Applied and Numerical Analysis Seminar

Heraklion, 06 December 2012

Weak solutions to rate-independent systems

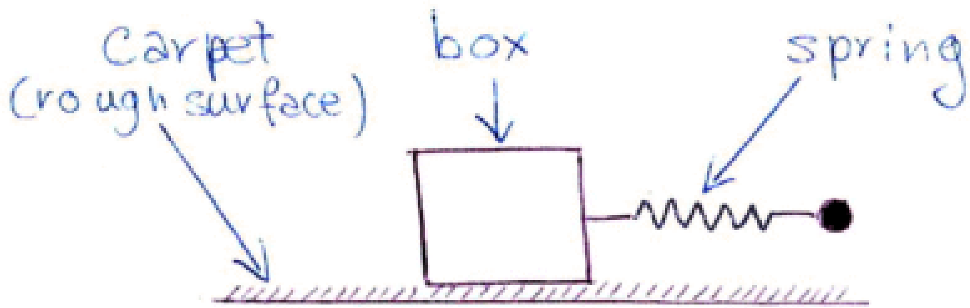
- Energetic solutions
- BV solutions constructed by vanishing viscosity
- BV solutions constructed by epsilon-neighborhood method

Contents

Rate-independent system

- Preserved under time rescaling.
- No own dynamics.
- No inertial effects.
- No kinetic energy.
- Dissipated energy doesn't depend on velocity.
- Application: dry friction, crack propagation, delamination, shape-memory alloys, etc.

Toy model



- Input: $y(t)$ the free end of the spring.
- Output: $x(t)$ the center of the box.

Two forces:

- The external force f_e due to the spring, $f_e := -\partial_x \mathscr{E}(t,x)$, here $\mathscr{E}(t,x) = \frac{c}{2}(x - y(t))^2$.
- The frictional force f_a due to the carpet (dry friction), $f_a := -k\frac{v}{|v|}$ if $v \neq 0$, $f_a := -f_e$ and $|f_a| \leq k$ if $v = 0$.

k : the frictional coefficient, $v = \dot{x}(t)$: the velocity of the box.

Toy model

- If $|f_e| < k$, then $f_a = -f_e$ and the box does not move.
- After reaching the critical value $|f_e| = k$, the box starts moving.

Equation of dynamics

$$m\ddot{x} = f_a + f_e,$$

m : the mass of the box.

Quasistatic evolution $\Rightarrow$ neglect the term $m\ddot{x}$

$$0 = f_a + f_e.$$

Define $\Psi(v) := k|v|$, then $-f_a \in \partial\Psi(v)$. We get

$$0 \in \partial\Psi(\dot{x}(t)) + \partial_x \mathscr{E}(t, x(t)).$$

Toy model

Explicitly, we can write

- $|c(x(t) - y(t)| \leq k$.
- $\dot{x}(t)[z - c(x(t) - y(t))] \leq 0$ for all $z \in [-k, k]$.

Easy to check that this is RIS: doubling the speed at which $y(t)$ moves, the effect on $x(t)$ is also doubled in speed.

Abstract framework

- X [finite-dimensional] normed vector space.
- $\mathscr{E}(t, x)$ smooth energy functional.
- $\Psi(x)$ dissipation functional, for simplicity, we assume $\Psi(x) = |x|$.
- The position x is stable $\Leftrightarrow$ it minimizes the total energy (total energy = external energy + dissipation).

Differential inclusion:

$$0 \in \partial|\dot{x}(t)| + \nabla_x \mathscr{E}(t, x(t)) \text{ for a.e. } t \in (0, T). \tag{1}$$

- $\mathscr{E}(t, .)$ convex $\Rightarrow$ existence of a unique strong solution.
- In general, strong solutions may not exist $\Rightarrow$ weak solutions are needed.

Simple example

- $\mathscr{E}(t,x) := x^2 - x^4 + 0.3\,x^6 + t\,(1 - x^2) - x,\ t \in [0,2],\ x \in \mathbb{R}.$
- Initial position: $x_0 := 0$.
- Strong solution: $x(t) = 0$ for $t \in [0,1)$.
- Strong solution cannot be extended continuously when $t \geq 1$, since it would violate the local minimality.

