorname{div} s)(U)$. We may reverse this argument to see that any section of $\mathcal{O}(D)(U)$ gives rise to a section of $\mathcal{L}(U)$. Thereby, the maps are inverse.

Clearly, the described isomorphism is the composition of the first two. □

Corollary 2.1.13: Let D and D' be Weil Divisors. Then

$$\mathcal{O}(D - D') \cong \mathcal{O}(D) \otimes \mathcal{O}(D')^{-1}$$

and $D \sim D'$ if and only if $\mathcal{O}(D) \cong \mathcal{O}(D')$.

Proof: This is nothing but a restatement of the fact that the above bijections are indeed morphisms. □

Remark 2.1.14: Observe that if X is non-singular, it is locally factorial. Indeed, in this case every local ring is regular and, by the Auslander-Buchsbaum Theorem [Eis95, Theorem 19.19], therefore factorial.

Remark 2.1.15: Let $f : X \longrightarrow Y$ be a morphism of curves, D a divisor on Y. Then, by Remark 2.1.7, f^*D is a divisor on X. On the other hand, by Proposition 1.4.16, $f^*\mathcal{O}_Y(D)$ is an invertible sheaf on X. Note that $\mathcal{O}_X(f^*D) \cong f^*\mathcal{O}_Y(D)$ as $f^*t_Q = t_Q \circ f$ for any $t_Q \in \mathfrak{m}_Q$.

Remark 2.1.16: Let $\mathcal{L}$ be any invertible sheaf and D a divisor. Consider the sheaf $\mathcal{L} \otimes \mathcal{O}(D)$. Locally, $\mathcal{L}$ will be of the form $g \cdot \mathcal{O}(U)$ for some generator g and $\mathcal{O}(D)(U)$ consists of rational functions $f \in k(X)$ with $\operatorname{div} f + D \geq 0$ on U. Therefore, sections of $\mathcal{L} \otimes \mathcal{O}(D)$ will, locally, be of the form $s := gf$ satisfying $\operatorname{div}(s) + D \geq 0$. Hence we may think of $\mathcal{L} \otimes \mathcal{O}(D)$ as "allowing poles at D" for the sections of the sheaf $\mathcal{L}$, i.e. any section of $\mathcal{O}(D)$ will do as a coefficient in the modules of $\mathcal{L}$. We call this process *twisting $\mathcal{L}$ by D* and write

$$\mathcal{L}(D) := \mathcal{L} \otimes \mathcal{O}(D).$$

In regard of Definition 2.1.11, this amounts to those rational sections of $\mathcal{L}$ being "constrained" by D and is hence in complete analogy to the definition of $\mathcal{O}(D)$.

2.1.2 Linear Systems

Specifically when studying the case of curves we want to concentrate on the linear equivalence class of a fixed divisor. Therefore, we introduce the notion of a linear system and study their relationship to embeddings into projective space.

For this purpose, let X be a *non-singular curve* for the entire section, unless otherwise stated.

Definition 2.1.17: Let X be a curve and D a divisor on X. Then we define the *complete linear system belonging to D* to be the set of all effective divisors D' that are linearly equivalent to D. We denote it by $|D|$.

It is often practical to consider the corresponding invertible sheaf instead of the divisor. In the case of linear systems, this point of view helps in giving them the structure of a projective vector space.

Proposition 2.1.18: Let D be a divisor on X, $|D|$ be the corresponding complete linear system, $\mathcal{O}(D)$ the corresponding invertible sheaf (cf. Theorem 1) and let $L(D) := \mathrm{H}^0(X, \mathcal{O}(D))$.

Then the map

$$\left(L(D) \setminus \{0\}\right)\big/k^\times \longrightarrow |D|$$
$$s \longmapsto D + \operatorname{div} s$$

is a one-to-one correspondence.

Proof: Let $s \in L(D)$. Then $\operatorname{div} s + D \geq 0$, so the divisor is effective and—as s is a rational function on X—by definition linearly equivalent to D.

Now for any $c \in k^\times$ we have $\operatorname{div} s = \operatorname{div}(cs)$, therefore the map in the proposition is well-defined. On the other hand, if $\operatorname{div} s = \operatorname{div} s'$ for $s, s' \in L(D)$, we have

$$0 = \operatorname{div} s - \operatorname{div} s' = \operatorname{div} \frac{s}{s'}$$

i.e. $\frac{s}{s'} \in \mathrm{H}^0(X, \mathcal{O}^\times) = k^\times$, as X is projective. Hence the map is also injective.

To see that it is also surjective, let $D' \in |D|$ be any effective divisor linearly equivalent to D. But this means that we find a $s \in k(X)^\times$ so that $D' - D = \operatorname{div} s$, i.e. $D + \operatorname{div} s = D' \geq 0$, as D' is an effective divisor. Thus we have $s \in L(D)$ and we have found a preimage of D'. □

Definition 2.1.19: A *linear system* $\mathfrak{d}$ on X is a subset of $|D|$ that corresponds to a projective linear subspace $V \subseteq L(D)$ via the map of Proposition 2.1.18.

The *dimension* of $\mathfrak{d}$ is its dimension as a linear projective variety, that is

$$\dim \mathfrak{d} := \dim V - 1.$$ [39]

The *degree* of a linear system is the degree of the corresponding divisor, in symbols:

$$\deg \mathfrak{d} := \deg D.$$

As all the divisors in the system are linearly equivalent, the degree is well-defined (cf. Remark 2.1.7).

Now we will describe a special type of linear system that will turn out to be very important.

Definition 2.1.20: A linear system $\mathfrak{d}$ is called a $\mathfrak{g}^r_d$ iff

$$\deg \mathfrak{d} = d \text{ and } \dim \mathfrak{d} = r.$$

As an even more special case, we shall call a $\mathfrak{g}^1_d$ a *pencil*.

Additionally, we shall refer to any curve X of genus greater than one that admits a $\mathfrak{g}^1_2$ as *hyperelliptic*.

[39] Recall that by Proposition 1.5.23 all these dimensions are finite and hence well-defined.

2.1.3 Morphisms into Projective Space

Our aim is to define morphisms of X into some projective space $\mathbb{P}^n$. The idea is to choose some line bundle that has "enough" global sections $s_0, \dots, s_n$ and to define a map

$$\begin{aligned} X &\longrightarrow \mathbb{P}^n \\ p &\longmapsto \big(s_0(p) : \dots : s_n(p)\big). \end{aligned}$$

An obvious prerequisite for this to work is that for every point $p \in X$ at least one of the sections $s_0, \dots, s_n$ must be non-zero. As we will see in a moment, we can express this as a condition on a linear system, more specifically the associated divisor.

Definition 2.1.21: Let $\mathfrak{d}$ be a linear system and $p \in X$. We call p a *base point of* $\mathfrak{d}$, iff $p \in \operatorname{supp} D$ for every $D \in \mathfrak{d}$, i.e. if $D = \sum n_q q$, then $n_p \neq 0$.

A linear system with no base points is called *base-point-free.*

Proposition 2.1.22 (cf. [Har77, Lemma II.7.8]): Let $\mathfrak{d}$ be a linear system and $V \subseteq \mathrm{H}^0(X, \mathcal{L})$ the corresponding vector space. Then $p \in X$ is a base point if and only if $s_p \in \mathfrak{m}_p\mathcal{L}_p$ for all $s \in V$ and $\mathfrak{d}$ is base-point-free if and only if the global sections of V generate $\mathcal{L}$.

Proof: Let $\mathfrak{d}$ be a subset of $|D|$ for some divisor $D = \sum n_q q$. By Proposition 2.1.18, $s \in V$ corresponds to $\operatorname{div} s + D \geq 0$. What happens at the point p?

Recall that by embedding $\mathcal{L}$ in $k(X)$, the ideal $\mathfrak{m}_p$ gives rise to a submodule $\mathfrak{m}_p\mathcal{L}_p$ of the germ $\mathcal{L}_p$ consisting of $s \in k(X)$ so that $v_p(s) \geq -n_p + 1$, where v_p is the valuation of the regular ring $\mathcal{O}_p$.

Now, for any base point p we have

$$v_p(s) + n_p \geq 1$$

if $s \in V$ is a section of $\mathcal{L}$ corresponding to an element of $\mathfrak{d}$. So we see that indeed p is a base point if and only if $s_p \in \mathfrak{m}_p\mathcal{L}_p$ for all $s \in V$.

If $\mathfrak{d}$ is base-point-free, then—by the above—at every point p there exists an $s \notin \mathfrak{m}_p\mathcal{L}_p$. But as $\mathcal{O}_p$ is a local ring, this implies that $s \in \mathcal{O}_p^\times$ and thus is a generator of $\mathcal{L}_p$, as $\mathcal{L}$ is locally free of rank one. On the other hand, any such generator must also be a unit in $\mathcal{O}_p$ and therefore not in $\mathfrak{m}_p\mathcal{L}_p$. □

In other words, a base point of $\mathcal{L}$ is a point of X at which all global sections of $\mathcal{L}$ vanish. Conversely, if $\mathcal{L}$ is base-point-free, we find—for any $p \in X$—a global section $s \in \mathrm{H}^0(X, \mathcal{L})$ that does not vanish at p.

Take any $s \in \mathrm{H}^0(X, \mathcal{L})$. Then we obtain an open[40] set $\mathfrak{D}(s) := \{p \in X \mid s_p \notin \mathfrak{m}_p\mathcal{L}_p\}$ and Proposition 2.1.22 tells us that a linear system V is base-point-free if and only if the $\mathfrak{D}(s)$, $s \in V$, cover X.

We are now in a situation to describe all maps into projective space.

[40] Indeed, we may cover X by trivialising neighbourhoods U_i. On each of these open sets we find a generator t_i so that $s|_{U_i} = f_i t_i$ for some regular function f_i. Then we clearly have $\mathfrak{D}(s) = \bigcup \mathfrak{D}(f_i)$.

Theorem 2: *Let X be any k-scheme. Then there is a bijection*

$$\{(\mathcal{L}, s_0, \dots, s_n)\} \longleftrightarrow \operatorname{Hom}_{\mathfrak{Sch}}(X, \mathbb{P}^n)$$

where $\mathcal{L}$ is an invertible sheaf on X and $s_0, \dots, s_n$ are global sections that generate $\mathcal{L}$, up to isomorphism of these data.

Proof: Let $\varphi\colon X \longrightarrow \mathbb{P}^n_k = \operatorname{Proj} k[x_0, \dots, x_n]$ be a morphism and consider the invertible sheaf $\mathcal{O}(1)$ on $\mathbb{P}^n$. By Proposition 1.4.16, $\varphi^*\mathcal{O}(1)$ is an invertible sheaf on X and the preimages of trivialising neighbourhoods trivialise the pullback. In particular, we have $\mathfrak{D}(\varphi^*(x_i)) = \varphi^{-1}(\mathfrak{D}(x_i))$, showing that $\bigcup \mathfrak{D}(\varphi^*(x_i)) = X$. Therefore, by Proposition 2.1.22, $s_i := \varphi^*(x_i)$ are $n+1$ global sections that generate the invertible sheaf $\mathcal{L} := \varphi^*\mathcal{O}(1)$.

Now let $\mathcal{L}$ be an invertible sheaf and let $s_0, \dots, s_n$ be generating global sections. Observe that $\mathcal{L}$ trivialises over $\mathfrak{D}(s_i)$: locally, s_i is of the form $f_j t_j$ where t_j is the generator of $\mathcal{L}$ over some trivialising U_j and f_j is a regular function. As[41] s_i has "no zeros" in $\mathfrak{D}(s_i)$, all the f_j are also invertible, hence patch to give some $f_i \in \mathcal{O}_X(\mathfrak{D}(s_i))^\times$ and this induces the local isomorphism. By abuse of notation, we identify the f_i and s_i. Consider the morphism

$$\varphi_i\colon\ \mathfrak{D}(s_i) \longrightarrow \mathfrak{D}(x_i) = \operatorname{Spec} k\left[\tfrac{x_0}{x_i}, \dots, \tfrac{x_n}{x_i}\right] \subset \mathbb{P}^n$$

that is induced by the ring homomorphism $\frac{x_j}{x_i} \longmapsto \frac{s_j}{s_i} \in \mathcal{O}_X(\mathfrak{D}(s_i))$ (via Proposition 1.2.12). Clearly, these maps glue to give a map $\varphi\colon X \longrightarrow \mathbb{P}^n$. Note that $\varphi^*\mathcal{O}(1) \cong \mathcal{L}$, as $\varphi^*(x_i) = s_i$ and the s_i trivialise $\mathcal{L}$ over $\mathfrak{D}(s_i)$.

Therefore, these constructions are inverse to one another, as claimed. See also [Har77, Theorem II.7.1] and [Vak12, Important Theorem 17.4.1]. □

In other words, the morphisms $X \longrightarrow \mathbb{P}^n$ correspond to the base-point-free linear systems of dimension n on X, cf. [Har77, Remark II.7.8.1]. Note that this really is the morphism we wanted: take any point $p \in X$. Then the stalk $\mathcal{O}_p$ is generated by the $(s_i)_p$ and $\mathcal{O}_{\varphi(p)}$ is generated by the $(x_i)_p$. The associated ring homomorphism sends $(x_i)_p$ to $(s_i)_p$ and therefore $\varphi(p) = (s_0(p) : \dots : s_n(p))$, as $s_i(p) = (s_i)_p \mod \mathfrak{m}_p$, i.e. the image in the residue field $\kappa(p)$ and φ respects this, being a morphism of schemes.

We now describe a criterion that determines if the morphism induced by some invertible sheaf $\mathcal{L}$ is a closed immersion. Recall that, in this case, we called $\mathcal{L}$ very ample.

Proposition 2.1.23: Let X be a variety, $\mathfrak{d}$ be a base-point-free linear system and $\varphi\colon X \longrightarrow \mathbb{P}^n$ the induced morphism. Then φ is a closed immersion if and only if φ is injective on closed points and tangent vectors at closed points. In the case that X is a non-singular curve, this translates to

(a) $\mathfrak{d}$ *separates points*, i.e. given distinct closed points $p, q \in X$ we find a $D \in \mathfrak{d}$ with $p \in \operatorname{supp} D$ and $q \notin \operatorname{supp} D$; and

(b) $\mathfrak{d}$ *separates tangent vectors*, i.e. for any closed $p \in X$ we find some $D \in \mathfrak{d}$ so that $n_p = 1$ in D.

[41] Indeed, restrict s_i to any affine subset of $\mathfrak{D}(s_i)$. Then this element is not contained in any maximal ideal, hence it must be a unit. These patch together giving an invertible element of $\mathcal{O}(\mathfrak{D}(s_i))$.

Proof: Let $s_0, \dots, s_n$ be a basis of $\mathfrak{d}$ so that φ is the morphism described above. Then φ is injective on points iff for any $p, q \in X$ there exists some s_i with $s_i(p) \neq s_i(q)$. But then $s := s_i - s_i(p)$ belongs to a divisor that is linearly equivalent to the divisor of s_i and whose support contains p but not q.

Consider the second condition. Any morphism $f\colon X \longrightarrow Y$ induces, for any $p \in X$, a local morphism $f_p^\sharp\colon \mathcal{O}_{Y,f(p)} \longrightarrow f_*\mathcal{O}_{X,p}$ and thereby a morphism of the dual vector spaces $\left(\mathfrak{m}_p/\mathfrak{m}_p^2\right)^* \longrightarrow \left(\mathfrak{m}_{f(p)}/\mathfrak{m}_{f(p)}^2\right)^*$, i.e. a morphism of the tangent spaces. Here, φ induces a morphism $\left(\mathfrak{m}_p\mathcal{L}_p/\mathfrak{m}_p^2\mathcal{L}_p\right)^* \longrightarrow \left(\mathfrak{m}_{\varphi(p)}/\mathfrak{m}_{\varphi(p)}^2\right)^*$. By linear algebra, this is injective iff the dual morphism $\mathfrak{m}_{\varphi(p)}/\mathfrak{m}_{\varphi(p)}^2 \longrightarrow \mathfrak{m}_p\mathcal{L}_p/\mathfrak{m}_p^2\mathcal{L}_p$ is surjective. But the space $\mathfrak{m}_{\varphi(p)}/\mathfrak{m}_{\varphi(p)}^2$ is generated by those sections s with $s_p \in \mathfrak{m}_p$ i.e. those with a zero of order one at p and in the case that X is a non-singular curve, the tangent space at p will be one dimensional. Hence, the map on tangent spaces is injective iff there exists a non-zero $(s_i)_p \in \mathfrak{m}_{\varphi(p)}/\mathfrak{m}_{\varphi(p)}^2$. But such an s_i corresponds to a divisor with $n_p = 1$.

Therefore, in the case of curves, the separation conditions are equivalent to φ being injective on closed points and on tangent vectors at closed points. The fact that this is equivalent to φ being a closed immersion may be found in [Vak12, Theorem 20.1.1] or [Har77, Proposition II.7.3]. □

Before we are able to express these conditions exclusively by the numerical data of the corresponding divisors, we must introduce more sophisticated techniques relating the dimension of a linear system to the degree of the corresponding divisor.

2.1.4 Differentials and the Canonical Sheaf

Beside the sheaf of regular functions, there is another sheaf (almost) every variety is equipped with: the sheaf of differential forms. Indeed, it is of such fundamental importance that it is known as the "canonical sheaf". To be able to define it correctly, however, we must start by encoding all information about "tangent spaces" into suitable algebraic objects.

To begin with, we must therefore collect some algebraic results. A classical reference for this is, e.g. [Mat89, §25].

In the following, let A be a k-algebra and M a module over A.

Definition 2.1.24: A *(k-)derivation* from A to M is a k-linear map $D\colon A \longrightarrow M$ that satisfies the *Leibniz formula*, i.e.

$$D(f+g) = D(f) + D(g) \text{ and } D(fg) = fD(g) + D(f)g$$

for any $f, g \in A$. The set of all such derivations is denoted by $\mathrm{Der}_k(A, M) =: \mathrm{Der}(A, M)$.

Remark 2.1.25: As any k-derivation D is k-linear, we have

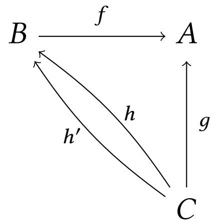
$$D(1) = D(1 \cdot 1) = 1D(1) + D(1)1$$

and thus $D(a) = 0$ for any $a \in k$.

Now, the set $\mathrm{Der}(A, M)$ may be considered as an A-module by defining

$$D + D' \text{ via } \big(D + D'\big)(f) := D(f) + D'(f) \text{ and } aD \text{ via } \big(aD\big)(f) := aD(f).$$

Example 2.1.26: There is a close relationship between lifts of morphisms and derivations. Consider the following commutative diagram in the category of k-algebras:



In this situation, we refer to h and h' as being (respectively) a *lift of g to B*. As the diagram commutes, $h - h'$ is a morphism from C to $\ker f =: I \lhd B$. Also, we may identify $f(B)$ with B/I, thereby obtaining an $f(B)$-module structure on I/I^2. As the diagram commutes, we have $f(B) \subseteq g(C)$ and therefore we may consider I/I^2 as a C-module. But then $h - h' \in \mathrm{Der}(C, I/I^2)$ as it is obviously k-linear and additive, and satisfies the Leibniz formula: to understand this, we must take a closer look at the C-module structure of I/I^2. We have

$$c \cdot (i + I^2) := h(c) \cdot i + I^2 = h'(c) \cdot i + I^2$$

as $g = f \circ h = f \circ h'$ and $h(c)$ does not differ from $h'(c)$, when both are viewed as elements of B/I. But now the multiplication in B yields

$$\begin{aligned}\big(h - h'\big)(cc') &= h(c)h(c') - h'(c)h'(c') = h(c)h(c') - h(c')h'(c) + h'(c)h(c') - h'(c)h'(c') \\ &= h(c') \cdot \big(h - h'\big)(c) + h'(c) \cdot \big(h - h'\big)(c')\end{aligned}$$

and this is just the Leibniz formula in I/I^2 considered as a C-module.

On the other hand, let $D \in \mathrm{Der}(C, I/I^2)$. Then $h + D$ is a lift of g to B/I^2. Indeed, the C-module structure of I/I^2 is again induced by h, i.e. the Leibniz rule for D amounts to

$$D(cc') = h(c)D(c') + h(c')D(c)$$

and thus we see that

$$\big(h + D\big)(c) \cdot \big(h + D\big)(c') = h(c)h(c') + h(c)D(c') + D(c)h(c') + D(c)D(c')$$

and as $D(c)D(c') \in I^2$ and $h + D$ is clearly additive, $h + D$ is a homomorphism of rings, as claimed. Also, as $\mathrm{im}\, D \subseteq \ker f$, $h + D$ is a lift of g for any lift h.

All in all, we obtain a bijection of lifts of g to B/I^2 and $\mathrm{Der}(C, I/I^2)$.

Fixing A, we obtain a covariant functor[42] $M \longmapsto \mathrm{Der}(A, M)$ from $\mathfrak{Mod}_A$ to itself, which, as it turns out, is representable:[43]

[42] Indeed, let $\varphi\colon M \longrightarrow M'$ be a morphism of modules, then it induces a map $\mathrm{Der}(A, M) \longrightarrow \mathrm{Der}(A, M')$ via $f \longmapsto \varphi \circ f$ and this respects the module structure of Der as φ itself is A-linear.

[43] The pedantic reader may wish in prior to compose with a forgetful functor.

Proposition 2.1.27: Consider $f\colon A \otimes_k A \longrightarrow A$ given by $a \otimes b \longmapsto ab$ and let $I := \ker f$.

