Some certain classes of weak solutions to rate-independent systems

Mach Nguyet Minh

machnt@mathematik.uni-stuttgart.de

Department of Mathematics, Chair of Optimization and Inverse Problems,
University of Stuttgart, Germany

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Rate-independent systems

- Is carried out on a time-scale much longer than the relaxation time of the system.
- Inertial can be ignored.
- Irreversible due to dissipation effect (Ψ).
- The changes of the systems are caused by the changes of the external conditions $(\mathscr{E})$.
- Rate-independent, i.e. the rate of change of the solutions to the system depends only on the change of the velocity of the external conditions $\Longrightarrow$ dissipation effect is positively 1-homogeneous.

Applications

Brittle fractures

Source: Internet.

Some references

- G. Francfort and C. J. Larsen, *Existence and convergence for quasistatic evolution in brittle fracture*, Comm. Pure. Appl. Math., 56 (2003), pp. 1465-1500.
- G. Dal Maso and G. Lazzaroni, *Quasistatic crack growth in finite elasticity with non-interpenetration*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 27 (2010), pp. 257-290.

Applications

Soil mechanics
(the Cam-Clay model)

Source: Internet.

Some references

- G. Dal Maso and A. DeSimone, *Quasistatic evolution for Cam-Clay plasticity: Examples of spatially homogeneous solutions*, Math. Models Methods Appl. Sci. 19 (2009), pp. 1-69.
- G. Dal Maso, A. DeSimone and F. Solombrino, *Quasistatic evolution for Cam-Clay plasticity: A weak formulation via viscoplastic regularization and time rescaling*, Calc. Var. PDEs, 40 (2011), pp. 125-181.

Applications

Capillary drops

Source: Internet.

Some references

- G. Alberti and A. DeSimone, *Quasistatic evolution of sessile drops and contact angle hysteresis*, Archives for Rational Mechanics & Analysis, 202 (2011), pp. 295-348.

Abstract framework

- X : finite-dimensional normed vector space.
- $x(t)$: a "lazy" particle in X.
- $\mathscr{E}(t,x)$: smooth energy functional (of class C^2).
- $\Psi(x)$: dissipation functional, convex and positively 1-homogeneous.

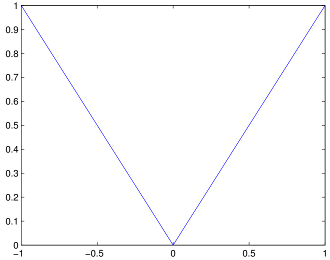


Figure 1. $\Psi(x) = |x|$ in 1D.

Rate-independent evolution

$$\nabla_x \mathscr{E}(t, x(t)) + \partial\Psi(\dot{x}(t)) \ni 0 \quad \text{in } X^* \qquad \text{for a.e. } t \in (0, T). \quad (1)$$

Notations:

- X^*: dual space of X.
- Subdifferential of convex function

$$\partial\Psi(x_0) := \{\eta \in X^* \mid \langle \eta, z\rangle \leq \Psi(z) \ \ \forall z \in X,\ \langle \eta, x_0\rangle = \Psi(x_0)\}$$

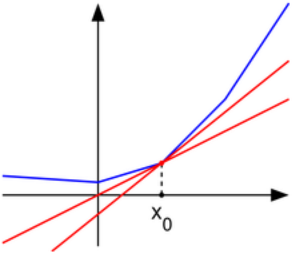


Figure 2. Subdifferentials (red) of a convex function (blue) at x_0.
Source: Wiki.

Solutions to rate-independent systems

- If $\mathscr{E}$ is strictly convex, Mielke and Theil have proved that (1) admits a unique solution which is Lipschitz continuous.

A. Mielke and F. Theil, *On rate-independent hysteresis models*, NoDEA Nonlinear Differential Equation Appl., 11 (2004), pp. 151-189.

- If $\mathscr{E}$ is not strictly convex, uniqueness may be lost, and solutions may have jumps in time $\Longrightarrow$ weak solutions are needed.

The Minimizing Movement Scheme

- Fix $\tau > 0$ time-step size. Define by $t_n := n\tau$, $n = 0, 1, \ldots, N$, $N\tau \geq T$.
- Technical assumption for $\mathscr{E}$:

  There exists $\lambda = \lambda(\mathscr{E})$ such that

$$|\partial_t \mathscr{E}(s,x)| \leq \lambda \mathscr{E}(s,x) \quad \text{for all } (s,x) \in [0,T] \times X.$$

- There exists a sequence $\{u_n^\tau\}$, $n = 0, \ldots, N$, such that
  1. $u_0^\tau := x_0$ (the initial datum).
  2. For $n = 1, \ldots, N$, u_n^τ minimizes the functional

$$u \mapsto \mathscr{E}(t_n, u) + \Psi\left(u - u_{n-1}^\tau\right). \tag{2}$$

The Minimizing Movement Scheme

$$\nabla_x \mathscr{E}(t, x(t)) + \partial\Psi(\dot{x}(t)) \ni 0 \quad \text{in } X^* \qquad \text{for a.e. } t \in (0, T).$$

$$\Downarrow$$

$$u_n^\tau \quad \text{minimizes} \quad u \mapsto \mathscr{E}(t_n, u) + \tau\Psi\left(\frac{u - u_{n-1}^\tau}{\tau}\right).$$

$$\Downarrow$$

$$u_n^\tau \quad \text{minimizes} \quad u \mapsto \mathscr{E}(t_n, u) + \Psi\left(u - u_{n-1}^\tau\right).$$

