

Orbifold Points on Prym-Teichmüller Curves

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Flat Surfaces & Teichmüller Curves

A **flat surface** is a pair (X, ω) where X is a compact Riemann surface or equivalently a smooth projective irreducible complex curve of genus g and ω is a holomorphic differential, i.e. a section of the canonical bundle, on X . Integrating ω endows X (outside of the zeros of ω) with an atlas where all chart changes are (locally) translations. We may therefore picture (X, ω) as a polygon in the plane, whose sides are identified by translations.

A flat surface admits a natural $\mathrm{SL}_2(\mathbb{R})$ action by affine shearing of the flat structure. Consider now $\Omega\mathcal{M}_g$, the **moduli space of flat surfaces**, which admits a natural projection $\pi: \Omega\mathcal{M}_g \rightarrow \mathcal{M}_g$ to the moduli space of genus g curves. In the rare case that the projection $\pi(\mathrm{SL}_2(\mathbb{R})(X, \omega))$ is a curve in \mathcal{M}_g , we call this image a **Teichmüller curve (generated by (X, ω))**.

The situation can be summarised by the following commutative diagram (note that $\mathrm{SO}(2)$ acts holomorphically):

$$\begin{array}{ccc} \mathrm{SL}_2(\mathbb{R}) & \xrightarrow{F} & \Omega\mathcal{M}_g \\ \downarrow & & \downarrow \\ \mathbb{H} \cong \mathrm{SO}(2) \backslash \mathrm{SL}_2(\mathbb{R}) & \xrightarrow{f} & \mathbb{P}\Omega\mathcal{M}_g \\ \downarrow & & \downarrow \pi \\ \mathcal{C} = \mathbb{H}/\Gamma & \longrightarrow & \mathcal{M}_g \end{array}$$

where the map F is given by the action $A \mapsto A \cdot (X, \omega)$. Note that a Teichmüller curve is never compact, but always admits a finite number of **cusps**.

Orbifold Points

An **orbifold point** of an orbifold \mathbb{H}/Γ is the projection of a fixed point of the action of Γ , i.e. a point $s \in \mathbb{H}$ such that $\mathrm{PStab}_\Gamma(s) \leq \mathrm{PSL}_2(\mathbb{R})$ is non-trivial. We call the cardinality of $\mathrm{PStab}_\Gamma(s)$ the **(orbifold) order** of s . For a Teichmüller curve, this can be expressed in terms of the flat structure:

Lemma. Let $\mathcal{C} = \mathbb{H}/\Gamma$ be a Teichmüller curve. Then (X, ω) corresponds to an orbifold point on \mathcal{C} if and only if X admits a holomorphic automorphism σ such that

$$\sigma^*\omega = \lambda\omega \text{ with } \lambda \in \mathbb{C}^* \setminus \{\pm 1\}.$$

For a curve \mathcal{C} , denote by χ the **orbifold Euler Characteristic**, by h_0 the number of connected components, by C the number of cusps and by e_d the number of points of order d . Then this determines the **genus** g :

$$2h_0 - 2g = \chi + C + \sum_d e_d \left(1 - \frac{1}{d}\right).$$

McMullen's Prym Construction

Not many infinite families of (primitive) Teichmüller curves are known. For low genus, the following construction by McMullen gives a rich set of examples.

Let D be a **(real) discriminant**, i.e. $D > 0$ is not a square and $D \equiv 0$ or $1 \pmod{4}$ and denote by \mathcal{O}_D the unique **quadratic order** of discriminant D in $\mathbb{Q}(\sqrt{D})$.

Let $A = \mathbb{C}^2/\Lambda$ be a (polarised) **abelian surface**. Then we say that A admits **real multiplication** by \mathcal{O}_D if there is an embedding $\iota: \mathcal{O}_D \hookrightarrow \mathrm{End}(A)$ that is self-adjoint with respect to the polarisation of A . Moreover, we say that real multiplication is **proper** if it cannot be extended to any larger order in $\mathbb{Q}(\sqrt{D})$.

Now, let (X, ω) be a flat surface admitting a holomorphic **prym involution** $\rho: X \rightarrow X$, such that the quotient X/ρ has genus 2. Then $\Omega(X)$, the space of differentials on X , splits into ρ -eigen spaces, $\Omega(X)^\pm$. We call

$$\mathcal{P}(X, \rho) = \ker(\mathrm{Jac}(X) \rightarrow \mathrm{Jac}(X/\rho)) = \frac{(\Omega(X)^*)^-}{\mathrm{H}_1(X, \mathbb{Z})^-}$$

the associated **Prym variety**.

