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Godeaux-Serre Varieties with Prescribed Arithmetic Fundamental Group

Diplomarbeit

vorgelegt von

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Introduction

In order to explain what "Godeaux-Serre varieties" are, we need the notion of étale fundamental groups. An étale fundamental group in Algebraic Geometry is an analogue to a fundamental group in Algebraic Topology. One of the questions in the study of étale fundamental groups is the following:

Which groups can be fundamental groups of smooth projective varieties?

As listed in [Ara95], there are several classes of groups for which the above question is answered positively, but also many that yield negative results. One of positive results is that every finite group is a fundamental group of a smooth projective variety. In fact, Serre proved the following statement in [Ser58, Prop.15]:

Proposition. *Let G be a finite group, $r \geq 1$ an integer and k be an algebraically closed field. Then there exists a smooth projective variety Y of dimension r which is a complete intersection in $\mathbb{P}^n_k$ for some $n \in \mathbb{N}$, on which G acts without fixed points.*

From this proposition, we obtain by Proposition 2.11 the following (non-canonical) exact sequence:

$$1 \longrightarrow \pi_1(Y) \longrightarrow \pi_1(Y/G) \longrightarrow G \longrightarrow 1. \tag{1}$$

Since Y is a complete intersection in $\mathbb{P}^n_k$, we also have $\pi_1(Y) = 1$, provided $r = \dim Y \geq 2$, by the Lefschetz Hyperplane Theorem. This implies that G is isomorphic to the fundamental group of the quotient variety Y/G, the so-called **Godeaux-Serre variety**.

Now to give an arithmetic viewpoint on Godeaux-Serre varieties, let us consider the following more general problem: Suppose that X is a geometrically connected variety (or more generally, a quasi-compact and geometrically connected scheme) defined over a field k that is not necessarily algebraically closed, and let $\bar{k}$ denote the algebraic closure of k. Then the fundamental group $\pi_1(X)$ is a profinite group that is an extension of the absolute Galois group Gal_k by the fundamental group $\pi_1(X \otimes_k \bar{k})$, i.e. we have the following exact sequence

$$1 \longrightarrow \pi_1(X \otimes_k \bar{k}) \longrightarrow \pi_1(X) \longrightarrow \mathrm{Gal}_k \longrightarrow 1, \tag{2}$$

compare Theorem 2.18. Consequently, the question stated at the beginning may be extended to the question whether there exists a smooth and geometrically connected projective variety X over k such that the fundamental group $\pi_1(X)$ is isomorphic to a given profinite group as an extension of Gal_k by a group which occurs as the fundamental group of a smooth projective variety over $\bar{k}$, and the exact sequence (2) is compatible with this group extension.

The aim of this diploma thesis is to show that the above question is answered positively for every group extension of Gal_k by a finite group, which will be demonstrated in Theorem 4.6.

Since its base extension from k to its algebraic closure $\bar{k}$ is a Godeaux-Serre variety whose fundamental group is the given finite group, it may be considered a **k-form of a Godeaux-Serre variety**.

The idea of this construction is somewhat similar to that of Godeaux-Serre varieties, i.e. one has to find a complete intersection in a projective space on which a finite group acts admissibly without fixed points. So we will give a review of group actions on schemes in Chapter 1. An important class of such group actions is the one of admissible actions[1], for which the quotient schemes always exist. A prominent example for this is an action on a projective spectrum induced by automorphisms of graded rings, which will be discussed in Section 1.2. In Section 1.3 we will deal with group actions on the geometric points of a scheme and introduce the notion of group actions avoiding fixed points.

As motivation for our next step, let us discuss the case $k = \mathbb{R}$ with $\mathrm{Gal}_{\mathbb{R}} = \mathrm{Gal}(\mathbb{C}|\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$, in which we have an extension of finite groups

$$1 \longrightarrow G \longrightarrow E \longrightarrow \mathrm{Gal}_{\mathbb{R}} \longrightarrow 1.$$

Two remarks should be given here. Firstly, the variety to be constructed shouldn't be a quotient variety of a complete intersection in $\mathbb{P}^n_{\mathbb{R}}$, since the fundamental group of a complete intersection in $\mathbb{P}^n_{\mathbb{R}}$ is $\mathrm{Gal}_{\mathbb{R}}$, and the sequence (1) would imply that $\mathrm{Gal}_{\mathbb{R}}$ is a normal subgroup of the fundamental group of the quotient variety, which is not desired. One possible approach is to consider a complete intersection in $\mathbb{P}^n_{\mathbb{C}}$ on which a group E acts without fixed points, so that the fundamental group of the quotient by E is actually E by (1).