Then $\Omega_{A/k} := I/I^2$ is the representing object of $\mathrm{Der}(A, -)$, i.e.

$$\mathrm{Der}(A, -) \cong \mathrm{Hom}(\Omega_{A/k}, -).$$

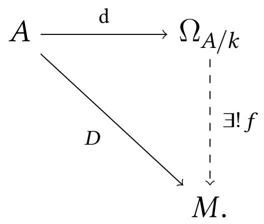
In particular, we obtain a distinguished derivation[44] $\mathrm{d}\colon A \longrightarrow \Omega_{A/k}$ and $\Omega_{A/k}$ is generated by $\mathrm{im}\,\mathrm{d}$ as an A-module.

Proof: The idea is to apply Example 2.1.26 to the above setting. See [Mat89, p. 192]. □

Definition 2.1.28: We call $\Omega_{A/k}$ the *Module of (Kähler) differentials of A over k*.

Of course, the representability of the functor implies that $\Omega_{A/k}$ satisfies a universal mapping property:

Remark 2.1.29: Let A and $\Omega_{A/k}$ be defined as above. Then for any A-module M and $D \in \mathrm{Der}(A, M)$ there exists a unique A-module homomorphism f so that the following diagram commutes:



Remark 2.1.30: As we never really made use of the fact that k is a field, we may conduct all these constructions in a relative setting: we may simply replace k by any commutative ring R in the entire section thus giving rise to the *R-derivations of A into M*, the A-module $\mathrm{Der}_R(A, M)$ and the module of *relative differential forms of A over R*, $\Omega_{A/R}$. Note however, that in the construction of $\Omega_{A/R}$ (Proposition 2.1.27), we must now tensor over R.

Our aim is now to place these constructions in a geometric setting. The map $A \otimes A \longrightarrow A$ is well-known and gives a first hint, as to how we shall accomplish this: in the affine case, this is the ring homomorphism that corresponds to the diagonal embedding

$$\Delta\colon X \longrightarrow X \times_{\mathrm{Spec}\,k} X. \tag{2.1}$$

The true importance of these objects reveals itself only in a local setting:

Proposition 2.1.31: Let A be a local ring with maximal ideal $\mathfrak{m}$ and containing a field k isomorphic to its residue field $A/\mathfrak{m}$. Then we have an isomorphism:

$$\delta\colon\ {}^{\mathfrak{m}}/_{\mathfrak{m}^2} \xrightarrow{\sim} \Omega_{A/k} \otimes_A k.$$

Proof: [Har77, Proposition II.8.7]. □

[44] Concretely, this is the map defined by $a \longmapsto 1 \otimes a - a \otimes 1 + I^2$.

As the cotangent space at a given point (of a given scheme) is just $\mathfrak{m}/\mathfrak{m}^2$ (where $\mathfrak{m}$ is the maximal ideal of the corresponding local ring), we have just seen that the module of differentials of the local ring of the point contains this information. As a consequence of this, we will see that our usual concept of regularity at a point (the dimension of the tangent and thus also the cotangent space equals the local dimension at that point) is also compatible with the concept of the differential module:

Proposition 2.1.32: In the setting of Proposition 2.1.31, assume further that k is perfect and A is a localisation of a finitely generated k-algebra. Then $\Omega_{A/k}$ is a free A-module of rank equal to $\dim A$ if and only if A is a regular local ring.

Proof: [Har77, Theorem II.8.8]. □

Having reassured ourselves that this algebraic construct has its uses in geometry, it is high time to transport it into the world of sheaves and schemes. The idea is as stated above:

Definition 2.1.33 (cf. [Har77, p. 175]): Let X, Y be schemes $f\colon X \longrightarrow Y$ a morphism, Δ the diagonal morphism (see (2.1)) and $\mathcal{I}$ the sheaf of ideals[45] of $\Delta(X)$. Then we define the *sheaf of relative differentials of X over Y* to be the following sheaf on X:

$$\Omega_{X/Y} \coloneqq \Delta^* \left(\mathcal{I} \big/ \mathcal{I}^2 \right)$$

Remark 2.1.34: The sheaf $\mathcal{I}/\mathcal{I}^2$ comes with a natural structure as $\mathcal{O}_{\Delta(X)}$-module and—since Δ is a homeomorphism on its image—$\Omega_{X/Y}$ inherits a natural $\mathcal{O}_X$-module structure. Additionally, $\Omega_{X/Y}$ is quasi-coherent and in the case that Y is Noetherian and f is a morphism of finite type, $\Omega_{X/Y}$ will even be a coherent sheaf [Har77, Remark II.8.9.1].

Furthermore, locally the sheaf $\Omega_{X/Y}$ comes from the module of differentials of the corresponding rings (cf. [Har77, Remark II.8.9.2]). Especially, note that if $Y = k$, the fibres of $\Omega_{X/k}$ correspond to the cotangent spaces, i.e. $\Omega_{X/k}|_p \cong \mathfrak{m}_p/\mathfrak{m}_p^2$ at every point of X via Proposition 2.1.31. Hence, we will occasionally refer to $\Omega_{X/k}$ as the *cotangent bundle of X*.

In light of Proposition 2.1.32, for any (integral, seperated) regular scheme X that is of finite type over k and of dimension n, the sheaf $\Omega_{X/k}$ will be locally free of rank n. This is good, but often not good enough for our purposes.

Definition 2.1.35: Let X be a non-singular variety of dimension n over k. Then we define the *canonical sheaf of X* to be the nth exterior power of the sheaf of differentials, i.e.

$$\omega_X \coloneqq \bigwedge^n \Omega_{X/k}.$$

If X is in addition projective, we define the *geometric genus* of X to be

$$g_g \coloneqq g_g(X) \coloneqq \dim \mathrm{H}^0(X, \omega_X).$$

Occasionally we shall refer to global sections of ω_X as *differential forms* or *1-forms*.

[45] While $\Delta(X)$ is only a closed subscheme of $X \times_Y X$ if f is a separated morphism, *locally* f is always a morphism of affine schemes and thus separated (the map $a \otimes a' \longmapsto aa'$ is always surjective!). Therefore, $\Delta(X)$ is a *locally closed* subscheme, i.e. a closed subscheme of an open subscheme of X and this is in fact all we need to construct our ideal sheaf.

Before moving on, let us quickly note a vital fact about the geometric genus:

Proposition 2.1.36: Let X and X' be two birationally equivalent non-singular projective varieties over k. Then $g_g(X) = g_g(X')$.

Proof: [Har77, Theorem II.8.19]. □

Recall that $\operatorname{rank} \bigwedge^k R^n = \binom{n}{k}$ for any free R-module of rank n. Hence ω_X is locally free of rank one, as $\Omega_{X/k}$ is locally free of rank n and therefore ω_X is always an invertible sheaf. This gives rise to the next definition.

Definition 2.1.37: Let X satisfy the prerequisites of Theorem 1. Then we denote the class of divisors corresponding to ω_X by $\mathcal{K}$ and call this the *canonical divisor of* X.

Note that if X is a curve, we simply have $\omega_X = \Omega_{X/k}$. Even better, all concepts of genus coincide, so there is no more reason to distinguish them. Indeed, when X is a curve, the arithmetic genus simplifies to

$$g_a(X) = (-1)^{\dim X}(\chi(\mathcal{O}_X) - 1) = \dim \mathrm{H}^1(X, \mathcal{O}_X)$$

as $\mathrm{H}^0(X, \mathcal{O}_X) = k$, i.e. the dimension is 1, and $\mathrm{H}^i(X, \mathcal{O}_X) = 0$ for $i > 1$ by Proposition 1.5.24.

Proposition 2.1.38: Let X be a curve. Then we have $g_a = g_g$, more explicitly:

$$g := g(X) = \dim \mathrm{H}^1(X, \mathcal{O}_X) = \dim \mathrm{H}^0(X, \omega_X).$$

We call g simply the *genus of* X. By Proposition 2.1.36, this is a birational invariant.

Proof: Actually, this equality follows from Serre Duality (Theorem 4). See e.g. [Har77, Proposition IV.1.1]. □

Additionally, it can even be shown that in the case $k = \mathbb{C}$, the genus g of a curve X is equal to the (topological) genus of the corresponding Riemann Surface [Mir95, p. 191].

2.2 The Riemann-Roch Theorem

Arguably the most important tool in the study of curves is the theorem of Riemann-Roch. It expresses the relationship of the dimension of a linear system to the degree of the corresponding divisor using only basic invariants of the curve: the genus (cf. Proposition 2.1.38) and the canonical divisor (cf. Definition 2.1.37). More precisely:

Theorem 3 (Riemann-Roch): *Let X be a curve, D any divisor on X and $\mathcal{K}$ the canonical divisor. Then we have*

$$\mathrm{l}(D) - \mathrm{l}(\mathcal{K} - D) = \deg D - g + 1 \tag{2.2}$$

where g denotes the genus of X and $\mathrm{l}(D) := \dim L(D) = \dim \mathrm{H}^0(X, \mathcal{O}(D))$.

2.2.1 Observations on $\mathrm{l}(D)$

Before we can sketch the proof, it is useful to make some general observations about the dimensions of the linear systems involved as well as to cite some technically more demanding results.

We start with some simple—but practical—observations on the degree and dimension of a linear system:

Lemma 2.2.1 (cf. [Har77, Lemma IV.1.2]): Let D be a divisor on a curve X. Then:

(a) $\mathrm{l}(D) \neq 0$ implies that $\deg D \geq 0$ and

(b) if $\mathrm{l}(D) \neq 0$ and $\deg D = 0$, we must have $D \sim 0$ and thus $\mathcal{O}(D) \cong \mathcal{O}_X$.

Proof: If $\mathrm{l}(D) \neq 0$, then the complete linear system $|D|$ is non-empty, i.e. there exists an effective divisor D' linearly equivalent to D. But the degree of a divisor is invariant under linear equivalence and surely an effective divisor will have non-negative degree so we must also have $\deg D \geq 0$.

Consider the case that $\deg D = 0$. Again, if $\mathrm{l}(D) \neq 0$, $|D|$ contains some effective divisor D' of degree 0. But the only effective divisor of degree 0 is the zero divisor, hence $D \sim 0$. □

Lemma 2.2.2: Let D be any effective divisor on a curve X and $P \notin \operatorname{supp} D$. Then

$$\mathrm{l}(D - P) \leq \mathrm{l}(D) - 1.$$

Proof: Indeed, as D is an effective divisor, sections in $L(D)$ include all (global) regular functions on X, i.e. k, as well as any function with poles bounded by D. All sections in $L(D-P)$ are, however, required to have a zero[46] at the point P thus definitely excluding all the constants (possibly even more[47]). Hence the dimension will decrease. □

2.2.2 Serre Duality and a Proof of the Riemann-Roch Theorem

The technically truly demanding aspect of the proof of the Riemann-Roch theorem lies in Serre's duality theorem, in its simplest form (for curves) it may be summarised as follows:

Theorem 4 (Serre Duality): *Let D be any divisor on the curve X, ω_X the canonical sheaf. Then the vector spaces*

$$\mathrm{H}^0(X, \omega_X \otimes \mathcal{O}(D)^{-1}) \text{ and } \mathrm{H}^1(X, \mathcal{O}(D))$$

are dual to one another.

Unfortunately, a proof of this statement would lead us too far astray! A proof involving sheaves of derivations of the Hom functor may be found in e.g. [Har77, §II.7], another using spectral sequences in e.g. [LAGI, §4.11]. Accepting this, however, we may give an otherwise complete proof of the Riemann-Roch theorem.

[46] I.e. the corresponding germ in the local ring $\mathcal{O}_P$ must lie in the maximal ideal $\mathfrak{m}_P$, cf. Definition 2.1.4.

[47] But this will not change the dimension, as we will see in a moment (Remark 2.2.3) using slightly fancier arguments.

Proof of Theorem 3: We are guided by [Har77, Theorem V.1.3]. Using Theorem 4, we see that—as the divisor $\mathcal{K} - D$ corresponds to the invertible sheaf $\omega_X \otimes \mathcal{O}(D)^{-1}$—the dimension of $\mathrm{H}^0(X, \mathcal{O}(\mathcal{K} - D)) = \mathrm{H}^0(X, \omega_X \otimes \mathcal{O}(D)^{-1})$ is the same as $\mathrm{H}^1(X, \mathcal{O}(D))$. Therefore, we may restate our formula to read

$$\dim \mathrm{H}^0(X, \mathcal{O}(D)) - \dim \mathrm{H}^1(X, \mathcal{O}(D)) = \deg D + 1 - g.$$

But we know the left hand side of the equation: it is the Euler characteristic $\chi(\mathcal{O}(D))$.

The aim is now to show this equation for all concievable divisors D. We start with the easiest case: for $D = 0$, we must verify

$$\dim \mathrm{H}^0(X, \mathcal{O}_X) - \dim \mathrm{H}^1(X, \mathcal{O}_X) = 0 + 1 - g$$

as $\mathcal{O}(0) \cong \mathcal{O}_X$. But as any projective variety admits only constant global sections and $g = \dim \mathrm{H}^1(X, \mathcal{O}_X)$ by definition, this formula does indeed hold.

Now comes the trick: We show that the formula holds for a divisor D *if and only if* it is true for $D + P$ for any point P. Thus, we can reach any divisor (in a finite number of steps) and we are done.

To accomplish this, consider P as a closed subscheme of X via ι. Then the structure sheaf $\mathcal{O}_{\{P\}}$ of P may be identified with its pushforward on X, i.e. the skyscraper sheaf $\iota_*\mathcal{O}_{\{P\}}$. This gives rise to the short exact sequence

$$0 \longrightarrow \mathcal{I}_{\{P\}} \hookrightarrow \mathcal{O}_X \twoheadrightarrow \iota_*\mathcal{O}_{\{P\}} \longrightarrow 0 \tag{2.3}$$

where the corresponding ideal sheaf $\mathcal{I}_{\{P\}}$ is just $\mathcal{O}(-P)$. Indeed, any section of $\mathcal{O}(-P)$ has a zero at P and is therefore in the kernel of the projection onto $\iota_*\mathcal{O}_{\{P\}}$. Conversely, this is the only condition imposed, as the skyscraper sheaf only cares for the point P.

Now we tensor this sequence with $\mathcal{O}(D + P)$. Since this is locally free of rank one and a sequence of sheaves is exact if it is exact on the stalks, the sequence remains exact[48]. Furthermore, it is clear what happens on the first two sheaves: we must simply add the divisors. But the stalks of the skyscraper sheaf $\mathcal{O}_{\{P\}}$ are all zero except at the point P; here the stalk is simply equal to our ground field k. Especially, we note, it remains indifferent to being tensored with $\mathcal{O}(D + P)$. All in all, this yields

$$0 \longrightarrow \mathcal{O}(D) \hookrightarrow \mathcal{O}(D + P) \twoheadrightarrow \mathcal{O}_{\{P\}} \longrightarrow 0. \tag{2.4}$$

Now is a good time to remember that the Euler characteristic is additive on short exact sequences (Remark 1.5.26). Also, as P is zero-dimensional, $\mathrm{H}^1(P, \mathcal{O}_{\{P\}}) = 0$ by Proposition 1.5.24 and thus $\chi(\mathcal{O}_{\{P\}}) = \dim \mathrm{H}^0(P, \mathcal{O}_{\{P\}}) = 1$, yielding

$$\chi(\mathcal{O}(D + P)) = \chi(\mathcal{O}(D)) + 1.$$

But—trivially—$\deg(D + P) = \deg D + 1$ so indeed the formula holds for D if and only if it holds for $D + P$. □

[48] See, for example, [Eis95, Chapter 6] for a discussion of flatness.

Remark 2.2.3: Similar techniques allow us to "generalise" Lemma 2.2.2 to *any* divisor D and *any* point $P \in X$: We apply equations (2.3) and (2.4) of the above proof to the situation of Lemma 2.2.2, obtaining an exact sequence

$$0 \longrightarrow \mathcal{O}(D-P) \hookrightarrow \mathcal{O}(D) \twoheadrightarrow \iota_* \mathcal{O}_{\{P\}} \longrightarrow 0.$$

Now, taking global sections is left exact, so we obtain

$$0 \longrightarrow L(D-P) \hookrightarrow L(D) \longrightarrow k.$$

Hence, as the image of $L(D)$ must be a subvector space of k, we see that, for any P, $\mathrm{l}(D) - \mathrm{l}(D-P) \leq 1$ and $\mathrm{l}(D) \geq \mathrm{l}(D-P)$.

Applying this result to Lemma 2.2.2 means that we may replace the "$\leq$" by an "$=$" in the formula.

2.2.3 Some First Consequences

As a first demonstration of the power of the theorem, we may now calculate the degree of the canonical divisor and the dimension of the corresponding complete linear system with seemingly no effort at all:

Corollary 2.2.4: Let X be a curve of genus g, let $\mathcal{K}$ denote the canonical divisor. Then:

(a) $\mathrm{l}(\mathcal{K}) = g$ and

(b) $\deg \mathcal{K} = 2g - 2$.

Proof: This is just (2.2) for the divisors $D = 0$ and $D = \mathcal{K}$, respectively. □

Now for a "true" geometric application of Theorem 3:

Corollary 2.2.5: Let X be a curve of genus 2. Then X is hyperelliptic.

Proof: Indeed, the canonical divisor $\mathcal{K}$ on X gives rise to a $\mathfrak{g}^1_2$: By Corollary 2.2.4, we have $\deg \mathcal{K} = 2$ and $\mathrm{l}(\mathcal{K}) = 2$. Hence—by Definition 2.1.19—the dimension of the corresponding complete linear system is 1 as required. □

Taking a closer look at the Riemann-Roch formula (2.2), the term $\mathrm{l}(\mathcal{K} - D)$ stands out: if it weren't for that number, we would have a direct linear relationship between the degree of a divisor and the dimension of the corresponding linear system. In fact, it is not uncommon for this term to vanish entirely. Indeed, it is a common enough occurrence to deserve its own name.

Definition 2.2.6: A divisor D is called *special* if $\mathrm{l}(\mathcal{K} - D) \neq 0$. Otherwise we say that D is *non-special.* We will refer to the number $\mathrm{l}(\mathcal{K} - D)$ as its *index of speciality.*

Example 2.2.7: As a first example, note that Lemma 2.2.1 together with Corollary 2.2.4 yields that any D with $\deg D > 2g - 2$ is non-special.

Remark 2.2.8: To give another notion of the concept of a special divisor, consider the following: by Theorem 1, $\mathcal{K} - D$ corresponds to the invertible sheaf $\mathcal{O}(\mathcal{K}) \otimes \mathcal{O}(-D) \cong \omega_X(-D)$, i.e. the sheaf ω_X twisted by D, consisting of differential forms with "coefficients" in $\mathcal{O}(-D)$ (cf. Remark 2.1.16). Thus, the index of speciality of D may be thought of as counting the number of linearly independent differentials that are locally of the form $f \cdot \mathrm{d}x$ where $\mathrm{d}x$ is a local generator of ω_X and $f \in \mathcal{O}(-D)$, i.e. locally, $\operatorname{div} f - D \geq 0$.

2.3 Intersection Theory

Picturing a curve as the zero locus of some polynomial automatically gives rise to some concept of the "degree" of a curve. Also, if we imagine curves embedded in, e.g. $\mathbb{A}^2$, there is some intuitive notion of "intersection multiplicity": two curves intersecting "transversally" such as $\mathfrak{V}(x)$ and $\mathfrak{V}(y)$ should intersect with "multiplicity 1" while curves meeting "tangentially" such as $\mathfrak{V}(y)$ and $\mathfrak{V}(y-x^2)$ should intersect with "multiplicity 2". Clearly, the concepts of multiplicities and degrees are interrelated. But giving adequate definitions that generalise properly and behave well turns out to be surprisingly complicated and technical.

2.3.1 The Degree of a Curve

Up to this point, we have successfully avoided mention of the degree of a curve. Clearly, however, when intersecting curves, their degree will play a part in studying the intersection. In the case that we are dealing with a *plane curve*, i.e. one that admits an embedding into $\mathbb{P}^2$, this curve is necessarily a divisor in $\mathfrak{Cl}(\mathbb{P}^2)$, hence comes with some notion of degree. These should agree. Also, given some hypersurface arising as the projective space of the quotient of some polynomial ring by a degree d polynomial, we expect this to also be of degree d.

Hence, we require a definition that is general enough to be applicable to various kinds of varieties, but specialises to the two mentioned cases. Let $\mathcal{F}$ be any coherent sheaf on some projective k-scheme $X \subseteq \mathbb{P}^n$. Then we define the *Hilbert polynomial of* $\mathcal{F}$ via

$$p_{\mathcal{F}}(m) := \chi(X, \mathcal{F}(m)) = \sum_{i \geq 0} (-1)^i \dim \mathrm{H}^i(X, \mathcal{F}(m)).$$

Note that it is a polynomial[49] of degree $\dim \operatorname{supp} \mathcal{F}$ (see, e.g. [Vak12, Theorem 19.5.1] or [Ser55, §80, Proposition 3]). In particular, in combination with Proposition 1.5.23 (b) this implies that $\dim \mathrm{H}^0(X, \mathcal{F}(m))$ is polynomial for large m. Further, we set $p_X := p_{\mathcal{O}_X}$ and call this the *Hilbert polynomial of* X. Consequently, it is of degree $\dim X$.

Definition 2.3.1: Let $X \subset \mathbb{P}^d$ be a projective k-scheme of dimension n. Then we define the *degree of* X, $\deg X$, as the leading coefficient of p_X multiplied with $n!$.