Energy plus dissipation function at the beginning

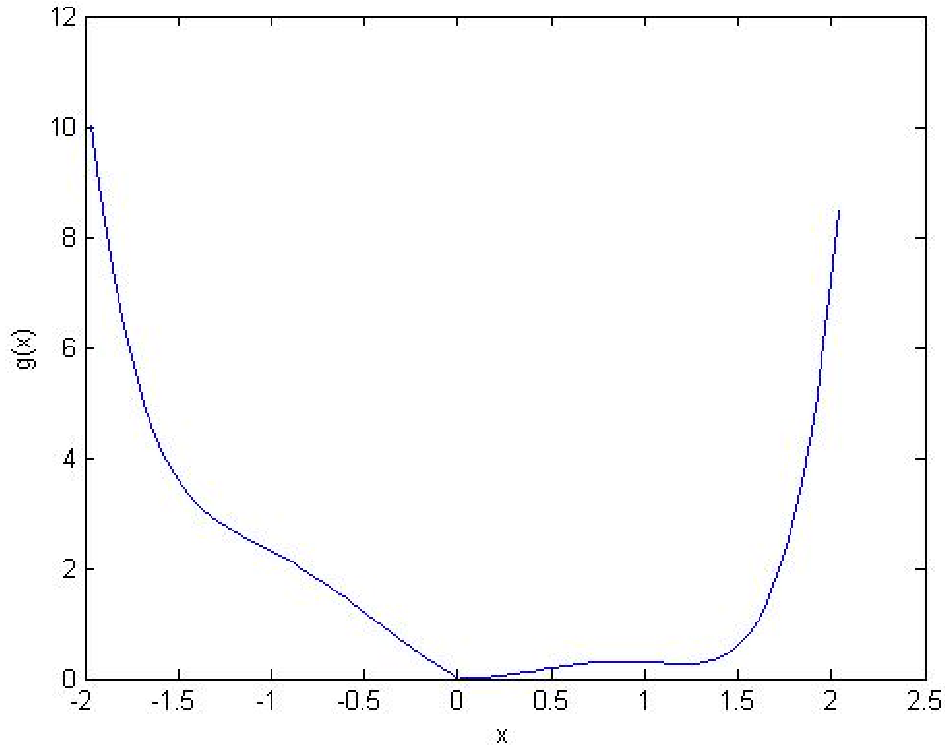


Figure 1. Function $\mathscr{E}(t, x) + |x - x_0|$ with $t = 0$ and $x_0 = 0$.
Unique global minimizer at $x = 0$.

Energy plus dissipation function at $t = 1/3$

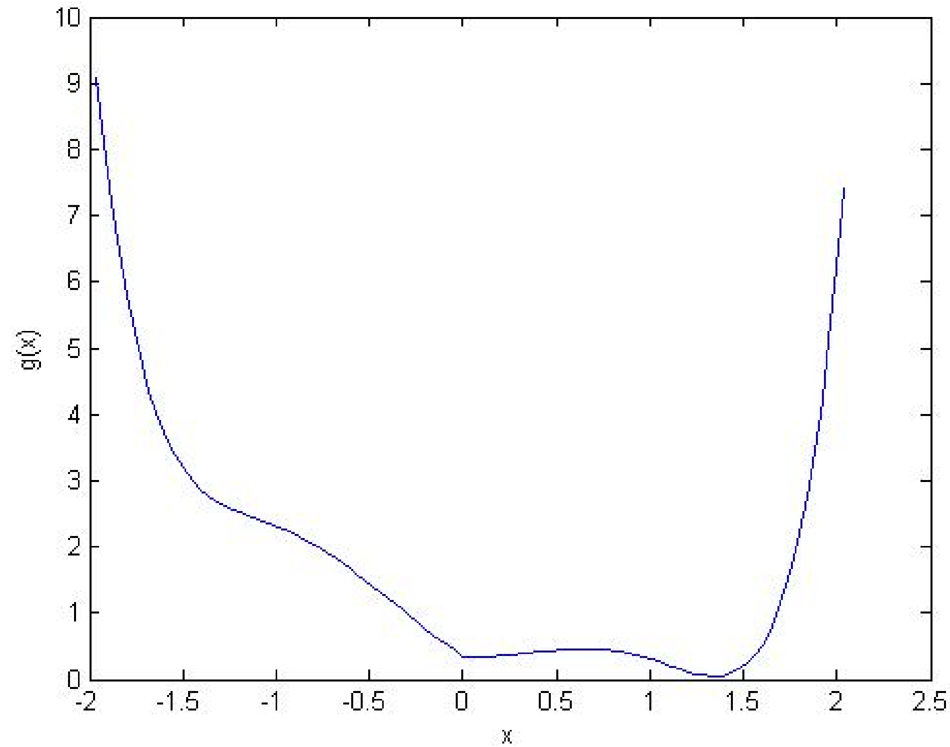


Figure 2. Function $\mathscr{E}(t,x) + |x - x_0|$ with $t = 1/3$ and $x_0 = 0$.

Unique global minimizer at $x = \frac{\sqrt{10+\sqrt{10+90t}}}{3}$.
One local minimizer at $x = 0$.