As a second remark, the group E should not act on this complete intersection as a scheme over $\mathbb{C}$, since otherwise the quotient scheme would be a scheme over $\mathbb{C}$, whence its fibre product over $\mathbb{R}$ with $\mathrm{Spec}\,\mathbb{C}$ consists of two copies of this scheme, i.e. the quotient scheme as a scheme over $\mathbb{R}$ is not geometrically connected. To avoid this problem, one may consider a group action of E such that the automorphism induced by an element of E with the non-trivial image in $\mathrm{Gal}_{\mathbb{R}}$ is compatible with the complex conjugation. This leads to the notion of semilinear actions on schemes, which will be introduced in Section 1.4.

This thesis could not be completed without the notion of étale fundamental groups, which will be discussed in Chapter 2. We will review the definition and some basic properties of fundamental groups in the first section. In Section 2.2 we will see that if G is a finite group acting on a scheme Y from the right without fixed points, then the quotient morphism is étale and we obtain the following (non-canonical) group homomorphism:

$$\pi_1(Y/G) \longrightarrow G,\ \alpha \mapsto g \quad \text{if} \quad \alpha_Y(\overline{y}) = \overline{y}g,$$

where $\overline{y} \in Y(\Omega)$ is a geometric point of Y. Afterwards we will discuss the compatibility of group homomorphisms obtained in this way with quotient schemes by subgroups and flat base extensions.

Now let $k'|k$ be a finite Galois field extension and $\pi : E \to \mathrm{Gal}(k'|k)$ be a surjective group homomorphism. What happens if we consider an admissible π-semilinear action of E on a scheme Y over k' avoiding fixed points? The scheme Y/E will be a scheme over k and we will get both homomorphisms $\pi_1(Y/E) \to \pi_1(\mathrm{Spec}\,k) = \mathrm{Gal}_k$ induced by the structure morphism

[1] i.e. a group G acts admissibly on a scheme X, or according to [SGA1, V §1]: G opère de façon admissible.

$Y/E \to \operatorname{Spec} k$ and $\pi_1(Y/E) \to E$ as mentioned above. As $\mathrm{Gal}(k'|k)$ can be considered a factor group of Gal_k as well as E, provided that π is surjective, one may ask whether both homomorphisms are compatible with the projections onto $\mathrm{Gal}(k'|k)$. The answer is yes and the proof will be given in Section 2.3.

One important step in the construction of Godeaux-Serre varieties is to find a linear subspace of the projective space whose intersection with a given smooth subscheme is again smooth. Its existence is assured by Bertini's Theorem if we work over an infinite field. In the case of a finite field we will have to allow hypersurfaces of arbitrary degree. This will be discussed in Chapter 3 after we have reviewed some basic facts about local rings in the first section.

Having acquired the tools to be used in the construction of (k-forms of) Godeaux-Serre varieties, we come to our main result in Chapter 4. In the first section we will construct for a given finite Galois extension $k'|k$ and an extension of finite groups

$$1 \longrightarrow G \longrightarrow \widetilde{E} \longrightarrow \mathrm{Gal}(k'|k) \longrightarrow 1 \tag{3}$$

a smooth and geometrically integral complete intersection in $\mathbb{P}^n_{k'}$ on which the group $\widetilde{E}$ acts admissibly and semilinearly without fixed points. Its consequence is that we get the main result for the case $k = \mathbb{R}$, where the absolute Galois group is finite. In order to deal with the general case, we need a tool from profinite group theory which will be treated in Section 4.3. Indeed, we will show that for a given profinite group extension

$$1 \longrightarrow G \longrightarrow E \longrightarrow \mathrm{Gal}_k \longrightarrow 1, \tag{4}$$

where G is a finite group, there exists a finite Galois extension $k'|k$ and a finite factor group $\widetilde{E}$ of E such that one has the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G & \longrightarrow & E & \longrightarrow & \mathrm{Gal}_k & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & G & \longrightarrow & \widetilde{E} & \longrightarrow & \mathrm{Gal}(k'|k) & \longrightarrow & 1 \end{array}$$

In other words, we can reduce the profinite group extension (4) to a finite group extension (3), so that we can use the construction given in Section 4.1 and show that the quotient variety obtained in this way has the fundamental group isomorphic to E and the exact sequences (2) and (4) are compatible, which is the main result of this thesis.