Note that $p_{\mathcal{F}}(m)$ is always in $\mathbb{Z}$ and, in particular, the degree of X is always a natural number ([Har77, Proposition I.7.3a]).

We should check that this agrees with our intuition of degree.

Example 2.3.2: Let $X = \mathbb{P}^n = \operatorname{Proj} k[x_0, \dots, x_n]$. What is p_X? Recall that, by Remark 1.5.21, $\dim \mathrm{H}^0(X, \mathcal{O}_X(m)) = \binom{m+n}{n}$ if $m \geq 0$. Apart from $\dim \mathrm{H}^n(X, \mathcal{O}_X(m)) = \binom{-m-1}{-n-m-1}$ if $m \leq -n-1$, all other cohomology groups vanish. Therefore, the hilbert polynomial is

$$p_{\mathbb{P}^n}(m) = \frac{(m+n)(m+n-1)\cdots(m+1)}{n!}$$

and in particular the leading coefficient (belonging to m^n) is $\frac{1}{n!}$. Hence, $\deg \mathbb{P}^n = 1$ which seems reasonable.

[49] In fact, observe that if $\mathcal{F} \cong \widetilde{M}$, by Proposition 1.4.13, it agrees with the Hilbert polynomial of M [Ser55, §80, Proposition 4]. See, for example, [Eis95, §1.9] for Hilbert polynomials of modules.

Example 2.3.3: Now let $X = \mathfrak{V}(f) \subseteq \mathbb{P}^n$ be a hypersurface for some homogeneous f of degree d, i.e. $X = \operatorname{Proj} k[x_0, \ldots, x_n]/(f)$. But the ideal (f) corresponds (as a module) to the polynomial ring twisted by $-d$ as the inclusion morphism sends any degree m element g to the degree $d + m$ element $fg \in R$. Via the $\tilde{\cdot}$-functor, we obtain a short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d) \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \iota_* \mathcal{O}_X \longrightarrow 0,$$

where ι is the inclusion of X in $\mathbb{P}^n$. Now recall that the Euler characteristic is additive on short exact sequences (Remark 1.5.26), hence the same holds for the Hilbert polynomial. In other words, we have $p_X(m) = p_{\mathbb{P}^n}(m) - p_{\mathbb{P}^n}(m - d)$. So what is the degree of X? By Example 2.3.2, we have

$$\begin{aligned} p_X &= \frac{(m+n)\cdots(m+1)}{n!} - \frac{(m-d+n)\cdots(m-d+1)}{n!} \\ &= \frac{1}{n!}\left(m^n + (m-d)^n + \frac{n(n+1)}{2}m^{n-1} - \frac{n(n+1)}{2}m^{n-1} + \cdots\right) \end{aligned}$$

where we ignore summands of degree (in m) less than $n - 1$. Indeed, the monomials of degree n cancel and the term m^{n-1} occurs in $p_{\mathbb{P}^1}(m)$ only in products where all but one factor is m and the remaining factor runs over $1, \ldots, n$. The same happens in $p_{\mathbb{P}^1}(m - d)$, so these coefficients cancel. But in this case, the term

$$(m-d)^n = \sum_{k=0}^{n} \binom{n}{k} m^k (-d)^{n-k} = m^n - ndm^{n-1} + \cdots$$

tells us the coefficient of m^{n-1}: $\frac{1}{n!}nd = \frac{1}{(n-1)!}d$. As we obtain $\deg X$ by multiplying with the factorial of $\dim X = n - 1$, we see that $\deg X = d$ as we wished.

Example 2.3.4: Next we consider the situation of section 2.1.3. Let X be some curve that admits a very ample divisor D. We set $n := \dim |D|$ and let $\varphi\colon X \hookrightarrow \mathbb{P}^n$ be the embedding induced by D. Recall that, by Remark 1.5.22, $\mathrm{H}^i(X, \mathcal{O}_X) \cong \mathrm{H}^i(\mathbb{P}^n, \varphi_* \mathcal{O}_X)$ for all i. Observe, however, what happens when we twist $\varphi_* \mathcal{O}_X$:

$$\big(\varphi_* \mathcal{O}_X\big)(m) = (\varphi_* \mathcal{O}_X) \otimes \mathcal{O}_{\mathbb{P}^n}(m) \cong \varphi_*\Big(\mathcal{O}_X \otimes \varphi^*\big(\mathcal{O}_{\mathbb{P}^n}(m)\big)\Big)$$

by the projection formula ([Stacks, Lemma 18.7.2 (01E8)] or [Vak12, 17.3.H]). As above, the cohomology does not care for φ_* and we have $\varphi^*\big(\mathcal{O}_{\mathbb{P}^n}(m)\big) \cong \big(\varphi^* \mathcal{O}_{\mathbb{P}^n}\big)(m)$, by Proposition 1.4.16. Recall that, by the construction of φ, we have $\varphi^* \mathcal{O}_{\mathbb{P}^n}(1) \cong \mathcal{O}(D)$ and therefore

$$\begin{aligned} \chi\big(\varphi_* \mathcal{O}_X(m)\big) &= \chi\big(\mathcal{O}_X \otimes \mathcal{O}(D)^{\otimes m}\big) = \chi\big(\mathcal{O}_X(mD)\big) \\ &= m \cdot \deg D + \chi(\mathcal{O}_X) \end{aligned}$$

by Riemann-Roch. Hence, p_X is linear and the leading coefficient is $\deg D$, i.e. $\deg X = \deg D$ which, again, appears reasonable.

In summary, we see that the Euler characteristic does not care for the embedding of X into projective space, while the Hilbert polynomial does in fact depend on it very much.

2.3.2 Intersections of Curves on a Surface

Let X be a *surface*, i.e. a non-singular projective variety of dimension 2. By Definition 2.1.1, the prime divisors on X correspond to the (possibly singular) curves on X. Hence, we may picture any divisor on X as a weighted sum of curves. In order to understand the intersection of curves on X it will therefore suffice to understand the intersection of any two divisors.

Furthermore, "counting intersection points" of two curves a priori only makes sense if they intersect "nicely", i.e. they have no components or singularities in common. Considering divisors, our intersection theory should be independent of linear equivalence and count points when representatives intersect appropriately.

We will now attempt to make these concepts precise.

Definition 2.3.5: Let C and D be curves on X and $p \in X$ a point. Then we say that C meets D *transversally* at p, iff $\mathfrak{m}_p = (f, g)$, where $\mathfrak{m}_p$ is the maximal ideal of the local ring $\mathcal{O}_{X,p}$ of p and f and g are local equations[50] of C and D, respectively.

Remark 2.3.6: Observe that both C and D must be regular at p to meet transversally: Indeed, we have $\mathcal{O}_{C,p} = \mathcal{O}_{X,p}/(g)$ and $\mathcal{O}_{D,p} = \mathcal{O}_{X,p}/(f)$. But a quotient of a regular ring by a generator of its maximal ideal is always regular [Mat89, Theorem 14.2].

Fortunately, an intersection theory fulfilling all our wishes exists:

Theorem 5: *There is a unique pairing* $\mathfrak{Div}(X) \times \mathfrak{Div}(X) \longrightarrow \mathbb{Z}$*, which we call the* intersection pairing *and denote by* $C \cdot D$ *for any two divisors* C, D*, such that*

(a) if C and D are non-singular curves meeting transversally, then $C \cdot D = \sharp(C \cap D)$;

(b) it is symmetric: $C \cdot D = D \cdot C$;

(c) it is additive: $(C_1 + C_2) \cdot D = C_1 \cdot D + C_2 \cdot D$; and

(d) it depends only on the linear equivalence classes: if $C_1 \sim C_2$ then $C_1 \cdot D = C_2 \cdot D$.

More concretely, if C is an irreducible non-singular curve on X and D is any effective divisor meeting C transversally, we have

$$C \cdot D = \sharp(C \cap D) = \deg_C(\mathcal{L}(D) \otimes \mathcal{O}_C), \tag{2.5}$$

where $\mathcal{L}(D) \otimes \mathcal{O}_C$ is an invertible sheaf on C and $\deg_C$ *refers to the degree of the associated Weil divisor.*

Proof: [Har77, Theorem V.1.1 and Lemma V.1.3]. □

This allows us to view divisors as elements of a (graded) commutative ring. It is a special case of the Chow Ring $A(X)$ (see section 2.3.4).

In practice, curves will not always intersect transversally and it is tedious and cumbersome to find the appropriate representatives in the divisor class group. The trick is to instead count intersection points "correctly", by adding an appropriate weight.

[50] I.e. generators of the maximal ideal in the local rings $\mathcal{O}_{C,p}$ and $\mathcal{O}_{D,p}$, respectively.

Definition 2.3.7: (a) Recall that, for any R-module M, we may form a *composition series*

$$M = M_0 \supset M_1 \supset \cdots \supset M_r = 0$$

so that the factors M_i/M_{i+1} are simple modules and we call r the *length*[51] of M.

(b) Let C and D be effective divisors on X with no common irreducible component. For $p \in C \cap D$, we define the *intersection multiplicity* $(C \cdot D)_p$ of C and D at p to be the length of $\mathcal{O}_{X,p}/(f,g)$ (as a k-module[52]). Note in particular that this is a local property.

Indeed, this definition does not disappoint.

Proposition 2.3.8 ([Har77, Proposition V.1.4]): Let C and D be effective divisors on X having no common irreducible component. Then

$$C \cdot D = \sum_{p \in C \cap D} (C \cdot D)_p.$$

Remark 2.3.9: Observe that the intersection multiplicity is a generalisation of the order of a regular function as defined in Definition 2.1.4: If f and g are polynomials defining plane curves with f irreducible and $\overline{g}$ is the rational function induced on $\mathfrak{V}(f)$ by projecting g to $k[x,y]/(f)$, we have

$$(\mathfrak{V}(f) \cdot \mathfrak{V}(g))_p = \operatorname{ord}_{\mathfrak{V}(f),p} \overline{g}$$

for any $p \in \mathfrak{V}(f)$, cf. [Ful84, Example 1.2.1].

Example 2.3.10: (a) Consider the intersection of the two coordinate axes in $\mathbb{A}^2_k$. At the origin, i.e. the point corresponding to the ideal $O := (x,y)$, we have an intersection multiplicity of $\dim_k k[x,y]_{(x,y)}/(x,y) = 1$.

(b) Now consider the intersection of $C_1 := \mathfrak{V}(y - x^2)$ and $C_2 := \mathfrak{V}(y)$ in $\mathbb{A}^2$, i.e. the intersection of a quadratic parabola and the x-axis in the plane. Again, we compute the intersection number at O and find

$$(C_1 \cdot C_2)_O = \dim_k \mathcal{O}_{\mathbb{A}^2,O}\big/(y, y - x^2) = \dim_k k[x]\big/x^2 = 2.$$

Hence the intersection number corresponds to our ad hoc concept of a tangent line versus a transversal intersection.

Given any curve C, we can now use (2.5) to calculate the *self-intersection* of C: by Theorem 5 we simply have to intersect two linear equivalent representatives of C (considered as a divisor).

Example 2.3.11: Recall that $\mathfrak{Div}(\mathbb{P}^2) = \mathbb{Z}$ by Example 2.1.6. Hence we may choose any line (i.e. curve of degree[53] one) as generator, say $C := \mathfrak{V}(x_0)$. Now, any two lines are linearly equivalent, as both are hypersurfaces of degree one and—by virtue of projective space[54]—any two lines meet in a single point. Therefore, Theorem 5 yields $C^2 = 1$.

[51] Recall that this is well-defined by the Jordan-Hölder theorem [Mat89, p. 12].

[52] Note that in this case the length corresponds to the k-vector-space dimension.

[53] Observe that this is in accordance with Definition 2.3.1 by Example 2.3.3.

[54] Indeed, two lines are given by linear polynomials. We fix a line $ax_0 + x_1$ and intersect it with either $cx_0 + x_1$ or $x_0 + bx_1$. But unless they are equal, two such polynomials will always generate a maximal ideal, i.e. the two lines intersect at precisely one point.

Again by Example 2.1.6, any curve of degree d is linearly equivalent to dC and thereby the intersection of a curve of degree n with one of degree m yields something of degree nm. Considering the local intersection numbers, we obtain a special case of Bézout's theorem (cf. Proposition 2.3.20).

Consider the canonical divisor $\mathcal{K}$ on X. The self-intersection $\mathcal{K}^2$ gives rise to another interesting invariant of our curve. Moreover, $\mathcal{K}$ turns out to contain vital geometric information for all curves on X.

Proposition 2.3.12 (Adjunction Formula): Let C be a non-singular curve of genus g on X and $\mathcal{K}$ the canonical divisor on X. Then we have

$$2g - 2 = C \cdot (C + \mathcal{K}).$$

Proof: This requires fiddling around with (2.5) in order to apply Serre Duality, cf. [Har77, Proposition V.1.5]. □

Example 2.3.13: Proposition 2.3.12 provides an easy way to compute the degree of the canonical divisor $\mathcal{K}$ of $\mathbb{P}^2$: again, let $C := \mathfrak{V}(x_0)$ be a line on $\mathbb{P}^2$. Then C is isomorphic to $\mathbb{P}^1$, in particular it has genus 0 and—with the help of Example 2.3.11—the Adjunction Formula yields

$$-2 = 1 + \deg \mathcal{K},$$

i.e. $\deg \mathcal{K} = -3$.

To further demonstrate the power of the Adjunction Formula, we prove a special case of the Plücker Formula, which turns out to be a handy criterion for testing if a curve is a plane curve.

Proposition 2.3.14 (Plücker Formula): Let C be a non-singular plane curve of degree d. Then

$$g = \frac{1}{2}(d-1)(d-2).$$

Proof: By Example 2.3.11, $\mathfrak{Cl}(\mathbb{P}^2)$ is generated by a line l and $C = dl$ in $\mathfrak{Cl}(\mathbb{P}^2)$. On the other hand, $\mathcal{K} = -3l$ by Example 2.3.13, hence Proposition 2.3.12 yields

$$2g - 2 = d(d-3),$$

as, Example 2.3.11, the intersection pairing is simply multiplication of degrees for $X = \mathbb{P}^2$. But this amounts to the Plücker Formula. □

2.3.3 Varieties in Projective Space

We now move to the case where X is a variety of dimension greater than 2. Unfortunately, describing the intersection of subvarieties becomes significantly more complicated and we must introduce more powerful techniques.

Sadly, going beyond a brief survey of the most elementary facts of this fascinating theory would lead us too far astray. Thus we must restrict ourselves to quickly gathering the most important results and then moving on.

We begin with the case of projective varieties intersecting hypersurfaces in a fixed $\mathbb{P}^n_k$. At first, we make some observations about dimensions. We follow [Har77, §I.7].

REMARK 2.3.15: Given two varieties $X, Y \subseteq \mathbb{P}^n$ the intersection will in general not be irreducible. Be that as it may, we observe that if $\dim X = r$ and $\dim Y = s$, every irreducible component of $X \cap Y$ has dimension greater or equal to $r + s - n$. Note that this is also true if we replace $\mathbb{P}^n$ by $\mathbb{A}^n$. In the projective case, however, we additionally have that if $r+s-n$ is non-negative, then $Y \cap Z$ is non-empty [Har77, Proposition I.7.1 and Theorem I.7.2].

REMARK 2.3.16: In addition, given some subscheme $Y \subseteq \mathbb{P}^n$ and $Y = Y_1 \cup Y_2$ with $\dim Y_i = \dim Y$ and $\dim(Y_1 \cap Y_2) < \dim Y$, the degree is additive, i.e. $\deg Y = \deg Y_1 + \deg Y_2$ [Har77, Proposition I.7.6(b)].

The greatest difficulty lies in generalising the notion of intersection multiplicity. Even in this rather special case, it is very technical.

Given a finitely generated graded module M over some Noetherian graded ring S, we define the *annihilator* of M as $\operatorname{Ann} M := \{s \in S \mid s \cdot M = 0\}$. Note that this is a homogeneous ideal in S. In this case, we call any prime $\mathfrak{p} \leq S$ that is minimal containing $\operatorname{Ann} M$ a *minimal prime of* M. Now, for any minimal prime $\mathfrak{p}$ of M, we may define the *multiplicity* of M at $\mathfrak{p}$ to be the length of $M_\mathfrak{p}$ over S_p. We write $\mu_\mathfrak{p}(M)$.

DEFINITION 2.3.17: Let $Y \subseteq \mathbb{P}^n$ be a variety of dimension r and $H \subseteq \mathbb{P}^n$ be a hypersurface not containing Y. Let $Y \cap H =: Z_1 \cup \cdots \cup Z_s$ be the irreducible components of the intersection and denote by $\mathfrak{p}_j$ the generic point of Z_j. Recall that, by Proposition 1.4.14 (a), Y and H correspond to homogeneous ideals I_Y and I_H in $S := k[x_0, \ldots, x_n]$. Then we define the *intersection multiplicity* of Y and H along Z_j as

$$i(Y, H; Z_j) := \mu_{\mathfrak{p}_j}\left(S/I_H + I_Y \right). \tag{2.6}$$

Note that, by Remark 2.3.15, $\dim Z_j = r - 1$ so the $\mathfrak{p}_j$ are in fact minimal primes as $\mathfrak{p}_j \supseteq I_H + I_Y$ is minimal, as Z_j is an irreducible component of $\mathfrak{V}(I_H + I_Y) = Y \cap H$.

EXAMPLE 2.3.18: Let $n = 2$. Given two distinct curves $C = \mathfrak{V}(f)$ and $D = \mathfrak{V}(g)$, the irreducible components of $C \cap D$ are points $p_1, \ldots, p_s$. At each point, the intersection multiplicity is

$$i(C, D; p_j) = \mu_{p_j}\left(k[x_0, x_1, x_2]/(f) + (g) \right) = \operatorname{length}\left(\mathcal{O}_{X,p}/(f, g) \right)$$

as localisation commutes with taking the quotient. Hence this notion of intersection multiplicity does in fact generalise Definition 2.3.7 (b).

EXAMPLE 2.3.19: Next, consider a curve $X \subseteq \mathbb{P}^n$ and an (irreducible) hypersurface $H := \mathfrak{V}(f) \subseteq \mathbb{P}^n$ for some polynomial[55] $f \in S$ that does not contain X. Again, by Remark 2.3.15, $\dim X \cap H \geq 1 + n - 1 - n = 0$, hence non-empty and of dimension 0, as X is irreducible. In analogy to Remark 2.3.9, the divisor (on X)

$$D := \sum_{p \in H \cap X} i(X, H; p) \cdot p$$

[55] Note that, as the polynomial ring is factorial, any height 1 prime ideal is principal so in fact every hypersurface is of this form [Eis95, Proposition 3.11b].

that we obtain corresponds to $(\operatorname{div}\overline{f})_0$, the divisor of zeros of $\overline{f}$, where $\overline{f}$ is the image of f in S/I where I is the ideal belonging to X by Proposition 1.4.14 (a).

We are now able to state the theorem of Bézout that relates intersection numbers to the degrees of the involved varieties. Note that this generalises Example 2.3.11.

Proposition 2.3.20 (Bézout): Let Y be a variety in $\mathbb{P}^n$ and H a hypersurface not containing Y. Let $Z_1, \dots, Z_s$ be the irreducible components of $Y \cap H$. Then

$$\sum_{j=1}^{s} i(Y, H; Z_j) \cdot \deg Z_j = (\deg Y)(\deg H).$$

Proof: [Har77, Thm I.7.7]. □

2.3.4 The Chow Ring and Chern Classes

In analogy to Theorem 5, it is possible to define an intersection theory on higher-dimensional schemes that give rise to a ring structure generalising the Picard group that is called the Chow ring. It bears striking similarities to the Homology ring in algebraic topology.

As we require it only peripherally, we must again revert to mentioning only a few basic facts, doing little justice to the complexity of this theory. We follow [Har77, Appendix A], for a comprehensive introduction, see e.g. [Ful84] or [Stacks, Chapter 29].

Let X be any k-variety. When intersecting two arbitrary subvarieties of, say, codimension r and s, we cannot expect to get an intersection *number*. Instead, we obtain a variety of codimension $r + s$ up to some form of "rational equivalence". This leads us to the following.

Definition 2.3.21: Consider $\hat{A}^r(X)$, the free abelian group generated by the subvarieties of codimension r of X. A *cycle of codimension r on X* is an element of $\hat{A}^r(X)$. Given any closed subscheme $Y \subseteq X$ of codimension r, we may associate a cycle $[Y] \in \hat{A}^r(X)$ to Y by defining $[Y] := \sum n_i Z_i$ where Z_i are the codimension r irreducible components of Y and $n_i := \operatorname{length} \mathcal{O}_{Y,\eta_i}$ for η_i, the generic point of Z_i. We call this the *cycle associated to* Y.

In analogy to Definition 2.1.5, we define rational equivalence: for any $(r + 1)$-dimensional subvariety $W \subseteq X$ and any $f \in k(W)^\times$, we define a $(n - r)$-cycle (where $n = \dim X$) by

$$[\operatorname{div} f] := \sum \operatorname{length} \mathcal{O}_{W,V} \big/ (f) \cdot V$$

where the sum is taken over all dimension r subvarieties of W.

An r-cycle Z is called *rationally equivalent to* 0 if there exist finitely many f_i so that $Z = \sum [\operatorname{div} f_i]$. These form a subgroup and we denote the quotient by $A^r(X)$ [Ful84, §1.3].

This gives rise to a graded group $A(X) := \oplus A^i(X)$ where the sum is taken from 0 to $\dim X$. As X is the only component of codimension 0, $A^0(X) = \mathbb{Z}$ and $A^r(X) = 0$ for $r > n$.