Energy plus dissipation function at $t = 1$

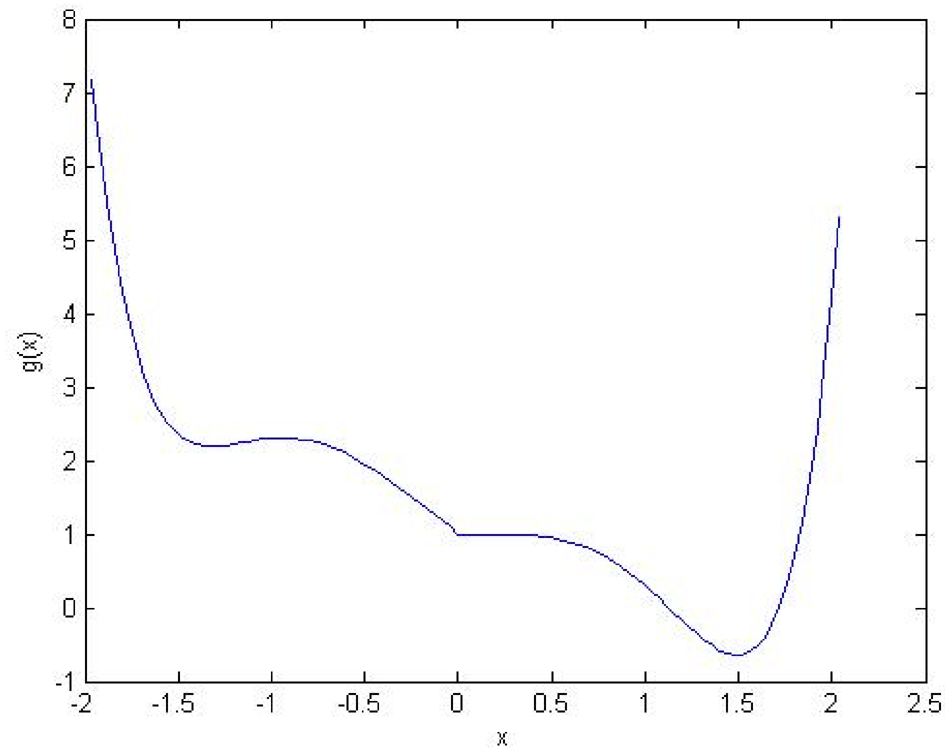


Figure 3. Function $\mathscr{E}(t, x) + |x - x_0|$ with $t = 1$ and $x_0 = 0$.

Unique global minimizer at $x = \frac{\sqrt{10+\sqrt{10+90t}}}{3}$.
One local minimizer at $x = 0$.

Energy plus dissipation function at $t = 1.2$

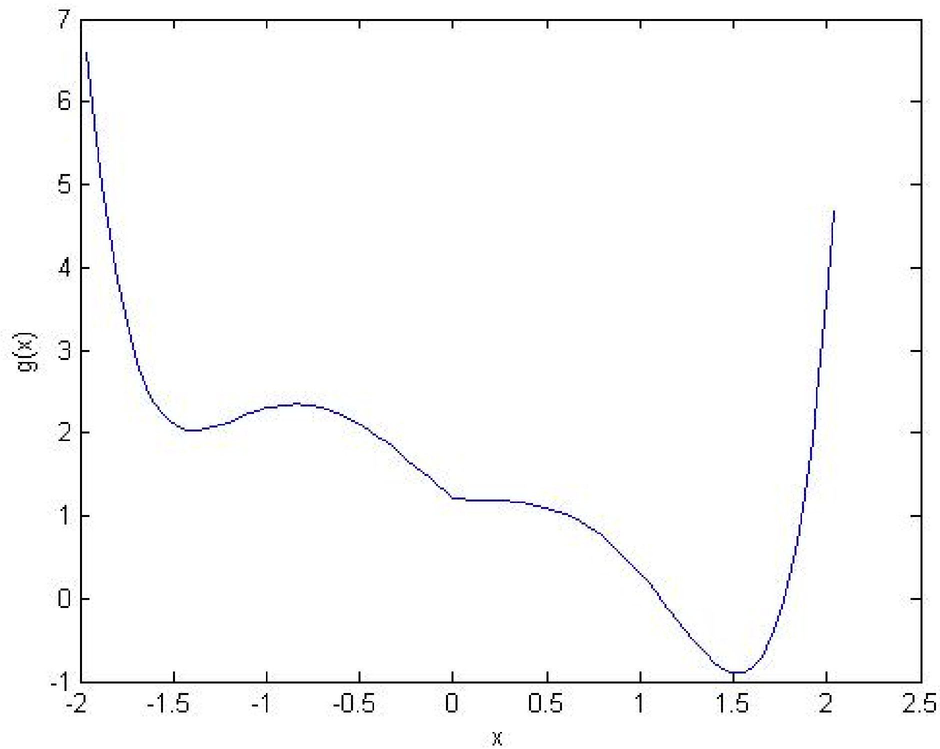


Figure 4. Function $\mathscr{E}(t, x) + |x - x_0|$ with $t = 1.2$ and $x_0 = 0$.

Unique global minimizer at $x = \frac{\sqrt{10+\sqrt{10+90t}}}{3}$.
$x = 0$ is neither local minimizer nor global minimizer.

