Regularization of an inverse nonlinear parabolic problem with time-dependent coefficient and locally Lipschitz source term

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Abstract

We consider a backward problem of finding a function $u$ satisfying a nonlinear parabolic equation in the form $u_t + a(t)Au(t) = f(t, u(t))$ subject to the final condition $u(T) = \varphi$. Here $A$ is a positive self-adjoint unbounded operator in a Hilbert space $H$ and $f$ satisfies a locally Lipschitz condition. This problem is ill-posed. Using quasi-reversibility method, we shall construct a regularized solution $u_\varepsilon$ from the measured data $a_\varepsilon$ and $\varphi_\varepsilon$. We show that the regularized problem are well-posed and that their solutions converge to the exact solutions. Error estimate is given.

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1. Introduction

Let $(H, \|\cdot\|)$ be a Hilbert space with the inner product $(\cdot, \cdot)$. Let $A$ be a positive self-adjoint operator defined on a dense subspace $D(A) \subset H$ such that $-A$ generates a compact contraction semi-group $S(t)$ on $H$. Let $f : [0, T] \times H \to H$ satisfy the locally Lipschitz condition: for each $M > 0$, there exists $k(M) > 0$ such that

$$
\|f(t, u) - f(t, v)\| \leq k(M)\|u - v\| \text{ if } \max \{\|u\|, \|v\|\} \leq M.
$$

(1)

We shall consider a backward problem of finding a function $u : [0, T] \to H$ such that

$$
u_t + a(t)Au(t) = f(t, u(t)), \quad 0 < t < T,
$$

$$
u(T) = \varphi,
$$

where $a \in C([0, T])$ is a given real-valued function and $\varphi \in H$ is a prescribed final value.

This nonlinear nonhomogeneous problem is severely ill-posed. In fact, the problem is extremely sensitive to measurement errors (see, e.g., [2]). The final data is usually the result of discrete experimental measurements and is subject to error. Hence, a solution corresponding to the data does not

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always exist, and in the case of existence, does not depend continuously on the given data. Thus one has to resort to a regularization.

The backward problem (2) has a long history. The linear homogeneous case \( f = 0 \) has been considered by many authors such as quasi-reversibility method \([7, 8, 6, 10, 1]\), quasi-boundary value method \([4, 5]\). The problem with constant coefficient and nonlinear source term, i.e. 

\[
\begin{align*}
u_t + Au(t) &= f(t, u(t)), & 0 < t < T, \\
u(T) &= \varphi, \\
\end{align*}
\]

was studied in \([3, 12, 13, 14]\). However, in these papers, the source function \( f \) is assumed to be globally Lipschitz, that is 

\[\|f(t, u) - f(t, v)\| \leq k\|u - v\|\]

where \( k \) is independent of \( t, u \). Recently, in \([15]\), a regularization method for locally Lipschitz source term has been established under an extra condition on the source term:

There exists a constant \( L \geq 0 \), such that \( \langle f(t, u) - f(t, v), u - v \rangle + \|u - v\|^2 \geq 0 \).

This condition holds for the source \( f(u) = u\|u\|^2_H \) (see \([15]\)). However, it is not satisfied in several cases, for example, \( f(u) = au - bu^3 \) \((b > 0)\) of the Ginzburg-Landau equation. Hence, another regularization method which can be applied to any locally Lipschitz source term is of interests. In this paper, we shall assume that the source term \( f \) is locally Lipschitz with respect to \( u \) (i.e. \( f \) satisfies (1)). Our main idea is approximating the function \( f \) by a sequence \( f_\varepsilon \) of Lipschitz functions

\[\|f_\varepsilon(t, u) - f_\varepsilon(t, v)\| \leq k_\varepsilon\|u - v\|\]

Then, we use the results in \([12, 14]\) to approximate problem (3) by the following problem

\[
\begin{align*}
d\nu^\varepsilon(t) + A_\varepsilon u^\varepsilon(t) &= B(\varepsilon, t)f_\varepsilon(t, u^\varepsilon(t)), & t \in [0, T], \\
u^\varepsilon(T) &= \varphi \\
\end{align*}
\]

where \( A_\varepsilon, B(\varepsilon, t) \) are defined appropriately.

When the perturbed coefficient \( a \) is time-dependent, the problems turns to be more complicated. Indeed, the strategies used for constant coefficient cannot be applied to the time-dependent coefficient case. The problem with time-dependent coefficient has been recently investigated in \([9]\). However, the methods proposed in \([9]\) can be merely applied either for zero source with perturbed time-dependent coefficient or for globally Lipschitz source with unperturbed time-dependent coefficient. We would like to emphasize that our regularization method for constant coefficient also works for unperturbed time-dependent coefficient.

The paper is organized as follows. In Section 2, we shall investigate a regularization method for the case of constant coefficient \( a \equiv 1 \). In particular, we shall give precise formulas of \( A_\varepsilon, B(\varepsilon, t) \) and \( f_\varepsilon(t, v) \); show that the regularized problem (4) is well-posed and prove the convergence of \( u^\varepsilon \) to the exact solution in \( C([0, T]; H) \) with explicit error estimates. Section 3 provides a regularization method for perturbed time-dependent coefficient \( a(t) \).
2. Regularization of backward parabolic problem with constant coefficient

2.1. The well-posedness of the regularized problem (4)

We shall first give the precise formula of the operator \( S(t) \). Assume that \( A \) is a positive self-adjoint operator in the separable Hilbert space \( (H, (\cdot, \cdot)) \) and 0 is in its resolvent set. Since \( A^{-1} \) is a compact self-adjoint operator, there is an orthonormal eigenbasis \( \{\phi_n\}_{n=1}^{\infty} \) of \( H \) corresponding to a sequence of its eigenvalues \( \{\lambda_n^{-1}\}_{n=1}^{\infty} \) in which

\[
0 < \lambda_1 \leq \lambda_2 \leq ... \lim_{n \to \infty} \lambda_n = \infty.
\]