Theorem ([McM03; McM06]). Let X be of genus 2, 3 or 4 admitting a Prym involution ρ and differential ω such that

- ω has only one zero,
- $\rho^*\omega = -\omega$, and
- $\mathcal{P}(X, \rho)$ admits real multiplication by some \mathcal{O}_D with ω as an eigenform.

Then (X, ω) generates a Teichmüller curve $W_D(2g-2)$.

The curves $W_D(2g-2)$ are known as **Prym-Teichmüller** or **Prym-Weierstraß** curves and are non-empty for every discriminant D unless $g=3$ and $D \equiv 5 \pmod{8}$.

Main Result

Theorem ([TTZ15]). For non-square discriminant $D > 12$, the Prym-Teichmüller curves $W_D(4)$ for genus 3 have orbifold points of order 2 and 3. More precisely:

- for odd D , there are no points of order 2, otherwise

$$e_2(D) = \#\{a, b, c \in \mathbb{Z} : a^2 + b^2 + c^2 = D\}/24;$$

- the number of orbifold points of order 3 is

$$e_3(D) = \#\{a, b, c \in \mathbb{Z} : 2a^2 - 3b^2 - c^2 = 2D, (*)\},$$

where condition $(*)$ restricts the set to those $a, b, c \in \mathbb{Z}$

that satisfy $f = \frac{c + b\sqrt{-3}}{2(a - \sqrt{D})} \in \triangle \subseteq \mathbb{C};$

- $W_8(4)$ has one point of order 3 and one point of order 4;
- $W_{12}(4)$ has a single orbifold point of order 12.

Moreover, let $D = f_0^2 D_0$ where D_0 is a fundamental discriminant. Then the above sets are subject to the condition $\mathrm{gcd}(a, b, c, f_0) = 1$.

Remark. By [LN14] and [Möll14], $W_D(4)$ is empty for $D \equiv 5 \pmod{8}$. Moreover, by [LN14], $W_D(4)$ has two components iff $D \equiv 1 \pmod{8}$.

Theorem ([Zac15]). If $D \equiv 1 \pmod{8}$ and D is not a square, the two components of $W_D(4)$ are homeomorphic.

Theorem ([TTZ16]). For non-square discriminant $D > 12$, the Prym-Teichmüller curves $W_D(6)$ for genus 4 have orbifold points of order 2 and 3. More precisely:

- for odd D , there are no points of order 2, otherwise

$$e_2(D) = \begin{cases} h(-D) + h(-\frac{D}{4}), & \text{if } D \equiv 12 \pmod{16}, \\ h(-D), & \text{if } D \equiv 0, 4, 8 \pmod{16}, \end{cases}$$

where $h(-D)$ is the class number of \mathcal{O}_{-D} ;