Energetic solutions (Mielke and Theil 1999)

Definition. $x(\cdot)$: energetic solutions

- (Initial condition) $x(0) = x_0$,
- (Global stability) $\mathscr{E}(t, x(t)) \leq \mathscr{E}(t, z) + |z - x(t)|, \;\; \forall z \in X$.
- (Energy-dissipation balance) For all $0 \leq s \leq t \leq T$,

$$\mathscr{E}(t, x(t)) - \mathscr{E}(s, x(s)) = \int_s^t \partial_t \mathscr{E}(r, x(r)) dr - \mathscr{D}iss(x(\cdot); [s, t]).$$

Dissipation energy = energy dissipated when the particle moves.

- Equals to the usual variation (or length), i.e.,

$$\mathscr{D}iss(x; [s, t]) :=$$
$$\sup \left\{ \sum_{n=1}^{N} |x(t_n) - x(t_{n-1})| \mid N \in \mathbb{N}, s = t_0 < t_1 < ... < t_N = t \right\}.$$

- The loss of energy along the jump = the jump step.

Construction of energetic solutions

- Time-partition $t_i = i\tau$, $\tau > 0$, $i = 0, 1, 2, \ldots$
- Let $x_0^\tau = x_0$ and $x_n^\tau \in \underset{x\in X}{\operatorname{argmin}}\{\mathscr{E}(t_n, x) + |x - x_{n-1}^\tau|\}$.
- Interpolation $x^\tau(t) = x_{n-1}^\tau$ for all $t \in [t_{n-1}, t_n)$.
- Pointwise limit $x^\tau(t) \to x(t)$ as $\tau \to 0$ for every $t \in [0, T]$.
- $x(\cdot)$ is an energetic solution of $(\mathscr{E}, |\cdot|, x_0)$.

Energy plus dissipation function at $t = 1/6$

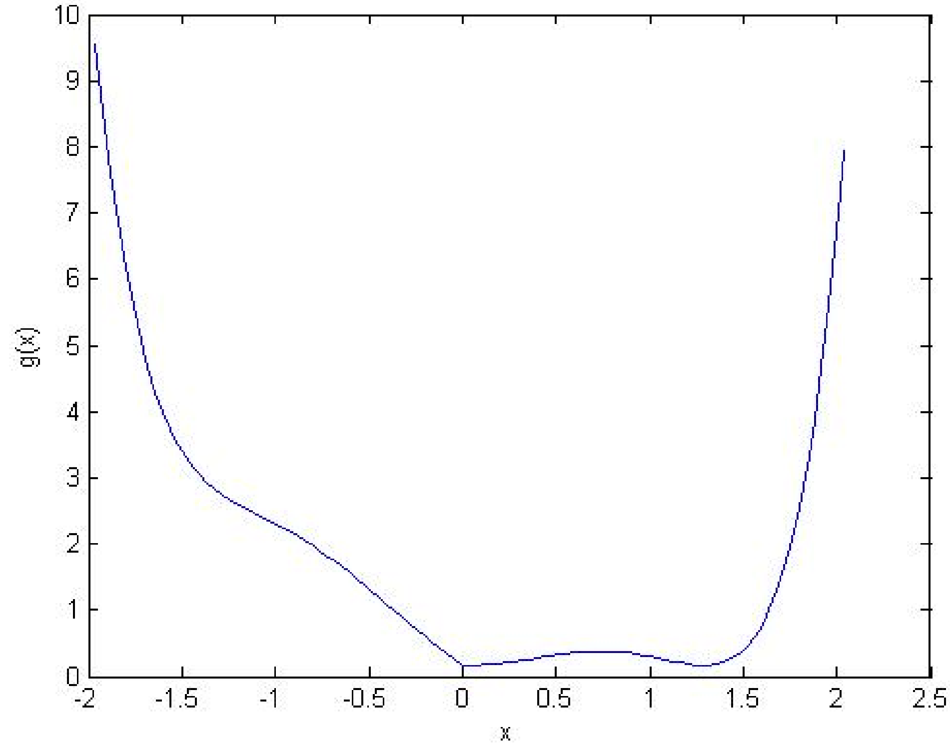


Figure 5. Function $\mathscr{E}(t, x) + |x - x_0|$ with $t = 1/6$ and $x_0 = 0$.

Two global minimizers at $x = 0$ and $x = \sqrt{5/3}$.