Thus \( A^{-1} \phi_n = \lambda_n^{-1} \phi_n \) and \( A \phi_n = \lambda_n \phi_n \) for each \( n \geq 1 \). The compact contraction semi-group \( S(t) \) corresponding to \( A \) is

\[
S(t)v = \sum_{n=1}^{\infty} e^{-t\lambda_n} (\phi_n, v) \phi_n, \quad v \in H.
\]

Problem (3) can be written in the language of semi-group as follows.

\[
u(t) = S(t - T) \varphi - \int_t^T S(t - s) f(s, u(s)) \, ds.
\] (5)

For each \( \varepsilon > 0 \), we define the bounded operator

\[
A_{\varepsilon}(v) = -\frac{1}{T} \sum_{n=1}^{\infty} \ln(\varepsilon + e^{-T\lambda_n})(\phi_n, v) \phi_n.
\] (6)

The compact contraction semi-group \( S_{\varepsilon}(t) \) corresponding to \( A_{\varepsilon} \) is

\[
S_{\varepsilon}(t)v = \sum_{n=1}^{\infty} \left( \varepsilon + e^{-T\lambda_n} \right)^{\frac{1}{T}} (\phi_n, v) \phi_n, \quad v \in H.
\]

Obviously, (4) can be written as

\[
u^{\varepsilon}(t) = S_{\varepsilon}(t - T) \varphi - \int_t^T S_{\varepsilon}(t - s) B(\varepsilon, s) f(\varepsilon, u^{\varepsilon}(s)) \, ds,
\] (7)

For each \( t \leq T \), define by \( B(\varepsilon, t) \) the bounded operator

\[
B(\varepsilon, t) := S_{\varepsilon}(t - T)S(T - t).
\]

The operator \( B(\varepsilon, t) \) can be written explicitly as

\[
B(\varepsilon, t)(v) = \sum_{n=1}^{\infty} (1 + \varepsilon e^{-T\lambda_n})^{\frac{1}{T} - 1} (\phi_n, v) \phi_n, \quad v \in H.
\] (8)

In particular,

\[
B(\varepsilon, t) \phi_n = S_{\varepsilon}(t - T)S(T - t) \phi_n = S_{\varepsilon}(t - T) \left( e^{-(T-t)\lambda_n} \phi_n \right)
\]

\[
= \left( \varepsilon + e^{-T\lambda_n} \right)^{\frac{1}{T}} e^{-(T-t)\lambda_n} \phi_n = (\varepsilon e^{T\lambda_n} + 1)^{\frac{1}{T}} \phi_n, \quad \forall n \geq 1.
\]

Our later calculations will be represented via operators \( S_{\varepsilon}(t) \) and \( B(\varepsilon, t) \). We shall need some upper bounds of these operators.
Lemma 1. Let $0 \leq t \leq T$. Then $S_{\varepsilon}(-t)$ and $B(\varepsilon, t)$ are bounded operators and

$$\|S_{\varepsilon}(-t)\| \leq e^{-\frac{t}{\varepsilon}}, \quad \|B(\varepsilon, t)\| \leq 1.$$ 

Moreover,

$$\|[B(\varepsilon, t) - I] \phi_n\| \leq \varepsilon e^{T \lambda_n}, \forall n \geq 1.$$ 

Proof. For each $n \geq 1$, one has

$$\|S_{\varepsilon}(t) \phi_n\| = \left(\varepsilon + e^{-T \lambda_n}\right)^{\frac{t}{2}} \leq e^{-\frac{t}{2}},$$

$$\|B(\varepsilon, t) \phi_n\| = (1 + \varepsilon e^{T \lambda_n})^{\frac{t}{2}} \leq 1$$

$$\|[I - B(\varepsilon, t)] \phi_n\| = 1 - (1 + \varepsilon e^{T \lambda_n})^{\frac{t}{2}}$$

$$\leq 1 - (1 + \varepsilon e^{T \lambda_n})^{-1} \leq e^{T \lambda_n}.$$ 

The desired result follows. \[\square\]

Next, we define an approximation $f_{\varepsilon}$ of $f$. Recall that $f : [0, T] \times H \rightarrow H$ satisfies the locally Lipschitz condition $(1)$:

For each $M > 0$, there exists $k(M) > 0$ such that $\|f(t, u) - f(t, v)\| \leq k(M) \|u - v\|$ if $\max\{\|u\|, \|v\|\} \leq M$.

It is obvious that the function $k$ is increasing on $[0, \infty)$. We can choose a set $\{M_{\varepsilon} > 0\}_{\varepsilon > 0}$ satisfying

$$\lim_{\varepsilon \to 0^+} M_{\varepsilon} = \infty \quad \text{and} \quad k(M_{\varepsilon}) \leq \ln(\varepsilon^{-1})/(4T).$$

Define

$$f_{\varepsilon}(t, v) = f\left(\min\left\{\frac{M_{\varepsilon}}{\|v\|^2}, 1\right\}, v\right), \quad \forall (t, v) \in [0, T] \times H,$$ 

(9)

in particular $f_{\varepsilon}(t, 0) = f(t, 0)$. With this definition, we claim that $f_{\varepsilon}$ is a Lipschitz function. In fact, we have

Lemma 2. For $\varepsilon > 0$, $t \in [0, T]$ and $v_1, v_2 \in H$, one has

$$\|f_{\varepsilon}(t, v_1) - f_{\varepsilon}(t, v_2)\| \leq k_{\varepsilon} \|v_1 - v_2\|,$$

where $k_{\varepsilon} = 2k(M_{\varepsilon}) \leq \ln(\varepsilon^{-1})/(2T)$.

