Enhancing residual-based techniques with shape reconstruction features in Electrical Impedance Tomography

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Electrical Impedance Tomography (EIT)

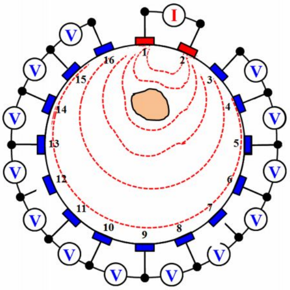


Figure: A circular domain with surface electrodes

(**Source:** Medical Imaging Study Group at Yonsei Univ.)

- reference body $\Omega \subset \mathbb{R}^n$, $n \geq 2$
- apply small currents $(g(x)|_{\partial\Omega})$
- measure induced voltages $(u(x)|_{\partial\Omega})$

$\rightsquigarrow$ reconstruction conductivity $\sigma(x)$ inside Ω

Mathematical Model

Recover σ from Neumann-to-Dirichlet Operator

$$\Lambda(\sigma) : L^2_\diamond(\partial\Omega) \to L^2_\diamond(\partial\Omega), \quad g|_{\partial\Omega} \mapsto u|_{\partial\Omega},$$

where u solves

$$\begin{cases} \nabla \cdot (\sigma \nabla u) & = 0, \qquad \text{in } \Omega, \\ \sigma \partial_\nu u|_{\partial\Omega} & = g|_{\partial\Omega}, \qquad \text{on } \partial\Omega. \end{cases}$$

D-bar Method; Factorization Method; Enclosure Method; ...

Linearization (NOSER, GREIT, ...). $\Lambda(\sigma), \Lambda'(\sigma)\kappa$: linear, compact, self-adjoint

Linearizing around a reference conductivity σ_0: $\Lambda(\sigma) - \Lambda(\sigma_0) \approx \Lambda'(\sigma_0)(\sigma - \sigma_0)$

NOSER

NOSER (Cheney et al., 1990), GREIT (Adler et al., 2009)

$$\text{Solve}\ \ \Lambda'(\sigma_0)\kappa \approx \Lambda(\sigma) - \Lambda(\sigma_0), \quad \text{then}\ \kappa \approx \sigma - \sigma_0$$

- multiple possibilities to measure residual norm and to regularize
- no rigorous theory for single linearization step
- almost no theory for Newton iteration:
  - Dobson (1992): (Local) convergence for regularized EIT eqn
  - Lechleiter/Rieder (2008): (Local) convergence for discretized setting

Exact (One-step) Linearization

Theorem (Harrach/Seo, SIAM J. Math. Anal. 2010)

Let κ, σ, σ_0 *piecewise analytic and* $\Lambda'(\sigma_0)\kappa = \Lambda(\sigma) - \Lambda(\sigma_0)$. *Then*

(a) $\operatorname{supp}_{\partial\Omega}\kappa = \operatorname{supp}_{\partial\Omega}(\sigma - \sigma_0)$

(b) $\frac{\sigma_0}{\sigma}(\sigma - \sigma_0) \leq \kappa \leq \sigma - \sigma_0$ *on the bdry of* $\operatorname{supp}_{\partial\Omega}(\sigma - \sigma_0)$

- Existence of exact solution is unknown
- In practice: finite-dimensional, noisy measurements

Monotonicity Test

- Monotonicity:

$$\tau \leq \sigma \quad \Longrightarrow \quad \Lambda(\tau) \geq \Lambda(\sigma)$$

- Idea: Simulate $\Lambda(\tau)$ for test conductivity τ and compare with $\Lambda(\sigma)$ (Tamburrino/Rubinacci 02, Lionheart, Soleimani, Ventre, ...)
- Inclusion detection: For $\sigma = 1 + \chi_D$ with unknown D, use $\tau = 1 + \chi_B$, with small ball B

$$B \subset D \quad \Longrightarrow \quad \tau \leq \sigma \quad \Longrightarrow \quad \Lambda(\tau) \geq \Lambda(\sigma)$$

- Algorithm: Mark all balls B with $\Lambda(1 + \chi_B) \geq \Lambda(\sigma)$
- Result: upper bound of D

Monotonicity Test

- Monotonicity test detects exact shape:

Theorem (Harrach/Ullrich, SIAM J. Math. Anal., 2013)

$\Omega \setminus \overline{D}$ *connected*, $\sigma = 1 + \chi_D$

$$B \subseteq D \quad \Longleftrightarrow \quad \Lambda(1 + \chi_B) \geq \Lambda(\sigma)$$