- the number of orbifold points of order 3 is

$$e_3(D) = \#\{a, i, j \in \mathbb{Z} : a^2 + 3j^2 + (2i - j)^2 = D\}/12,$$

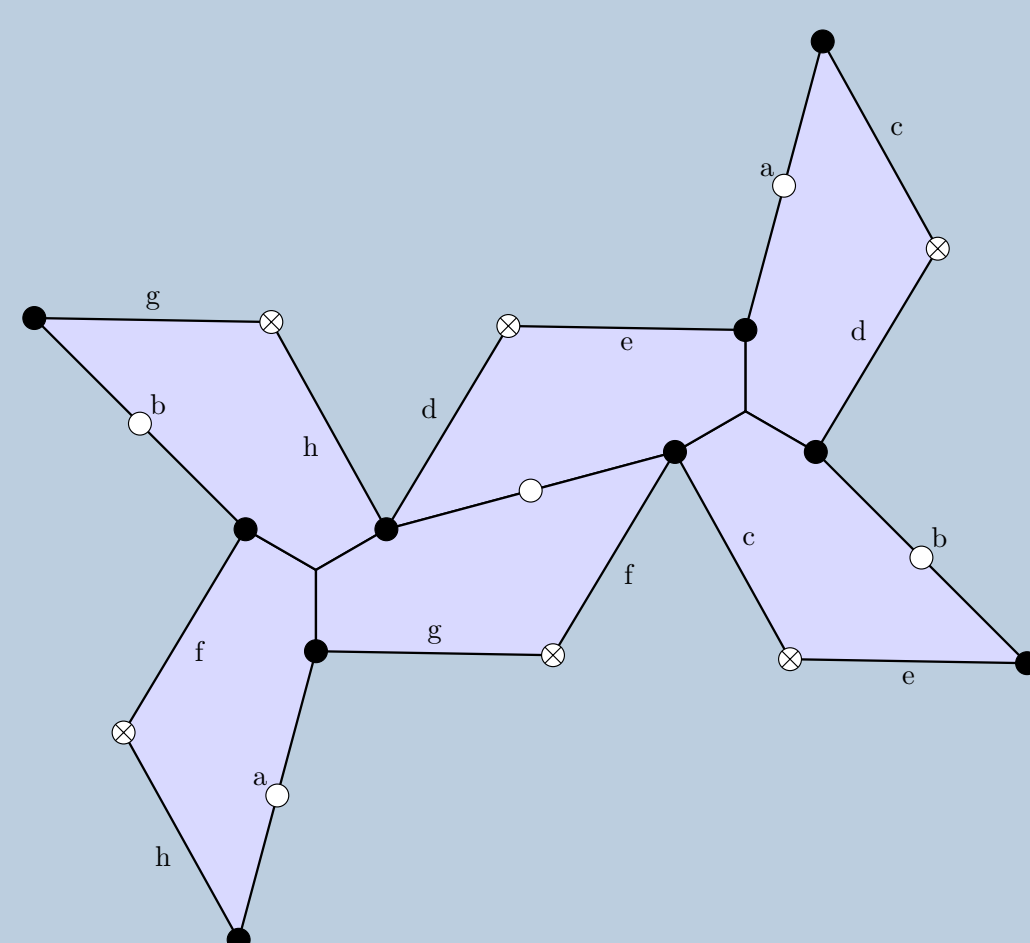
again subject to the condition that $\mathrm{gcd}(a, i, j) = 1$;

- $W_5(6)$ has one point of order 3 and one point of order 5;
- $W_8(6)$ has one point of order 2 and one point of order 3;
- $W_{12}(6)$ has one point of order 2 and one point of order 6.

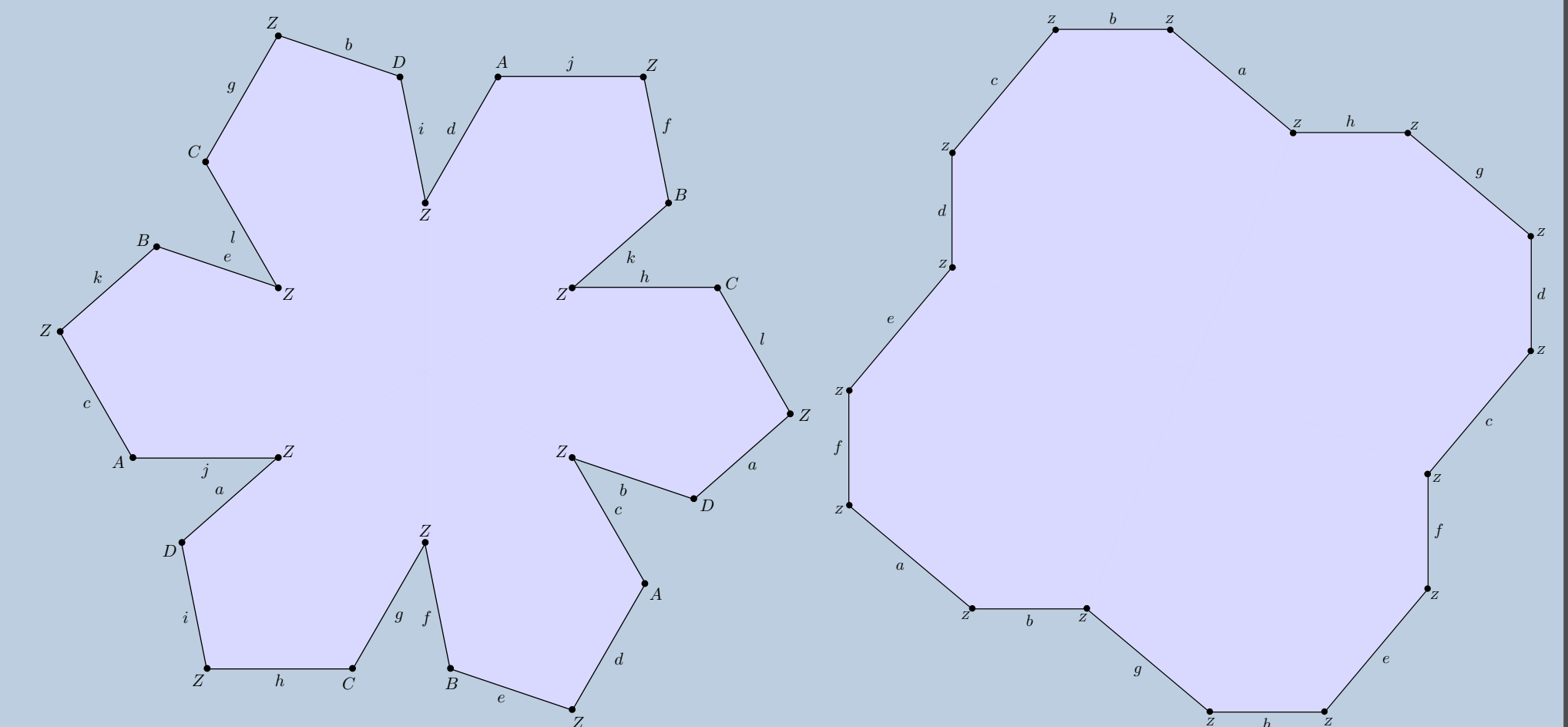
Remark. The cusps and connected components of $W_D(4)$ and $W_D(6)$ are described in [LN14], while the Euler characteristics are computed in [Möll14]. Thus, this completes the topological classification of the Prym-Teichmüller curves.