Proof. Due to the continuity, it is enough to prove Lemma 2 for non-zero vectors $v_1$, $v_2$. We can assume that $\|v_1\| \geq \|v_2\| > 0$. Using the locally Lipschitz property of $f$, one has

$$\|f_{\varepsilon}(t, v_1) - f_{\varepsilon}(t, v_2)\| = \left\|f\left(\min\left\{\frac{M_{\varepsilon}}{\|v_1\|^2}, 1\right\}, v_1\right) - f\left(\min\left\{\frac{M_{\varepsilon}}{\|v_2\|^2}, 1\right\}, v_2\right)\right\|$$

$$\leq k(M_{\varepsilon}) \min\left\{\frac{M_{\varepsilon}}{\|v_1\|^2}, 1\right\} \|v_1 - \min\left\{\frac{M_{\varepsilon}}{\|v_2\|^2}, 1\right\} v_2\|.$$ 

It remains to show that

$$\min\left\{\frac{M_{\varepsilon}}{\|v_1\|^2}, 1\right\} v_1 - \min\left\{\frac{M_{\varepsilon}}{\|v_2\|^2}, 1\right\} v_2 \leq 2\|v_1 - v_2\|.$$
This inequality is trivial if \( M_\varepsilon \geq ||v_1|| \geq ||v_2|| \). When \( ||v_1|| \geq ||v_2|| \geq M_\varepsilon \), one has
\[
\left\| \frac{M_\varepsilon}{||v_1||} v_1 - \frac{M_\varepsilon}{||v_2||} v_2 \right\| = M_\varepsilon \left( \left\| \frac{v_1 - v_2}{||v_1||} + \frac{||v_2|| - ||v_1||}{||v_1|| \cdot ||v_2||} \right\| \right)
\leq M_\varepsilon \left( \left\| \frac{v_1 - v_2}{||v_1||} \right\| + \frac{||v_2|| - ||v_1||}{||v_1|| \cdot ||v_2||} \right)
= \left( ||v_1 - v_2|| + ||v_2|| - ||v_1|| \right) \leq 2 ||v_1 - v_2||.
\]

Finally, if \( ||v_1|| \geq M_\varepsilon \geq ||v_2|| \) then
\[
\left\| \frac{M_\varepsilon}{||v_1||} v_1 - v_2 \right\| = \left\| \frac{M_\varepsilon}{||v_1||} v_1 + v_1 - v_2 \right\|
\leq \left\| \frac{M_\varepsilon}{||v_1||} v_1 \right\| + ||v_1 - v_2||
= ||M_\varepsilon - ||v_1|| + ||v_1 - v_2|| \leq 2 ||v_1 - v_2||.
\]
Here we have used the inequality \( |M_\varepsilon - ||v_1||| \leq ||v_2|| - ||v_1|| \leq ||v_1 - v_2||. \)

We now study the existence, the uniqueness and the stability of a (weak) solution of problem (4).

**Theorem 1.** Let \( \varepsilon > 0 \). For each \( \varphi \in H \), problem (4) has a unique solution \( u^\varepsilon \in C([0,T]; H) \). Moreover, the solutions depend continuously on the data in the sense that if \( u_j^\varepsilon \) is the solution corresponding to \( \varphi_j \), \( j = 1,2 \), then
\[
||u_j^\varepsilon(t) - u^\varepsilon(t)|| \leq e^{k_\varepsilon(T-t)||\varphi_1 - \varphi_2||} ||u_j^\varepsilon - u^\varepsilon||.
\]

**Proof. Step 1: Uniqueness**

Fix \( \varphi \in H \). For each \( w \in C([0,T]; H) \), define by
\[
F(w)(t) := S_\varepsilon(t-T)\varphi - \int_t^T S_\varepsilon(t-s) B(\varepsilon,s) f_\varepsilon(s,w(s)) \, ds.
\]
It is sufficient to show that \( F \) has a unique fixed point in \( C([0,T]; H) \). This fact will be proved by contraction principle.

We claim by induction with respect to \( m = 1,2,... \) that, for all \( w,v \in C([0,T]; H) \),
\[
||F^m(w)(t) - F^m(v)(t)|| \leq \left( \frac{k_\varepsilon}{\varepsilon} \right)^m \frac{(T-t)^m}{m!} ||w(s) - v(s)||, \tag{10}
\]
where \( ||.|| \) is the sup norm in \( C([0,T]; H) \). For \( m = 1 \), using lemmas 1 and 2, we have
\[
||F(w)(t) - F(v)(t)|| = \left\| \int_t^T S_\varepsilon(t-s) B(\varepsilon,s) [f_\varepsilon(s,w(s)) - f_\varepsilon(s,v(s))] \, ds \right\|
\leq \int_t^T ||S_\varepsilon(t-s)|| \cdot ||B(\varepsilon,s)|| \cdot ||f_\varepsilon(s,w(s)) - f_\varepsilon(s,v(s))|| \, ds
\leq k_\varepsilon \int_t^T \varepsilon^\frac{m-1}{m} ||w - v|| \, ds \leq \frac{k_\varepsilon}{\varepsilon} \int_t^T ||w - v|| \, ds
\leq \frac{k_\varepsilon}{\varepsilon}(T-t)||w(s) - v(s)||. \]
Suppose that (10) holds for \( m = j \). We prove that (10) holds for \( m = j + 1 \). Infact, we have
\[
\left\| F^{j+1}(w)(t) - F^{j+1}(v)(t) \right\| = \left\| F(F^j(w))(t) - F(F^j(v))(t) \right\|
\]
\[
\leq \frac{k_e}{\varepsilon} \int_t^T \left\| F^j(w)(s) - F^j(v)(s) \right\| \, ds
\]
\[
\leq \frac{k_e}{\varepsilon} \int_t^T \left( \frac{k_e}{\varepsilon} \right)^j \frac{(T-s)^j}{j!} \left\| w(s) - v(s) \right\| \, ds
\]
\[
= \left( \frac{k_e}{\varepsilon} \right)^{j+1} \frac{(T-t)^{j+1}}{(j+1)!} \left\| w(s) - v(s) \right\|.
\]
Therefore (11) holds for all \( m = 1, 2, \ldots \) by the induction principle. In particular, one has
\[
\left\| F^m(w)(t) - F^m(v)(t) \right\| \leq \left( \frac{k_e T}{\varepsilon} \right)^m \frac{1}{m!} \left\| w(s) - v(s) \right\|.
\]
Since
\[
\lim_{m \to \infty} \left( \frac{k_e T}{\varepsilon} \right)^m \frac{1}{m!} = 0,
\]
there exists a positive integer number \( m_0 \) such that \( F^{m_0} \) is a contraction mapping. It follows that \( F^{m_0} \) has a unique fixed point \( u^e \) in \( C([0, T]; H) \). Since \( F^{m_0}(F(u^e)) = F(F^{m_0}(u^e)) = F(u^e) \), we obtain \( F(u^e) = u^e \) due to the uniqueness of the fixed point of \( F^{m_0} \). The uniqueness of the fixed point of \( F \) also follows the uniqueness fixed point of \( F^{m_0} \). The unique fixed point \( u^e \) of \( F \) is the solution of (7) corresponding to final value \( \varphi \).

