Weak solutions to rate-independent systems

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Rate-independent systems

- Perceive as a limit problem in many physical and mechanics problems.
- No inertial effects.
- No own dynamics.
- Preserved under time rescaling.
- Application: dry friction, crack propagation, delamination, shape-memory alloys, etc.

Simple example:

$$0 \in \mathrm{Sign}(\dot{y}(t)) + \mathrm{D}\mathcal{U}(y(t)) - \ell(t), \qquad y(0) = y_0.$$

Abstract framework

- X Banach space. Here: $X = \mathbb{R}^d$.
- $\Psi(x)$ dissipation functional (convex and positively 1-homogeneous). Here: $\Psi(v) = |v|$.
- $\mathscr{E}(t, x)$ smooth energy functional, for instance, $\mathscr{E} \in \mathrm{C}^2([0, T] \times X)$.

Differential inclusion:

$$0 \in \partial\Psi(\dot{x}(t)) + \nabla_x \mathscr{E}(t, x(t)) \text{ for a.e. } t \in (0, T). \tag{1}$$

The position x is stable $\Leftrightarrow$ it minimizes the total energy (total energy = external energy + dissipation).

Solutions to RIS

$$0 \in \partial\Psi(\dot{x}(t)) + \nabla_x \mathscr{E}(t, x(t)) \text{ for a.e. } t \in (0, T).$$

- $\mathscr{E}(t, .)$ convex $\Rightarrow$ existence of a unique strong solution.
- In general, strong solutions may not exist $\Rightarrow$ weak solutions are needed.

There are many notions of weak solutions:

- Energetic solutions (Francfort-Marigo 1998, Mielke-Theil 1999)
- Local solutions (Toader-Zanini 2009)
- Epsilon-stable solutions (Larsen 2010)
- Parametrized solutions (Mielke-Rossi-Savaré 2010)
- BV solutions (Mielke-Rossi-Savaré 2010-2012)

An example in 1-dim

- $\mathscr{E}(t,x) := x^2 - x^4 + 0.3\,x^6 + t\,(1-x^2) - x,\ t \in [0,2],\ x \in \mathbb{R}$.
- Dissipation $\Psi(v) := |v|$. Initial position $x(0) := 0$.

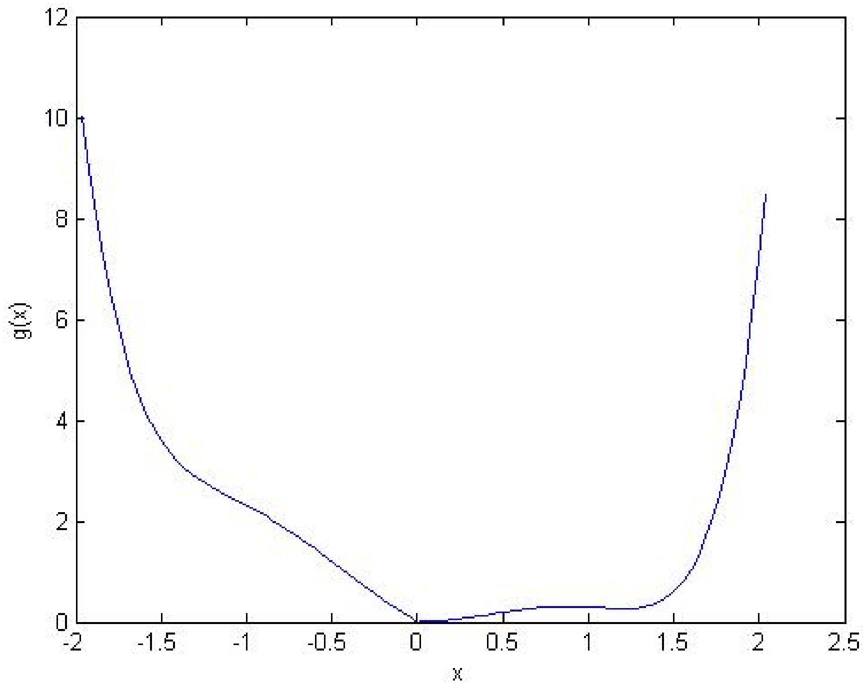

Figure 1. Function $\mathscr{E}(t,x) + |x - x_0|$ with $t = 0$ and $x_0 = 0$.
Unique global minimizer at $x = 0$.