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Notation

In this thesis, k will always denote a field. Its algebraic closure will be denoted by $\bar{k}$. If $k'|k$ is a Galois field extension in $\bar{k}$, then its Galois group is denoted by $\mathrm{Gal}(k'|k)$. In particular, $\mathrm{Gal}_k := \mathrm{Gal}(k^{\mathrm{sep}}|k)$, where $k^{\mathrm{sep}} \subseteq \bar{k}$ denotes the separable closure of k in $\bar{k}$, will denote the absolute Galois group of k. Furthermore, Ω will always be an algebraically closed field.

All rings and algebras over a field will be commutative with unit. If A is a ring and $\mathfrak{p} \trianglelefteq A$ is a prime ideal, then $A_\mathfrak{p}$ denotes the localisation of A by $A \setminus \mathfrak{p}$. For $f \in A$, A_f will denote the one of A by $\{f^n \mid n \in \mathbb{N}_0\}$. If G is a group acting on a ring A by ring automorphisms, then $A^G := \{a \in A \mid g(a) = a \text{ for all } g \in G\}$ denotes the subring of invariants under G.

If G is a group, then the opposite group G^{op} is defined as a group whose underlying set is G and whose operation $* : G \times G \to G$ is given by

$$g * h := hg \quad \text{for all } g, h \in G.$$

By this notation, a right group action of G on a scheme X is induced by a group homomorphism $G^{\mathrm{op}} \to \mathrm{Aut}(X)$. The automorphism on X induced by an element $g \in G$ in this way will be denoted by ρ_g. Note that if $g, h \in G$, then we have $\rho_{gh} = \rho_h \circ \rho_g$.

A ring homomorphism will always send the unit to the unit. If $\varphi : A \to B$ is a ring homomorphism, then $\varphi^* : \operatorname{Spec} B \to \operatorname{Spec} A$ denotes the corresponding morphism of schemes.

If $(X, \mathcal{O}_X)$ is a scheme and $x \in X$, we will denote by $\mathcal{O}_{X,x}$ the local ring of X in x, $\mathfrak{m}_{X,x}$ its maximal ideal and $\kappa_X(x) = \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$ its residue field. For each field k, $X(k)$ will denote the set of k-valued points of X, i.e. the set $\mathrm{Hom}(\operatorname{Spec} k, X)$. In particular, $X(\Omega)$ is the set of geometric points of X with an embedding $\kappa(x) \hookrightarrow \Omega$, where $x \in X$ denotes the underlying point. If X is a scheme over k and $k \to A$ is a k-algebra, then the fibre product $X \times_{\operatorname{Spec} k} \operatorname{Spec} A$ will be denoted by $X \otimes_k A$.

By **sets**, we mean the category of finite sets, and if X is a connected scheme, and by $\mathbf{Fet}_X$ the category of finite étale coverings of X, see Definition 2.1. If $k'|k$ is a field extension, we will denote by $\mathrm{Aut}(k'|k)$ the group of field automorphisms $\sigma : k' \to k'$ such that $\sigma|_k = \mathrm{id}_k$. If $X \xrightarrow{\varphi} S$ is a morphism of schemes, then $\mathrm{Aut}(X|S)$ denotes the group of automorphisms of X over S, i.e. those automorphisms $\psi : X \to X$ such that $\varphi \circ \psi = \varphi$.

Finally, every algebraic variety will always be integral, i.e. reduced and irreducible.

Chapter 1

Group actions on schemes

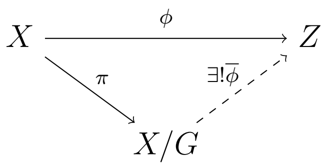
In this chapter we will mainly consider certain group actions on schemes which also define a quotient scheme. We will begin with the class of admissible group actions and study some of their properties. In the last section, we will investigate the class of group actions on schemes over a field which are in some sense compatible with the field automorphisms.