**Step 2: Continuous dependence on the data**

We now let \( u_1^e \) and \( u_2^e \) be two solutions corresponding to final values \( \varphi_1 \) and \( \varphi_2 \), respectively. In the same manner as Step 1, we have for every \( w, v \in C([0, T]; H) \)
\[
\left\| F(w)(t) - F(v)(t) \right\| \leq k_e \int_t^T \varepsilon^{\frac{t-s}{T}} \left\| w(s) - v(s) \right\| \, ds.
\]
Hence
\[
\left\| u_1^e(t) - u_2^e(t) \right\| = \left\| S_e(t-T) \left( \varphi_1 - \varphi_2 \right) + F(u_1^e)(t) - F(u_2^e)(t) \right\|
\]
\[
\leq \left\| S_e(t-T) \right\| \cdot \left\| \varphi_1 - \varphi_2 \right\| + \left\| F(u_1^e)(t) - F(u_2^e)(t) \right\|
\]
\[
\leq \varepsilon^{\frac{t-T}{T}} \left\| \varphi_1 - \varphi_2 \right\| + k_e \int_t^T \varepsilon^{\frac{t-s}{T}} \left\| u_1^e(s) - u_2^e(s) \right\| \, ds.
\]
The latter inequality can be written as
\[
\varepsilon^{\frac{t}{T}} \left\| u_1^e(t) - u_2^e(t) \right\| \leq \varepsilon^{-1} \left\| \varphi_1 - \varphi_2 \right\| + k_e \int_t^T \varepsilon^{-\frac{s}{T}} \left\| u_1^e(s) - u_2^e(s) \right\| \, ds.
\]
It follows from Gronwall’s inequality that
\[
\varepsilon^{\frac{t}{T}} \left\| u_1^e(t) - u_2^e(t) \right\| \leq \varepsilon^{-1} e^{k_e(T-t)} \left\| \varphi_1 - \varphi_2 \right\|, \ t \in [0, T].
\]
This completes the proof of Theorem 1. 

\( \square \)
2.2. Regularization of problem (3)

Our purpose in this section is to construct a regularized solution of the ill-posed problem (3). We mention that the existence of a solution of (3) is not considered here. Instead, we assume that there is an exact solution $u$ corresponding to the exact datum $\varphi$, and our aim is to construct, from the given datum $\varphi_\varepsilon$ approximating $\varphi$, a regularized solution $U_\varepsilon$ which approximates $u$.

Denote by $u^\varepsilon$ the solution of problem (4) corresponding to the final condition $\varphi_\varepsilon$. We shall show that for each fixed time $t > 0$, the function $u^\varepsilon(t)$ gives a good approximation of $u(t)$, where the order of approximation is $\varepsilon^{\frac{2}{T}}$. However, it is difficult to derive an approximation at $t = 0$. We therefore need an adjustment in choosing the regularized solution. The main idea is that we first use the continuity of $u$ to approximate the initial value $u(0)$ by $u(t_\varepsilon)$ for some suitable small time $t_\varepsilon > 0$, and then approximate $u(t_\varepsilon)$ by $u^\varepsilon(t_\varepsilon)$.

**Lemma 3.** Let $T > 0$ and let $\varepsilon > 0$ small enough. There exists a unique $t_\varepsilon > 0$ such that $\varepsilon^{\frac{2}{T}} = t_\varepsilon$. Moreover,

$$t_\varepsilon \leq \frac{2T \ln(\ln(\varepsilon^{-1}))}{\ln(\varepsilon^{-1})}.$$  

**Proof.** Note that each solution $t > 0$ of $\varepsilon^{\frac{2}{T}} = t$ is a zero of the function

$$h(t) = \ln(t) + \frac{\ln(\varepsilon^{-1})}{2T} t, \quad t > 0.$$  

We have $h$ is strictly increasing as $h'(t) > 0$. Moreover, $\lim_{t \to 0^+} h(t) = -\infty$ and

$$h\left(\frac{2T \ln(\ln(\varepsilon^{-1}))}{\ln(\varepsilon^{-1})}\right) = \ln\left[2T \ln(\ln(\varepsilon^{-1}))\right] > 0$$  

for $\varepsilon > 0$ small enough. Thus the equation $h(t) = 0$ has a unique solution $t_\varepsilon > 0$ such that

$$t_\varepsilon \leq \frac{2T \ln\left(\ln\left(\frac{1}{\varepsilon}\right)\right)}{\ln\left(\frac{1}{\varepsilon}\right)}.$$

We have the following regularization result.