Energy plus dissipation function at $t = 1/6$

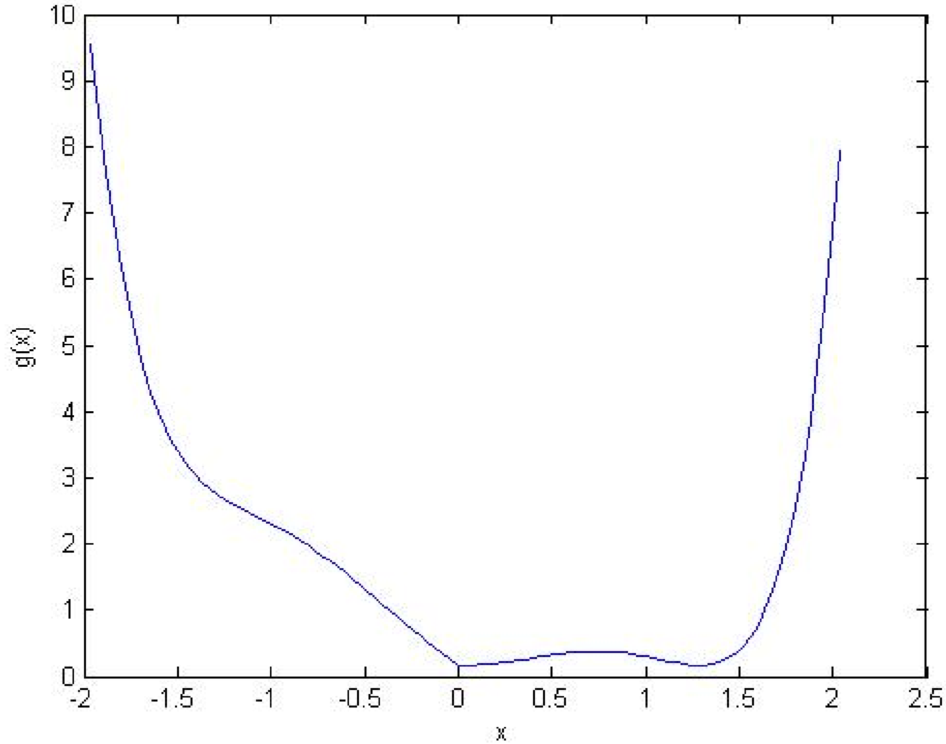

Figure 2. Function $\mathscr{E}(t,x) + |x - x_0|$ with $t = 1/6$ and $x_0 = 0$.

Two global minimizers at $x = 0$ and $x = \sqrt{5/3}$.

Energy plus dissipation function at $t = 1/3$

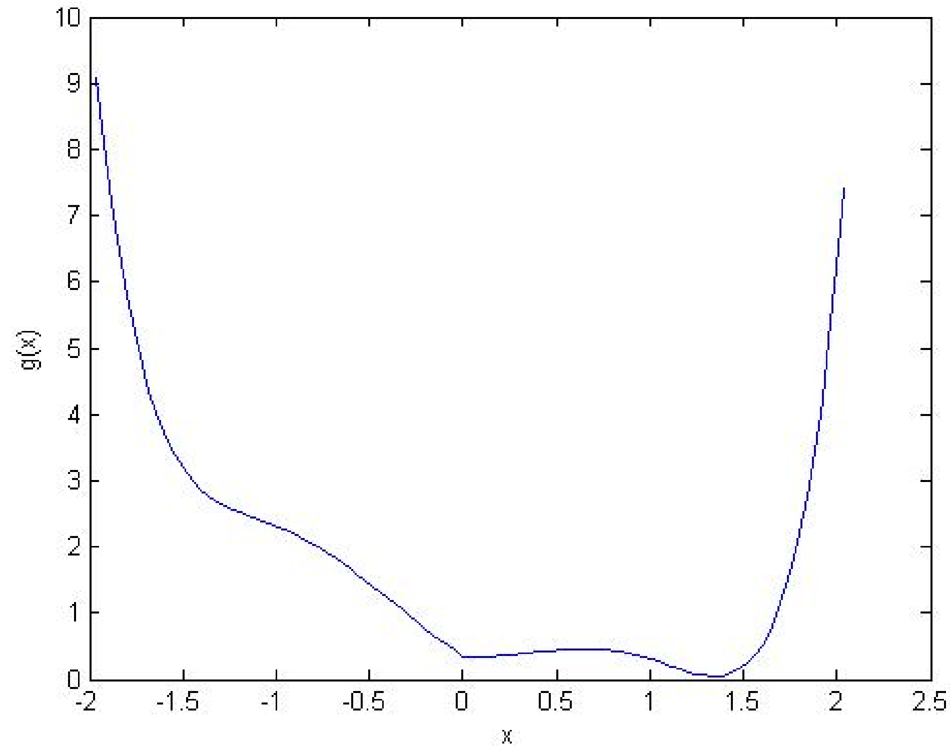

Figure 3. Function $\mathscr{E}(t, x) + |x - x_0|$ with $t = 1/3$ and $x_0 = 0$.

Unique global minimizer at $x = \frac{\sqrt{10+\sqrt{10+90t}}}{3}$.
One local minimizer at $x = 0$.