1.1 Admissible group actions

The main reference in this section will be [SGA1, V §1]. We begin by recalling the definition of a quotient object in an arbitrary category as follows:

Definition 1.1. *Let $\mathcal{C}$ be a category, X be an object of $\mathcal{C}$ and $G \leq \mathrm{Aut}(X)$ be a subgroup of the automorphisms of X. A* quotient object of X by G *is an object X/G in $\mathcal{C}$, together with a morphism $\pi : X \to X/G$, such that $\pi \circ \sigma = \pi$ for all $\sigma \in G$, which satisfies the following universal mapping property:*

Given any morphism $\phi : X \to Z$ in $\mathcal{C}$ such that $\phi \circ \sigma = \phi$ for all $\sigma \in G$, there exists a unique morphism $\overline{\phi} : X/G \to Z$ such that $\phi = \overline{\phi} \circ \pi$.



Note that by the universal mapping property, the morphism $\pi : X \to X/G$ of the quotient object by a finite group is always an epimorphism, i.e. if $\varphi, \psi : X/G \to Z$ are morphisms in $\mathcal{C}$ such that $\varphi \circ \pi = \psi \circ \pi : X \to Z$, then $\varphi = \psi$, and if a quotient object exists, it is unique up to unique isomorphism.

Remark 1.2. If $\rho : H^{\mathrm{op}} \to \mathrm{Aut}(X), g \mapsto \rho_g$ defines a group action on X from the right, we may define the quotient object of X by H by the above universal mapping property. Also note that in this case, the quotient object X/H is the same as X/G, where $G := \mathrm{im}\,\rho \leq \mathrm{Aut}(X)$.

In the following, we will deal with the group actions on schemes which are "admissible", for which the quotient schemes will always exist.

Definition 1.3. *A right group action of a finite group G on a scheme X is said to be* admissible *if there exists a scheme Z and an affine morphism $p : X \to Z$ invariant under G such that the homomorphism $p^\flat : \mathcal{O}_Z \to p_*\mathcal{O}_X$ induces an isomorphism*

$$\mathcal{O}_Z \xrightarrow{\sim} (p_*\mathcal{O}_X)^G.$$

Proposition 1.4. *Assume that a finite group G acts admissibly on a scheme X and let $p : X \to Z$ be the morphism as in the definition above, then the following holds:*

(i) *The morphism p is integral.*

(ii) *The morphism p is surjective and induces the quotient map of the underlying topological spaces. Its fibres are the G-orbits of X.*

(iii) *If $x \in X$ and $z := p(x) \in Z$, then $\kappa(x)|\kappa(z)$ is an algebraic normal extension and the canonical group homomorphism*

$$G_x := \{g \in G \mid \rho_g(x) = x\} \longrightarrow \mathrm{Aut}(\kappa(x)|\kappa(z))$$

is surjective.

(iv) *(Z, p) is the quotient scheme of X by G, i.e. $Z \cong X/G$. In particular, the quotient scheme always exists if the group G acts admissibly on the scheme X.*

In the special case $X = \operatorname{Spec} A$, where A is a ring, we have $X/G \cong \operatorname{Spec} A^G$, where G also acts on A as a ring automorphism corresponding to $\operatorname{Spec} A \to \operatorname{Spec} A$ and

$$A^G := \{a \in A \mid \forall g \in G : g(a) = a\}.$$

Furthermore, if X is a scheme of finite type over a scheme S and $G \leq \mathrm{Aut}(X|S)$, then so is X/G as a scheme over S and the morphism $p : X \to X/G$ is finite.

Proof. [SGA1, V Prop.1.1, 1.3 and Cor.1.5] □

Corollary 1.5. *If a finite group G acts admissibly on a scheme X with the quotient morphism $p : X \to X/G$, then for each open subscheme $U \subseteq X/G$, G acts admissibly on $p^{-1}(U)$ and the morphism $p^{-1}(U) \to U$ induced by p is also the quotient morphism of the subscheme $p^{-1}(U) \subseteq X$ by G.*

Proof. [SGA1, V Cor.1.4] □

Proposition 1.6. *A right group action of a finite group G on a scheme X is admissible if and only if X can be covered by open affine subschemes invariant under G.*

Proof. [SGA1, V Prop.1.8] □

Corollary 1.7. *If a finite group G acts admissibly on a scheme X, then so does every subgroup $H \leq G$. In particular, the quotient scheme X/H exists for every subgroup $H \leq G$.*

Proof. [SGA1, V Cor.1.7] □

Under the condition as in the corollary above, we may consider the quotient scheme X/H for every subgroup $H \leq G$. The following proposition states that we can let the group G act again on the quotient scheme X/H if $H \trianglelefteq G$ is a normal subgroup.