- For faster implementation:

$$B \subseteq D \quad \Longleftrightarrow \quad \Lambda(1) + \frac{1}{2}\Lambda'(1)\chi_B \geq \Lambda(\sigma)$$

- Does not behave well in high level of noise

Mathematical Setting

Assumptions:

- Continuum model
- Ω: bounded
- Homogeneous background: $\sigma_0 \equiv 1$
- True conductivity $\sigma = 1 + \gamma\chi_D$, $\gamma \in L^\infty(\Omega)$ (implies definiteness)
- $\Omega \setminus D$ connected

Combination idea of NOSER and MT

$$\min_{\{\text{constraints}\}} \|\Lambda(\sigma) - \Lambda(1) - \Lambda'(1)\kappa\| \tag{1}$$

- Approximate by
  $\mathbf{A}(\kappa) := (\langle g_i, (\Lambda(\sigma) - \Lambda(1) - \Lambda'(1)\kappa)\, g_j\rangle)_{i,j=1}^{N}$
- Minimize $\|\mathbf{A}(\kappa)\|_F$
- Approximate κ by $\sum_{k=1}^{T} a_k \chi_{P_k}$
- Use Monotonicity Test to define constraints

Monotonicity-based Constraints

$\Omega \setminus D$ connected:

(i) If P_k lies inside the true inclusion

$$\Lambda(1) - \Lambda(\sigma) + \alpha\Lambda'(1)\chi_k \geq 0$$

for (at least) all $\alpha \in [0, a]$, where $a = 1 - \frac{1}{1+\inf_D \gamma}$.

(ii) If P_k lies outside the inclusions then

$$\Lambda(1) - \Lambda(\sigma) + \alpha\Lambda'(1)\chi_k \not\geq 0$$

for all $\alpha \geq 0$.

$\rightsquigarrow$ Find β_k biggest

$$\Lambda(1) - \Lambda(\sigma) + \alpha\Lambda'(1)\chi_k \geq 0 \quad \forall\alpha \in [0, \beta_k]. \tag{2}$$

Monotonicity-based Constraints

Lemma

$\Omega \setminus D$ connected: $\beta_k > 0$ iff $P_k \subseteq D$.

- We allow $\beta_k = \infty$
- Constraints defined by Monotonicity Test:

$$\sum_{k=1}^{T} a_k \chi_{P_k}, \quad 0 \leq a_k \leq \min(a, \beta_k)$$

- $\Omega \setminus D$ connected: $\min(a, \beta_k) = a$ if $P_k \subset D$, $\min(a, \beta_k) = 0$ if $P_k \not\subseteq D$

Monotonicity-based Constraints

- Use β_k alone: also improve shape reconstruction

  [Zhou, Harrach and Seo, Monotonicity-based Electrical Impedance Tomography Lung Imaging, submitted]
- Appearance of a:
  - make sure minimizer exists
  - produce much improved results (see last figure)
  - numerical results more stable in high level of noise

Existence of Minimizer

Theorem

$$\min\left\{\|\mathbf{A}(\kappa)\|_F : \kappa = \sum_{k=1}^{T} a_k \chi_{P_k}, 0 \le a_k \le \min(a, \beta_k)\right\} \qquad (3)$$

(i) *Unique minimizer* $\hat{\kappa}$.

(ii) $P_k \subset \operatorname{supp}\hat{\kappa}$ *iff* $P_k \subset D$.
Moreover, $\hat{\kappa} = \sum_{k=1}^{T} \min(a, \beta_k)\chi_{P_k}$.

Use operator norm in (1) $\rightsquigarrow$ same result, numerical results not good as Frobenius norm

Sketch of proof

(i) $\kappa \mapsto \|\mathbf{A}(\kappa)\|_F^2 := \sum_{i,j=1}^N \langle g_i, r(\kappa) g_j \rangle^2$ continuous $\rightsquigarrow$ a minimizer in compact sets.

(ii) **Step 1:** If $\kappa = \sum_{k=1}^T \alpha_k \chi_k, 0 \leq \alpha_k \leq \min(a, \beta_k)$, then $r(\kappa) \leq 0$

- if $\alpha_k > 0$: Lemma 3 $\rightsquigarrow P_k \subseteq D$
- $\alpha_k \leq a$: $\forall g \in L_\diamond^2(\partial\Omega)$

$$\langle g, r(\kappa) g \rangle = \left\langle g, (\Lambda(\sigma) - \Lambda(1) - \Lambda'(1)\kappa) g \right\rangle \leq -\int_D a |\nabla u_g^0|^2 \, \mathrm{d}x + \sum_k \int_{P_k} \alpha_k |\nabla u_g^0|^2 \, \mathrm{d}x \leq 0$$

Step 2: $\hat{\kappa} = \sum_{k=1}^T \hat{\alpha}_k \chi_k$ is a minimizer of (3). Then $\operatorname{supp} \hat{\kappa} \subseteq D$.