Energy plus dissipation function at $t = 1$

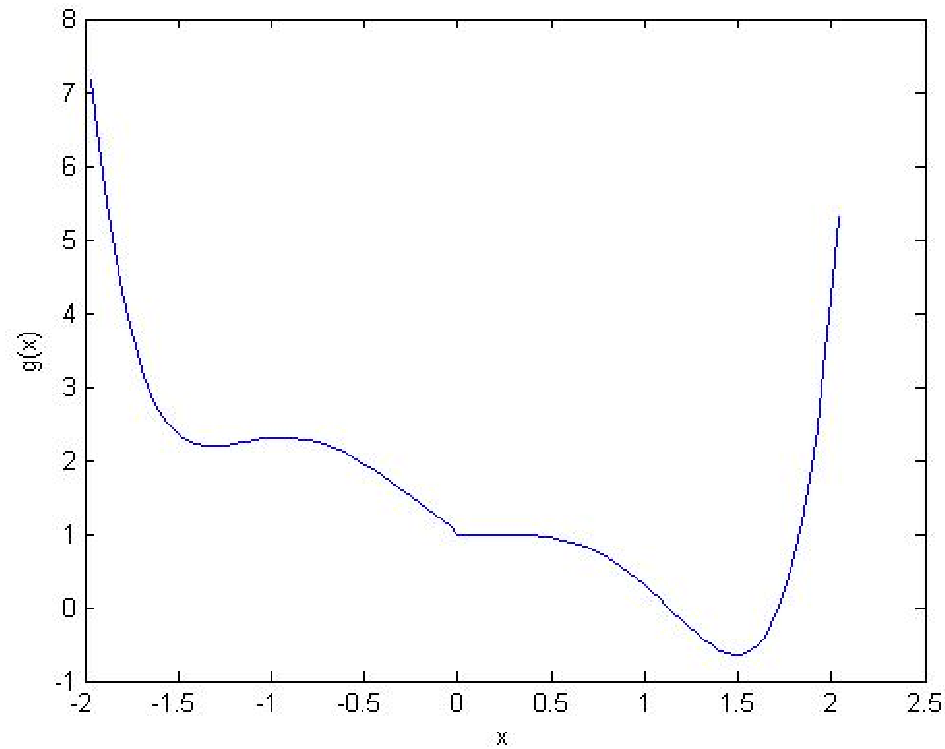

Figure 4. Function $\mathscr{E}(t, x) + |x - x_0|$ with $t = 1$ and $x_0 = 0$.

Unique global minimizer at $x = \frac{\sqrt{10+\sqrt{10+90t}}}{3}$.

$x = 0$ is no longer a local minimizer.

Energy plus dissipation function at $t = 1.2$

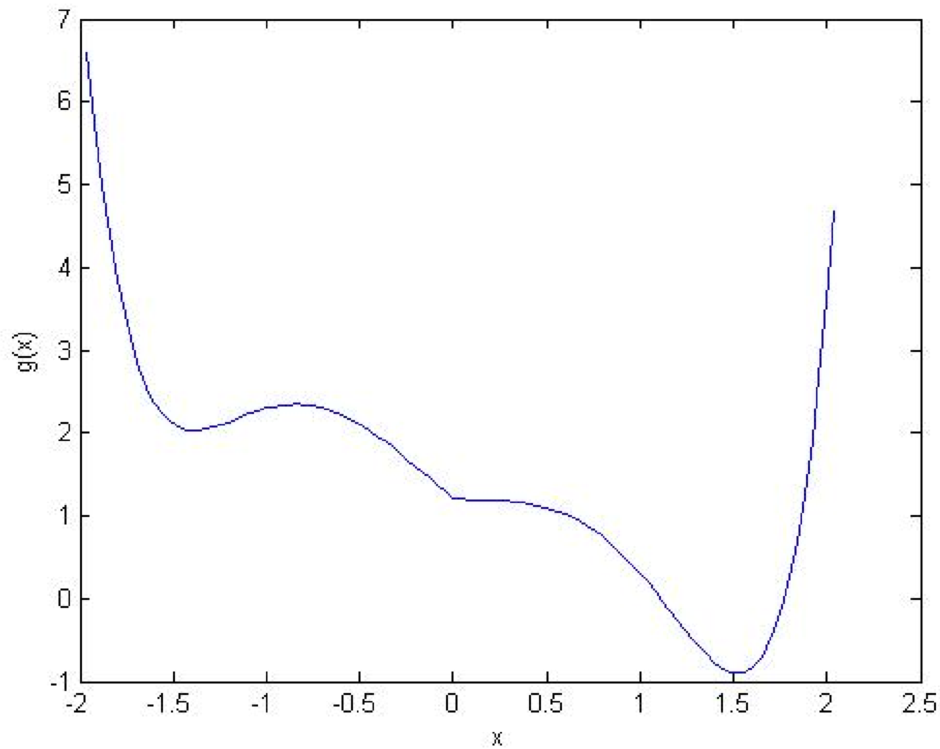

Figure 5. Function $\mathscr{E}(t, x) + |x - x_0|$ with $t = 1.2$ and $x_0 = 0$.

Unique global minimizer at $x = \frac{\sqrt{10+\sqrt{10+90t}}}{3}$.
$x = 0$ is neither local minimizer nor global minimizer.

An example in 1-dim

- **Reasonable solution**

$$x(t) = \begin{cases} 0 & \text{if } t \in [0,1); \\ \frac{\sqrt{10+\sqrt{10+90t}}}{3} & \text{if } t \in (1,2]. \end{cases}$$

and $x(1) \in \{0, \sqrt{20/3}\}$.

- **Strong solution** $x^{strong}(t) = 0$ for $t \in [0,1)$. Strong solution cannot be extended continuously when $t \geq 1$, since it would violate the local minimality.