The Minimizing Movement Scheme

- Piecewise constant interpolation

$$u^{\tau}(t) := u^{\tau}_{n-1} \quad \text{for } t \in (t_{n-1}, t_n].$$

- Up to subsequence

$$u^{\tau_k}(t) \to u(t) \quad \text{for all } t \in [0, T].$$

E. De Giorgi, *New problems on minimizing movements*, in Boundary Value Problems for PDE and Applications, C. Baiocchi and J. L. Lions, eds., Masson, (1993), pp. 81-98.

Energetic solutions by the minimization scheme

Follow the minimization scheme (2), we get the limit $u(t)$ satisfying the following conditions

- the initial condition: $u(0) = u_0$;
- the global stability:

$$\mathscr{E}(t, u(t)) \leq \mathscr{E}(t, x) + \Psi(x - u(t)) \quad \text{for all } (t, x) \in [0, T] \times X;$$

- the energy-dissipation balance:

$$\mathscr{E}(t_2, u(t_2)) - \mathscr{E}(t_1, u(t_1)) = \int_{t_1}^{t_2} \partial_t \mathscr{E}(s, u(s))\, \mathrm{d}s - \mathscr{D}iss_\Psi(u; [t_1, t_2]) \ \ \forall\, 0 \leq t_1 \leq t_2 \leq T.$$

Remark: The energy-dissipation balance makes solutions irreversible.

Notations:

$$\mathscr{D}iss_\Psi(u(t); [t_1, t_2]) := \sup \left\{ \sum_{i=1}^{N} \Psi(u(s_i) - u(s_{i-1})) \ \middle|\ N \in \mathbb{N},\ t_1 = s_0 < s_1 < \cdots < s_N = t_2 \right\}.$$

Energetic solutions: Some references

- Energetic solutions in brittle fracture.

  G. A. Francfort and J.-J. Marigo, *Revisiting brittle fracture as an energy minimization problem*, J. Mech. Phys. Solids, 46 (2) (1998), pp. 1319-1342.

- Energetic solutions for shape-memory alloys.

  A. Mielke and F. Theil, *A mathematical model for rate-independent phase transformations with hysteresis*, Models of Continuum Mechanics in Analysis and Engineering, H.-D. Alber, R. Balean, and R. Farwig, eds., Shaker Ver., Aachen, (1999), pp. 117-129.

- Abstract framework.

  A. Mainik and A. Mielke, *Existence results for energetic models for rate-independent systems*, Calc. Var. Partial Differential Equations, 22 (2005), pp. 73-99.

Existence of Energetic solutions

- Energy estimates: for all $n \in \{1, \ldots, N\}$, we have

$$\begin{aligned} \mathscr{E}(t_n, u_n^\tau) &\leq \mathscr{E}(0, x_0)\, e^{\lambda t_n}, \\ \mathscr{E}(0, u_n^\tau) &\leq \mathscr{E}(0, x_0)\, e^{2\lambda t_n}. \end{aligned}$$

Proof: Global minimality of u_n^τ, smoothness and technical assumption of $\mathscr{E}$.

- Energy-dissipation inequality: for all $0 \leq s \leq t \leq T$, it holds that

$$\mathscr{E}(t, u^\tau(t)) - \mathscr{E}(s, u^\tau(s)) \leq \int_s^t \partial_t \mathscr{E}(r, u^\tau(r))\, \mathrm{d}r - \mathscr{D}iss_\Psi(u^\tau; [s,t]).$$

Proof: Global minimality of u_n^τ.

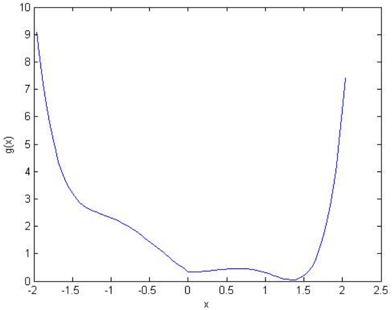
- Existence of the limit: Helly's selection theorem.

Existence of Energetic solutions

- Global stability:
  - global minimality of u_n^τ;
  - continuity of $\mathscr{E}$.
- Energy-dissipation upper bound:
  - energy-dissipation inequality for u^τ;
  - smoothness of $\mathscr{E}$;
  - dominated convergence theorem;
  - the lower-semicontinuity of $\mathscr{D}iss_\Psi$.
- Energy-dissipation lower bound:
  - a "clever" partition of $[0, T]$;
  - smoothness of $\mathscr{E}$;
  - global stability.