- if $\hat{\alpha}_k > 0$:
  $\Lambda(\sigma) - \Lambda(1) - \Lambda'(1)\alpha\chi_k \leq \Lambda(\sigma) - \Lambda(1) - \Lambda'(1)\hat{\kappa} \leq 0$ for all $\alpha \leq \hat{\alpha}_k$.
- Lemma 3 $\rightsquigarrow P_k \subseteq D$.

Sketch of proof

Step 3: $\hat{\kappa} = \sum_{k=1}^{T} \hat{\alpha}_k \chi_k$ is a minimizer of (3). If $P_k \subseteq D$, then $P_k \subseteq \mathrm{supp}\hat{\kappa}$.

- If $P_k \not\subseteq \mathrm{supp}\hat{\kappa} \rightsquigarrow \hat{\alpha}_k = 0$.
- $P_k \subseteq D$: Lemma 3 $\rightsquigarrow \exists \beta_k > 0$: $\Lambda(\sigma) - \Lambda(1) - \Lambda'(1)\alpha\chi_k \leq 0 \quad \forall \alpha \in [0, \beta_k]$.
- Claim: for any $\alpha, 0 \leq \alpha \leq \min(a, \beta_k)$: $\|\mathbf{A}(\hat{\kappa} + \alpha\chi_k)\|_F < \|\mathbf{A}(\hat{\kappa})\|_F$

$$\|\mathbf{A}(\hat{\kappa} + \alpha\chi_k)\|_F^2 - \|\mathbf{A}(\hat{\kappa})\|_F^2 = \sum_{i=1}^{N} |\lambda_i(\hat{\kappa} + \alpha\chi_k)|^2 - \sum_{i=1}^{N} |\lambda_i(\hat{\kappa})|^2 =$$

$$= \sum_{i=1}^{N} (\lambda_i(\hat{\kappa} + \alpha\chi_k) + \lambda_i(\hat{\kappa})) \cdot (\lambda_i(\hat{\kappa} + \alpha\chi_k) - \lambda_i(\hat{\kappa}))\,.$$

- Step 3 $\rightsquigarrow r(\hat{\kappa}) \leq 0$: $\mathrm{x}^\top \mathbf{A}(\hat{\kappa})\mathrm{x} = \sum_{i,j=1}^{N} x_i x_j \langle g_i, r(\hat{\kappa}) g_j \rangle = \langle g, r(\hat{\kappa}) g \rangle \leq 0$
- $-\mathbf{A}(\hat{\kappa}), -\mathbf{A}(\hat{\kappa} + \alpha\chi_k)$: positive semi-definite symmetric matrices $\rightsquigarrow$ all of eigenvalues are non-negative.
- $\mathbf{B}_k := (-\langle g_i, \Lambda'(1)\chi_k g_j \rangle)_{i,j=1}^{N} \rightsquigarrow \mathbf{A}(\hat{\kappa} + \alpha\chi_k) = \mathbf{A}(\hat{\kappa}) + \alpha\mathbf{B}_k$. Weyl's Inequalities:

$$\lambda_i(\hat{\kappa} + \alpha\chi_k)) \geq \lambda_i(\hat{\kappa}) + \alpha\lambda_N(\mathbf{B}_k) \geq \lambda_i(\hat{\kappa}) \qquad \text{for all} \qquad i \in \{1, \ldots, N\}. \qquad (4)$$

Sketch of proof

Step 4: $\hat{\kappa}$ is a minimizer of (3) $\rightsquigarrow \hat{\kappa} = \sum_{k=1}^{T} \min(a, \beta_k)\chi_k$.

- $\hat{\kappa} \leq \sum_{k=1}^{T} \min(a, \beta_k)\chi_k$
- If there exists P_k: $\hat{\kappa}(x) < \min\{a, \beta_k\}$ a.e. $\exists h > 0$: $\hat{\kappa} + h\chi_k \leq \min(a, \beta_k)$ a.e.
- Step 1 $\rightsquigarrow r(\hat{\kappa}) \leq 0$, $r(\hat{\kappa} + h\chi_k) \leq 0$.
- Similar to Step 3: $\|\mathbf{A}(\hat{\kappa} + h\chi_k)\|_F < \|\mathbf{A}(\hat{\kappa})\|_F$