Energy plus dissipation function at $t = 0.5$

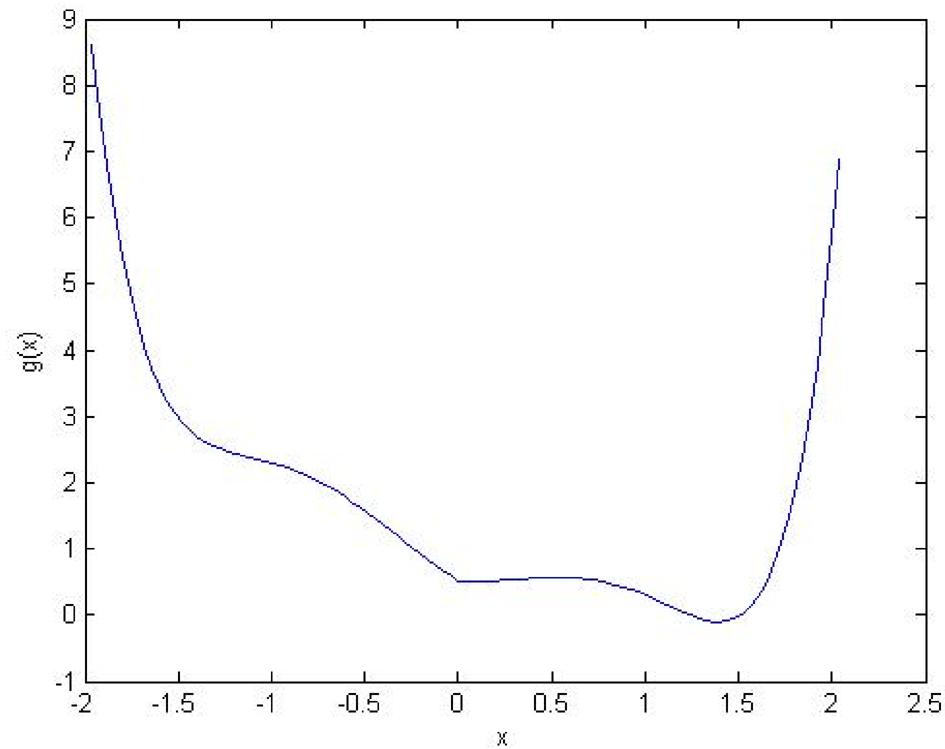


Figure 6. Function $\mathscr{E}(t, x) + |x - x_0|$ with $t = 0.5$ and $x_0 = 0$.

Unique global minimizer at $x = \frac{\sqrt{10+\sqrt{10+90t}}}{3}$.
One local minimizer at $x = 0$.

Simple example

- $\mathscr{E}(t,x) := x^2 - x^4 + 0.3\,x^6 + t\,(1-x^2) - x,\ \ t \in [0,2],\ \ x \in \mathbb{R}.$
- Initial position: $x_0 := 0$.
- When $t > 1/6, x = 0$ is no longer a global minimizer. Thus, energetic solution must jump at $t = 1/6$.
- One energetic solution:

$$x(t) = \begin{cases} 0 & \text{if } t \in [0, 1/6); \\ \frac{\sqrt{10+\sqrt{10+90t}}}{3} & \text{if } t \in (1/6, 2]. \end{cases}$$

  and $x(1/6) \in \{0, \sqrt{5/3}\}$.
- This solution is not good since it does not agree with strong solution.

Vanishing viscosity (Mielke, Rossi, and Savaré 2012)

- Add a small viscosity into the dissipation, e.g. $\varepsilon|x|^2$.
- With time step $\tau > 0$ and viscous term $\varepsilon|x|^2$, choose $x_0^{\tau,\varepsilon} = x_0$ and

$$x_n^{\tau,\varepsilon} \in \operatorname*{argmin}_{x \in X} \left\{ \mathscr{E}(t_n, x) + |x - x_{n-1}^{\tau,\varepsilon}| + \frac{\varepsilon}{\tau} |x - x_{n-1}^{\tau,\varepsilon}|^2 \right\}.$$

- Interpolation + pointwise limit ($\tau/\varepsilon \to 0$) $\Rightarrow$ BV function $x(\cdot)$.

Properties: $x(0) = x_0$ and

- (Weak local stability) $|\nabla_x \mathscr{E}(t, x(t))| \leq 1$ if $t \notin J$.
- (New energy-dissipation balance) For all $0 \leq s \leq t \leq T$,

$$\mathscr{E}(t, x(t)) - \mathscr{E}(s, x(s)) = \int_s^t \partial_t \mathscr{E}(r, x(r)) dr - \mathscr{D}iss_{new}(x(\cdot); [s, t]).$$

New dissipation energy

- Another computation for the loss of energy along the jump

$$
\begin{aligned}
& \mathscr{D}iss_{new}(x;[s,t]) := \\
= \quad & \mathscr{D}iss(x;[s,t]) - \sum_{t\in J} \left[|x(t^-) - x(t)| + |x(t) - x(t^+)| \right] \\
& + \sum_{t\in J} \left[\Delta_{new}(t, x(t^-), x(t)) + \Delta_{new}(t, x(t), x(t^+)) \right],
\end{aligned}
$$

where

$$
\Delta_{new}(t,a,b) := \inf_{\substack{v\in AC([0,T];\mathbb{R}^d) \\ v(0)=a, v(1)=b}} \left\{ \int_0^1 |\dot{v}(r)| \cdot \max\{1, |\partial_x \mathscr{E}(t, v(r))|\} \right\}.
$$

- In general, $\mathscr{D}iss_{new}(x;[s,t]) \geq \mathscr{D}iss(x;[s,t])\ \forall x \in \mathrm{BV}$.

- If $r \mapsto x(r)$ is continuous on $[s,t]$, then $\mathscr{D}iss_{new}(x;[s,t]) = \mathscr{D}iss(x;[s,t])$.