Remark. For the Weierstraß curves in genus 2, the cusps were described by McMullen [McM05], the orbifold points by Mukamel [Muk14], and the Euler characteristic was computed by Bainbridge [Bai07].

Flat Pictures



$g=3$: Choosing the side a as a complex parameter yields differentials with a $\mathbb{Z}/6$ and $\mathbb{Z}/4$ action, respectively, and a single 4-fold zero.



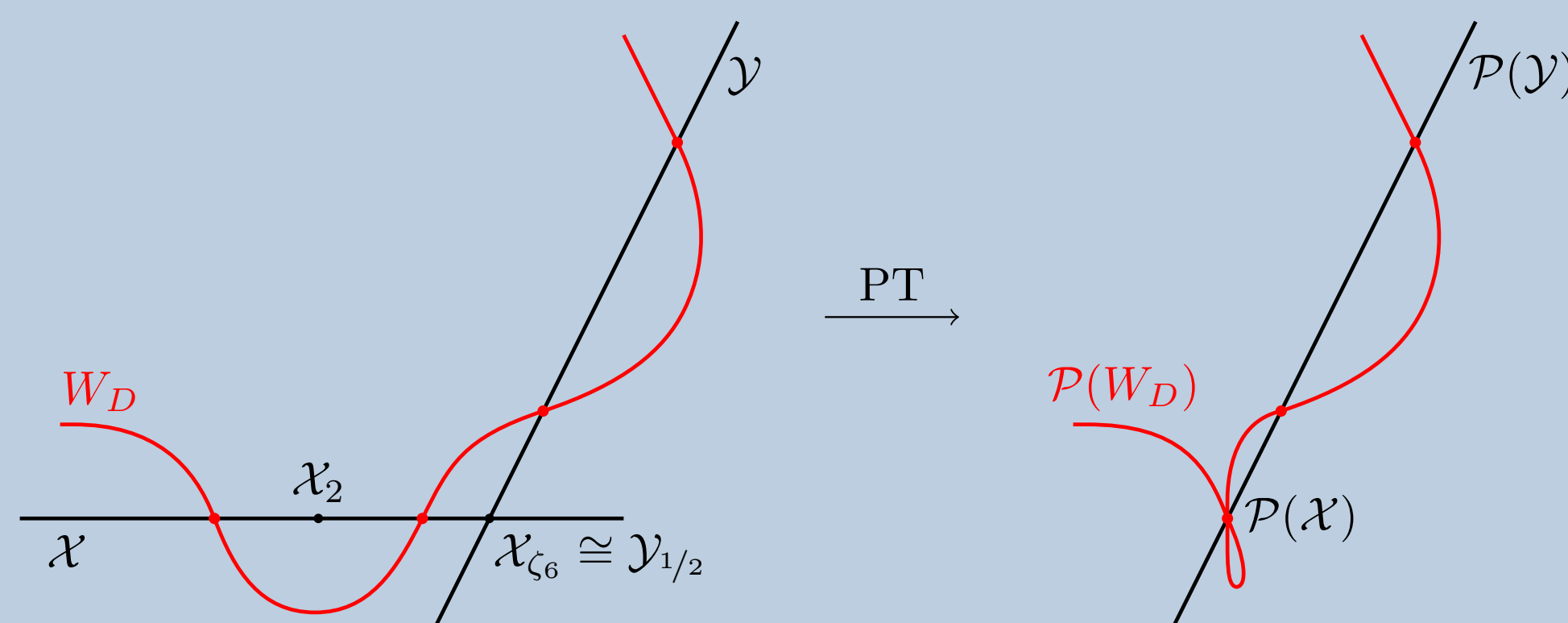
$g=4$: Choosing the side d as a complex parameter yields differentials with a $\mathbb{Z}/6$ and $\mathbb{Z}/4$ action, respectively, and a single 6-fold zero.