Energetic solutions

(Francfort-Marigo 1998, Mielke-Theil 1999)

Definition. $x(\cdot)$ is an energetic solution if

- (Initial condition) $x(0) = x_0$,
- (Global stability) $\mathscr{E}(t, x(t)) \leq \mathscr{E}(t, z) + \Psi(z - x(t)), \quad \forall z \in X$.
- (Energy-dissipation balance) For all $0 \leq s \leq t \leq T$,

$$\mathscr{E}(t, x(t)) - \mathscr{E}(s, x(s)) = \int_s^t \partial_t \mathscr{E}(r, x(r)) dr - \mathscr{D}iss(x(\cdot); [s, t]).$$

Dissipation energy = usual variation (or length), i.e.,

$$\mathscr{D}iss(x; [s, t]) :=$$
$$\sup \left\{ \sum_{n=1}^{N} \Psi(x(t_n) - x(t_{n-1})) \mid N \in \mathbb{N}, s = t_0 < t_1 < ... < t_N = t \right\}.$$

Construction of energetic solutions

- Time-partition $t_i := i\tau$, $\tau > 0$, $i = 0, 1, 2, \ldots$
- Write equation (1) into discrete form

$$0 \in \nabla_x \mathscr{E}(t_n, x_n^\tau) + \partial\Psi\left(\frac{x_n^\tau - x_{n-1}^\tau}{\tau}\right).$$

- Let $x_0^\tau := x_0$ and $x_n^\tau \in \operatorname*{argmin}_{x \in X}\{\mathscr{E}(t_n, x) + \Psi(x - x_{n-1}^\tau)\}$.

Impose one technical assumption on the loading: There exists λ such that $|\partial_t \mathscr{E}(s,x)| \le \lambda\, \mathscr{E}(s,x)$ for all $(s,x) \in [0,T] \times \mathbb{R}$.

- Energy bound $\mathscr{E}(t_n, x_n^\tau) \le \mathscr{E}(0, x_0)\, e^{\lambda t_n}$.
- Interpolation $x^\tau(t) := x_{n-1}^\tau$ for all $t \in [t_{n-1}, t_n)$.
- Integral bound for $x^\tau, 0 \le s \le t \le T$

$$\mathscr{E}(t, x^\tau(t)) - \mathscr{E}(s, x^\tau(s)) \le \int_s^t \partial_t \mathscr{E}(r, x^\tau(r))\, dr - \mathscr{D}iss(x^\tau; [s,t]).$$

Construction of energetic solutions

- Uniform bound for $\mathscr{D}iss(x^\tau; [0,T])$ for every τ.
- Helly's Principle $\Rightarrow$ Exists subsequence $\{\tau_k\}$ st $x^{\tau_k}(t) \to x(t)$ pointwise as $\tau_k \to 0$ for every $t \in [0,T]$.
- Claim: $x(t)$ is an energetic solution of $(\mathscr{E}, \Psi, x_0)$.

(1) Global stability: $\mathscr{E}(t, x(t)) \leq \mathscr{E}(t,z) + \Psi(z - x(t)), \ \ \forall z \in X$.

(2) (Energy-dissipation balance) For all $0 \leq s \leq t \leq T$,

$$\mathscr{E}(t, x(t)) - \mathscr{E}(s, x(s)) = \int_s^t \partial_t \mathscr{E}(r, x(r))dr - \mathscr{D}iss(x(\cdot); [s,t]).$$

An example in 1-dim

- **Reasonable solution**

$$x(t) = \begin{cases} 0 & \text{if } t \in [0,1); \\ \frac{\sqrt{10+\sqrt{10+90t}}}{3} & \text{if } t \in (1,2]. \end{cases}$$

and $x(1) \in \{0, \sqrt{20/3}\}$.

- **Strong solution** $x^{strong}(t) = 0$ for $t \in [0,1)$. Strong solution cannot be extended continuously when $t \geq 1$, since it would violate the local minimality.

- **Energetic solution**

$$x^{ener}(t) = \begin{cases} 0 & \text{if } t \in [0,1/6); \\ \frac{\sqrt{10+\sqrt{10+90t}}}{3} & \text{if } t \in (1/6,2]. \end{cases}$$

and $x^{ener}(1/6) \in \{0, \sqrt{5/3}\}$.