**Theorem 2.** Let $u \in C^1([0, T]; H)$ be a solution of problem (3) corresponding to $\varphi \in H$. Assume that

$$\sup_{t \in [0, T]} \left[\sum_{n=1}^{\infty} e^{2Tt_n} |(\phi_n, u(t_n))|^2 + \|u'(t)\|\right] = M < \infty.$$  

Let $\varphi_\varepsilon$ be a measured datum satisfying $\|\varphi_\varepsilon - \varphi\| \leq \varepsilon$ with $\varepsilon > 0$, and let $u^\varepsilon$ be the solution of problem (4) corresponding to $\varphi_\varepsilon$. Choose $t_\varepsilon > 0$ as in Lemma 3. Define the regularized solution $U^\varepsilon : [0, T] \to H$ by

$$U^\varepsilon(t) = u^\varepsilon(\max\{t, t_\varepsilon\}), \quad t \in [0, T].$$  

Then one has the error estimate, for $\varepsilon > 0$ small enough, $t \in [0, T]$,

$$\|U^\varepsilon(t) - u(t)\| \leq (2M + 1) \min\left\{\varepsilon^{\frac{2}{T}}, \frac{2T \ln(\ln(\varepsilon^{-1}))}{\ln(\varepsilon^{-1})}\right\}.$$
Proof. We have in view of (5)
\[ u(t) = S(t - T) \varphi - \int_0^T S(t - s)f(s, u(s)) \, ds. \]

Using \( B(\varepsilon, t) = S_\varepsilon(t - T)S(T - t) \), one has
\[ B(\varepsilon, t)u(t) = S_\varepsilon(t - T)\varphi - \int_0^T S_\varepsilon(t - s)B(\varepsilon, s)f(s, u(s)) \, ds. \]

We have in view of (7)
\[ u_\varepsilon(t) = S_\varepsilon(t - T)\varphi_\varepsilon - \int_0^T S_\varepsilon(t - s)B(\varepsilon, s)f_\varepsilon(s, u_\varepsilon(s)) \, ds. \]

Thus
\[ u_\varepsilon(t) - u(t) = S_\varepsilon(t - T)(\varphi_\varepsilon - \varphi) + [B(\varepsilon, t) - I]u(t) + \]
\[ - \int_0^T S_\varepsilon(t - s)B(\varepsilon, s)[f_\varepsilon(s, u_\varepsilon(s)) - f(s, u(s))] \, ds. \]

Using Lemma 1 and noting that \( f(s, u(s)) = f_\varepsilon(s, u(s)) \) for \( \varepsilon > 0 \) small enough, \( M_\varepsilon \geq \sup_{t \in [0, T]} \|u(t)\| \), we get
\[ \|u_\varepsilon(t) - u(t)\| \leq \|S_\varepsilon(t - T)\| \cdot \|\varphi_\varepsilon - \varphi\| + \|[B(\varepsilon, t) - I]u(t)\| + \]
\[ + \int_0^T \|S_\varepsilon(t - s)\| \cdot \|B(\varepsilon, s)\| \cdot \|f_\varepsilon(s, u_\varepsilon(s)) - f(s, u(s))\| \, ds \]
\[ \leq e^{\|\varepsilon\|_T} \cdot \varepsilon + \varepsilon \sqrt{\sum_{n=1}^\infty e^{2\varepsilon T \lambda_n} |(\phi_n, u)|^2} + k_\varepsilon \int_0^T e^{\|\varepsilon\|_T} ||u_\varepsilon(s) - u(s)|| \, ds \]
\[ \leq (M + 1)e^{\|\varepsilon\|_T} + k_\varepsilon \int_0^T e^{\|\varepsilon\|_T} ||u_\varepsilon(s) - u(s)|| \, ds. \]

The latter inequality can be written as
\[ e^{-\frac{T}{2}} \|u_\varepsilon(t) - u(t)\| \leq (M + 1) + k_\varepsilon \int_0^T e^{-\frac{T}{2}} \|u_\varepsilon(s) - u(s)|| \, ds. \]

It follows from Gronwall’s inequality that
\[ e^{-\frac{T}{2}} \|u_\varepsilon(t) - u(t)\| \leq (M + 1)e^{k_\varepsilon T}, \ \forall t \in (0, T]. \]
In particular, if $t \in [t_\varepsilon, T]$ then
\[
\|U^\varepsilon(t) - u(t)\| = \|u^\varepsilon(t) - u(t)\| \leq (M + 1)e^{k_T t_\varepsilon} \\
\leq (M + 1)e^{\frac{2T(M + 1)\ln(\varepsilon^{-1})}{\ln(\varepsilon^{-1})}},
\]
where we have used
\[
e^{k_T t_\varepsilon} \leq \sqrt{\ln(\varepsilon^{-1})} \leq \frac{\ln(\varepsilon^{-1})}{2T\ln(\varepsilon^{-1})} \leq t_\varepsilon^{-1} = \varepsilon^{-\frac{1}{2}} \leq \varepsilon^{-\frac{3}{4}}.
\] (11)

Let us now consider $t \in [0, t_\varepsilon]$. One has
\[
\|U^\varepsilon(t) - u(t)\| = \|u^\varepsilon(t_\varepsilon) - u(t)\| \leq \|u^\varepsilon(t_\varepsilon) - u(t_\varepsilon)\| + \|u(t_\varepsilon) - u(t)\|.
\]
Due to the continuity of $u_\varepsilon$, we get for $\varepsilon$ small enough
\[
\|u(t_\varepsilon) - u(t)\| = \left\| \int_t^{t_\varepsilon} u_\varepsilon(s)ds \right\| \leq \int_0^{t_\varepsilon} \|u_\varepsilon(s)\| ds \leq Mt_\varepsilon.
\]
Thus, for $t \in [0, t_\varepsilon]$,
\[
\|U^\varepsilon(t) - u(t)\| \leq (M + 1)e^{\frac{2T\ln(\varepsilon^{-1})}{\ln(\varepsilon^{-1})}} + Mt_\varepsilon = (2M + 1)t_\varepsilon
\]
\[
\leq (2M + 1) \min \left\{ \varepsilon^{-\frac{1}{2}}, \frac{2T\ln(\varepsilon^{-1})}{\ln(\varepsilon^{-1})} \right\}.
\]
This completes the proof of Theorem 2.