Proposition 1.8. *Let G be a finite group acting admissibly on a scheme X from the right via a group homomorphism $\rho : G^{\mathrm{op}} \to \mathrm{Aut}(X)$, and $H \trianglelefteq G$ be a normal subgroup. Then the action of G on X induces a well-defined action of G as well as one of G/H on X/H. Moreover, the action of G/H on X/H is admissible and induces the isomorphism*

$$X/G \cong (X/H)/(G/H).$$

Proof. In the following proof, we will denote by $p_G : X \to X/G$ and $p_H : X \to X/H$ the quotient maps of X onto X/G and X/H respectively. Since for each $h \in H$, we have $p_G \circ \rho_h = p_G$ (because if $h \in H$, then $h \in G$), and by the universal mapping property of the quotient scheme X/G, there exists a unique map $\overline{p}_G : X/H \to X/G$ such that $p_G = \overline{p}_G \circ p_H$.

In order to show first that the group action of G (and G/H) on X/H is well-defined, consider an arbitrary element $g \in G$. This induces an automorphism $\rho_g : X \to X$. Now for each $h \in H$, there is an element $h' \in H$ such that $gh' = hg$ (since H is a normal subgroup), so that

$$\begin{aligned}(p_H \circ \rho_g) \circ \rho_h &= p_H \circ (\rho_g \circ \rho_h) = p_H \circ \rho_{hg} = p_H \circ \rho_{gh'}\\ &= p_H \circ (\rho_{h'} \circ \rho_g) = (p_H \circ \rho_{h'}) \circ \rho_g = p_H \circ \rho_g.\end{aligned}$$

So by the universal mapping property of the quotient X/H, there exist a unique morphism $\overline{\rho}_g : X/H \to X/H$ such that $p_H \circ \rho_g = \overline{\rho}_g \circ p_H$. Furthermore, one has $\overline{\rho}_1 = \mathrm{id}_{X/H}$ since $\overline{\rho}_1 \circ p_H = p_H \circ \rho_1 = p_H \circ \mathrm{id}_X = p_H = \mathrm{id}_{X/H} \circ p_H$ and for each $g, g' \in G$:

$$\begin{aligned}(\overline{\rho}_g \circ \overline{\rho}_{g'}) \circ p_H = \overline{\rho}_g \circ (\overline{\rho}_{g'} \circ p_H) &= \overline{\rho}_g \circ (p_H \circ \rho_{g'}) = (\overline{\rho}_g \circ p_H) \circ \rho_{g'} = (p_H \circ \rho_g) \circ \rho_{g'}\\ &= p_H \circ (\rho_g \circ \rho_{g'}) = p_H \circ \rho_{g'g} = \overline{\rho}_{g'g} \circ p_H,\end{aligned}$$

i.e. $\overline{\rho}_g \circ \overline{\rho}_{g'} = \overline{\rho}_{g'g}$. Using this fact, we see that for each $g \in G$, $\overline{\rho}_g \circ \overline{\rho}_{g^{-1}} = \overline{\rho}_{g^{-1}g} = \overline{\rho}_1 = \mathrm{id}_{X/H}$, and similiarly, $\overline{\rho}_{g^{-1}} \circ \overline{\rho}_g = \mathrm{id}_{X/H}$, i.e. $\overline{\rho}_g$ is invertible and hence an automorphism of the scheme X/H. Hence we get a well-defined group homomorphism

$$\overline{\rho} : G^{\mathrm{op}} \longrightarrow \mathrm{Aut}(X/H), g \mapsto \overline{\rho}_g,$$

which induces a well-defined group action of G on the scheme X/H. Moreover, we see that for each $h \in H$, $\overline{\rho}_h \circ p_H = p_H \circ \rho_h = p_H = \mathrm{id}_{X/H} \circ p_H$, i.e. $\rho_h = \mathrm{id}_{X/H}$, or in other words $h \in \ker \rho$, so ρ induces a well-defined group homomorphism $\overline{\rho} : (G/H)^{\mathrm{op}} \to \mathrm{Aut}(X/H), gH \mapsto \overline{\rho}_g$, hence also a well-defined group action of G/H on X/H.