Energy plus dissipation function at $t = 1/6$

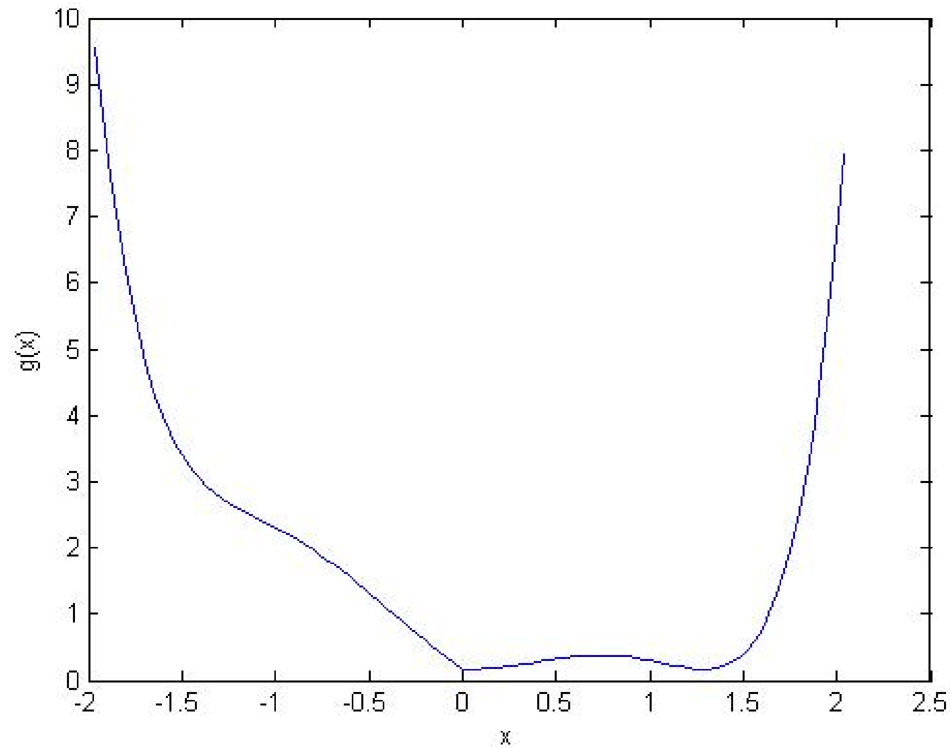

Figure 2. Function $\mathscr{E}(t,x) + |x - x_0|$ with $t = 1/6$ and $x_0 = 0$.

Two global minimizers at $x = 0$ and $x = \sqrt{5/3}$.

(Global stability: $\mathscr{E}(t, x(t)) \leq \mathscr{E}(t, z) + \Psi(z - x(t)), \;\; \forall z \in X$.)

BV solutions (Mielke-Rossi-Savaré 2010)

Definition. $x(\cdot)$ is a BV solution if

- (Initial condition) $x(0) = x_0$,
- (Local stability) $|\nabla_x \mathscr{E}(t, x(t))| \leq 1$ for a.e. t,
- (New energy-dissipation balance) For all $0 \leq s \leq t \leq T$,

$$\mathscr{E}(t,x(t)) - \mathscr{E}(s,x(s)) = \int_s^t \partial_t \mathscr{E}(r,x(r))dr - \mathscr{D}iss_{new}(x(\cdot);[s,t]).$$

For every $x(\cdot) \in \mathrm{BV}([0,\mathrm{T}])$: $\mathscr{D}iss_{new}(x;[s,t]) \geq \mathscr{D}iss(x;[s,t])$.
Thus,

$$\mathscr{E}(t,x^{\mathrm{BV}}(t)) - \mathscr{E}(s,x^{\mathrm{BV}}(s)) \leq \int_s^t \partial_t \mathscr{E}(r,x^{\mathrm{BV}}(r))dr - \mathscr{D}iss(x^{\mathrm{BV}}(\cdot);[s,t]).$$

In some cases, the inequality is **strict**!

New dissipation vs Old dissipation

- Old dissipation:

$$\mathscr{D}iss(x;[s,t]) :=$$
$$\sup\left\{\sum_{n=1}^{N}\Psi(x(t_n)-x(t_{n-1}))\mid N\in\mathbb{N}, s=t_0<t_1<...<t_N=t\right\}.$$

- Energy released at jump of energetic solution:

$$\begin{aligned}\mathscr{E}(t,x^{\mathrm{ener}}(t^+))-\mathscr{E}(t,x^{\mathrm{ener}}(t^-)) \quad &= \quad -|x^{\mathrm{ener}}(t^+)-x^{\mathrm{ener}}(t)| \\ &\quad\;\; -|x^{\mathrm{ener}}(t)-x^{\mathrm{ener}}(t^-)|.\end{aligned}$$

- New dissipation

$$
\begin{aligned}
& \mathscr{D}iss_{new}(x;[s,t]) := \\
= \quad & \mathscr{D}iss(x;[s,t]) - \sum_{t\in J} \left[|x(t^-) - x(t)| + |x(t) - x(t^+)| \right] \\
& + \sum_{t\in J} \left[\Delta_{new}(t, x(t^-), x(t)) + \Delta_{new}(t, x(t), x(t^+)) \right],
\end{aligned}
$$

where

$$
\Delta_{new}(t,a,b) := \inf_{\substack{v\in AC([0,T];\mathbb{R}^d)\\ v(0)=a, v(1)=b}} \left\{ \int_0^1 |\dot{v}(r)| \cdot \max\{1, |\nabla_x \mathscr{E}(t, v(r))|\} \right\}.
$$

Optimal transition

Proposition. *(Mielke-Rossi-Savaré 2012)*

- $\exists$ *an optimal transition between* u_- *and* u_+ *iff*

$$\mathscr{E}(t, u_+) - \mathscr{E}(t, u_-) = -\Delta_{new}(t; u_-, u(t)) - \Delta_{new}(t; u(t), u_+).$$

- *Absolutely continuous curve* $\gamma : [0, 1] \to \mathbb{R}^d$ *connecting* u_- *and* u_+ *and satisfying*

  (i) $|\nabla_x \mathscr{E}(t, \gamma(s))| \geq 1$ *for all* $s \in (0, 1)$.

  (ii) $\nabla_x \mathscr{E}(t, \gamma(s)) \cdot \dot{\gamma}(s) = -|\nabla_x \mathscr{E}(t, \gamma(s))| \cdot |\dot{\gamma}(s)|$.

In 1-dim, optimal transition is the linear path connecting u_- and u_+.