Energetic solutions: Advantage and Drawback

- construction: not so complicated $\Longrightarrow$ favorite solution for engineers.
- Model works well for convex energy. For non-convex energy: unexpected jumps.



BV solutions constructed by vanishing viscosity

- Idea: Replace Ψ by Ψ_ε. Ψ_ε has super-linear growth and converges to Ψ in some appropriate sense as $\varepsilon \to 0$.
- Repeat the Minimizing Movement Scheme for Ψ_ε to get the limit u^ε.
- Take $\varepsilon \to 0$ to get the limit u.

u satisfies the definitions of BV solutions, which is

+ initial condition: $u(0) = x_0$;
+ weak local stability: If u is continuous at t, then $-\nabla_x \mathscr{E}(t, u(t)) \in \partial\Psi(0)$;
+ new energy-dissipation balance: For all $0 \leq t_1 \leq t_2 \leq T$

$$\mathscr{E}(t_2, u(t_2)) - \mathscr{E}(t_1, u(t_1)) = \int_{t_1}^{t_2} \partial_t \mathscr{E}(s, u(s))\, \mathrm{d}s - \mathscr{D}iss_{new}(u; [t_1, t_2]).$$

BV solutions constructed by vanishing viscosity:

- Finite-dimensional state space:

A. Mielke, R. Rossi and G. Savaré, *BV solutions and viscosity approximations of rate-independent systems*, ESAIM Control Optim. Calc. Var., 18 (2012), pp. 36-80.

- Infinite-dimensional state space:

A. Mielke, R. Rossi and G. Savaré, *BV solutions to infinite-dimensional rate-independent systems*, Submitted Paper, (2013).

BV solutions constructed by eps-neighborhood

- Repeat the Minimizing Movement Scheme with the minimizing problem

$$u_n^{\tau,\varepsilon} \quad \text{minimizes } u \mapsto \mathscr{E}(t_n, u) + \Psi(u - u_{n-1}^{\tau,\varepsilon}) \quad \text{among all } \|u - u_{n-1}^{\tau,\varepsilon}\| \leq \varepsilon$$

  to get a limit u^ε.
- Take $\varepsilon \to 0$ to get the limit u.

u satisfies the definition of BV solutions.

Finite-dimensional state space:

M., *BV solutions constructed by epsilon-neighborhood method*, Submitted Paper (2013).

Approximate optimal transition

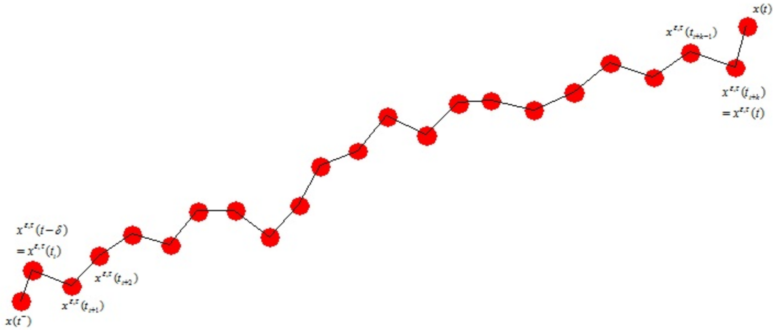


Figure 3. Approximate optimal transition between $x(t^-)$ and $x(t)$.

$$x(t^-) \to x^{\varepsilon,\tau}(t-\delta) = x^{\varepsilon,\tau}(t_i) \to x^{\varepsilon,\tau}(t_{i+1})$$
$$\to x^{\varepsilon,\tau}(t_{i+2}) \to \cdots \to x^{\varepsilon,\tau}(t_{i+k}) = x^{\varepsilon,\tau}(t) \to x(t).$$

BV solutions constructed by eps-neighborhood

New energy-dissipation upper bound

Lemma (Approximate optimal transition): Denote $x_j := x^{\tau,\varepsilon}(t_j)$, it holds that

$$\langle -\nabla_x \mathscr{E}(t_i, x_i), x_i - x_{i-1} \rangle = \Psi(x_i - x_{i-1}) + \min_{\eta \in \partial\Psi(0)} \|\eta + \nabla_x \mathscr{E}(t_i, x_i)\|_* \cdot \|x_i - x_{i-1}\|.$$

Consequently, if $\delta \geq \varepsilon + |t - t_i|$ and v is the linear curve connecting x_{i-1} and x_i, there exists $g(\delta)$ such that $g(\delta) \to 0$ as $\delta \to 0$ and

$$\begin{aligned} \mathscr{E}(t, x_{i-1}) - \mathscr{E}(t, x_i) \quad \geq \quad & \int_a^b \Psi(\dot{v}(s)) + \min_{\eta \in \partial\Psi(0)} \|\eta + \nabla_x \mathscr{E}(t, v(s))\|_* \cdot \|\dot{v}(s)\| \, ds \\ & -(b-a) g(\delta) \|x_i - x_{i-1}\|. \end{aligned}$$

Thank you very much for your attention!