Now we will proof that the morphism $\overline{p}_G : X/H \to X/G$ is affine. So let $U \subseteq X/G$ be an open affine subscheme. Then $p_G^{-1}(U) \subseteq X$ is an open subscheme X, say $p_G^{-1}(U) \cong \operatorname{Spec} A$ for a ring A. Then we have $\mathcal{O}_{X/G}(U) \cong \mathcal{O}_X(p^{-1}(U))^G \cong A^G$, i.e. $U \cong \operatorname{Spec} A^G$. Furthermore, the morphism p_H maps the open subscheme $p_G^{-1}(U) \cong \operatorname{Spec} A$ onto the subscheme of X/H isomorphic to $\operatorname{Spec} A^H$. Since p_H is surjective, we have

$$p_H(p_G^{-1}(U)) = p_H((\overline{p}_G \circ p_H)^{-1}(U)) = p_H(p_H^{-1}(\overline{p}_G^{-1}(U))) = \overline{p}_G^{-1}(U).$$

Hence $\overline{p}_G^{-1}(U) \cong \operatorname{Spec} A^H$, i.e. $\overline{p}_G$ is an isomorphism. Moreover, this shows that if $U \subseteq X/G$ is an open affine subscheme with $p_G^{-1}(U) \cong \operatorname{Spec} A$, then $\overline{p}_G^{\flat} : \mathcal{O}_{X/G} \to p_*\mathcal{O}_{X/H}$ induces the isomorphism

$$\mathcal{O}_{X/G}(U) \cong \mathcal{O}_X(U)^G \cong A^G \cong (A^H)^{G/H} \cong \mathcal{O}_{X/H}(\overline{p}_G^{-1}(U))^{G/H} = (\overline{p}_{G,*}\mathcal{O}_{X/H})^{G/H}(U).$$

Since the open affine subschemes of X/G form a basis of topology of X/G, we obtain the isomorphism

$$\mathcal{O}_{X/G} \cong (\overline{p}_{G,*}\mathcal{O}_{X/H})^{G/H}.$$

Hence the action of G/H on X/H is admissible and induces by Proposition 1.4 the isomorphism $X/G \cong (X/H)/(G/H)$ as desired. □

The last proposition of this section deals with stability under a flat base change:

Proposition 1.9. *Let S be a scheme and G be a finite group acting admissibly on a scheme X over S. Then for an arbitrary morphism $S' \to S$, G also acts on the scheme $X \times_S S'$ via base change of the automorphisms of X induced by the action of G on X. Moreover, if $S' \to S$ is flat, then the action of G on $X \times_S S'$ is also admissible and there is a canonical isomorphism*

$$(X/G) \times_S S' \cong (X \times_S S')/G.$$

Proof. [SGA1, V Prop.1.9] □

1.2 Quotient schemes of a projective spectrum

An important example of the quotient scheme for our purpose is the one of a projective scheme. Recall that if $A = \bigoplus_{l \geq 0} A_l$ is a graded ring, we can define the scheme $\operatorname{Proj} A$ to be the scheme whose underlying set is the set of all homogeneous prime ideals $\mathfrak{p} \trianglelefteq A$ such that $\mathfrak{p} \not\supseteq A_+ := \bigoplus_{d>0} A_d$, and for each homogeneous $f \in A_d$ $(d > 0)$, the set

$$D_+^A(f) = D_+(f) := \{\mathfrak{p} \in \operatorname{Proj} A \mid f \notin \mathfrak{p}\}$$

defines an open subscheme of $\operatorname{Proj} A$ which is isomorphic to $\operatorname{Spec} A_{(f)}$, where

$$A_{(f)} := \{a/f^l \in A_f \mid l \in \mathbb{N}_0 \text{ and } a \in A_{dl}\}.$$

For a later purpose, we state the following proposition:

Proposition 1.10. *Let $A = \bigoplus_{l \geq 0} A_l$ be a graded ring and $d \geq 1$ be an integer. Define the graded ring $A^{(d)} := \bigoplus_{l \geq 0} (A^{(d)})_l$ by*

$$(A^{(d)})_l := A_{dl} \quad \textit{for each } \ l \geq 0.$$

Then the morphism $\operatorname{Proj} A \to \operatorname{Proj} A^{(d)}, \mathfrak{p} \mapsto \mathfrak{p} \cap A^{(d)}$ is an isomorphism.