Vanishing viscosity method (Mielke-Rossi-Savaré 2012)

- Add a small viscosity into the dissipation to result Ψ_ε.

$$\Psi_\varepsilon(v) = |v| + \text{viscosity},$$

i.e., viscosity $= \varepsilon\Psi_0, \Psi_0$ is any convex function vanishes at 0 and has super-linear growth at infinity.

- The new incremental problem is

$$x_n^{\tau,\varepsilon} \in \operatorname*{argmin}_{x\in X} \left\{ \mathscr{E}(t_n, x) + \tau\Psi_\varepsilon \left(\frac{x - x_{n-1}^{\tau,\varepsilon}}{\tau} \right) \right\}.$$

- Energy bound: $\mathscr{E}(t_n, x_n^{\tau,\varepsilon}) \le \mathscr{E}(0, x_0)e^{\lambda t_n}$.
- Interpolation $x^{\tau,\varepsilon}(t) := x_{n-1}^{\tau,\varepsilon}$ for $t \in [t_{n-1}, t_n)$.
- Uniform bound for $\mathscr{D}iss(x^{\tau,\varepsilon}; [0, T])$ for every τ, ε.
- Helly's Principle $\Rightarrow$ Exist subsequences $\{\tau_k\}$, $\{\varepsilon_k\}$ st $x^{\tau_k,\varepsilon_k}(t) \to x(t)$ pointwise as $k \to \infty$ for every $t \in [0, T]$.

Vanishing viscosity method (Mielke-Rossi-Savaré 2012)

Claim: $x(t)$ is a BV solution of $(\mathscr{E}, |\cdot|, x_0)$ provided that $\tau/\varepsilon^2 \to 0$, i.e. $x(t)$ satisfies

- (Initial condition) $x(0) = x_0$.
- (Local stability) $|\nabla_x \mathscr{E}(t, x(t))| \leq 1$ if $t \notin J$.
- (New energy-dissipation balance) For all $0 \leq s \leq t \leq T$,

$$\mathscr{E}(t, x(t)) - \mathscr{E}(s, x(s)) = \int_s^t \partial_t \mathscr{E}(r, x(r)) dr - \mathscr{D}iss_{new}(x(\cdot); [s, t]).$$

An example in 1-dim

- $\mathscr{E}(t,x) := x^2 - x^4 + 0.3\,x^6 + t\,(1-x^2) - x,\ t \in [0,2],\ x \in \mathbb{R}$.
  Initial position: $x(0) := 0$.
- Choose viscosity as $\varepsilon|x|^2$. Corresponding BV solution $x(t) = 0$ for all $t \in [0,2]$.
- Choose viscosity as $\varepsilon^5\,x^6$. The corresponding BV solution:

$$x(t) = \begin{cases} 0 & \text{if } t \in [0,1); \\ \frac{\sqrt{10+\sqrt{10+90t}}}{3} & \text{if } t \in (1,2]. \end{cases}$$

  and $x(1) \in \{0, \sqrt{20}/3\}$.

Epsilon-neighborhood method

- Fix $\varepsilon > 0$. With time-partition $\tau > 0$, choose $x_0^{\varepsilon,\tau} = x_0$ and

$$x_n^{\varepsilon,\tau} \in \underset{|x - x_{n-1}^{\varepsilon,\tau}| \le \varepsilon}{\operatorname{argmin}} \left\{ \mathscr{E}(t_n, x) + |x - x_{n-1}^{\varepsilon,\tau}| \right\}.$$

- Interpolation + pointwise limit ($\tau \to 0$) $\Rightarrow$ BV function $x^\varepsilon(\cdot)$.

(i) (Epsilon local stability) If $x^\varepsilon(\cdot)$ is right-continuous at t, then

$$\mathscr{E}(t, x^\varepsilon(t)) \le \mathscr{E}(t, z) + |z - x^\varepsilon(t)| \quad \text{for all } |z - x^\varepsilon(t)| \le \varepsilon.$$

(ii) (Energy-dissipation inequalities) $-\mathscr{D}iss_{new}(x^\varepsilon; [s,t]) \le$ $\mathscr{E}(t, x^\varepsilon(t)) - \mathscr{E}(s, x^\varepsilon(s)) - \int_s^t \partial_t \mathscr{E}(r, x^\varepsilon(r))dr \le -\mathscr{D}iss(x^\varepsilon; [s,t])$.

- Pointwise limit of $x^\varepsilon(\cdot)$ ($\varepsilon \to 0$)$\Rightarrow$ BV function $x(\cdot)$.

Claim: $x(\cdot)$ satisfies the definition of BV solutions.

An example in 1-dim

- $X = \mathbb{R}$, $\Psi(x) = |x|$, $x(0) = 0$,

$$\mathscr{E}(t,x) = x^2 - x^4 + 0.3\,x^6 + t(1-x^2) - x, \quad t \in [0,2].$$

- BV solution by epsilon-neighborhood

$$x(t) = 0 \quad \text{if} \;\; t < 1, \;\; x(t) = \frac{\sqrt{10 + \sqrt{10 + 90t}}}{3} \;\text{if}\; t > 1.$$

- This solution jumps at time $t = 1$, from $x = 0$ to $x = \sqrt{20}/3$.
  This is a reasonable solution!