Optimal transition

- $\exists$ an optimal transition between u_- and u_+ iff

$$\mathscr{E}(t, u_+) - \mathscr{E}(t, u_-) = -\Delta_{new}(t; u_-, u(t)) - \Delta_{new}(t; u(t), u_+).$$

- Absolutely continuous curve $\gamma : [0, 1] \to \mathbb{R}^d$ connecting u_- and u_+ and satisfying

  (i) $|\nabla_x \mathscr{E}(t, \gamma(s))| \geq 1$ for all $s \in (0, 1)$.

  (ii) $\nabla_x \mathscr{E}(t, \gamma(s)) \cdot \dot{\gamma}(s) = -|\nabla_x \mathscr{E}(t, \gamma(s))| \cdot |\dot{\gamma}(s)|$.

- In 1-dim, optimal transition is the linear path connecting u_- and u_+.

- In n-dim, the existence of optimal transition is much more complicated, and it is obtained by using time rescaling technique.

Simple example

- $\mathscr{E}(t,x) := x^2 - x^4 + 0.3\,x^6 + t\,(1-x^2) - x,\ t \in [0,2],\ x \in \mathbb{R}$.
- Initial position: $x_0 := 0$.
- Choose viscosity as $\varepsilon^5\, x^6$.
- The corresponding BV solution:

$$x(t) = \begin{cases} 0 & \text{if } t \in [0,1); \\ \frac{\sqrt{10+\sqrt{10+90t}}}{3} & \text{if } t \in (1,2]. \end{cases}$$

  and $x(1) \in \{0, \sqrt{20}/3\}$.

- This solution is good since it agrees with strong solution up to strong solution exists.

Drawback of BV solutions constructed by vanishing viscosity

$x(\cdot)$ depends heavily on the viscosity. Inappropriate choice of viscosity $\Rightarrow$ solution jumps later than expected!

Example. $X = \mathbb{R}$, $\Psi(x) = |x|$, $x_0 = 0$,

$$\mathscr{E}(t, x) = x^2 - x^4 + 0.3\, x^6 + t(1 - x^2) - x, \quad t \in [0, 2].$$

- Choose viscousity as $\varepsilon |x|^2$.
- Corresponding BV solution $x(t) = 0$ for all $t \in [0, 2]$.
- Unreasonable solution. Since local minimality is violated when $t \geq 1$.

Epsilon-neighborhood solutions

Construction:

- Fix $\varepsilon > 0$. With time-partition $\tau > 0$, choose $x_0^{\varepsilon,\tau} = x_0$ and

$$x_n^{\tau,\varepsilon} \in \underset{|x - x_{n-1}^{\varepsilon,\tau}| \leq \varepsilon}{\operatorname{argmin}} \left\{ \mathscr{E}(t_n, x) + |x - x_{n-1}^{\varepsilon,\tau}| \right\}.$$

- Interpolation + pointwise limit ($\tau \to 0$) $\Rightarrow$ BV function $x^\varepsilon(\cdot)$.

(i) (Epsilon local stability) If $x^\varepsilon(\cdot)$ is right-continuous at t, then

$$\mathscr{E}(t, x^\varepsilon(t)) \leq \mathscr{E}(t, z) + |z - x^\varepsilon(t)| \quad \text{for all } |z - x^\varepsilon(t)| \leq \varepsilon.$$

(ii) (Energy-dissipation inequalities) $-\mathscr{D}iss_{new}(x^\varepsilon; [s,t]) \leq$
$\mathscr{E}(t, x^\varepsilon(t)) - \mathscr{E}(s, x^\varepsilon(s)) - \int_s^t \partial_t \mathscr{E}(r, x^\varepsilon(r)) dr \leq -\mathscr{D}iss(x^\varepsilon; [s,t])$.

- Pointwise limit of $x^\varepsilon(\cdot)$ ($\varepsilon \to 0$)$\Rightarrow$ BV function $x(\cdot)$.

Properties: Weak-local stability and new energy-dissipation balance **hold**.

Simple example

- $X = \mathbb{R}$, $\Psi(x) = |x|$, $x_0 = 0$,

$$\mathscr{E}(t,x) = x^2 - x^4 + 0.3\,x^6 + t(1-x^2) - x, \quad t \in [0,2].$$

- BV solution by epsilon-neighborhood

$$x(t) = 0 \quad \text{if} \;\; t < 1, \; x(t) = \frac{\sqrt{10 + \sqrt{10 + 90t}}}{3} \text{ if } t > 1.$$

- This solution jumps at time $t = 1$, from $x = 0$ to $x = \sqrt{20}/3$.
  This is a reasonable solution!