Proposition 2.3.22 ([Har77, Appendix A, Theorem 1.1]): In the case that X is a non-singular quasi-projective variety over k, there exists a unique bilinear intersection pairing $A^r \times A^s \longrightarrow A^{r+s}$ making $A(X)$ into a graded ring that we call the *Chow ring*.

This pairing is functorial in the following sense: any morphism of varieties $f\colon X \longrightarrow X'$ induces a ring homomorphism $f^*\colon A(X') \longrightarrow A(X)$ and this is compatible with composition.

It should come as no surprise that a useful definition of intersection numbers is again difficult. The intersection pairing comes together with a local intersection multiplicity: given subvarieties $Y, Z \subseteq X$ so that every irreducible component W_j of $Y \cap Z$ is of codimension equal to $\operatorname{codim} Y + \operatorname{codim} Z$, there exist integers $i(Y, Z; W_j)$ depending only on local data that satisfy $Y \cdot Z = \sum i(Y, Z; W_j) \cdot W_j$.

Serre provided a formula for this multiplicity. Recall that for any R-module M the functor $- \otimes_R M$ is right exact. By section 1.5.1, we may therefore form the left derived functors that we denote by $\mathfrak{Tor}_n^R(-, M)$. Then we have, in the above situation,

$$i(Y, Z; W) = \sum (-1)^i \operatorname{length} \mathfrak{Tor}_i^{\mathcal{O}_{X,\eta_W}} \left(\mathcal{O}_{X,\eta_W}\big/\mathfrak{p} \,,\, \mathcal{O}_{X,\eta_W}\big/\mathfrak{q} \right)$$

where $\mathfrak{p}$ and $\mathfrak{q}$ are the (local) ideals of Y and Z and η_W is the generic point of W.

We finish this section by drawing attention to a special construction of elements of $A(X)$. Let $\mathcal{F}$ be a locally free sheaf of rank r on X and $\mathbb{P}(\mathcal{F})$ the associated projective space bundle. If π is the structure morphism $\mathbb{P}(\mathcal{F}) \longrightarrow X$, π^* endows $A(\mathbb{P}(\mathcal{F}))$ with an $A(X)$-module structure.

Now observe that $A^1(X)$ corresponds to closed subschemes of codimension 1, i.e. Weil divisors, up to linear equivalence. Hence, by Theorem 1, $A^1(X) \cong \mathfrak{Pic}(X)$. This gives us an element $\xi := [\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)] \in A^1(\mathbb{P}(\mathcal{F}))$ as $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$ is an invertible sheaf. Note that in this case, $A(\mathbb{P}(\mathcal{F}))$ is a *free* $A(X)$ module of rank r and that we may choose $1, \xi, \xi^2, \dots, \xi^{r-1}$ as generators [Har77, Appendix A, A11].

Definition 2.3.23: We define the i-th *Chern class* $c_i(\mathcal{F}) \in A^i(X)$ by setting $c_0(\mathcal{F}) = 1$ and requiring

$$\sum_{i=0}^{r} (-1)^i \pi^* c_i(\mathcal{F}) \cdot \xi^{r-i} = 0$$

in $A(\mathbb{P}(\mathcal{F}))$. Note that as $A(\mathbb{P}(\mathcal{F}))$ is free of rank r, this determines the c_i uniquely.

Example 2.3.24: Let $\mathcal{F}$ be an invertible sheaf, i.e. $\mathcal{F} = \mathcal{O}(D)$ for some divisor D. Then we find an affine cover U_i of X and $t_i \in \mathcal{F}(U_i)$ so that $\mathcal{F}|_{U_i} = t_i \cdot \mathcal{O}_{U_i}$. In particular, $\mathcal{F}(U_i)$ is a free $\mathcal{O}(U_i)$-module of rank 1 and

$$\operatorname{Sym} t_i \mathcal{O}(U_i) \cong \mathcal{O}(U_i)[t_i],$$

i.e. a polynomial ring over $\mathcal{O}(U_i)$. Therefore, $\mathbb{P}(\mathcal{F}) \cong X$ and $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1) = \mathcal{F}$, as $\mathcal{F}$ is locally generated by the t_i, the degree 1 elements of $\mathcal{O}(U_i)[t_i]$. Consequently, we have $c_1(\mathcal{F}) = D$.

2.4 The Canonical Embedding

Recall that in section 2.1.3 we established that a base-point-free linear system induces a morphism into projective space and saw a criterion for this morphism to be an immersion.

We will now apply the techiques of section 2.2 to translate this into a more "workable" condition.

In particular, given a curve, this will allow us to determine if the canonical sheaf is very ample and in this case the Riemann-Roch theorem will give us a deeper understanding of the geometry of the image of our curve in projective space.

Throughout this section, X is a non-singular curve of genus g over k.

2.4.1 Linear Systems and Projective Embeddings

We would like to determine whether the map to projective space induced by a linear system is a morphism and more specifically an embedding only by virtue of the dimension of its linear system. We begin by giving a criterion for a linear system to be base-point-free only in terms of its dimension. See also [Har77, Proposition IV.3.1] or [Vak12, §20.2].

Lemma 2.4.1: Let D be a divisor on X. Then $|D|$ has no base points if and only if $\dim|D-p| = \dim|D| - 1$ for every closed $p \in X$.

Proof: By Lemma 2.2.2 and Remark 2.2.3, if p is no base point, there exists some $D' \in |D|$ so that $\mathrm{l}(D'-p) = \mathrm{l}(D') - 1$. Therefore, $\dim|D-p| = \dim|D| - 1$, as $\dim|D| = \dim|D'| = \mathrm{l}(D') - 1$. On the other hand, by Remark 2.2.3, if $\dim|D-p| \neq \dim|D| - 1$, then $\dim|D| = \dim|D-p|$ and therefore $p \in \operatorname{supp} D'$ for every $D' \in |D|$ by Lemma 2.2.2.□

Recall that in Proposition 2.1.23, we provided a criterion to determine whether a line bundle is very ample, i.e. gives rise to a closed embedding into some $\mathbb{P}^n$. We are now able to restate this in a form that again requires only knowledge of the dimension of linear systems. Note that we say that a divisor D is *very ample* iff $\mathcal{O}(D)$ is.

Proposition 2.4.2: D is very ample if and only if $\dim|D-p-q| = \dim|D| - 2$ for every (not necessarily different) points $p, q \in X$.

Proof: In any case, we may assume that $|D|$ has no base points. Indeed, if D is very ample, in particular it induces a morphism into some $\mathbb{P}^n$ and hence must be base-point-free. On the other hand, as Remark 2.2.3 implies that $\dim|D| - \dim|D-p| \in \{0,1\}$, Lemma 2.4.1 assures that any $|D|$ satisfying $\dim|D-p-q| = \dim|D| - 2$ for all $p, q \in X$ has no base points.

We now test the criterion established in Proposition 2.1.23. Note that the linear system $|D-p|$ consists of all effective divisors $D' = D - p + \operatorname{div} f$ for some $f \in k(X)^\times$. In particular, $|D|$ separates points if and only if for any distinct $p, q \in X$, q is no base point of $|D-p|$, as this is the case iff there exists some $D' \in |D|$ so that $D' - p$ is effective but $D' - p - q$ is not. In this case, $\dim|D-p-q| = \dim|D-p| - 1 = \dim|D| - 2$ by Lemma 2.4.1 as p is no base point of $|D|$.

$|D|$ separates tangent vectors iff, given any p, we find a $D' \in |D|$ so that $n_p = 1$ in D', i.e. $D' - p$ is effective but $D' - 2p$ is not. In other words, p is no base point of $|D-p|$. Again, by Lemma 2.4.1, this is equivalent to $\dim|D-2p| = \dim|D| - 2$ as $|D|$ is base-point-free.□

Example 2.4.3: Let D be a divisor. Recall that if D is non-special, by Riemann-Roch, $\dim|D|$ depends only on $\deg D$. Therefore, $\dim|D-p| = \dim|D|-1$ and $\dim|D-p-q| = \dim|D|-2$ for any p, q if the divisors $D-p$ and $D-p-q$ are non-special, respectively. Now, by Example 2.2.7, any D' is non-special if $\deg D' > 2g-2$ and thus $D-p$ is non-special if $\deg D \geq 2g$ and $D-p-q$ is non-special if $\deg D \geq 2g+1$. Hence, the first is a criterion for $|D|$ to be base-point-free and the second condition implies that $|D|$ is very ample.

2.4.2 Constructing the Canonical Embedding

What happens if we choose $D = \mathcal{K}$, the canonical divisor, in section 2.1.3? The observations of section 2.2 combined with section 2.4.1 will yield very concrete answers to this question. See also [Har77, §IV.5] and [Vak12, §20.7 and 20.8.A].

By Corollary 2.2.4, $\dim|K| = 2g-1$. Therefore, if $g = 0$, $|\mathcal{K}|$ is empty and if $g = 1$, $\dim|\mathcal{K}| = 0$, i.e. it determines the constant map to a point. The other cases, however, are more fruitful.

We begin with a technical observation.

Lemma 2.4.4: Let $p \in X$. If $\dim|p| > 0$ then $X \cong \mathbb{P}^1$.

Proof: Indeed, let p be any point with $\dim|p| > 0$. Then $|p|$ must contain some effective divisor $q \neq p$ of degree 1. But this means there exists some rational function $f \in k(X)^\times$ with $\operatorname{div} f = p - q$ and this induces a morphism $\varphi\colon X \longrightarrow \mathbb{P}^1$ as in Remark 2.0.2 (note that this is the same morphism that is induced by the 2-dimensional linear system $\mathfrak{d}$ spanned by p and q that is base-point-free as $p \neq q$). Hence, $\varphi(p) = (s_0(p) : s_1(p))$ with s_0 and s_1 corresponding to elements of $|p|$ via Proposition 2.1.18. In particular, $\deg \operatorname{div} s_i = \deg p = 1$, i.e. each s_i has exactly one zero.

Now consider $0 := (0 : 1) \in \mathbb{P}^1$. Then $\deg 0 = 1$ as 0 is a closed point, but $\varphi^*(\{0\}) = p = \deg\varphi \cdot \deg 0$ (see Remark 2.1.7), as s_0 has only one zero by the above. Therefore, $1 = \deg\varphi = [k(\mathbb{P}^1) : k(X)]$, i.e. the function fields are isomorphic. But then, by Remark 2.0.2, $X \cong \mathbb{P}^1$. See also [Vak12, Proposition 20.4.1] and [Har77, Example II.6.10.1]. □

Proposition 2.4.5: If X is of genus $g \geq 2$ then $|\mathcal{K}|$ is base-point-free.

Proof: We use Lemma 2.4.1. Again, by Corollary 2.2.4, $\dim|K| = g-1$. Consider any $p \in X$. By Theorem 3, we have

$$\dim|p| - \dim|\mathcal{K} - p| = \deg|p| - g + 1 = 2 - g.$$

But as $g \geq 2$, X is not rational, hence $\dim|p| = 0$ for all p by Lemma 2.4.4 and therefore $\dim|\mathcal{K} - p| = \dim|\mathcal{K}| - 1$. □

We can even say in which cases this map is an embedding.

Proposition 2.4.6: If X is of genus $g \geq 2$. Then $\mathcal{K}$ is very ample if and only if X is not hyperelliptic.

Proof: We now apply the criterion for $\mathcal{K}$ to be very ample, Proposition 2.4.2. Given any $p, q \in X$, $\mathcal{K}$ is very ample iff $\dim|\mathcal{K} - p - q| = \dim|\mathcal{K}| - 2$. We know $\dim|\mathcal{K}| = g - 1$. What can we say about $\dim|\mathcal{K} - p - q|$? Riemann-Roch yields

$$\dim|p + q| - \dim|\mathcal{K} - p - q| = \deg(p + q) - g + 1 = -(g - 1 - 2).$$

Hence, it remains to show that $\dim|p + q| = 0$ iff X is not hyperelliptic. As $p + q$ is effective, $\dim|p+q| \geq 0$. But if $\dim|p+q| \geq 1$, it admits a one-dimensional subspace, i.e. a $\mathfrak{g}^1_2$. In other words, X is hyperelliptic (Definition 2.1.20). Conversely, any $\mathfrak{g}^1_2$ contains a divisor of the form $p + q$ with $\dim|p + q| = 1$ so we are done. □

Recall that by Corollary 2.2.5 any genus 2 curve is hyperelliptic. Hence, we may conclude for $g \geq 3$ and any non-hyperelliptic X, we obtain an embedding $\varphi_{\mathcal{K}} : X \longrightarrow \mathbb{P}^{g-1}$ that we call the *canonical embedding*. We call the image of X under this embedding a *canonical curve*. Note that, by Example 2.3.4, a canonical curve is of degree $\deg \mathcal{K} = 2g - 2$.

Remark 2.4.7: Observe that, conversely, any curve of genus g and degree $2g - 2$ in $\mathbb{P}^{g-1}$ is a canonical curve. Indeed, if $\iota\colon X \hookrightarrow \mathbb{P}^{g-1}$ is the corresponding embedding, $\iota^*\mathcal{O}_{\mathbb{P}^{g-1}}(1)$ corresponds to a $\mathfrak{g}^{g-1}_{2g-2}$ on X. But the only $\mathfrak{g}^{g-1}_{2g-2} =: |D|$ on any curve is $\mathcal{K}$, as Riemann-Roch yields

$$\dim|D| - \dim|\mathcal{K} - D| = \deg D - g + 1,$$

i.e. $\dim|\mathcal{K} - D| = 0$. Hence $|\mathcal{K} - D|$ is nonempty so $\mathcal{K} - D$ is linearly equivalent to an effective divisor of degree 0, i.e. the zero divisor. Therefore, $\mathfrak{g}^{g-1}_{2g-2} = |\mathcal{K}|$ and X is in fact a canonical curve.

2.4.3 The Geometric Version of the Riemann-Roch Theorem

It will often be significantly easier (or at least more practical) to study our curve X embedded in some projective space $\mathbb{P}^n$. By Proposition 2.4.6, if $g \geq 3$ and X is not hyperelliptic, the canonical divisor $\mathcal{K}$ gives rise to the canonical embedding

$$\varphi_{\mathcal{K}} : X \hookrightarrow \mathbb{P}^{g-1}$$

and—more generally—every base-point-free linear system $\mathfrak{d}$ induces some morphism into projective space as seen in section 2.1.3.

The idea is therefore to fix some morphism φ of X into some projective space $\mathbb{P}^n$ and to use this to "transport" Theorem 3 into projective space in order to work with the image of X under φ. We follow the lead of [ACGH85, §I.2] to make this notion of "transporting" more precise.

Let D be any effective divisor on X and $\varphi\colon X \longrightarrow \mathbb{P}^n$ any non-constant morphism. Furthermore, let $H =: \mathfrak{V}(f) \subset \mathbb{P}^n$ be a hyperplane, i.e. $f = a_0x_0 + \ldots + a_nx_n$ is a linear polynomial. Then the image of φ is a subscheme of dimension at most one[56] and thus, if $\varphi(X) \not\subset H$, the intersection of $\varphi(X)$ and H can be thought of as a divisor on the image of φ, which we shall

[56] Indeed, $\varphi(X)$ is irreducible and we obtain an inclusion of functions fields $k(\varphi(X)) \hookrightarrow k(X)$.

denote by H'. By Remark 2.1.7, the pullback $\varphi^*(H')$ is then a divisor on X (counting the intersection points of the image of X and H with their apropriate multiplicities, cf. Example 2.3.19) and this gives rise to the following definition:

Definition 2.4.8: The *image of D under φ* is the intersection of all hyperplanes $H \subset \mathbb{P}^n$ so that either $\varphi(X) \subset H$ or $\varphi^*(H') \geq D$ and we shall denote it by $\overline{\varphi(D)}$.

Remark 2.4.9: In the case that φ is an embedding and $D = p_1 + \cdots + p_m$ is a sum of distinct points, $\overline{\varphi(D)}$ is simply the linear span of the points $\varphi(p_i)$ in $\mathbb{P}^n$: again, by Example 2.3.19, $\varphi^*(H) \geq D$ iff H' is generated by a linear form that vanishes at the $\varphi(p_i)$ and the intersection of all these is the span. Similarly, if the points are not distinct, only the H are admitted that intersect the canonical curve with the appropriate multiplicities.

If we apply this to the canonical morphism $\varphi_{\mathcal{K}}$, what does this do to the Riemann-Roch formula (2.2)? Recall that $\mathrm{I}(\mathcal{K} - D)$ counted the number of (linearly independent) differential forms $\omega \in \mathrm{H}^0(X, \Omega^1_X)$ with $\operatorname{div} \omega \geq D$ (see, e.g. Remark 2.1.16). But this is exactly complementary to the space cut out by the hyperplanes H involved in creating $\overline{\varphi_{\mathcal{K}}(D)}$: indeed, by the construction of Theorem 2, the canonical curve is cut out by the sections ω. Hence, we have $\varphi^*(H') \geq \mathcal{K}$ for the hyperplanes H lying above $\overline{\varphi_{\mathcal{K}}(D)}$ by Example 2.3.19. See also [ACGH85, p. 12]. Therefore, they combine to give the entire space $\mathbb{P}^{g-1}$, yielding

$$\mathrm{I}(\mathcal{K} - D) = g - 1 - \dim \overline{\varphi_{\mathcal{K}}(D)}.$$

And plugging this into Theorem 3 completes our result:

Theorem 6 (Geometric Riemann-Roch): *Let D be an effective divisor on a non-hyperelliptic curve X with genus $g \geq 3$. Then*

$$\mathrm{I}(D) = \deg D - \dim \overline{\varphi_{\mathcal{K}}(D)}.$$

3 Moduli Spaces and Teichmüller Curves

Up to this point, we have only been concerned with the case of a single curve. However, seeing as many families of algebraic curves vary naturally in an algebraic way, the techniques introduced allow us to study entire families at once by parametrising them via a suitable scheme. The idea is to endow the family with a relative scheme structure, which we may then analyse. Unfortunately, things will seldom be that easy. For example, this does not work if our curves admit non-trivial automorphisms.

Having succeeded in parametrising our families via some geometric object, we may study the geometry of the resulting space. In this context, we will be interested in "nice" curves on our parameter space.

3.1 Moduli Problems

These classification problems are generally known as *moduli problems* and may be formalised in the following way[57]. First, we must define the notion of a "family of objects" over a base scheme and observe what happens when the base scheme changes.

Definition 3.1.1: Let B be a scheme.

(a) A *family over B* is a morphism $\mathfrak{X} \longrightarrow B$ (of schemes).

(b) A morphism between two families $\mathfrak{X} \longrightarrow B$ and $\mathfrak{X}' \longrightarrow B'$ consists of a pair of morphisms f and g making the following square commute.

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{\ f\ } & \mathfrak{X}' \\ \downarrow & & \downarrow \\ B & \xrightarrow{\ g\ } & B' \end{array}$$

This yields a category of families over schemes, albeit not a very useful one. Observe what happens when we change the base: Consider any morphism of schemes $\alpha\colon A \longrightarrow B$. Given

[57] A standard reference for this is [HM98, Chapter 1].

any family $\mathfrak{X} \longrightarrow B$, we may pull it back along α, obtaining a family $\mathfrak{X} \times_B A$ over A together with a morphism of families:

$$\begin{array}{ccc} \mathfrak{X} \times_B A & \overset{\alpha^*}{\dashrightarrow} & \mathfrak{X} \\ \downarrow & & \downarrow \\ A & \overset{\alpha}{\longrightarrow} & B. \end{array}$$

However, this *does not* yield a functor from the category of schemes to the *category of families* (as defined above), as this would require some consistent choice of family for each scheme. Rather, for every choice of base change $A \longrightarrow B$, we obtain a morphism of the *set of all* families over B to the *set of all* families over A. We solve this by instead considering all families over a fixed base B bundled together into a single object, $S(B)$, of our category; thereby we may view the category thus obtained as a subcategory of $\mathfrak{Set}$. In fact, this is an example of a *fibred category*, explaining the peculiar morphisms and behaviours. In particular, this allows us to retrieve some of the structure lost by retreating to $\mathfrak{Set}$. Unfortunately, delving deeper into this aspect would lead us too far astray; for further information see e.g. [Stacks, §4.30 (02XJ)].

In the cases we shall be concerned with, the base scheme B should possess "nice" geometric properties, i.e. be Noetherian and of finite type over an algebraically closed field k of characteristic 0. Thus the objects of our moduli problem are now the elements of $S(\operatorname{Spec} k)$, i.e. the fibres over the closed points of B.

Finally, we are able to express our moduli problem functorially.

Definition 3.1.2: A *moduli functor* is a contravariant functor $\mathcal{F}$ from $\mathfrak{Sch}$, the category of schemes over k, to the subcategory of $\mathfrak{Set}$ described above, obtained by the association

$$B \longmapsto S(B)/\sim$$

where $\sim$ is some equivalence relation[58] on the set of families over B.

In other words: in order to understand the moduli problem, it suffices to understand the moduli functor[59]. If our functor is representable, this translates into understanding the representing object. The reason is the following. Say that $\mathcal{F}$ is representable. Then $\mathcal{F}$ is naturally equivalent to the *functor of points* $\operatorname{Hom}_{\mathfrak{Sch}}(-, \mathcal{M})$ for some scheme $\mathcal{M}$. But this means that—for any base scheme B—there is a one-to-one correspondence between families $\mathfrak{X} \longrightarrow B$ (up to equivalence) and morphisms $B \longrightarrow \mathcal{M}$. In particular, any closed point $b \in B$ is sent to the point in $\mathcal{M}$ corresponding to the fibre $\mathfrak{X}_b$ over b.