3. Regularization of backward parabolic problem with time-dependent coefficient

In this section, we consider the following backward nonlinear parabolic problem with time-dependent coefficient
\[
\begin{align*}
u_t + a(t)Au(t) &= f(t, u(t)), \quad 0 < t < T, \\
u(T) &= \varphi, (12)
\end{align*}
\]
where $a \in C([0, T])$ is given. The function $a$ is noised by the perturbed data $a_\varepsilon \in C[0, T]$ such that
\[
\|a_\varepsilon - a\|_{C([0, T])} \leq \varepsilon. (13)
\]
where the norm $\|\cdot\|_{C([0, T])}$ is given by the sup norm, i.e., $\|v\|_{C([0, T])} = \sup_{0 \leq t \leq T} |v(t)|$ for every continuous function $v : [0, T] \to \mathbb{R}$. We would like to emphasize that it is impossible to apply the technique in Section 2 to solve problem (12) when the time-dependent coefficient is perturbed by noise. Therefore, we investigate a new regularized problem as follows
\[
\begin{align*}
\frac{dv_\varepsilon(t)}{dt} + a_\varepsilon(t)\widetilde{A}_\varepsilon v_\varepsilon(t) &= f_\varepsilon(t, v_\varepsilon(t)), \quad 0 < t < 1, \\
v_\varepsilon(T) &= \varphi_\varepsilon,
\end{align*}
\] (14)

where $\widetilde{A}_\varepsilon$ is defined by
\[
\widetilde{A}_\varepsilon(\nu) := -\frac{1}{QT} \sum_{n=1}^{\infty} \ln \left( \varepsilon + e^{QT\lambda_\varepsilon} \right) \langle \nu, \phi_n \rangle \phi_n
\]

and $Q = \|a_\varepsilon\|_{C([0, T])}$.

The regularization result for time-dependent perturbed coefficient is given in the following theorem.
Theorem 3. Let \( u \in C^1([0, T]; H) \) be a solution of problem (12) corresponding to \( \varphi \in H \). Assume that
\[
\sup_{t \in [0, T]} \left( \sum_{n=1}^{\infty} e^{2QT_n} |(\phi_n, u(t))|^2 + \|u'(t)\|^2 \right) = E_Q < \infty.
\]
Let \( \varphi_e \) and \( a_e \) be measured data satisfying \( \|\varphi_e - \varphi\| \leq \varepsilon \) and \( \|a_e - a\|_{C([0, T])} \leq \varepsilon \) for \( \varepsilon > 0 \). We denote by \( v_e \) the solution of problem (14) corresponding to \( \varphi_e \) and \( a_e \). Choose \( t_e > 0 \) as in Lemma 3. Define the regularized solution \( W^e : [0, T] \rightarrow H \) by
\[
W^e(t) = v_e(\max \{t, t_e\}), \quad t \in [0, T].
\]

Then one has the following error estimate for \( \varepsilon > 0 \) small enough and \( t \in [0, T] \),
\[
\|W^e(t) - u(t)\| \leq 2E_Q \sqrt{2 \left( \frac{1}{Q} + 1 \right) e^{2T} \min \left\{ \varepsilon, \frac{2T \ln(\varepsilon^{-1})}{\ln(\varepsilon^{-1})} \right\}}.
\]

Proof. The existence of solutions to problem (12) can be proved in the same manner as Theorem 1. It remains to prove the error estimation between \( W^e \) and \( u \). To this end, we first need the error estimation between \( u_e \) and \( u \). The technique we use here is different from Theorem 2. The problem (12) can be written as
\[
\begin{aligned}
\left\{ \begin{array}{l}
u'(t) + a_e(t) \tilde{A}_e u(t) = a_e(t) \tilde{A}_e u(t) - a(t)Au(t) + f(t, u(t)), \\
u(T) = \varphi.
\end{array} \right.
\end{aligned}
\]
(16)

Recall that \( v_e \) solves the following equation
\[
\begin{aligned}
\left\{ \begin{array}{l}
v'_e(t) + a_e(t) \tilde{A}_e v_e(t) = f_e(t, v_e(t)), \\
v_e(T) = \varphi_e.
\end{array} \right.
\end{aligned}
\]
(17)

Substituting (17) into (16) bothsides, we obtain
\[
\begin{aligned}
\left\{ \begin{array}{l}
v'_e(t) - u'(t) = -a_e(t) \tilde{A}_e v_e(t) - u(t)) - a_e(t) \tilde{A}_e u(t) + a(t)Au(t) \\
v_e(T) - u_e(T) = \varphi_e - \varphi.
\end{array} \right.
\end{aligned}
\]
(18)

For \( b > 0 \), we define by
\[
\tilde{z}_e(t) := e^{b(t-T)}(v_e(t) - u(t)).
\]
By differentiating \( \tilde{z}_e(t) \) with respect \( t \) and combining to (18) gives
\[
\begin{aligned}
\tilde{z}'_e(t) &= \tilde{b}e^{b(t-T)}(v_e(t) - u(t)) + e^{b(t-T)}(v'_e(t) - u'(t)) \\
&= \tilde{b} \tilde{z}_e(t) + e^{b(t-T)} \left[ -a_e(t) \tilde{A}_e v_e(t) - u(t)) + f(t, v_e(t)) - f(t, u(t)) \right] \\
&\quad - e^{b(t-T)} \left[ (a_e(t) - a(t))Au(t) + a_e(t)(\tilde{A}_e - A)u(t) \right] \\
&= \tilde{b} \tilde{z}_e(t) - \tilde{A}_e \tilde{z}_e(t) + e^{b(t-T)} \left[ f(t, v_e(t)) - f(t, u(t)) \right] \\
&\quad - e^{b(t-T)}(a_e(t) - a(t))Au(t) - e^{b(t-T)}a_e(t)(\tilde{A}_e - A)u(t).
\end{aligned}
\]
(19)
By taking the inner product (19) with \( z_e(t) \), we get
\[
\langle z'_e(t) + a_e(t)\overline{A}_e z_e(t) - \overline{b}z_e(t), z_e(t) \rangle = \left\langle e^{\overline{b}(t-T)} \left[ f(t, v_e(t)) - f(t, u(t)) \right], z_e(t) \right\rangle 
- \left\langle e^{\overline{b}(t-T)}((a_e(t) - a(t))Au(t), z_e(t)) \right\rangle 
- \left\langle e^{\overline{b}(t-T)}((\overline{A}_e - A)u(t), z_e(t)) \right\rangle.
\] (20)