Idea of Proof

Let (X, ω) be an orbifold point of order $2d$. Then X admits a holomorphic automorphism σ of order $2d$ that fixes the (single) zero of ω and descends to X/ρ .

Idea: study families of curves with such an automorphism admitting an eigenform with a single zero and check when the Prym part of such a curve admits real multiplication, i.e. **count intersections with the Teichmüller curve**.

For any such family \mathcal{X} , consider therefore the **Prym-Torelli image** $\mathrm{PT}(\mathcal{X})$, i.e. the family of abelian surfaces with fibres $\mathcal{P}(\mathcal{X}_t, \sigma^d)$. We must check these fibres for **real multiplication**. A sketch of the situation in \mathcal{M}_3 :



More precisely, we obtain:

- For $g(X) = 3$, we have $g(X/\rho) = 1$, hence $\bar{\sigma}$ is of order $d = 2, 3, 4$ or 6 and $g(X/\sigma) = 0$. We thus obtain families of **cyclic covers of \mathbb{P}^1** and in these cases the σ -eigenspace decomposition of $\Omega(X)$ is understood. In fact, all orders occur ($d = 4$ and 6 give 0-dimensional families).

- For $g(X) = 4$, we have $g(X/\rho) = 1$, hence $\bar{\sigma}$ is of order $d = 2, 3, 4, 5, 6, 8$ or 10 . Using Riemann-Hurwitz, one can show that only $d = 2, 3, 5$ and 6 occur. The case $d = 2$ is special, because in this case the quotient is an elliptic curve. But this family can be constructed as a **fibre-product of the quotient elliptic curves**. Again, $d = 5$ and 6 give 0-dimensional families.

As the curves have many automorphisms, one can use **Bolza's method** and other tricks to calculate the endomorphism rings of the Prym part explicitly.

The following positive-dimensional families occur:

$g(\mathcal{X}_t)$	d	$\dim \mathcal{X}$	$\dim \mathrm{PT}(\mathcal{X})$	$\mathrm{Aut}(\mathcal{X}_t)$	$\mathrm{End}(\mathcal{P}(\mathcal{X}_t, \sigma^d))$
3	2	1	0	G	order in $M_2(\mathbb{Q}[i])$
3	3	1	1	$\mathbb{Z}/6$	order in $(\frac{2, -3}{\mathbb{Q}})$
4	2	2 (1)	1	D_8	$M_2(\mathrm{End}(E_t))$
4	3	1	0	$\mathbb{Z}/6 \times \mathbb{Z}/2$	$M_2(\mathbb{Z}[\zeta_6])$

Here, $G = \mathbb{Z}/2 \times (\mathbb{Z}/2 \times \mathbb{Z}/4)$, $E_t: y^2 = x(x-1)(x-t)$, and $(\frac{2, -3}{\mathbb{Q}})$ denotes the Quaternion Algebra over \mathbb{Q} .

Remark. Note that $g = 3 = d$ gives the Shimura curve uniformised by $\Delta(2, 6, 6)$, explaining the hyperbolic triangle in the theorem.

Remark. Observe that for $g = 4$, $d = 2$, the D_8 -family is 2-dimensional. However, restricting to curves that admit an eigendifferential with a sixfold zero reduces the dimension.

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