Therefore, $\mathcal{M}$ is precisely the parameter space we had in mind. This motivates the following definition.

Definition 3.1.3: A *fine moduli space* (for the moduli problem $\mathcal{F}$) is a scheme $\mathcal{M}$ satisfying $\operatorname{Hom}_{\mathfrak{Sch}}(-, \mathcal{M}) \cong \mathcal{F}$, i.e. a representing object for the moduli functor $\mathcal{F}$.

[58] Usually this will encode the isomorphisms of the objects of our moduli problem.
[59] It is therefore practical to use the terms "moduli problem" and "moduli functor" interchangeably.

Representable functors have the additional advantage of possessing a *universal object.* Recall that in this situation, there is one morphism that stands out: denote the natural isomorphism by $\Psi\colon \mathcal{F} \longrightarrow \mathrm{Hom}(-, \mathcal{M})$ and consider $\mathrm{id}_{\mathcal{M}} \in \mathrm{Hom}(\mathcal{M}, \mathcal{M})$. This yields the universal object, i.e. the family

$$\Psi_{\mathcal{M}}^{-1}(\mathrm{id}_{\mathcal{M}}) =: \{C \longrightarrow \mathcal{M}\} \in \mathcal{F}(\mathcal{M}).$$

Now, given any morphism $\alpha\colon B \longrightarrow \mathcal{M}$, we obtain a family $\{D \longrightarrow B\} := \Psi_B^{-1}(\alpha)$. Alternatively, we obtain a family $D' := \{C \times_{\mathcal{M}} B \longrightarrow B\}$ by pulling C back along α. But then $\Psi_B(D') = \alpha$ also and as Ψ is an isomorphism, the two families must be equal.

Hence, we obtain *every family* $D \longrightarrow B$ in $\mathcal{F}(B)$ as the pullback of C along a *unique* morphism $\alpha\colon B \longrightarrow \mathcal{M}$. This justifies referring to C as the *universal family.*

Example 3.1.4: (a) Consider lines in the plane passing through the origin. An obvious choice for parameter space is $\mathbb{P}^1$. Let us see if we can translate this into the present situation.

Such a line will be given as the zero locus of $ax_0 - bx_1$ for $a, b \in k$ not simultaneously 0. Hence, two such equations $ax_0 - bx_1$ and $a'x_0 - b'x_1$ describe the same line iff $(a : b)$ and $(a' : b')$ are the same point in $\mathbb{P}^1$. Thus, the objects we wish to parametrise are

$$\mathfrak{V}(ax_0 - bx_1) = \mathrm{Spec}\ k[x_0, x_1]\big/(ax_0 - bx_1) \subset \mathbb{A}^2$$

subject to the above mentioned restrictions. Over $\mathbb{P}^1$, our family will therefore be

$$\{(x_0, x_1), (a : b) \mid ax_0 - bx_1 = 0\} \subset \mathbb{A}^2 \times \mathbb{P}^1.$$

Indeed, this is a scheme as $\mathbb{A}^2 \times \mathbb{P}^1$ consists of 2 copies of $\mathbb{A}^3 = \mathbb{A}^2 \times \mathbb{A}^1$ that are glued along $\mathbb{A}^2 \times \mathbb{A}^1 \setminus \{0\}$; as our polynomial is homogeneous in a and b, the local affine subschemes patch together. Additionally, this yields a natural projection from C onto $\mathbb{P}^1$ and the fibre over a point $(a : b) \in \mathbb{P}^1$ is precisely the subspace $\mathfrak{V}(ax_0 - bx_1)$ of $\mathbb{A}^2$.

Now let $\mathfrak{X} \longrightarrow B$ be any family of lines in the plane passing through the origin over some scheme[60] B. For every $p \in B$, the fibre $\mathfrak{X}_p$ is then some line

$$\mathfrak{X}_p = \kappa(p) \times_B \mathfrak{X} = \mathfrak{V}(a_p x_0 - b_p x_1) \subset \mathbb{A}^2$$

for some $(a_p : b_p) \in \mathbb{P}^1$. But then $p \longmapsto (a_p : b_p)$ is a map from B into $\mathbb{P}^1$ and this is a morphism because of the scheme structure of $\mathfrak{X}$. In turn, this ensures that $\mathfrak{X}$ is the pull-back of C along this morphism.

Conversely, any morphism $\alpha\colon B \longrightarrow \mathbb{P}^1$ gives rise to a family of plane lines through the origin over B by pulling back the family C.

Therefore, $\mathbb{P}^1$ is indeed a fine moduli space for this problem and C is the corresponding universal family.

[60] Recall our conventions about "nice" schemes!

(b) In fact, this is a special case of a *Hilbert scheme*. The Hilbert Scheme is a fine moduli space parametrising subschemes with a fixed Hilbert polynomial[61] in a fixed projective space. It has the advantage that it can be constructed concretely as a subscheme of the *Grassmannian scheme*. For an elaborate construction, see [ACG11, Chapter 1]. In particular, the above is a special case of [ACG11, Chapter 1, example 2.6].

Unfortunately, most "real-life" moduli functors are not representable (in the category of schemes). For a discussion of several examples and the exact obstructions to their admitting a fine moduli space, see [HM98, Chapter 1A and 2A].

There are two canonical resolutions to these obstructions. The first is to enlarge our category. For example, any moduli functor will be representable in the category $\mathfrak{Fun}(\mathfrak{Sch}, \mathfrak{Set})$ of functors. Indeed, according to the Yoneda Lemma [Mac98, p. 61], any such $\mathcal{F}$ is its own representing[62] object. However, endowing these more abstract objects with a suitable geometric structure requires considerable efforts and will sometimes not be possible or at least viable. Nevertheless, it is an approach we will return to shortly.

The other option is, of course, to relax our conditions on the moduli space $\mathcal{M}$. The most natural approach is to drop the requirement of admitting a natural isomorphism and be satisfied instead with Ψ being "only" a natural transformation. However, this no longer guarantees that the points of our moduli space correspond to our moduli objects, so we must require this explicitly to acquire a useful theory.

Definition 3.1.5: Let $\mathcal{M}$ be a scheme and $\Psi\colon \mathcal{F} \longrightarrow \mathrm{Hom}(-, \mathcal{M})$ a natural transformation. Then $\mathcal{M}$ is a *coarse moduli space* for $\mathcal{F}$ iff

(a) $\Psi_{\mathrm{Spec}\,k}\colon \mathcal{F}(\mathrm{Spec}\,k) \longrightarrow \mathrm{Hom}(\mathrm{Spec}\,k, \mathcal{M})$ is a bijection of sets, i.e. the closed points of $\mathcal{M}$ correspond to the objects of the moduli problem; and

(b) $\mathcal{M}$ is universal in the following sense: for any scheme $\mathcal{M}'$ and natural transformation $\Psi'\colon \mathcal{F} \longrightarrow \mathrm{Hom}(-, \mathcal{M}')$, we obtain a unique natural transformation τ making the following diagram commute:

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\Psi} & \mathrm{Hom}(-, \mathcal{M}) \\ & {}_{\Psi'}\searrow & \Big\downarrow {\scriptstyle \exists!\, \tau} \\ & & \mathrm{Hom}(-, \mathcal{M}'). \end{array}$$

Of course, the second condition is equivalent to requiring the existence of a unique morphism $\tilde{\tau}\colon \mathcal{M} \longrightarrow \mathcal{M}'$, as these stand in one-to-one correspondence[63] to the natural transformations

[61] In particular, this implies that genus, degree and Euler characteristic are fixed.

[62] Recall that $\mathfrak{Sch}$ becomes a subcategory of $\mathfrak{Fun}(\mathfrak{Sch}, \mathfrak{Set})$ via the Yoneda embedding. What happens to $\mathcal{F}$? Well, again by Yoneda, $\mathrm{Hom}_{\mathfrak{Fun}}(\mathrm{Hom}_{\mathfrak{Sch}}(-, S), \mathcal{F}) \cong \mathcal{F}(S)$ is a natural isomorphism where $\mathrm{Hom}_{\mathfrak{Sch}}(-, S)$ is just the image of S under the Yoneda embedding. Hence, $\mathcal{F}$ turns out to "be" the Hom functor $\mathrm{Hom}_{\mathfrak{Fun}}(-, \mathcal{F})$ and is therefore representable by definition.

[63] Observe that the association $\mathcal{M} \longrightarrow \mathcal{M}' \longmapsto \mathrm{Hom}(-, \mathcal{M}) \longrightarrow \mathrm{Hom}(-, \mathcal{M}')$ is again a covariant functor.

τ, again, by the Yoneda Lemma [Mac98, p. 61]. Hence the second condition ensures that a coarse moduli space—if it exists—is unique up to a unique isomorphism.

3.2 $\overline{\mathcal{M}_g}$ and $\overline{\mathcal{M}_{g,n}}$

The moduli spaces we are most interested in are $\mathcal{M}_g$, the space of non-singular, complete, connected curves of genus g and the closely related space $\mathcal{M}_{g,n}$ of curves that are additionally equipped with *n distinct* marked points. In other words, a point in $\mathcal{M}_{g,n}$ may be thought of as a tuple $(C; p_1, \dots, p_n)$ where C is a curve and the $p_i \in C$ are distinct points. Note in particular that any morphism is required to fix the *set* $\{p_1, \dots, p_n\}$. As $\mathcal{M}_g$ may be canonically identified with $\mathcal{M}_{g,0}$, the spaces $\mathcal{M}_{g,n}$ can be viewed as a direct generalisation.

Unfortunately, these spaces fail to be fine moduli spaces for a number of reasons, most notably because there are curves with non-trivial automorphisms. See also the examples in [HM98, §2A].

As mentioned in the previous section, there are two naïve solutions to this problem: either we change our conditions or we change our category. The first solution may be realised by simply "throwing away"[64] the points that don't suit us (i.e. have automorphisms) and consider instead the space $\mathcal{M}_g^0$ of remaining points. However, as this often results in counter-intuitive geometric consequences, this is often no feasible approach [HM98, p. 37].

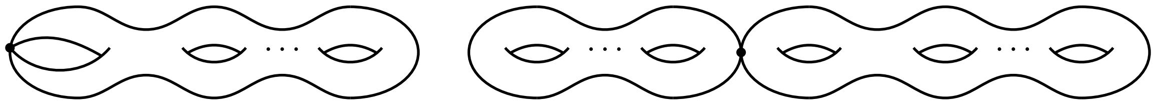
The other option is to turn to the language of *algebraic stacks*. On the downside, this approach is highly technical (see [Stacks, Definition 60.12.1 (026O)] or [DM69, Definition 4.1] for the "proper" definition of a stack). Delving into the depths of this theory leads us too far astray; we therefore choose to follow [HM98] and their approach of "speaking of stacks without defining stacks", i.e. attempting to remain in a logical framework that is as self-contained as possible. For our purposes, it will suffice to "know"[65] that there is such a thing as a *moduli stack* and that it behaves (more or less) as we expect it to, cf. also the discussion at the beginning of [HM98, §3D].

There exist multiple approaches to constructing the coarse moduli space $\mathcal{M}_g$, an overview may be found in [HM98, §2C] or [Lo000]. In particular, it can be shown that $\mathcal{M}_g$ is neither affine nor projective. The strategy best-suited for our needs consists of embedding curves in some suitable $\mathbb{P}^N$ where the images are parametrised by suitable Hilbert schemes. To retrieve $\mathcal{M}_g$, one must then take the quotient by the automorphism group $\mathrm{PGL}(N+1)$. Note that it is a priori all but clear that the resulting quotient is "geometrically well-behaved". This is a result of the so called *geometric invariant theory*, see [HM98, §4A] for a short introduction or [Mum65] for an extensive one. The procedure for $\mathcal{M}_{g,n}$ is similar, see [Knu83].

The main advantage—besides being purely algebraic—of this approach is that it allows us to also construct the projective closure $\overline{\mathcal{M}_{g,n}}$ of $\mathcal{M}_{g,n}$. The essential observation (in [DM69]) was to admit a slightly more general type of curves.

64 Alternatively, we may consider $\mathcal{M}_{g,n}$ with n large enough to fix all automorphisms. This is possible as the number of automorphisms is bounded by $84(g-1)$ (for $g \geq 2$, see e.g. [GH78, pp. 275–276]) for any genus g curve.

65 Perhaps "believe" is the more accurate term in this context.



(a) (b)

Figure 3.1: The curves pictured lie dense in the boundary $\overline{\mathcal{M}_g} \setminus \mathcal{M}_g$. Curves of type (a) are irreducible of genus $g-1$ with a single node and lie dense in δ_0. On the other hand, curves of type (b) are reducible consisting of non-singular components of genus i and $g-i$, respectively, joined at a node and lie dense in each δ_i.

Definition 3.2.1: A *stable curve* is a complete connected curve that has only nodes[66] as singularities and admits only finitely many automorphisms. A *stable n-pointed curve* is a complete connected curve with only nodes as singularities together with distinct smooth points $p_1, \ldots, p_n \in C$ so that $(C; p_1, \ldots, p_n)$ has only finitely many automorphisms.

Recall that any curve of genus $g \geq 2$ has at most $84(g-1)$ automorphisms. Therefore a connected curve can only fail to have a finite automorphism group if it contains a rational component with less than 3 marked/singular points or a genus 1 component with no marked/singular points.

Proposition 3.2.2: There exist coarse moduli spaces $\overline{\mathcal{M}_g}$ and $\overline{\mathcal{M}_{g,n}}$ of stable curves and n-pointed stable curves for all g, n so that $3g-3+n > 0$. These spaces are projective varieties of dimension $3g-3+n$.

Proof: A proof is outlined in [HM98, Chapter 4] or [Lo000]. Some of the omitted technical details may be found in [ACG11, Chapters XII and XIV]. □

Note in particular that in the language of stacks, it is also possible to define the Picard group. To avoid torsion problems, we limit this discussion to the rational Picard group (where $\mathfrak{Pic}_{\mathbb{Q}}(\cdot) := \mathfrak{Pic}(\cdot) \otimes \mathbb{Q}$). Fortunately, the Picard group of the moduli stack is isomorphic to $\mathfrak{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}_g})$, see [HM98, §3D], notably [HM98, Proposition 3.88].

What can we say about the *boundary* $\Delta := \overline{\mathcal{M}_g} \setminus \mathcal{M}_g$? Clearly it must consist of all stable curves that have at least one node. Now, it can be shown that the locus of curves with exactly δ nodes is of codimension δ in $\overline{\mathcal{M}_g}$ and deformation theory tells us that the locus of curves with more than δ nodes lies in the closure of those with exactly δ nodes, see e.g. [HM98, §3A and §3B] or [DM69]. Therefore, Δ is a divisor and consists of components that are the closures of curves with *exactly* one node. If we restrict our attention to the generic elements, we see that they are easy to categorise: either it is irreducible or consists of two irreducible non-singular components, one of genus i and one of genus $g-i$, joined together at a node (see[67] figure 3.1). We denote these by Δ_i for $i \in \{0, \ldots, \lfloor \frac{g}{2} \rfloor\}$. In particular, these give rise to elements δ_i and $\delta := \sum \delta_i$ of $\mathfrak{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}_g})$.

[66] Intuitively, it should be clear what a node or "double point" is. The precise definition is rather technical, involving formal completion, and may be found in, e.g., [EH00, Definition V-31].

[67] Note that these pictures are a bit misleading. The nodes occur where (components of) curves meet transversally and this cannot be pictured in less than 4 real dimensions. Nonetheless, it may be helpful to picture boundary points in this way, cf. also the discussion on [HM98, p. 50].

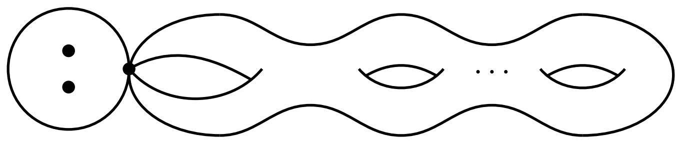


Figure 3.2: A point of $\delta_{0;\{1,2\}}$. The locus of all such points lies dense in the component.

Another class of divisors we shall be concerned with is the following. For any family

$$\pi\colon \mathfrak{X} \longrightarrow B$$

of stable curves, consider the *relative dualising sheaf*[68] $\omega_\pi := \omega_{\mathfrak{X}/B}$. Then we define the *Hodge bundle*

$$\Lambda_B := c_1\Big(\pi_*(\omega_{\mathfrak{X}/B})\Big),$$

where c_1 is the first Chern class (Definition 2.3.23), and this gives rise to a line bundle on $\overline{\mathcal{M}_g}$, see [HM98, p. 61] or [AC87, §2]. We denote the associated class by λ.

This allows a complete description of the Picard group.

Proposition 3.2.3: For $g \geq 3$, $\mathfrak{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}_g})$ is generated freely by λ and the δ_i while $\mathfrak{Pic}_{\mathbb{Q}}(\mathcal{M}_g)$ is generated freely by λ.

Proof: [AC87, Theorem 1]. □

The case of pointed curves is more technical. We adopt the notation of [CM11].

Observe that the space $\mathcal{M}_{g,1}$ has a special significance: in a way, it fulfills the purpose of the universal curve C_g over $\mathcal{M}_g$—at least for those points in $\mathcal{M}_g$ with no non-trivial automorphisms, cf. also the discussion in [HM98, §2B]. In any case, we have a forgetful map $\pi\colon \mathcal{M}_{g,1} \longrightarrow \mathcal{M}_g$ sending any $(C;p)$ to C, if $g \geq 2$. Again, this gives rise to a dualising sheaf $\omega_\pi := \omega_{\mathcal{M}_{g,1}/\mathcal{M}_g}$. Now, starting in $\mathcal{M}_{g,n}$, we have n different maps

$$\pi_i\colon \mathcal{M}_{g,n} \longrightarrow \mathcal{M}_{g,1}, \quad (C;p_1,\ldots,p_n) \longmapsto (C;p_i).$$

By Proposition 1.4.16, this allows us to define

$$\omega_{i,\mathrm{rel}} := \pi_i^*(\omega_\pi) \in \mathfrak{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}_{g,n}}).$$

Additionally, when talking of boundary components in $\overline{\mathcal{M}_{g,n}}$, we must keep track of the marked points. So take any subset $S \subset \{1,\ldots,n\}$ and let $\delta_{i;S}$ be the boundary divisor with a generic element consisting of a genus i and a genus $g-i$ component joined at a node with the additional requirement that the points p_j, $j \in S$ lie on the genus i component. Observe, in particular, that besides the "classical" (generic) δ_0 boundary component, we now have additional $\delta_{0;S}$ components (see figure 3.2), if $|S| \geq 2$, as stable curves are permitted to contain rational components with three marked or singular points.

[68] For "nice" $\mathfrak{X}$ and B this is just the canonical sheaf. However, to cope with singularities etc., the definition needs to be generalised. See [HM98, pp. 82–85] for a short summary or [Vak12, Chapter 31].

Proposition 3.2.4: $\mathfrak{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}_{g,n}})$ is generated by λ, $\omega_{i,\mathrm{rel}}$ for $i = 1, \dots, n$, δ_0 and $\delta_{i;S}$ for $i = 0, \dots, \lfloor \frac{g}{2} \rfloor$ and $S \subseteq \{1, \dots, n\}$, with the additional requirement that $|S| \geq 2$ if $i = 0$ and $1 \in S$ if $i = \frac{g}{2}$.

Proof: Originally, this result is [AC87, Theorem 2] but with slightly different generators; this version and the relationship between the generating sets may be found in [CM11, Theorem 2.2] or [Log03, §2]. □

We conclude this section by defining the *slope* of a divisor: given

$$\mathfrak{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}_g}) \ni D = a\lambda - \sum b_i \delta_i,$$

with $b_0 \neq 0$, we define the slope of D as $s(D) := \frac{a}{b_0}$.

3.3 The Divisor $\mathfrak{Lin}_3^1$

We now want to turn our attention to specific divisors on moduli spaces. Given a certain geometric property P, it is natural to study the locus of curves in some $\mathcal{M}_{g,n}$ that satisfy P. If this locus is in fact a divisor, we speak of a *geometric divisor* and say that P is a *codimension 1 property*. In most cases, P shall consist of requiring the admittance of some $\mathfrak{g}^r_d$.

Geometric divisors have been a vital tool in the study of the geometry of moduli spaces. Notably, Harris and Mumford used the Hurwitz divisor consisting of curves admitting a $\mathfrak{g}^1_k$ in their proof that $\mathcal{M}_g$ is of general type for odd $g \geq 25$ [HM82]. More recently, Logan used a series of geometric divisors to determine the Kodaira dimension of various pointed moduli spaces [Log03].

As seen in section 3.2, it is often easier to work in the projective closure, $\overline{\mathcal{M}_{g,n}}$ and in this case, the (free) generators of the divisor class group are known. Hence, we may express any divisor by giving coefficients with respect to the generating classes. Note also that we may view any divisor as an element of the Chow ring and in that respect it seems very natural to intersect divisors and curves as this yields intersection *numbers* (cf. Proposition 2.3.22). By checking the intersection of a curve with the generators of the Picard group, we may then compute the coefficients of the divisor in $\mathfrak{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}_{g,n}})$. We refer to the curves being used in such a procedure as *test curves*. Examples may be found in, e.g., [HM98, §3F], [Log03] or [Far09].

It remains to check whether requiring curves to admit a specific $\mathfrak{g}^r_d$ is indeed a codimension 1 property. This can be achieved using results of the classical Brill-Noether theory. A brief introduction may be found in, e.g., [HM98, §5A and §6F].