New energy-dissipation balance via epsilon-neighborhood

For all $t > s$,

$$\mathscr{E}(t, x(t)) - \mathscr{E}(s, x(s)) = \int_s^t \partial_t \mathscr{E}(r, x(r))dr - \mathscr{D}iss_{new}(x(\cdot); [s,t]).$$

At jumps:

$$\mathscr{E}(t, x(t^+)) - \mathscr{E}(t, x(t^-)) = -\Delta_{new}(t; x(t^-), x(t)) - \Delta_{new}(t; x(t), x(t^+))$$

$$\Delta_{new}(t, a, b) := \inf_{\substack{v \in AC([0,T];\mathbb{R}^d) \\ v(0)=a, v(1)=b}} \left\{ \int_0^1 \max\{1, |\nabla_x \mathscr{E}(t, v(s))|\} \cdot |\dot{v}(s)| \right\}.$$

Proposition (Lower bound - Mielke, Rossi, and Savaré 2009)**.** *Let $d \geq 1$ and $\mathscr{E} \in C^1([0,T] \times \mathbb{R}^d, \mathbb{R})$. For any BV function $u : [0,T] \to \mathbb{R}^d$, then*

$$\mathscr{E}(t, u(t^+)) - \mathscr{E}(t, u(t^-)) \geq -\Delta_{new}(t; u(t^-), u(t)) - \Delta_{new}(t; u(t), u(t^+)).$$

New energy-dissipation balance: Lower bound

To prove Lower Bound, write

$$\mathscr{E}(t,u(t^+))-\mathscr{E}(t,u(t^-)) = \mathscr{E}(t,u(t^+))-\mathscr{E}(t,u(t))+\mathscr{E}(t,u(t))-\mathscr{E}(t,u(t^-)).$$

If $v \in AC([0,1],\mathbb{R}^d)$ such that $v(0) = u(t)$ and $v(1) = u(t^+)$, then

$$\begin{aligned}
\mathscr{E}(t,u(t^+)) - \mathscr{E}(t,u(t)) &= \int_0^1 \nabla_x \mathscr{E}(t,v(s)) \cdot \dot{v}(s) ds \\
&\geq -\int_0^1 \max\{1, |\nabla_x \mathscr{E}(t,v(s))|\} \cdot |\dot{v}(s)| ds.
\end{aligned}$$

Thus

$$\begin{aligned}
\mathscr{E}(t,u(t^+)) - \mathscr{E}(t,u(t)) &\geq -\inf_{\substack{v \in AC([0,T];\mathbb{R}^d) \\ v(0)=a, v(1)=b}} \left\{ \int_0^1 ... \right\} \\
&= -\Delta_{new}(t; u(t), u(t^+)).
\end{aligned}$$

Discretized solutions

Lemma (Discretized solutions)**.** *Write* $x_j = x^{\varepsilon,\tau}(t_j)$*. Then*

$$-\nabla_x \mathscr{E}(t_i, x_i) \cdot (x_i - x_{i-1}) = \max\{1, |\nabla_x \mathscr{E}(t_i, x_i)|\} \cdot |x_i - x_{i-1}|.$$

Consequently, if $\delta \geq \max\{|t - t_i|, \varepsilon, \tau\}$ *and* $v : [a, b] \to \mathbb{R}^d$ *is the linear curve connecting* x_{i-1} *and* x_i*, then*

$$\int_a^b \max\{1, |\nabla_x \mathscr{E}(t, v(s))|\}.|\dot{v}(s)| ds \leq \mathscr{E}(t, x_{i-1}) - \mathscr{E}(t, x_i) + g(\delta) \cdot |x_i - x_{i-1}|$$

where $g(\delta) \to 0$ *as* $\delta \to 0$*.*

Discretized solutions

Recall that x_i is a minimizer for

$$\inf_{|z-x_{i-1}|\le\varepsilon} h(z) = \inf_{|z-x_{i-1}|\le\varepsilon} \left\{\mathscr{E}(t_i,z) + |z-x_{i-1}|\right\}.$$

1. Denote $c := |x_i - x_{i-1}|$; then x_i is also a minimizer for

$$\inf_{|z-x_{i-1}|=c} h(z).$$

By Lagrange multiplier, there exists $\lambda \in \mathbb{R}$ such that

$$\nabla_x \mathscr{E}(t_i,x_i) = \lambda(x_i - x_{i-1}).$$

2. Using $\partial_t \left(h(x_{i-1} + t(x_i - x_{i-1}))\right) \le 0$ at $t = 1$, we obtain

$$\nabla_x \mathscr{E}(t_i,x_i)\cdot(x_i - x_{i-1}) + |x_i - x_{i-1}| \le 0.$$

Thus either $x_i = x_{i-1}$, or $|\nabla_x \mathscr{E}(t_i,x_i)| \ge 1$ and $\lambda < 0$.