Proof. [GW10, Remark 13.7] □

We will show that if a finite group G acts on A as a graded ring, then G acts admissibly on $\operatorname{Proj} A$ and we have $\operatorname{Proj} A/G = \operatorname{Proj} A^G$. For this we need the following lemma:

Lemma 1.11. *Let $A = \bigoplus_{d \geq 0} A_d$ be a graded ring and G be a finite group acting on A in such a way that each element $g \in G$ defines a graded ring automorphism of A. Then for each homogeneous ideal $\mathfrak{p} \not\supseteq A_+$, there exists a G-invariant homogeneous element $c \in A_+^G$ such that $c \notin \mathfrak{p}$.*

Proof. Since $\mathfrak{p} \not\supseteq A_+$, there exists a homogeneous element $a \in A_+$, such that $a \notin \mathfrak{p}$. Now consider the polynomial

$$\prod_{g\in G}(T - g(a)) = T^n + c_{n-1}T^{n-1} + \cdots + c_0 \in A[T],$$

where $n := \#G$. Since this polynomial is invariant under the action of G and for each $g \in G$, $g(a)$ is homogeneous of the same degree as a, the coefficients $c_0, \ldots, c_{n-1}$ are homogeneous elements in A_+^G. Moreover, a is a zero of this polynomial, i.e. we have

$$a^n + c_{n-1}a^{n-1} + \cdots + c_0 = 0.$$

By the assumption that $a \notin \mathfrak{p}$, we have $a^n \notin \mathfrak{p}$, i.e. there must be a $k \in \{0, \ldots, n-1\}$ such that $c_k a^k \notin \mathfrak{p}$, i.e. $c_k \notin \mathfrak{p}$, hence the claim follows. $\square$

Proposition 1.12. *Under the same assumptions regarding G and A as in the previous lemma, the action of G on $X := \operatorname{Proj} A$ is admissible and we have $X/G \cong \operatorname{Proj} A^G$.*

Proof. Consider the morphism

$$p : X = \operatorname{Proj} A \longrightarrow Z := \operatorname{Proj} A^G, \mathfrak{p} \mapsto \mathfrak{p} \cap A^G,$$

which is induced by the graded ring homomorphism $A^G \hookrightarrow A$. This is well-defined since for each $x \in X$ with the corresponding homogeneous prime ideal $\mathfrak{p} \trianglelefteq A$, there exists by the previous lemma a homogeneous element $c \in A_+^G$ such that $c \notin \mathfrak{p}$ i.e. $c \notin \mathfrak{p} \cap A^G$. Hence we have $\mathfrak{p} \cap A^G \in \operatorname{Proj} A^G$, i.e. p is well-defined. We will show that p is an affine morphism inducing the isomorphism $\mathcal{O}_Z \xrightarrow{\sim} (p_*\mathcal{O}_X)^G$:

We first show that p is affine. Since $Z = \operatorname{Proj} A^G$ can be covered by affine open subschemes $D_+^{A^G}(f) \cong \operatorname{Spec}(A^G)_{(f)}$, where f runs over the homogeneous elements of A_+^G, we show that the preimage of each $D_+^{A^G}(f) \subseteq \operatorname{Spec} A^G$ is also affine. But since $f \in A^G$, we have

$$\mathfrak{p} \in p^{-1}(D_+^{A^G}(f)) \Leftrightarrow \mathfrak{p} \cap A^G = p(\mathfrak{p}) \in D_+^{A^G}(f) \Leftrightarrow f \notin \mathfrak{p} \cap A^G \Leftrightarrow f \notin \mathfrak{p} \Leftrightarrow \mathfrak{p} \in D_+^A(f).$$

So the preimage of $D_+^{A^G}(f)$ is just $D_+^A(f)$ which is an open subscheme of $\operatorname{Proj} A$ isomorphic to $\operatorname{Spec} A_{(f)}$. Hence p is an affine morphism.