First we define the set of points in $\mathcal{M}_{g,n}$ (where $n := r + 1$)

$$\mathfrak{Lin}^r_d := \{(C; p_1, \dots, p_n) \mid C \text{ admits a } \mathfrak{g}^r_d \text{ with sections vanishing at the } p_i\}.$$

In other words, any invertible sheaf $\mathcal{L}$ associated to a divisor of the $\mathfrak{g}^r_d$ satisfies

$$\dim \mathrm{H}^0(\mathcal{L} \otimes \mathcal{O}(-p_1 - \cdots - p_n)) \geq 1.$$

We will use the following criterion to check in which cases its projective closure gives rise to a divisor.

PROPOSITION 3.3.1: Let $\rho(g,r,d) := g-(r+1)(g-d+r)$ be the *Brill-Noether number*. Then $\overline{\mathfrak{Lin}^r_d}$ is a divisor on $\overline{\mathcal{M}_{g,n}}$ if $\rho(g,r,d)=0$.

Proof: This is the first part of [Far09, Theorem 4.6]. Note that our requirements are indeed equivalent to those of [Far09]: if $\rho(g,r,d)=0$, we have $gr=(r+1)(d-r)$ and therefore r divides d. Call the quotient $d/r =: s+1$ and clearly $g=s(r+1)$ as in the theorem. On the other hand, for any integers $r,s\geq 1$, setting $g:=s(r+1)$ and $d:=r(s+1)$ yields

$$\rho(g,r,d)=g-(r+1)(g-d+r)=(r+1)(s-sr-s+rs+r-r)=0. \qquad \square$$

EXAMPLE 3.3.2: Consider the locus of curves admitting a $\mathfrak{g}^1_3$. By Proposition 3.3.1, this yields a divisor if $g=2\cdot(3-1)=4$ and $n=2$. In other words, we obtain a divisor $\overline{\mathfrak{Lin}^1_3}$ on $\overline{\mathcal{M}_{4,2}}$ of curves that admit a $\mathfrak{g}^1_3$ with the non-zero section vanishing at the marked points.

3.4 Teichmüller Curves

As mentioned in the previous section, the study of the geometry of moduli spaces naturally leads not only to the study of divisors but also of curves. More specifically, we consider algebraic curves lying in $\mathcal{M}_{g,n}$ that may be extended to the boundary, see e.g. [Möl11, §3.1] for details.

Finding "interesting" curves is no easy task. We choose to concentrate on one particular class: Teichmüller curves. These have been studied intensively since their initial appearance in [Vee89]. Constructing them is quite technical at first, but the advantage is that there are explicit formulas for the intersections with the generators of $\mathfrak{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}_{g,n}})$ making them ideal candidates for test curves. For an extensive introduction, see [Möl11] and the references therein.

To obtain Teichmüller curves, we must endow our curves with additional structure. Therefore, we consider $\Omega\mathcal{M}_g$, the vector bundle of differential forms over $\mathcal{M}_g$ without the zero section, i.e. a point in $\Omega\mathcal{M}_g$ may be thought of as a pair (X,ω), where $X\in\mathcal{M}_g$ and ω is a global section of the canonical sheaf[69] on X. We call the pair (X,ω) a *flat surface*. See e.g. [Zor06] for a comprehensive introduction to the theory of flat surfaces.

Observe that there is a natural *stratification* of the space $\Omega\mathcal{M}_g$: by Corollary 2.2.4, ω has exactly $2g-2$ zeros; hence we may choose any partition $\mu := (m_1,\dots,m_k)$ of positive integers so that $\sum m_i = 2g-2$ and consider the subspace $\Omega\mathcal{M}_g(m_1,\dots,m_k)$ of points (X,ω) so that the coefficients of $\operatorname{div}\omega$ correspond to μ. In addition, we occasionally subdivide these strata into a *hyperelliptic* component $\Omega\mathcal{M}_g(\mu)^{\text{hyp}}$ consisting of those points (X,ω) for which X is hyperelliptic, i.e. admits a $\mathfrak{g}^1_2$. We denote the complement by $\Omega\mathcal{M}_g(\mu)^{\text{non-hyp}}$, see [Möl11, §2.3] for details.

As Teichmüller theory's origins lie in the study of Riemann surfaces (see e.g. [Hau+92, §6]), we must restrict ourselves to the case $k=\mathbb{C}$. In this case, we have an equivalence of the

[69] Take care not to confuse ω and ω_X! This awful notation is due only to historical reasons.

category of compact Riemann surfaces with holomorphic maps and non-singular projective curves (cf. e.g. [LAGI, §5] or [Mir95]). Therefore, we use the terms "curve" and "Riemann surface" interchangeably and we may think of $(X, \omega) \in \Omega\mathcal{M}_g$ as a Riemann surface of genus g with a non-zero holomorphic 1-form ω. If Z is the set of zeros of ω, this endows $X \setminus Z$ with a translation structure, i.e. an atlas of complex charts all whose transition functions are locally translations, and the group $\mathrm{SL}_2(\mathbb{R})$ acts on these translation structures, see e.g. [HS07, §2.1] for details. Note that this action respects the stratification, see e.g. [Möl11, Corollary 2.2]. Now, a *Teichmüller curve* is the image of the projection of a closed $\mathrm{SL}_2(\mathbb{R})$-orbit from $\Omega\mathcal{M}_g$ to $\mathcal{M}_{g,n}$, where the n marked points correspond to the points of Z on X, cf. [Möl11, §5].

Arguably, the "correct" way to define Teichmüller curves is as the image of a Teichmüller disk in $\mathcal{M}_g$, see e.g. [HS07, §2.4] for this approach and [Möl11, §5] to see that the two definitions coincide. Exploring which points give rise to Teichmüller curves yields a rich and interesting theory.

A strength of Teichmüller curves is that they provide a comparatively easy way of constructing many interesting curves in $\mathcal{M}_g$. We consider a special class of flat surfaces that turns out to be particularly fruitful: we say that (X, ω) is a *square-tiled surface* if X may be obtained as a covering of the torus, ramified over 1 point only and ω is the pull-back of the[70] differential form on the torus. This allows us to picture X as a series of squares where parallel sides are glued together. In particular, every square-tiled surface gives rise to a Teichmüller curve, cf. [Sch05, §1.5] or [Möl11, Proposition 5.3].

As mentioned, our interest in Teichmüller curves lies in intersecting them with divisors. Fortunately, given a Teichmüller curve C, we may give a very explicit description of the intersection with the generators of the Picard group. Note first that C is determined solely by points from a *single* stratum of $\Omega\mathcal{M}_g$, say $\mu := (m_1, \ldots, m_k)$. We may therefore define

$$\kappa_\mu := \frac{1}{12}\sum_{j=1}^{k}\frac{m_j(m_j+2)}{m_j+1} \tag{3.1}$$

for any Teichmüller curve. Additionally, given any such C, we denote its sum of *Lyapunov exponents* by $L(C)$. For the precise definition of these, we refer to [Möl11, §6] or [Kap11] and the references provided therein. The important fact for us is that $L(C)$ may be calculated by a formula provided by [EKZ12] for any given Teichmüller curve that is generated by a square-tiled surface.

Now, given a Teichmüller curve C generated by $(X, \omega) \in \Omega\mathcal{M}_g(m_1, \ldots, m_k)$, we denote its closure in $\overline{\mathcal{M}_{g,k}}$ by $\overline{C}$ and define the *slope* of C by

$$s(C) := \frac{\overline{C}\cdot\delta_0}{\overline{C}\cdot\lambda}. \tag{3.2}$$

Note that $\overline{C}$ avoids all boundary components except δ_0 so this definition can be seen as dual to the slope of a divisor, cf. e.g. [Möl11, Corollary 5.11].

[70] Note that as $g = 1$, we have $\mathfrak{l}(\mathcal{K}) = 1$ so there is only one differential form up to multiplication with scalars.

Proposition 3.4.1: We have the intersection numbers

$$\overline{C} \cdot \omega_{i,\mathrm{rel}} = \frac{\overline{C} \cdot \lambda - (\overline{C} \cdot \delta_0)/12}{(m_i + 1)\kappa_\mu}$$

as well as the identity

$$s(C) = 12 - \frac{12\kappa_\mu}{L(C)}.$$

Proof: [Möl11, Proposition 5.12] and [Möl11, Proposition 6.4]. □

4 Intersections of Curves and Divisors

Our goal is to study intersections of curves and divisors in $\mathcal{M}_{g,n}$, more precisely in $\mathcal{M}_{4,2}$. But before we can do this, we must take a detailed look at genus four curves. As we will see, these are closely related to certain quadrics in $\mathbb{P}^3$, so we will actually begin by studying those.

4.1 A Quadric in $\mathbb{P}^3$

Consider quadric surfaces in $\mathbb{P}^3$.
Definition 4.1.1: A *quadric surface* is a variety of dimension and degree 2.

In particular, any $\mathfrak{V}(f)$ for an irreducible polynomial $f \in k[x_0, \dots, x_3]$ that is homogeneous of degree two yields a quadric in $\mathbb{P}^3$.

Remark 4.1.2: Recall that, given any algebraically closed field that is not of characteristic 2, we are fortunate enough to possess a complete classification of quadrics, cf. e.g. [Küh06, §13.2] for an introduction. Indeed, for any quadric $Q \subset \mathbb{P}^n$, we may perform some change of coordinates so that Q is of the form $x_0^2 + \cdots + x_r^2$ for $0 \leq r \leq n$. In particular, for $n = 3$, this yields:

(a) $Q \cong \mathfrak{V}(x_0^2)$; or

(b) $Q \cong \mathfrak{V}(x_0^2 + x_1^2)$; or

(c) $Q \cong \mathfrak{V}(x_0^2 + x_1^2 + x_2^2)$; or

(d) $Q \cong \mathfrak{V}(x_0^2 + x_1^2 + x_2^2 + x_3^2)$.

Note, in particular, that in case (a), topologically, $\mathfrak{V}(x_0^2) = \mathfrak{V}(x_0)$ so we may identify Q with $\mathbb{P}^2$. As k is algebraically closed, in case (b), Q is reducible as $x_0^2 + x_1^2 = (x_0 + \sqrt{-1}x_1)(x_0 - \sqrt{-1}x_1)$. Clearly, in all other cases, Q is irreducible, as any factor that is no unit would have to be of degree 1 and this would result in mixed terms. Observe also that an irreducible quadric is singular if and only if it is in the isomorphism class of (c): Indeed, by the Jacobian criterion (cf. e.g. [GW10, §6.3 and §6.5]), at any point $p := (a : b : c : d) \in Q$, the tangent space T_p is isomorphic (as a vector space over k) to the kernel of the matrix

$$\mathcal{J}_p := \begin{pmatrix} 2a \\ 2b \\ 2c \\ 0 \end{pmatrix}$$

and this is of dimension[71] $3 = \dim Q + 1$ if and only if $p \neq (0 : 0 : 0 : 1)$, i.e. the "origin"

[71] Note that we are projective!

in $\mathfrak{D}(x_3)$. Indeed, in this case, $\mathcal{J}_p$ is the zero matrix, hence the kernel is of rank 4. For any other $p \in Q$, $\mathcal{J}_p$ is a rank 1 matrix, hence the kernel is of rank 3, i.e. Q is regular at p.

Conversely, the same argument shows that the quadrics of cases (a) and (d) are regular as the Jacobian matrix amounts to

$$\mathcal{J}_p := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathcal{J}_p := \begin{pmatrix} 2a \\ 2b \\ 2c \\ 2d \end{pmatrix},$$

respectively, and these are both always of rank 1 as $(0:0:0:0) \notin \mathbb{P}^3$.

4.1.1 The Regular Case

We now want to examine the case that our quadric is irreducible and regular. If Q amounts to $\mathbb{P}^2$ the situation is quite clear. Therefore, we concentrate on quadrics of type (d). Concretely, let $Q := \mathfrak{V}(xy - wz) \subset \mathbb{P}^3$, i.e. we apply a change of coordinates by substituting

$$w := \sqrt{-1}x_3 - x_2, \quad x := x_0 + \sqrt{-1}x_1, \quad y := x_0 - \sqrt{-1}x_1 \quad \text{and} \quad z := x_2 + \sqrt{-1}x_3.$$

In particular, Q is non-singular, hence a surface in the sense of section 2.3.2.

Lemma 4.1.3: $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ via the Segre embedding.

Proof: Recall that the Segre embedding (Proposition 1.2.32) describes a closed embedding of the product $\mathbb{P}^n \times \mathbb{P}^m$ into $\mathbb{P}^{mn+m+n}$. Concretely, in this case we have

$$\Psi\colon \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3, \quad \big((a:b),(c:d)\big) \longmapsto (ac:ad:bc:bd)$$

What is the image of this morphism? Clearly,

$$\operatorname{im}\Psi = \Big\{(w:x:y:z) \in \mathbb{P}^3 \;\Big|\; \exists (a:b),(c:d) : w = ac,\ x = ad,\ y = bc,\ z = bd\Big\}$$

But this amounts to Q as the four relations imply $xy = wz$. And as one of the coordinates must be non-zero for every such tuple, each of the four relations may be regained from the one equation. □

Remark 4.1.4: Next, we look at lines[72] on Q. By Lemma 4.1.3, there are two canonical projections

$$\pi_1, \pi_2\colon Q \cong \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1,$$

one for each factor. Now, over any point $p \in \mathbb{P}^1$, the fibres $\pi_i^{-1}(p)$ are $p \times \mathbb{P}^1$ and $\mathbb{P}^1 \times p$, respectively. In other words, we have two families of curves, each parametrised by $\mathbb{P}^1$. More concretely: fix a point $(a_0:b_0) \in \mathbb{P}^1$. What is the fibre over this point with respect to π_1? By Lemma 4.1.3,

$$(a_0:b_0) \times \mathbb{P}^1 = \{(a_0c : a_0d : b_0c : b_0d) \mid (c:d) \in \mathbb{P}^1\} \subset Q \subset \mathbb{P}^3. \tag{4.1}$$

[72] For a discussion of this using Fano schemes, see [EH00, §IV.3.2].

Without loss of generality, we may assume that $a_0 \neq 0$, i.e. $(a_0 : b_0) = \left(1 : \frac{b_0}{a_0}\right)$. This translates the points of (4.1) into those satisfying

$$\left(c : d : \tfrac{b_0}{a_0}c : \tfrac{b_0}{a_0}d\right) \subset \mathbb{P}^3,$$

i.e. those cut out by the two planes $x_2 - \frac{b_0}{a_0}x_0$ and $x_3 - \frac{b_0}{a_0}x_1$ in $\mathbb{P}^3$. Of course, fixing $(c_0 : d_0) \in \mathbb{P}^1$ yields a line $\mathfrak{V}(c_0x_1 - d_0x_0, c_0x_3 - d_0x_2) \subset Q$ by the same argument.

On the other hand, as any given point $p \in Q$ corresponds to a point

$$\big((a_0 : b_0), (c_0 : d_0)\big) \in \mathbb{P}^1 \times \mathbb{P}^1,$$

we find one line from each family going through p: $(a_0 : b_0) \times \mathbb{P}^1$ and $\mathbb{P}^1 \times (c_0 : d_0)$. Clearly[73], two lines in the same family do not intersect, while a pair of lines coming from both families will intersect in exactly one point.

Are there any other lines on Q? Note that any line is an intersection of two (unequal) hyperplanes by Remark 2.3.15. By Example 2.3.3, these hyperplanes have degree 1, hence by Bézout's theorem (Proposition 2.3.20) their intersection is irreducible and of degree 1. Now, clearly[74], any line on Q through p must lie in the projective closure H_p of the tangent plane T_p at p (cf. e.g. [Vak12, §13.3.2] or [GW10, §6.7]). But this is a hyperplane as Q is a regular surface. Therefore, by Example 2.3.3, it is of degree 1 and Bézout yields

$$\sum_j i(Q, H_p; Z_j) \cdot \deg Z_j = (\deg Q)(\deg H_p) = 2$$

for the irreducible components Z_j of $Q \cap H_p$. But the two lines constructed above both lie in this intersection and as they are each irreducible components of degree 1, there can be no third line passing through p.

Example 4.1.5: Concretely, we consider the point $p := (1 : 0 : 0 : 0)$. The corresponding point in $\mathbb{P}^1 \times \mathbb{P}^1$ is $\big((1 : 0), (1 : 0)\big)$. Therefore, by the observations regarding (4.1), we have $(1 : 0) \times \mathbb{P}^1 \cong \mathfrak{V}(x_2, x_3)$ and $\mathbb{P}^1 \times (1 : 0) \cong \mathfrak{V}(x_1, x_3)$. How does this relate to the tangent space? Note that $p \in \mathfrak{D}(x_0)$. Hence, we may give local coordinates of Q (cf. Remark 1.2.25):

$$Q \cap \mathfrak{D}(x_0) = \operatorname{Spec} k[x_1, x_2, x_3]\big/ x_3 - x_1x_2 \,.$$

Now, recall that the (affine) tangent space is given by

$$T_p = \mathfrak{V}\Big(\sum_i \tfrac{\partial(x_3 - x_1x_2)}{\partial x_i}(p)x_i\Big) = \mathfrak{V}(x_3)$$

and clearly $H_p \cap Q = \mathfrak{V}(x_3) \cap \mathfrak{V}(x_3 - x_1x_2) = \mathfrak{V}(x_1x_2, x_3) = \mathfrak{V}(x_1, x_3) \cup \mathfrak{V}(x_2, x_3)$ by Remark 1.2.4.

This is an example of a more general occurrence. We call any surface S that contains some family of lines l a *ruled surface*. Any family of lines on S is then called a *ruling of* S. Observe that Q is in fact a *doubly ruled surface*. The two rulings are the two families of lines described in Remark 4.1.4.

[73] As k is algebraically closed, all fibred products "behave well" by Remark 1.2.20.

[74] Indeed, any line on Q is cut out by linear homogeneous polynomials, hence these generate some ideal $I \supset (x_0x_3 - x_1x_2)$. In other words, they give rise to elements of $\left(\mathfrak{m}_p/\mathfrak{m}_p^2\right)^* \cong T_p$.

4.1.2 The Group $\mathfrak{Cl}\, Q$

Our next aim is to calculate $\mathfrak{Cl}\, Q$, the divisor class group of Q. It turns out that it has a pleasantly simple structure:
Proposition 4.1.6: $\mathfrak{Cl}\, Q \cong \mathbb{Z} \oplus \mathbb{Z}$.

To prove this we require a slightly technical result.
Lemma 4.1.7: Let X be a Noetherian, integral, separated scheme over k that is regular in codimension one. Then:

(a) if Y is an irreducible subset of codimension one, then there is an exact sequence

$$\mathbb{Z} \longrightarrow \mathfrak{Cl}\, X \longrightarrow \mathfrak{Cl}(X - Y) \longrightarrow 0,$$

where the first map is defined by $1 \longmapsto 1 \cdot Y$, the second one by assigning $P \cap (X - Y)$ to every prime divisor P on X;

(b) $\mathfrak{Cl}\, X \cong \mathfrak{Cl}(X \times_k \mathbb{A}^1_k)$.

Proof: cf. e.g. [Har77, Proposition II.6.5 and II.6.6]. □

Proof of Proposition 4.1.6: We follow [Har77, Example II.6.6.1]. The idea is to generalise Remark 2.1.7 to fit this case: again, denote by π_i the projections $Q \longrightarrow \mathbb{P}^1$. These give rise to homomorphisms

$$\pi_i^*\colon\ \mathfrak{Div}\,\mathbb{P}^1 \longrightarrow \mathfrak{Div}\, Q, \quad \sum n_p p \longmapsto \sum n_p \pi_i^{-1}(p).$$

Indeed, let $p \in \mathbb{P}^1$ be any point, then $\pi_i^{-1}(p)$ is a line in one of the two rulings (cf. Remark 4.1.4), in particular a closed integral subscheme of codimension one, i.e. a prime divisor, as required.

Note that for any $f \in k(\mathbb{P}^1)^\times = k(x)^\times$, $\pi_i^*(\operatorname{div} f)$ corresponds to the divisor $\operatorname{div} f$ of f considered as an element of $k(Q)^\times = k(x,y)^\times$. Thus the morphisms π_i^* descend to morphisms $\mathfrak{Cl}\,\mathbb{P}^1 \longrightarrow \mathfrak{Cl}\, Q$ and we will consequently treat them as such.

Now we use Lemma 4.1.7: To that end, we let $Y := p \times \mathbb{P}^1 = \pi_1^*(p)$, for some $p \in \mathbb{P}^1$, so that $X - Y = \mathbb{A}^1 \times \mathbb{P}^1$ and therefore $\mathfrak{Cl}(X - Y) \cong \mathfrak{Cl}\,\mathbb{P}^1 \cong \mathbb{Z}$ (cf. Example 2.1.6). Note that the first isomorphism is in fact given by π_2^* (via the correspondence of affine lines $\mathbb{A}^1 \times p'$ and points $p' \in \mathbb{P}^1$, see the proof of [Har77, Proposition II.6.6]) and it factors through $\mathfrak{Cl}\, Q$ by composition with the restriction map, as the function field does not care for Y. Hence, π_2^* (and by the same argument also π_1^*) is injective.