A direct computation implies that
\[
\frac{d}{dt}\|z_e(t)\|_H^2 = 2\langle - a_e(t)\overline{A}_e z_e(t), z_e(t) \rangle + 2\overline{b}\langle z_e(t), z_e(t) \rangle 
+ 2\left\langle e^{\overline{b}(t-T)} \left[ f(t, v_e(t)) - f(t, u(t)) \right], z_e(t) \right\rangle 
- 2\overline{b}\langle (a_e(t) - a(t))Au(t), z_e(t) \rangle 
- 2\overline{b}\langle (\overline{A}_e - A)u(t), z_e(t) \rangle.
\] (21)

where
\[
\overline{I}_1 = \langle - a_e(t)\overline{A}_e z_e(t), z_e(t) \rangle + \overline{b}\langle z_e(t), z_e(t) \rangle,
\overline{I}_2 = \left\langle e^{\overline{b}(t-T)} \left[ f_e(t, v_e(t)) - f(t, u(t)) \right], z_e(t) \right\rangle,
\overline{I}_3 = -\overline{b}\langle (a_e(t) - a(t))Au(t), z_e(t) \rangle,
\overline{I}_4 = -\overline{b}\langle (\overline{A}_e - A)u(t), z_e(t) \rangle.
\]

Since \( Q = \sup_{e \in [0, T]} |a_e(t)| \), we have
\[
\left| \langle - a_e(t)\overline{A}_e z_e(t), z_e(t) \rangle \right| \leq \sup_{e \in [0, 1]} |a_e(t)\|\overline{A}_e z_e(t)\|_H\|z_e(t)\|_H 
\leq \frac{Q}{ QT} \ln \left( \frac{1}{\varepsilon} \right) \|z_e(t)\|_H^2 
\leq \frac{1}{ T} \ln \left( \frac{1}{\varepsilon} \right) \|z_e(t)\|_H^2,
\]
which gives
\[
\langle - a_e(t)\overline{A}_e z_e(t), z_e(t) \rangle \geq -\frac{1}{ T} \ln \left( \frac{1}{\varepsilon} \right) \|z_e(t)\|_H^2.
\]

Then the term \( \overline{I}_1 \) is estimated by
\[
\overline{I}_1 = \langle - a_e(t)\overline{A}_e z_e(t), z_e(t) \rangle + \overline{b}\langle z_e(t), z_e(t) \rangle 
\geq -\frac{1}{ T} \ln \left( \frac{1}{\varepsilon} \right) \|z_e(t)\|_H^2 + \overline{b}\|z_e(t)\|_H^2.
\] (22)

Using Lemma 1 and noting that \( f(s, u(s)) = f_e(s, u(s)) \) for \( \varepsilon > 0 \) small enough, \( M_e \geq \sup_{e \in [0, T]} \|u(t)\| \), we have the following estimate
\[
\overline{I}_2 = \left\langle e^{-\overline{b}(t-T)} \left[ f_e(t, v_e(t)) - f(t, u(t)) \right], z_e(t) \right\rangle 
= e^{\overline{b}(t-T)}\left\langle f_e(v_e(t), t) - f_e(t, u(t)), v_e(t) - u(t) \right\rangle 
\geq -k_e e^{-\overline{b}(t-T)} \|v_e(t) - u(t)\|_H^2 
= -k_e\|z_e\|_H^2.
\] (23)
Thus, (21), (22), (23), (24) and (25) yields

\[ \tilde{T}_3 = \langle e^{-\tilde{T}(t)}(a_e(t) - a(t))Au(t), z_e(t) \rangle \]
\[ \leq e^{-2\tilde{T}(t)}|a_e(t) - a(t)|^2 \| Au(t) \|_H^2 + \| z_e(t) \|_H^2 \]
\[ \leq e^{-2\tilde{T}(t)}|a_e(t) - a(t)|^2 \left( \sum_{n=1}^{\infty} |\lambda_n^2| \| u(t), \phi_n \| \right) + \| z_e(t) \|_H^2 \]
\[ \leq e^{-2\tilde{T}(t)}|a_e(t) - a(t)|^2 \left( \sum_{n=1}^{\infty} \frac{1}{Q^2T^2} e^{2QT \lambda_n} |\langle u(t), \phi_n \rangle |^2 \right) + \| z_e(t) \|_H^2 \]
\[ \leq \frac{e^{-2\tilde{T}(t)}e^2E_Q^2}{QT} + \| z_e(t) \|_H^2. \tag{24} \]