Approximate optimal transition

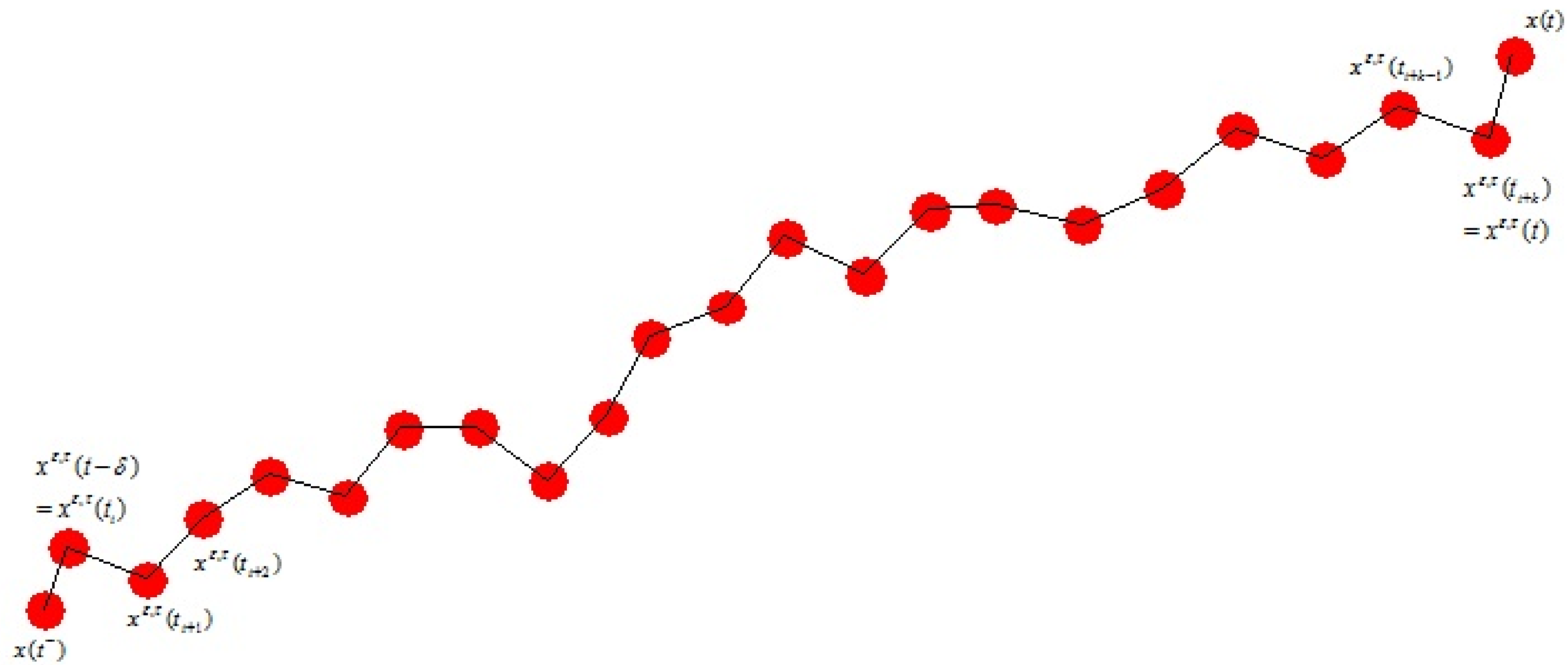


Figure 7. Approximate optimal transition between $x(t^-)$ and $x(t)$.

$$x(t^-) \to x^{\varepsilon,\tau}(t-\delta) = x^{\varepsilon,\tau}(t_i) \to x^{\varepsilon,\tau}(t_{i+1})$$

$$\to x^{\varepsilon,\tau}(t_{i+2}) \to \cdots \to x^{\varepsilon,\tau}(t_{i+k}) = x^{\varepsilon,\tau}(t) \to x(t).$$

New energy-dissipation balance: Upper bound

By linear interpolation, construct a curve $v : [0,1] \to \mathbb{R}^d$ connecting the points

$$x(t^-), x^{\varepsilon,\tau}(t-\delta) = x^{\varepsilon,\tau}(t_i), x^{\varepsilon,\tau}(t_{i+1}), ..., x^{\varepsilon,\tau}(t_{i+k}) = x^{\varepsilon,\tau}(t), x(t).$$

Then

$$\begin{aligned} \Delta_{new}(t, x(t^-), x(t)) &\leq \int_0^1 \max\{1, |\nabla_x \mathscr{E}(t, v(s))|\} \cdot |\dot{v}(s)|\, ds \\ &\leq \mathscr{E}(t, x^{\varepsilon,\tau}(t-\delta)) - \mathscr{E}(t, x^{\varepsilon,\tau}(t)) + Cg(\delta) \\ &\quad +C|x(t^-) - x^{\varepsilon,\tau}(t-\delta)| + C|x^{\varepsilon,\tau}(t) - x(t)|. \end{aligned}$$

Taking the limit $\tau \to 0$, then $\varepsilon \to 0$, then $\delta \to 0$, we conclude that

$$\Delta_{new}(\mathscr{E}, t, x(t^-), x(t)) \leq \mathscr{E}(t, x(t^-) - \mathscr{E}(t, x(t)).$$

Future works

Problem 1: Improve the weak local stability for BV solutions constructed by epsilon-neighborhood.

Problem 2: Prove the existence of BV solutions constructed by epsilon-neighborhood for capillary drops.