But now Lemma 4.1.7 yields the exact sequence

$$\mathfrak{Cl}\,\mathbb{P}^1 \cong \mathbb{Z} \longrightarrow \mathfrak{Cl}\, Q \longrightarrow \mathfrak{Cl}(\mathbb{A}^1 \times \mathbb{P}^1) \cong \mathbb{Z} \longrightarrow 0$$

and if we fix p as a generator of $\mathfrak{Cl}\,\mathbb{P}^1$, the first map is π_1^*, i.e. the sequence is in fact short and exact. However, as we have seen, $\mathfrak{Cl}(\mathbb{A}^1 \times \mathbb{P}^1)$ may be identified with $\mathfrak{Cl}\,\mathbb{P}^1$ via π_2^*. Therefore, sending $p' \in \mathbb{P}^1$ to $\mathbb{P}^1 \times p' \in \mathfrak{Cl}\, Q$, is a one-sided inverse to the restriction map $\mathfrak{Cl}\, Q \longrightarrow \mathfrak{Cl}(\mathbb{A}^1 \times \mathbb{P}^1)$. Indeed, $\pi_2^*(p') = \mathbb{P}^1 \times p'$ is restricted to the affine line $\mathbb{A}^1 \times p' \in \mathfrak{Cl}(\mathbb{A}^1 \times \mathbb{P}^1)$ and this corresponds to $p' \in \mathfrak{Cl}\,\mathbb{P}^1$ under the isomorphism.

Hence the sequence splits, which implies $\mathfrak{Cl}\, Q \cong \mathbb{Z} \oplus \mathbb{Z}$. □

Note that this implies that Q is not isomorphic to $\mathbb{P}^2$. In particular, compared to Remark 2.0.2, this shows that the correspondence of curves with their function fields cannot be extended to surfaces!

This result permits the following classification of curves on Q.

Definition 4.1.8: Let D be a divisor on Q. Then we say that D is *of type* (a, b), where (a, b) is the corresponding element in $\mathbb{Z} \oplus \mathbb{Z}$.

Taking this a step further, we may connect these results to section 2.3.2. Recall that Theorem 5 yields a unique pairing

$$\mathfrak{Cl}\, Q \times \mathfrak{Cl}\, Q = \mathbb{Z}^2 \times \mathbb{Z}^2 \longrightarrow \mathbb{Z}.$$

Now, as is evident from the proof of Proposition 4.1.6, lines contained in the two rulings of Remark 4.1.4, are of type $(1, 0)$ and $(0, 1)$, respectively. Hence, if we pick lines l of type $(1, 0)$ and m of type $(0, 1)$, the intersection pairing equates to $m \cdot l = \sharp(m \cap l) = 1$, as these meet in exactly one point by construction, and $l^2 = m^2 = 0$ as we may pick parallel linearly equivalent representatives and the intersection product only distinguishes up to linear equivalence. But as l and m generate $\mathfrak{Cl}\, Q$, this fixes the entire intersection product.

In particular, given divisors C and D of type (a, b) and (a', b'), respectively, the intersection product yields

$$C \cdot D = (al + bm) \cdot (a'l + b'm) = aa'l^2 + ab'l \cdot m + a'bm \cdot l + bb'm^2 = ab' + a'b.$$

4.1.3 The Singular Case

Let us briefly compare the regular quadric of type (d) to the singular quadric of type (c). Again, we perform a base change to obtain $Q' := \mathfrak{V}(xy - z^2) \subset \mathbb{P}^3$.

Remark 4.1.9: What can we say about the lines on Q'? As in the regular case, given a point $p \in Q'$, any line through p on Q' must lie in the intersection of the tangent plane T_p at p and Q'. By Remark 4.1.2, Q' has only one singular point: $O := (1 : 0 : 0 : 0)$ and we claim that *all* lines on Q' pass through O. Consider any $Q' \ni p =: (p_0 : p_1 : p_2 : p_3) \neq O$. Then the tangent plane at p is $T_p = \mathfrak{V}(p_2x + p_1y - 2p_3z) \subset \mathbb{P}^3$ (cf. [GW10, Example 6.4 and Proposition 6.10]). Clearly, $O \in T_p$. Now, calculating $T_p \cap Q'$ is no fun[75], so we revert to a trick: consider the hyperplane $\mathfrak{V}(z) \subset \mathbb{P}^3$. By virtue of $\mathbb{P}^3$ (cf. Remark 2.3.15), given any line and any hyperplane, the two must intersect, in particular any line in $T_p \cap Q'$ must also intersect $\mathfrak{V}(z)$. But we can calculate this intersection:

$$T_p \cap Q' \cap \mathfrak{V}(z) = \mathfrak{V}(p_2x + p_1y - 2p_3z, xy - z^2, z) = O$$

as $p \neq O$ implies that $p_1 \neq 0 \neq p_2$ (as $p \in Q'$). Therefore, any line on Q' through p must also pass through O and the existence of such a line is guaranteed by Bézout. But as a line is determined by two points, there can be at most one line through any point of Q' (with the obvious exception of O).

To summarise, the singular irreducible quadric Q' admits exactly one ruling.

[75] The reason is that the intersection is a line of multiplicity 2 and (2.6) suggests that this will be nasty.

4.2 Non-Hyperelliptic Genus 4 Curves

Let X be a non-hyperelliptic curve of genus 4. Then, by Theorem 2 and Example 2.3.4, the corresponding canonical curve has degree 6 and lives in $\mathbb{P}^3$. We shall now explore the relationship to quadric surfaces.

First, however, we should observe that it is indeed necessary to work in $\mathbb{P}^3$. As $g = 4$, X cannot be rational, so $\mathbb{P}^1$ is obviously disqualified.

Lemma 4.2.1: There is no closed immersion of X into $\mathbb{P}^2$, i.e. X is not a plane curve.

Proof: By Proposition 2.3.14, a plane curve of genus g must satisfy $2g = (d-1)(d-2)$, where d is the degree of C. But our curve X has genus 4 and therefore X cannot be a plane curve, as there is no integer d satisfying $8 = 2^3 = (d-1)(d-2)$. □

Now, X is no plane curve, so the obvious question is: what kind of geometric object in $\mathbb{P}^3$ contains X? It should come as no surprise that the answer is a quadric surface. In relevant cases, it will in fact be the quadric surface of section 4.1.

Proposition 4.2.2 (cf. [Har77, Example IV.5.2.2]): The image of X is contained in a unique irreducible quadric surface Q.

Proof: Let $\mathcal{I}$ denote the ideal sheaf of X in $\mathbb{P}^3$, i.e. let $\varphi\colon X \hookrightarrow \mathbb{P}^3$ be the canonical morphism and $\varphi^\sharp\colon \mathcal{O}_{\mathbb{P}^3} \twoheadrightarrow \varphi_*\mathcal{O}_X$ the corresponding morphism of sheaves. Then $\varphi^\sharp$ is surjective because φ is an immersion and we have

$$0 \longrightarrow \mathcal{I} \hookrightarrow \mathcal{O}_{\mathbb{P}^3} \twoheadrightarrow \varphi_*\mathcal{O}_X \longrightarrow 0. \tag{4.2}$$

We will now use a classical cohomological argument to locate our quadric. The idea is to pick non-zero elements of the vector space $\mathrm{H}^0(\mathbb{P}^3, \mathcal{I}(2))$, as these will be polynomials of degree 2 whose zero locus contains X. The reason is that any projective subscheme X' that contains X as a closed subscheme must have an ideal sheaf $\mathcal{I}'$ so that

$$\mathcal{I}' \hookrightarrow \mathcal{I} \quad \text{and hence} \quad \varphi'_*\mathcal{O}_{X'} \twoheadrightarrow \varphi_*\mathcal{O}_X$$

where φ' is the immersion of X' into $\mathbb{P}^3$ (as $\varphi'_*\mathcal{O}_{X'} \cong \mathcal{O}_{\mathbb{P}^3}/\mathcal{I}'$ and $\varphi_*\mathcal{O}_X \cong \mathcal{O}_{\mathbb{P}^3}/\mathcal{I}$ respectively).

Now—according to Serre (cf. Proposition 1.4.13)—we have

$$\mathcal{I} \cong \widetilde{\bigoplus_{n=0}^{\infty} \mathrm{H}^0(\mathbb{P}^3, \mathcal{I}(n))}$$

and each of these groups may be embedded[76] into $\mathrm{H}^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(n)) = k[x_0, \dots, x_3]_n$, that is the homogeneous polynomials of degree n, so any element of $\mathrm{H}^0(\mathbb{P}^3, \mathcal{I}(2))$ will yield a defining equation for a quadric surface containing X.

[76] Recall that taking global sections is left exact and twisting a sequence is exact. Indeed, a sequence of sheaves is exact if and only if it is exact on the stalks. Twisting the sequence amounts to tensoring with $\mathcal{O}(n)$ and since that sheaf is locally free of rank one, it's stalk is a flat module.

To show that such an element exists, we count dimensions. Taking cohomology of the twisted sequence yields the long exact sequence

$$0 \longrightarrow \mathrm{H}^0(\mathbb{P}^3, \mathcal{I}(2)) \longrightarrow \mathrm{H}^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) \longrightarrow \mathrm{H}^0(\mathbb{P}^3, \varphi_*\mathcal{O}_X(2)) \longrightarrow \cdots$$

As above, $\mathrm{H}^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) = k[x_0, x_1, x_2, x_3]_2$ and simply counting[77] (cf. Example 2.3.2) shows that this space has dimension 10.

As φ is the canonical embedding, $\big(\varphi_*\mathcal{O}_X\big)(m) \cong \varphi_*\big(\mathcal{O}_X \otimes \varphi^*\mathcal{O}_{\mathbb{P}^3}(m)\big) \cong \varphi_*\omega_X^{\otimes m}$ (cf. Example 2.3.4). Again, the cohomology disregards the embedding so we have

$$\mathrm{H}^0\Big(\mathbb{P}^3, \big(\varphi_*\mathcal{O}_X\big)(m)\Big) \cong \mathrm{H}^0(X, \omega_X^{\otimes m}).$$

In particular, $\dim \mathrm{H}^0(\mathbb{P}^3, \varphi_*\mathcal{O}_X(2)) = \dim \mathrm{H}^0(X, \omega_X^{\otimes 2}) = \mathrm{l}(2\mathcal{K})$, and we can use the Riemann-Roch Theorem (Theorem 3) to calculate this:

$$\mathrm{l}(2\mathcal{K}) - \mathrm{l}(\mathcal{K} - 2\mathcal{K}) = 12 - 4 + 1 = 9.$$

As $\deg \mathcal{K} = 2g - 2 = 6$, $\deg 2\mathcal{K} = 12$. But this implies that the divisor $2\mathcal{K}$ is non-special (cf. Example 2.2.7), so $\mathrm{l}(\mathcal{K} - 2\mathcal{K}) = 0$. Hence, $\dim \mathrm{H}^0(\mathbb{P}^3, \varphi_*\mathcal{O}_X(2)) = 9$ and therefore we find that

$$\dim \mathrm{H}^0(\mathbb{P}^3, \mathcal{I}(2)) \geq 1,$$

meaning that we find an element $0 \neq q \in \mathrm{H}^0(\mathbb{P}^3, \mathcal{I}(2))$ so that $Q := \mathfrak{V}(q)$ is indeed a quadric surface containing X.

It remains to show that Q is irreducible and unique. Say that (q) is no prime ideal. Then we would find polynomials a, b of degree one with $q = ab$. As the curve X is irreducible, that would imply a or $b \in \mathcal{I}$ and that in turn would mean that (without loss of generality) $X \subseteq \mathfrak{V}(a)$. But—$a$ being a linear polynomial—$\mathfrak{V}(a)$ is a hyperplane in $\mathbb{P}^3$, i.e. a $\mathbb{P}^2$, and that violates Lemma 4.2.1. Therefore (q) is a prime ideal and thereby Q is irreducible.

The fact that Q is unique follows from Bézout's theorem (Proposition 2.3.20): Say X is contained in another quadric Q'. By the same argument as above, Q' is irreducible and of degree 2, and X must be contained in an irreducible component of $Q \cap Q'$. But by Bézout, any such component is a curve of degree at most four. X being irreducible, it must lie completely in one of these components, Z. But as Z is also irreducible, we have $X = Z$ and this is a contradiction, as the two curves have different degrees. □

We can be even more precise in our locating of X:

Proposition 4.2.3: X is also contained in an irreducible cubic surface C. Furthermore, X is equal to the intersection $C \cap Q$.

Proof: To find the cubic surface, we use once more our cohomological trick: Of course, this time we must twist the sequence (4.2) three times before taking the cohomology:

$$0 \longrightarrow \mathrm{H}^0(\mathbb{P}^3, \mathcal{I}(3)) \longrightarrow \mathrm{H}^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) \longrightarrow \mathrm{H}^0(\mathbb{P}^3, \varphi_*\mathcal{O}_X(3)) \longrightarrow \cdots$$

[77] We choose two from four variables and do not care for the order: $\binom{4+2-1}{2} = 10$.

$\mathrm{H}^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$ is again simply the space of degree three polynomials so it has dimension $\binom{4+3-1}{3} = 20$ and—again, as above—$\dim \mathrm{H}^0(\mathbb{P}^3, \varphi_*\mathcal{O}_X(3)) = \mathrm{l}(3\mathcal{K})$, as X is a canonical curve. By Riemann-Roch, we have

$$\mathrm{l}(3\mathcal{K}) - \mathrm{l}(\mathcal{K} - 3\mathcal{K}) = 18 - 4 + 1 = 15$$

and—since $3\mathcal{K}$ is also non-special—we conclude that

$$\dim \mathrm{H}^0(\mathbb{P}^3, \mathcal{I}(3)) \geq 5.$$

This means that we have a five dimensional vector space to choose our cubic generators from. Now, Proposition 4.2.2 yielded a unique quadric $Q = \mathfrak{V}(q)$ containing X. Multiplying q with any linear polynomial l, it is evident that the $l \cdot q$'s span only a four dimensional vector space (as l is of the form $ax_0 + bx_1 + cx_2 + dx_3$), and that means that we find a cubic form c that is not divisible by q, i.e. so that $X \subset \mathfrak{V}(c) =: C$ and $Q \not\subset C$.

In particular, this implies that C too is irreducible. Indeed, if C were reducible, X would be contained either in a $\mathbb{P}^2$ or in a quadric that is not Q, both of which is impossible. Additionally, we are once again permitted to apply Bézout's theorem. The sum of weighted degrees of the irreducible components of the intersection $Q \cap C$ must equal $\deg Q \cdot \deg C = 2 \cdot 3 = 6$. But as X is also of degree 6 and must be contained in one of these irreducible components, X must inevitably equal the complete intersection. □

Now, as we observed in Remark 4.1.2, there are only three types (up to isomorphism) of irreducible quadrics in $\mathbb{P}^3$: the first is $\mathbb{P}^2$, the second is the regular quadric Q from Lemma 4.1.3 and the third is the singular quadric Q' from section 4.1.3. By Lemma 4.2.1, X does not lie in $\mathbb{P}^2$ and by Remark 4.1.4 and Remark 4.1.9, we can try to distinguish the other two cases by checking how many lines lie on the canonical curve.

Note that if X lies in a regular quadric, it is necessarily of type $(3, 3)$. Indeed, given one of the lines of the quadric, observe that it cannot be contained in the intersection $C \cap Q$, as this is an irreducible curve of degree 6. Hence, by Bézout's theorem Proposition 2.3.20, it must intersect C in 3 points (with multiplicity).

Remark 4.2.4: We can make this observation even more precise. Let l be any line in $\mathbb{P}^3$ that intersects X. As $X \subset Q$, l also intersects Q and, by Bézout, either $l \subset Q$ or l intersects Q in two points (with multiplicities). But we also know $X \subset C$ and by the same argument, l lies in C or intersects C in three points (with multiplicities). As l cannot lie in $Q \cap C = X$, any such l can intersect X in *at most* 3 points and if $X \cap l$ consists of exactly 3 points, $l \subset Q$, i.e. it must lie in a ruling.

4.3 $\mathfrak{Lin}_3^1$ and $\Omega\mathcal{M}_4(3,3)^{\text{non-hyp}}$

Finally, we are able to show that $\mathfrak{Lin}_3^1$ contains no points in $\Omega\mathcal{M}_4(3,3)^{\text{non-hyp}}$. To see that this is syntactically correct, note that a point $(X, \omega) \in \Omega\mathcal{M}_4(3,3)^{\text{non-hyp}}$ defines a point $(X, p, q) \in \mathcal{M}_{4,2}$ where p and q are the two zeros (each of order 3) of ω, cf. section 3.4. The following theorem is a minute generalisation of [CM11, Prop. 6.4].

Theorem 7: *For any $(X,\omega) \in \Omega\mathcal{M}_4(3,3)^{non\text{-}hyp}$ the corresponding point $(X,p,q) \in \mathcal{M}_{4,2}$ does not admit a $\mathfrak{g}_3^1$ with sections vanishing at p,q,r for some $r \in X$, i.e. (X,p,q) does not lie on $\mathfrak{Lin}_3^1$.*

Proof: As $(X,\omega) \in \Omega\mathcal{M}_4(3,3)^{\text{non-hyp}}$, the canonical divisor on X is $\mathcal{K} = \operatorname{div}\omega = 3p+3q$ where p,q are the zeros of ω, hence we see—cf. Definition 2.1.37—$\omega \cong \mathcal{O}_X(\mathcal{K}) = \mathcal{O}_X(3p+3q)$ for the canonical sheaf ω.

Additionally, X is of genus 4 and not hyperelliptic. Therefore, by Proposition 2.4.6 and Proposition 4.2.2, we may embed X into a uniquely determined quadric $Q \subset \mathbb{P}^3$ which must be of type (c) or (d) (in the language of Remark 4.1.2).

Let us assume that the corresponding point (X,p,q) of $\mathcal{M}_{4,2}$ lies on $\mathfrak{Lin}_3^1$. Then we find a $\mathfrak{g}_3^1$ on X containing a section s vanishing at p, q and r, i.e. $\operatorname{div} s \geq p+q+r =: D$. But this implies $\mathfrak{l}(D) \geq 2$ as we definitely find the constant functions and the (non-constant) section s in this space. Now we may apply Theorem 3 (Riemann-Roch), yielding

$$\mathfrak{l}(D) - \mathfrak{l}(\mathcal{K}-D) = \mathfrak{l}(p+q+r) - \mathfrak{l}(2p+2q-r) = 3-4+1 = 0$$

i.e. $\mathfrak{l}(2p+2q-r) = \mathfrak{l}(p+q+r) \geq 2$. Assume that $r \in \{p,q\}$, say $r = p$. Then $\mathfrak{l}(p+2q) = \mathfrak{l}(2p+q) \geq 2$.

On the other hand, if $r \neq p,q$, we have $\mathfrak{l}(2p+2q) \geq 3$ by Lemma 2.2.2. If we remove a point from this divisor, in the "worst case" the dimension will decrease by 1 (cf. Remark 2.2.3), yielding $\mathfrak{l}(2p+q) \geq 2$ and $\mathfrak{l}(p+2q) \geq 2$ respectively. Therefore we are in the same situation as above.

But this means nothing more than that the corresponding linear systems are of dimension at least 1 and hence we have found two $\mathfrak{g}_3^1$s consisting of the non-constant sections vanishing at $2p+q$ and $p+2q$, respectively.

What is the image (in the sense of Definition 2.4.8) of these linear systems on our quadric Q? The answer is provided by the geometric version of Riemann-Roch (Theorem 6): For any such divisor D, we have

$$\dim \overline{\varphi_{\mathcal{K}}(D)} = \deg D - \mathfrak{l}(D) = 3 - \mathfrak{l}(D)$$

and as $\mathfrak{l}(D) \geq 2$, we see that $\dim \overline{\varphi_{\mathcal{K}}(D)} \leq 1$. Assume for a moment that the dimension were 0. Then $\mathfrak{l}(D) = \deg D = 3$. But now, for any point P, the divisor $D-P$ is of degree 2 and—by Remark 2.2.3—$\mathfrak{l}(D-P) \geq 2$, in other words: $D-P$ gives rise to a $\mathfrak{g}_2^1$ and that contradicts the fact that X is non-hyperelliptic.

Therefore we see that $\dim \overline{\varphi_{\mathcal{K}}(D)} = 1$ and hence the image of the divisor D is a line in $\mathbb{P}^3$ for any D belonging to one of our $\mathfrak{g}_3^1$s. Now, as this line is spanned by the images (under $\varphi_{\mathcal{K}}$) of points belonging to the divisor, it must intersect Q in three points, counting multiplicities. By Bézout's theorem (Proposition 2.3.20), however, the sum of intersection multiplicities of Q and any line not contained in Q is 2 and therefore any such line must lie on Q, i.e. be a line in a ruling of Q.

But then we would have $l := \overline{\varphi_{\mathcal{K}}(2p+q)} = \overline{\varphi_{\mathcal{K}}(p+2q)}$ as both these lines pass through the points $\varphi_{\mathcal{K}}(p)$ and $\varphi_{\mathcal{K}}(q)$ and lines are determined by two points. However, this is a contradiction as any l is a ruling and any ruling meets X in exactly 3 points (cf. Remark 4.2.4). Therefore, both $p + 2q$ and $q + 2p$ would be equivalent to the intersection divisor $l \cap \varphi(X)$ and this is impossible as X is not rational (cf. Lemma 2.4.4). □

This result now permits us to use a Teichmüller curve contained in $\Omega\mathcal{M}_4(3,3)^{\text{non-hyp}}$ as a test curve to calculate the divisor class of $\mathfrak{Lin}_3^1$ in $\mathfrak{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}_{4,2}})$. In the following, we let δ_{other} denote some linear combination of the boundary components not involving δ_0. Recall that Teichmüller curves avoid δ_{other} altogether, justifying this notational laxity.