Employing Hölder inequality, we can bound \( \tilde{T}_4 \) as follows

\[ \tilde{T}_4 = \langle e^{-\tilde{T}(t)}a_e(t)(\tilde{A}_e(t) - A(t))u(t), z_e(t) \rangle \]
\[ \leq e^{-2\tilde{T}(t)}|a_e(t)|^2 \| (\tilde{A}_e - A)u(t) \|_H^2 + \| z_e(t) \|_H^2 \]
\[ \leq e^{-2\tilde{T}(t)}|a_e(t)|^2 \sum_{n=1}^{\infty} \left( \frac{1}{QT} \ln \left( \frac{1}{\epsilon + e^{-QT \lambda_n}} \right) - \frac{1}{QT} \ln(e^{QT \lambda_n}) \right) |\langle u(t), \phi_n \rangle |^2 + \| z_e(t) \|_H^2 \]
\[ \leq \frac{1}{T^2} e^{-2\tilde{T}(t)} e^2 \sum_{n=1}^{\infty} \ln^2 \left( e^{QT \lambda_n} + 1 \right) |\langle u(t), \phi_n \rangle |^2 + \| z_e(t) \|_H^2 \]
\[ \leq \frac{1}{T^2} e^{-2\tilde{T}(t)} e^2 \sum_{n=1}^{\infty} e^{2QT \lambda_n} |\langle u(t), \phi_n \rangle |^2 + \| z_e(t) \|_H^2 \]
\[ \leq \frac{1}{T^2} e^{-2\tilde{T}(t)} e^2 E_Q^2 + \| z_e(t) \|_H^2. \tag{25} \]

Thus, (21), (22), (23), (24) and (25) yields

\[ \frac{d}{dt} \| z_e(t) \|_H^2 \geq \left( -\frac{2}{T} \ln \left( \frac{1}{\epsilon} \right) + 2b - 2k_e - 4 \right) \| z_e(t) \|_H^2 \]
\[ -2e^{-2\tilde{T}(t)} e^2 E_Q^2 \left( \frac{1}{QT} + \frac{1}{T} \right). \tag{26} \]

Since \( b = \frac{1}{T} \ln \left( \frac{1}{\epsilon} \right) \) we obtain

\[ \frac{d}{dt} \| z_e(t) \|_H^2 \geq \left( -2k_e - 4 \right) \| z_e(t) \|_H^2 - 2e^2 E_Q^2 \left( \frac{1}{QT} + \frac{1}{T} \right). \]

Integrating the above inequality from \( t \) to \( T \), we get

\[ \| z_e(T) \|_H^2 - \| z_e(t) \|_H^2 \geq \left( -2k_e - 4 \right) \int_t^T \| z_e(s) \|_H^2 ds \]
\[ -2e^2 E_Q^2 \left( \frac{1}{QT} + \frac{1}{T} \right) (T - t). \]
Since \( \|z_\varepsilon(T)\|_H^2 = \|\varphi_\varepsilon - \varphi\| \leq \varepsilon \), we have
\[
\|z_\varepsilon(t)\|_H^2 \leq (2k_\varepsilon + 4) \int_t^1 \|z_\varepsilon(s)\|_H^2 ds + 2E_\varepsilon^2 \varepsilon^2 \left( \frac{1}{Q} + 1 \right) + \varepsilon^2.
\]
This implies that
\[
e^{-2\tilde{b}(T-t)} \|v_\varepsilon(t) - u(t)\|_H^2 \leq (2k_\varepsilon + 4) \int_t^T e^{-2\tilde{b}(T-s)} \|v_\varepsilon(s) - u(s)\|_H^2 ds + 2E_\varepsilon^2 \varepsilon^2 \left( \frac{1}{Q} + 1 \right) + \varepsilon^2.
\]
Multiplying bothside to \( e^{2\tilde{b}T} \), we obtain
\[
e^{2\tilde{b}t} \|v_\varepsilon(t) - u(t)\|_H^2 \leq (2k_\varepsilon + 4) \int_t^T e^{2\tilde{b}s} \|v_\varepsilon(s) - u(s)\|_H^2 ds + 2E_\varepsilon^2 \left( \frac{1}{Q} + 1 \right).
\]
Applying Grönwall’s inequality, we get
\[
e^{2\tilde{b}t} \|v_\varepsilon(t) - u(t)\|_H^2 \leq 2E_\varepsilon^2 \left( \frac{1}{Q} + 1 \right) e^{(2k_\varepsilon + 4)(T-t)}.
\]
Hence
\[
\|v_\varepsilon(t) - u(t)\|_H^2 \leq 2E_\varepsilon^2 \left( \frac{1}{Q} + 1 \right) e^{(2k_\varepsilon + 4)(T-t) - \frac{T}{2} \ln\left( \frac{1}{\varepsilon} \right)}.
\]
In particular, if \( t \in [t_\varepsilon, T] \) then
\[
\|W_\varepsilon(t) - u(t)\| = \|v_\varepsilon(t) - u(t)\| \leq E_\varepsilon \sqrt{2 \left( \frac{1}{Q} + 1 \right) e^{2T} e^{k_\varepsilon T} e^{\frac{T}{2} \ln\left( \frac{1}{\varepsilon} \right) - \frac{T}{2} \ln\left( \frac{1}{\varepsilon} \right)}}
\]
where we have used (11).
Let us now consider \( t \in [0, t_\varepsilon] \). One has
\[
\|W_\varepsilon(t) - u(t)\| = \|v_\varepsilon(t_\varepsilon) - u(t)\| \leq \|v_\varepsilon(t_\varepsilon) - u(t_\varepsilon)\| + \|u(t_\varepsilon) - u(t)\|.
\]
Due to the continuity, we get for $\varepsilon$ small enough

$$\|u(t_\varepsilon) - u(t)\| = \left\| \int_t^{t_\varepsilon} u_t(s)ds \right\| \leq \int_0^{t_\varepsilon} \|u_t(s)\| ds \leq E_Q t_\varepsilon.$$ 

Thus, for $t \in [0, t_\varepsilon]$,

$$\|W_\varepsilon(t) - u(t)\| \leq E_Q \sqrt{\frac{1}{Q + 1}} e^{2T \varepsilon^{2\varepsilon}} + E_Q t_\varepsilon$$

$$\leq 2E_Q \sqrt{\frac{1}{Q + 1}} e^{2T} \min\left\{ \varepsilon^{2\varepsilon}, \frac{2T \ln(\ln(\varepsilon^{-1}))}{\ln(\varepsilon^{-1})} \right\}.$$ 

This completes the proof of Theorem 3. 

References