Epsilon-neighborhood method

- Energy bound: $\mathscr{E}(t_n, x_n^{\varepsilon,\tau}) \leq \mathscr{E}(0, x_0)\, e^{\lambda t_n}$.
- Interpolation $x^{\varepsilon,\tau}(t) := x_{n-1}^{\varepsilon,\tau}$ for all $t \in [t_{n-1}, t_n)$.
- Integral bound: For all $0 \leq s \leq t \leq T$

$$\mathscr{E}(t, x^{\varepsilon,\tau}(t)) - \mathscr{E}(s, x^{\varepsilon,\tau}(s)) \leq \int_s^t \partial_t \mathscr{E}(r, x^{\varepsilon,\tau}(r))\, dr - \mathscr{D}iss(x^{\varepsilon,\tau}; [s,t]).$$

- Uniform bound for $\mathscr{D}iss(x^{\varepsilon,\tau}; [0,T])$ for every τ, ε.
- For every ε fixed: Helly's principle $\Rightarrow$ exists subsequence $\{\tau_k\}$ st $x^{\varepsilon,\tau_k}(t) \to x^{\varepsilon}(t)$ pointwise as $k \to \infty$ for every $t \in [0,T]$.
- Helly's principle $\Rightarrow$ exists a subsequence $\{\varepsilon_k\}$ st $x^{\varepsilon_k}(t) \to x(t)$ pointwise as $k \to \infty$ for every $t \in [0,T]$.

New energy-dissipation balance: Lower bound

Proposition (Lower bound, Mielke-Rossi-Savaré 2010)**.** *For any BV function u, for any energy functional $\mathscr{E} \in C^1$ satisfying $|\nabla_x \mathscr{E}(t, u(t))| \leq 1$ for a.e. $t \in (0, T)$, it holds that*

$$\mathscr{E}(t_1, u(t_1)) - \mathscr{E}(t_0, u(t_0)) \geq \int_{t_0}^{t_1} \partial_t \mathscr{E}(s, u(s))\, ds - \mathscr{D}iss_{new}(u; [t_0, t_1]).$$

New energy-dissipation balance: Upper bound

Claim: At jump points

$$\mathscr{E}(t, x(t^+)) - \mathscr{E}(t, x(t^-)) \leq -\Delta_{new}(t, x(t^-), x(t)) - \Delta_{new}(t, x(t), x(t^+)).$$

Lemma (Approximate optimal transition). *Write* $x_j = x^{\varepsilon,\tau}(t_j)$. *Then*

$$-\nabla_x \mathscr{E}(t_i, x_i) \cdot (x_i - x_{i-1}) = \max\{1, |\nabla_x \mathscr{E}(t_i, x_i)|\} \cdot |x_i - x_{i-1}|.$$

Consequently, if $\delta \geq \max\{|t - t_i|, \varepsilon, \tau\}$ *and* $v : [a, b] \to \mathbb{R}^d$ *is the linear curve connecting* x_{i-1} *and* x_i*, then*

$$\int_a^b \max\{1, |\nabla_x \mathscr{E}(t, v(s))|\}.|\dot{v}(s)| ds \leq \mathscr{E}(t, x_{i-1}) - \mathscr{E}(t, x_i) + g(\delta) \cdot |x_i - x_{i-1}|$$

where $g(\delta) \to 0$ *as* $\delta \to 0$.

Proof of Approximate Optimal Transition Lemma

Recall that x_i is a minimizer for

$$\inf_{|z-x_{i-1}|\le\varepsilon} h(z) = \inf_{|z-x_{i-1}|\le\varepsilon} \left\{\mathscr{E}(t_i,z) + |z-x_{i-1}|\right\}.$$

1. Denote $c := |x_i - x_{i-1}|$; then x_i is also a minimizer for

$$\inf_{|z-x_{i-1}|=c} h(z).$$

By Lagrange multiplier, there exists $\lambda \in \mathbb{R}$ such that

$$\nabla_x \mathscr{E}(t_i, x_i) = \lambda(x_i - x_{i-1}).$$

2. Using $\partial_t \left(h(x_{i-1} + t(x_i - x_{i-1}))\right) \le 0$ at $t = 1$, we obtain

$$\nabla_x \mathscr{E}(t_i, x_i) \cdot (x_i - x_{i-1}) + |x_i - x_{i-1}| \le 0.$$

Thus either $x_i = x_{i-1}$, or $|\nabla_x \mathscr{E}(t_i, x_i)| \ge 1$ and $\lambda < 0$.