Proposition 4.3.1 (cf. [CM11, Proposition 2.6]): $\overline{\mathfrak{Lin}_3^1} = k(\omega_{1,\text{rel}} + \omega_{2,\text{rel}} + 8\lambda - \delta_0 - \delta_{\text{other}})$ in $\mathfrak{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}_{4,2}})$, for some constant k.

Proof: We pick an arbitrary Teichmüller curve C lying in $\Omega\mathcal{M}_4(3,3)^{\text{non-hyp}}$: take the one generated by

$$\pi_r = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10) \text{ and } \pi_u = (1\ 9\ 5\ 6\ 8), \tag{4.3}$$

i.e. corresponding to the flat surface obtained from

	a	b	c	d	e	f	g	h	j	k	
l	1	2	3	4	5	6	7	8	9	10	l
	h	b	c	d	j	e	g	f	a	k	

by gluing the sides with matching letters. Note that as all but 4 vertices, 20 edges and 10 faces are identified, the (topological) Euler characteristic is -6 and, consequently, the surface is indeed of genus 4. Furthermore, we obtain a (canonical) cover of the torus ramified at two points and as each ramification point is a conic singularity with cone angle 8π, the pullback of the canonical differential on the torus has two zeros, each of order 3. See also [Zor06, §3.3] for an introduction on flat surfaces with a complex structure. Clearly, our polygon does not admit enough symmetries to glue to a hyperelliptic curve (cf. [Zor06, §9.4]). In conclusion, the flat surface described is a point in $\Omega\mathcal{M}_4(3,3)^{\text{non-hyp}}$, as required.

By Theorem 7, we know that $C \cap \mathfrak{Lin}_3^1$ must be empty. What happens at the boundary?

Recall that the boundary $\overline{\mathcal{M}_{g,n}} \setminus \mathcal{M}_{g,n}$ consists of pointed nodal curves and [HM98, §3A] explains the geometry of these: we consider the *normalisation* (cf. e.g. [GW10, §12.11]) and "replace" the canonical sheaf by the dualising sheaf. Then such a nodal curve is of arithmetic genus g, related to that of its normalisation by the number of nodes (cf. [HM98, formula (3.1)]). While the geometric genus is no longer defined, Serre duality is still valid for the dualising sheaf. In other words, the space of global sections of the dualising sheaf is also of dimension g. Moreover, the dualising sheaf is invertible and of degree $2g - 2$ and, in particular, the Riemann-Roch theorem applies as in the regular case.

Note that, as $\overline{C}$ avoids all boundary components except δ_0 (cf. [Möl11, Corollary 5.11]), this is the only place where $\overline{\mathfrak{Lin}_3^1}$ and $\overline{C}$ could meet. Now, curves in δ_0 that could lie on $\overline{C}$ contain no separating nodes and at most two irreducible components, each containing at least one zero of the differential (cf. [Möl11, Corollary 5.11]). Indeed, if (X, ω) corresponds to some point on $\overline{C} \cdot \delta_0$ consisting of two irreducible components X_i, joined

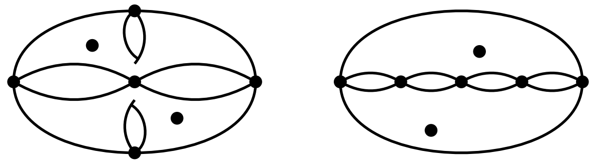


Figure 4.1: Examples of reducible curves in δ_0. In (a), we have $g_i = 0$, $n = 3$ and $m_i = 1$, while in (b), we have $g_i = 0$, $n = 5$ and $m_i = 0$.

together by n nodes, consider the normalisation $X_i' \longrightarrow X_i$ and denote by g_i the genus and by m_i the number of nodes contained only in X_i. Then the pullback of ω to the normalisation X' is a section of the canonical sheaf of X' and the restriction to some X_i' is consequently a rational section of the canonical sheaf with poles at the (preimages of) the n connecting nodes and double poles at the (preimages of) the m_i nodes contained in X_i, cf. [Art04, formula (1.2) and (1.3)]. Furthermore, each component must contain a threefold zero of ω (as we are in the stratum $(3,3)$, cf. [Möl11, Corollary 5.11]). Then [HM98, formula (3.1)] yields

$$2g_i - 2 = 3 - n - 2m_i, \text{ i.e. } 0 \leq 2(g_i + m_i) = 5 - n,$$

as the restriction of the dualising sheaf is of degree $2g_i - 2$. In particular, this requires that $n \in \{1, 3, 5\}$; examples are depicted in figure 4.1 and figure 4.2.

In any case, as no such X is hyperelliptic (cf. [Möl11, Proposition 5.13]), this implies that we satisfy the prerequisites of [Art04, Theorem 1.2] which asserts that the dualising sheaf is very ample and yields an embedding of X into $\mathbb{P}^3$. This—together with the fact that the degree of the dualising sheaf is again 6—shows that all arguments in section 4.2 can be applied to this more general situation. In other words, X is again contained in the intersection of a quadric and a cubic surface.

Describing the extension of $\mathfrak{Lin}_3^1$ to the boundary is also quite technical. Details may be found in, e.g., [HM98, §3G]. The correct generalisation of a $\mathfrak{g}_3^1$ to stable curves can be achieved in terms of *admissible covers* (see [HM98, Definitions 3.149 and 3.159] for a proper definition). As we are interested only in the intersection with $\overline{C}$, we need only consider curves X in δ_0 with no separating nodes. In this case, the admissible cover corresponding to a $\mathfrak{g}_3^1$ is a degree three map from X to $\mathbb{P}^1$ with one marked point, cf. [HM98, Theorem 3.160]; by definition of $\mathfrak{Lin}_3^1$, the preimages of the marked points on $\mathbb{P}^1$ must be the two marked points on X. Moreover, as we are in $\mathcal{M}_{4,2}$, the two marked points stay away from the nodes, see [Knu83, §1]. Therefore, we may apply geometric Riemann-Roch, just as we did in section 4.2, to show that any $\mathfrak{g}_3^1$ corresponds to a line on the quadric via the embedding into $\mathbb{P}^3$.

In summary, the arguments of section 4.2 and Theorem 7 are all applicable to this case and permit us to conclude that $\mathfrak{Lin}_3^1 \cdot \overline{C} = 0$.

But by Proposition 3.2.4, we know the generators of the Picard group. Therefore, there exist coefficients a_1, a_2, b and c so that

$$0 = \overline{C} \cdot \overline{\mathfrak{Lin}_3^1} = \overline{C} \cdot (a_1\omega_{1,\text{rel}} + a_2\omega_{2,\text{rel}} + b\lambda - c\delta_0 - \delta_{\text{other}}).$$

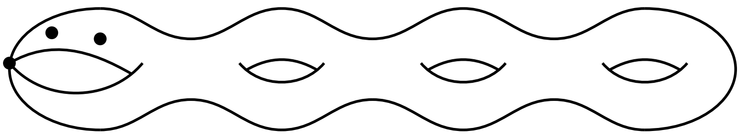


Figure 4.2: An example of an irreducible curve in δ_0. We have $g_1 = 3$, $n = 0$ and $m_1 = 1$.

As $\overline{\mathcal{M}_{4,2}}$ admits an automorphism σ, interchanging the two marked points and, by definition, $\overline{\mathfrak{Lin}_3^1}$ is invariant under this automorphism, the role of $\omega_{1,\mathrm{rel}}$ and $\omega_{2,\mathrm{rel}}$ cannot be distinguished, as they are permuted by σ. Therefore, we have $a_1 = a_2$.

Recall that we defined the slope of a Teichmüller curve to be

$$s(C) = \frac{\overline{C} \cdot \delta_0}{\overline{C} \cdot \lambda}$$

and in this case, by [CM11, Proposition 2.6], the formula from [EKZ12] yields $L(C) = 2$ for the sum of Lyapunov exponents (cf. section 3.4). As in (3.1), we obtain

$$\kappa_{(3,3)} = \tfrac{2}{12}\left(\tfrac{3\cdot(3+2)}{3+1}\right) = \tfrac{5}{8}$$

and with this, Proposition 3.4.1 allows us to calculate the slope of C:

$$s(C) = 12 - \frac{12\kappa_\mu}{L(C)} = 12 - \frac{15}{4} = \frac{33}{4}.$$

Hence we obtain the additional relation

$$\overline{C} \cdot \lambda = \tfrac{4}{33}\overline{C} \cdot \delta_0.$$

We can also calculate the intersection of $\overline{C}$ and the $\omega_{i,\mathrm{rel}}$ classes, again by Proposition 3.4.1:

$$\begin{aligned} 2 \cdot \overline{C} \cdot \omega_{1,\mathrm{rel}} &= 2 \cdot \frac{\overline{C} \cdot \lambda - \frac{1}{12}\overline{C} \cdot \delta}{4 \cdot \frac{5}{8}} = \tfrac{4}{5}\overline{C} \cdot \lambda - \tfrac{4}{5} \cdot \tfrac{1}{12}\overline{C} \cdot \delta_0 \\ &= \tfrac{4}{5}\overline{C} \cdot \lambda - \tfrac{1}{15}\overline{C} \cdot \delta_0. \end{aligned}$$

Combining all of this leads us to

$$\begin{aligned} \overline{C} \cdot \overline{\mathfrak{Lin}_3^1} = 0 &= 2a_1\overline{C} \cdot \omega_{1,\mathrm{rel}} + b\overline{C} \cdot \lambda - c\overline{C} \cdot \delta_0 \\ &= a_1\left(\tfrac{4}{5}\overline{C} \cdot \lambda - \tfrac{1}{15}\overline{C} \cdot \delta_0\right) + b\overline{C} \cdot \lambda - c\overline{C} \cdot \delta_0 \\ &= a_1\left(\tfrac{16}{5\cdot 33}\overline{C} \cdot \delta_0 - \tfrac{1}{15}\overline{C} \cdot \delta_0\right) + \tfrac{4}{33}b\overline{C} \cdot \delta_0 - c\overline{C} \cdot \delta_0 \\ &= \overline{C} \cdot \delta_0\left(\tfrac{a_1}{33} + \tfrac{4b}{33} - c\right) \end{aligned} \tag{4.4}$$

and therefore $a_1 + 4b - 33c = 0$.

Unfortunately, our Teichmüller curve only leads us this far. Picking another curve as test curve yields the additional relation

$$a_1 + 2b - 17c = 0,$$

see [CM11, Proposition 2.6] or [Möl11, Proposition 3.2][78]. Up to a constant, this shows

$$a = 1, \quad b = 8, \quad \text{and} \quad c = 1,$$

as claimed. □

Note that a more general calculation was done in [Far09, Theorem 4.6]. In particular, he shows that $k = \frac{1}{2}$ in our formula.

4.4 Teichmüller Curves and Divisors in Hyperelliptic Strata

In a sense, we want to reverse the above argument: in section 4.3, we used the fact that we knew that a certain divisor did not intersect a Teichmüller curve to find coefficients for that divisor in the Picard group. Now, we want to construct a family of divisors that avoids Teichmüller curves.

Given a single Teichmüller curve, we may proceed exactly as in Proposition 4.3.1.

Corollary 4.4.1: Any divisor in $\mathfrak{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}_{4,2}})$ of the form

$$D = a_1\omega_{1,\text{rel}} + a_2\omega_{2,\text{rel}} + b\lambda - c\delta_0 - \delta_{\text{other}}$$

with $a_1 + a_2 + 8b - 66c = 0$ avoids the Teichmüller curve $\overline{C}$ (cf. (4.3)).

Proof: Let $D \in \mathfrak{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}_{4,2}})$ be any divisor. By Proposition 3.2.4 (and the above arguments regarding boundary components), we find coefficients $a, b, c \in \mathbb{Q}$ so that

$$D = a_1\omega_{1,\text{rel}} + a_2\omega_{2,\text{rel}} + b\lambda - c\delta_0 - \delta_{\text{other}}.$$

Of course, there is no more reason to assume symmetry in the a_i. But in the course of Proposition 4.3.1, we calculated

$$a_i\overline{C} \cdot \omega_{i,\text{rel}} = \frac{a_i}{2 \cdot 33}\overline{C} \cdot \delta_0$$

and by simply ignoring the left hand side of the equation (4.4), we still obtain

$$D \cdot \overline{C} = \overline{C} \cdot \delta_0\left(\tfrac{a_1+a_2}{66} + \tfrac{4b}{33} - c\right).$$

In other words, any D with coefficients satisfying

$$a_1 + a_2 + 8b - 66c = 0$$

does not intersect $\overline{C}$. □

[78] Note that these include some confusing sign errors.

Taking a closer look at the proof, however, the only information used that is *specific to the chosen Teichmüller curve* is the slope. As the constant κ_μ depends only on the stratum we are in, we observe that any Teichmüller curves in the same stratum with the same slope intersect the same divisors. As the slope and the sum of Lyapunov exponents are related only by κ_μ (cf. Proposition 3.4.1), we may switch freely between the two when speaking of only one stratum.

Now, in [CM11], Chen and Möller study strata of $\Omega\mathcal{M}_g$ in which all Teichmüller curves have the same sum of Lyapunov exponents (hence also the same slope). By the above, if we are in any such stratum, we may pick *any* Teichmüller curve, calculate the sum of Lyapunov exponents with the help of [EKZ12] and perform the above calculation to find a family of divisors avoided by *all* Teichmüller curves in this stratum.

We conclude by performing this calculation in a classical case.

Theorem 8: *Any divisor of the form*

$$D = a\omega_{rel} + b\lambda - c\delta_0 - \delta_{other},$$

where $a\omega_{rel}$ is either $a_1\omega_{1,rel}$ or $a_1\omega_{1,rel} + a_2\omega_{2,rel}$, depending on the stratum, satisfying

$$a_1 + bg^2 - 4cg(2g+1) = 0$$

avoids all Teichmüller curves in the stratum $\Omega\mathcal{M}_g(2g-2)^{hyp}$, while

$$4(a_1 + a_2) + bg(g+1) - 4c(g+1)(2g+1) = 0$$

implies that D avoids all Teichmüller curves of the stratum $\Omega\mathcal{M}_g(g-1,g-1)^{hyp}$.

Proof: For these strata the sum of Lyapunov exponents may be calculated with the help of [EKZ12, Corollary 1], see also [Möl11, Theorem 6.9]. For a Teichmüller curve C generated by some (X,ω) this yields

$$L(C) = \frac{g^2}{2g-1}, \text{ if } (X,\omega) \in \Omega\mathcal{M}_g(2g-2)^{\text{hyp}}, \text{ and}$$
$$L(C) = \frac{g+1}{2}, \text{ if } (X,\omega) \in \Omega\mathcal{M}_g(g-1,g-1)^{\text{hyp}}.$$

Additionally, we may calculate κ_μ according to (3.1) for each stratum:

$$\kappa_{(2g-2)} = \frac{1}{12}\frac{(2g-2)2g}{2g-1} \text{ and } \kappa_{(g-1,g-1)} = \frac{2}{12}\frac{(g-1)(g+1)}{g} = \frac{g^2-1}{6g}.$$

Fortunately, this implies that Teichmüller curves from *both* strata have the same slope: for $(X,\omega) \in \Omega\mathcal{M}_g(2g-2)^{\text{hyp}}$, Proposition 3.4.1 yields

$$s(C) = 12 - \frac{12\kappa_{(2g-2)}}{L(C)} = 12 - \frac{(2g-2)2g}{2g-1}\frac{2g-1}{g^2} = \frac{12g-4g+4}{g} = 8 + \frac{4}{g},$$

while for $(X,\omega) \in \Omega\mathcal{M}_g(g-1,g-1)^{\text{hyp}}$ we have

$$s(C) = 12 - \frac{2(g^2-1)}{g}\frac{2}{(g+1)} = 12 - \frac{4(g-1)}{g} = 8 + \frac{4}{g}.$$

Therefore, by (3.2), for any such C we have

$$\overline{C}\cdot\lambda = \frac{1}{s(C)}\overline{C}\cdot\delta_0 = \frac{g}{8g+4}\overline{C}\cdot\delta_0. \tag{4.5}$$

Unfortunately, however, the κ_μ differ so we are forced to make a case distinction for the most tedious part of the calculation. We start with the case $(X,\omega)\in\Omega\mathcal{M}_g(2g-2)^{\mathrm{hyp}}$. Then, again by Proposition 3.4.1,

$$\overline{C}\cdot\omega_{1,\mathrm{rel}} = \frac{\overline{C}\cdot\lambda-\overline{C}\cdot\delta_0/12}{\frac{1}{12}(2g-1)\frac{(2g-2)2g}{2g-1}} = \frac{3\overline{C}\cdot\lambda-\overline{C}\cdot\delta_0/4}{g(g-1)}.$$

With (4.5), we obtain

$$\overline{C}\cdot\lambda\frac{3}{g(g-1)} = \frac{3}{4(g-1)(2g+1)}\overline{C}\cdot\delta_0$$

and combining all this yields the intersection with the divisor D:

$$D\cdot\overline{C} = \overline{C}\cdot(a_1\omega_{1,\mathrm{rel}}+b\lambda-c\delta_0) = \overline{C}\cdot\delta_0\left(\frac{3a_1}{4(g-1)(2g+1)} - \frac{a_1}{4g(g-1)} + \frac{gb}{4(2g+1)} - c\right)$$

And as

$$\frac{a_1}{4(g-1)}\left(\frac{3}{2g+1}-\frac{1}{g}\right) = \frac{a_1}{4(g-1)}\frac{g-1}{g(2g+1)} = \frac{a_1}{4g(2g+1)},$$

this shows that any D with coefficients so that

$$a_1 + bg^2 - 4cg(2g+1) = 0,$$

avoids all Teichmüller curves C generated by some $(X,\omega)\in\Omega\mathcal{M}_g(2g-2)^{\mathrm{hyp}}$. The other case works the same way: let C be generated by some $(X,\omega)\in\Omega\mathcal{M}_g(g-1,g-1)^{\mathrm{hyp}}$. Again, as we are in a stratum with symmetric partition, the intersection $C\cdot\omega_{i,\mathrm{rel}}$ will not depend on i. We have (by Proposition 3.4.1 and (4.5))

$$2\overline{C}\cdot\omega_{i,\mathrm{rel}} = \frac{12\overline{C}\cdot\lambda-\overline{C}\cdot\delta_0}{g^2-1} = \overline{C}\cdot\delta_0\left(\frac{3g}{(g^2-1)(2g+1)} - \frac{1}{g^2-1}\right),$$

for $i\in\{1,2\}$. Note that

$$\frac{3g}{(g^2-1)(2g+1)} - \frac{1}{g^2-1} = \frac{1}{(g+1)(2g+1)}$$

and the intersection with D is consequently

$$\overline{C}\cdot D = (a_1+a_2)\overline{C}\cdot\omega_{1,\mathrm{rel}}+b\overline{C}\cdot\lambda-c\overline{C}\cdot\delta_0 = \overline{C}\cdot\delta_0\left(\frac{a_1+a_2}{2(g+1)(2g+1)} + \frac{bg}{4(2g+1)} - c\right).$$

This implies that any D with coefficients satisfying

$$2(a_1+a_2) + bg(g+1) - 4c(g+1)(2g+1) = 0$$

avoids all Teichmüller curves C generated by some $(X,\omega)\in\Omega\mathcal{M}_g(g-1,g-1)^{\mathrm{hyp}}$. □

Conclusion

Clearly, the calculation performed in Theorem 8 may be done to equal effect for any stratum of $\Omega\mathcal{M}_g$ for which the sum of Lyapunov exponents is known and known to be non-varying. In their paper [CM11], Chen and Möller show that the sum is non-varying in almost all strata for genus 3 and 4 and in some for genus 5 and give counter-examples for many other cases (cf. [CM11, Theorems 1.1–1.3]). The sum is known to vary in most higher genus strata, with the notable exception of the hyperelliptic locus, which is known to be non-varying. The formulas provided by [EKZ12] permit a concrete computation of the sum and the method of Theorem 8 then yields families of divisors avoiding all curves in the stratum.

The divisor families thus obtained are quite large. Indeed, Theorem 8 lets us choose four parameters, but as the Teichmüller curves avoid the "higher" boundary components altogether, we have even more choices. This is remarkable insofar as the geometry of the image of a stratum of $\Omega\mathcal{M}_g$ in $\mathcal{M}_{g,n}$ (or $\overline{\mathcal{M}_{g,n}}$) is not yet well-understood. Recall that the stratification—while arising very naturally in $\Omega\mathcal{M}_g$—is quite artificial on the space of curves. But now we have a fairly large class of curves (cf. e.g. [Möl11, Proposition 5.3]) of which we know that it avoids a fairly large class of codimension one sub-spaces of $\overline{\mathcal{M}_{g,n}}$. On the other hand, the geometry of the divisors obtained is hard to understand. Indeed, it is not even immediately clear which—or how many—of the divisors are effective, let alone if the corresponding codimension 1 subvarieties parametrise interesting sets of curves.

Conversely, in [CM11], Chen and Möller developed Proposition 4.3.1 together with other techniques to *show* that the sum of Lyapunov exponents in certain strata is indeed non-varying. Unfortunately, it is not at all clear, how the method described in Proposition 4.3.1 could be generalised to other strata than $\Omega\mathcal{M}_4(3,3)^{\text{non-hyp}}$. Indeed, the techniques developed for showing Theorem 7 are extremely specific to this situation. We made repeated use of the fact that genus four curves embed into quadrics in $\mathbb{P}^3$ and that a $\mathfrak{g}^1_3$ corresponds to a line on such a quadric. As the reasoning used relied heavily on this classical geometric situation, changing any parameters puts us in a situation that is much less understood and it is unlikely that the same arguments can be used.

Index of Definitions

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