Numerical results

- Minimizing problem (3): quadratic minimizing problem + linear bounds
- **cvx**: can be solve directly, same complexity as LP
- **cvx**: (3) equiv. to

$$\min \left\{ \|\mathbf{A}_0 - \sum_{k=1}^{T} a_k \mathbf{A}_k\|_F : 0 \le a_k \le \min(a, \beta_k) \right\}$$

$\mathbf{A_0} := (\langle g_i, (\Lambda(1) - \Lambda(\sigma)) g_j \rangle)_{i,j=1}^N$; $\mathbf{A_k} := (-\langle g_i, \Lambda'(1)\chi_k g_j \rangle)_{ij=1}^N$

- $\kappa \mapsto \|\mathbf{A}(\kappa)\|^2$: convex $\rightsquigarrow$ no local min
- $\delta\%$ noise: *tol* (in finding β_k) can be chosen by default: $tol = -\delta$ (MT: *tol* has to choose by hand, in order to get good reconstruction)

Numerical results

Note: All pictures have different color scale. No regularization at this moment!

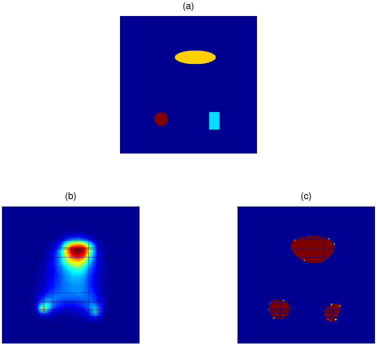


Figure: Reconstruction of conductivity change: (a) true distribution of conductivity change; (b) a plot of $\sum_k \beta_k \chi_k$; (c) minimizing residuum with constraint $\min(a, \beta_k)$. Relative noise $= 0.1\%$ (corresponding absolute noise $= 5.97 \times 10^{-5}$)

Numerical results

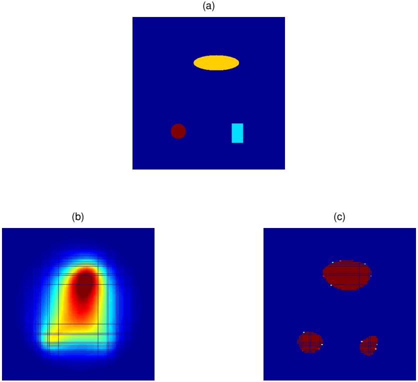


Figure: Reconstruction of conductivity change: (a) true distribution of conductivity change; (b) a plot of $\sum_k \beta_k \chi_k$; (c) minimizing residuum with constraint $\min(a, \beta_k)$. Relative noise = 1% (corresponding absolute noise $= 5.97 \times 10^{-4}$)

Numerical results

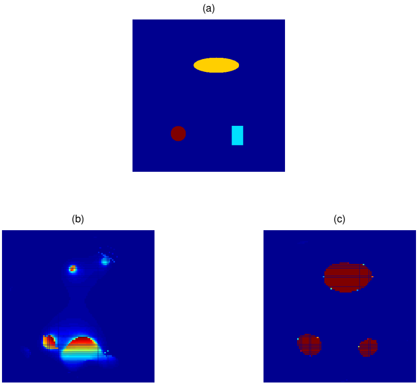
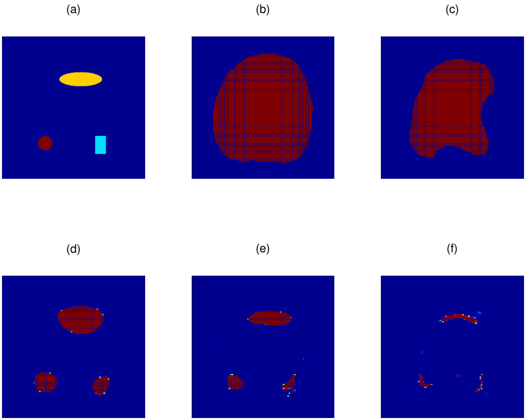


Figure: Reconstruction of conductivity change: (a) true distribution of conductivity change; (b) a plot of $\sum_k \beta_k \chi_k$; (c) minimizing residuum with constraint $\min(a, \beta_k)$. Relative noise $= 5\%$ (corresponding absolute noise $= 1.12 \times 10^{-3}$)

Compare to MT and to different constraints

Reconstruction of conductivity change



- (a) true cond change
- (b) MT: *tol* $= -\delta$
- (c) MT with better *tol* manually chosen
- (d) min residuum $\min(a, \beta_k)$
- (e) min residuum $\min(\sup(\sigma - 1), \beta_k)$
- (f) min residuum β_k.

Figure: Relative noise $= 0.1\%$