New energy-dissipation balance: Upper bound

Lemma. *If $x(t)$ jumps at t, then*

$$\begin{aligned}
\mathscr{E}(t,x(t))-\mathscr{E}(t,x(t^-)) &\leq -\Delta_{new}(t,x(t^-),x(t)),\\
\mathscr{E}(t,x(t^+))-\mathscr{E}(t,x(t)) &\leq -\Delta_{new}(t,x(t),x(t^+)).
\end{aligned}$$

Proof: By linear interpolation, construct a curve $v:[0,1]\to\mathbb{R}^d$ connecting the points

$$x(t^-),x^{\varepsilon,\tau}(t-\delta)=x^{\varepsilon,\tau}(t_i),x^{\varepsilon,\tau}(t_{i+1}),...,x^{\varepsilon,\tau}(t_{i+k})=x^{\varepsilon,\tau}(t),x(t).$$

Approximate optimal transition

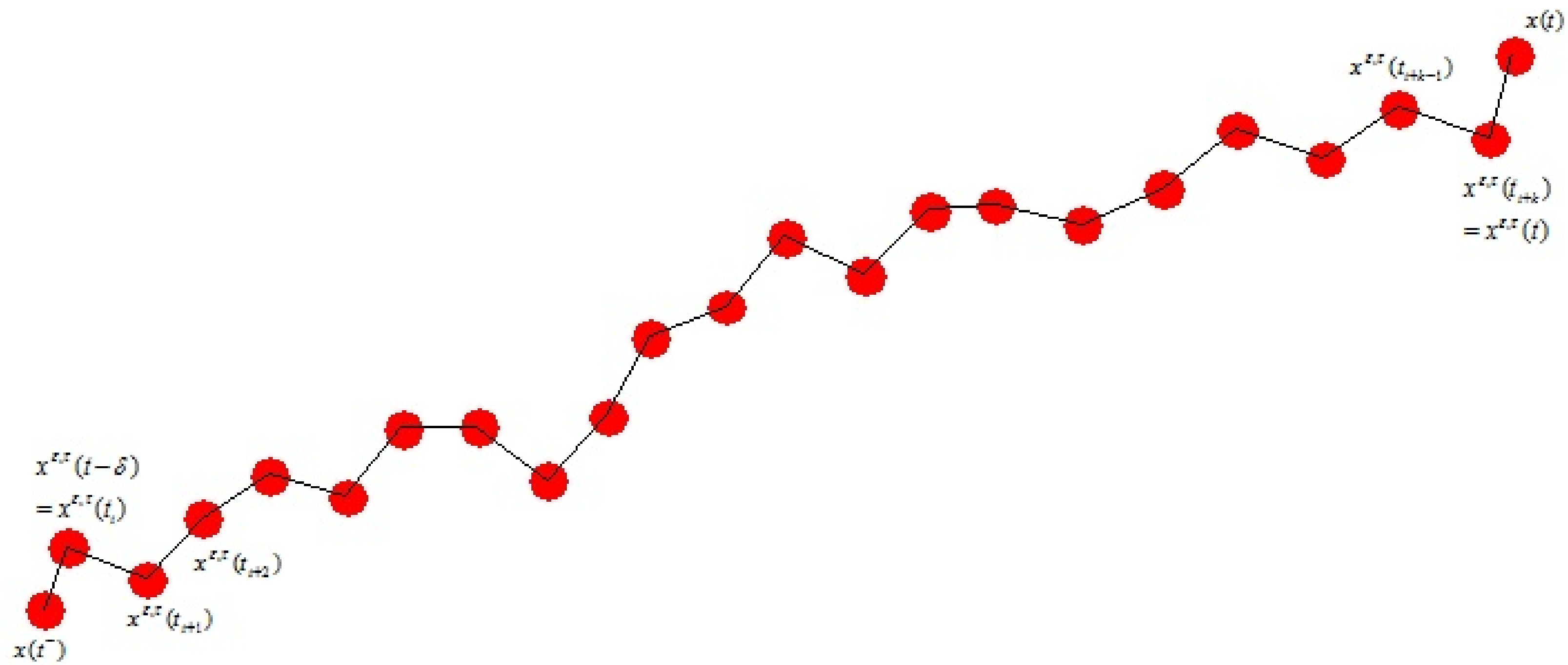

Figure 6. Approximate optimal transition between $x(t^-)$ and $x(t)$.

$$x(t^-) \to x^{\varepsilon,\tau}(t-\delta) = x^{\varepsilon,\tau}(t_i) \to x^{\varepsilon,\tau}(t_{i+1})$$
$$\to x^{\varepsilon,\tau}(t_{i+2}) \to \cdots \to x^{\varepsilon,\tau}(t_{i+k}) = x^{\varepsilon,\tau}(t) \to x(t).$$

New energy-dissipation balance: Upper bound

Then

$$
\begin{aligned}
\Delta_{new}(t, x(t^-), x(t)) &\leq \int_0^1 \max\{1, |\nabla_x \mathscr{E}(t, v(s))|\} \cdot |\dot{v}(s)|\, ds \\
&\leq \mathscr{E}(t, x^{\varepsilon,\tau}(t-\delta)) - \mathscr{E}(t, x^{\varepsilon,\tau}(t)) + Cg(\delta) \\
&\quad + C|x(t^-) - x^{\varepsilon,\tau}(t-\delta)| + C|x^{\varepsilon,\tau}(t) - x(t)|.
\end{aligned}
$$

Taking the limit $\tau \to 0$, then $\varepsilon \to 0$, then $\delta \to 0$, we conclude that

$$
\Delta_{new}(\mathscr{E}, t, x(t^-), x(t)) \leq \mathscr{E}(t, x(t^-) - \mathscr{E}(t, x(t)).
$$

Future works

Problem 1: Improve the weak local stability for BV solutions constructed by epsilon-neighborhood method.

Problem 2: Prove the existence of BV solutions constructed by epsilon-neighborhood for capillary drops.