To show that p induces the isomorphism $\mathcal{O}_Z \xrightarrow{\sim} (p_*\mathcal{O}_X)^G$, it suffices to show that for each homogeneous element $f \in A_+^G$, we have an isomorphism

$$p_{U_f}^\flat : \mathcal{O}_Z(U_f) = (A^G)_{(f)} \to (p_*\mathcal{O}_X)^G(U_f) = (A_{(f)})^G$$

where $U_f := D_+^{A^G}(f) \subseteq Z$ and $p_{U_f}^\flat$ is a canonical ring homomorphism induced by $A^G \hookrightarrow A$, i.e. $p_{U_f}^\flat(\frac{a}{f^d}) = \frac{a}{f^d} \in A_{(f)}$ for each $a \in A_{nd}$ ($d \in \mathbb{N}_0$ and $n := \deg f$). But this follows from the fact that if $S \subseteq A^G$ is a multiplicative subset, then $S^{-1}(A^G) = (S^{-1}A)^G$. So we have here $(A^G)_f = (A_f)^G$, and the claim above follows by considering the subrings of $(A^G)_f$ and $(A_f)^G$ consisting of elements whose numerator and denominator have the same degree. Hence the group action of G on X is admissible and we have the isomorphism $X/G \cong \operatorname{Proj} A^G$ as desired. $\square$

1.3 Group actions on geometric points of schemes

Now consider again a right group action of a finite group G on an arbitrary scheme X induced by a group homomorphism

$$\rho : G^{\mathrm{op}} \to \mathrm{Aut}(X), g \mapsto \rho_g.$$

Let Ω be an algebraically closed field. Then the set $X(\Omega) = \mathrm{Hom}(\mathrm{Spec}\,\Omega, X)$ is actually the set of geometric points of X corresponding to a point $x \in X$ together with an embedding $\kappa(x) \hookrightarrow \Omega$. On this set we have the following right group action of G via

$$\overline{x}g := \rho_g \circ \overline{x} \in X(\Omega) \quad \text{for } g \in G \text{ and a geometric point } \overline{x} \in X(\Omega).$$

By Proposition 1.4, we obtain the following fact about geometric points.

Proposition 1.13. *Assume that a finite group G acts admissibly on a scheme X from the right with the quotient morphism $p : X \to X/G$. Furthermore, let Ω be an algebraically closed field. Then two geometric points $\overline{x}, \overline{y} : \mathrm{Spec}\,\Omega \to X$ lie in the same G-orbit if and only if*

$$p \circ \overline{x} = p \circ \overline{y} \in (X/G)(\Omega),$$

i.e. they define the same geometric point in X/G.

Proof. If there is an element $g \in G$ such that $\overline{x} = \overline{y}g = \rho_g \circ \overline{y}$, then $p \circ \overline{x} = p \circ \rho_g \circ \overline{y} = p \circ \overline{y}$ (since $p \circ \rho_g = p$). To show that the converse is also true, suppose that $\overline{x}, \overline{y} \in X(\Omega)$ are geometric points such that

$$p \circ \overline{x} = p \circ \overline{y} : \mathrm{Spec}\,\Omega \to X/G.$$

Let $x, y \in X$ be the underlying points of $\overline{x}$ and $\overline{y}$ respectively and $z := p(x) = p(y) \in X/G$. Then by Proposition 1.4(ii), there is an element $g \in G$ such that $\rho_g(y) = x$, i.e. $\overline{y}g = \rho_g \circ \overline{y}$ has the same underlying point as $\overline{x}$ and since $p \circ \rho_g = p$, we have $p \circ \overline{y}g = p \circ \overline{x}$. Hence we can reduce the statement to the case that $\overline{x}$ and $\overline{y}$ have the same underlying point $x \in X$.

So let $\overline{x}^\sharp, \overline{y}^\sharp : \kappa(x) \hookrightarrow \Omega$ denote the embeddings corresponding to $\overline{x}, \overline{y} : \mathrm{Spec}\,\Omega \to X$ respectively. Since $p \circ \overline{x} = p \circ \overline{y}$, both $\overline{x}^\sharp$ and $\overline{y}^\sharp$ may be considered field homomorphisms over $\kappa(z)$. But by Proposition 1.4(iii), $\kappa(x)|\kappa(z)$ is a normal extension, so there exists an automorphism $\sigma \in \mathrm{Aut}(\kappa(x)|\kappa(z))$ such that $\overline{x}^\sharp = \overline{y}^\sharp \circ \sigma$. By loc.cit. there exists an element $g \in G_x$ corresponding to this automorphism σ, so $\rho_g \circ \overline{y} : \mathrm{Spec}\,\Omega \to X$ corresponds to the embedding $\overline{y}^\sharp \circ \sigma = \overline{x}^\sharp : \kappa(x) \hookrightarrow \Omega$,which also corresponds to $\overline{x} : \mathrm{Spec}\,\Omega \to X$. From this it follows that $\overline{x} = \rho_g \circ \overline{y}$ and we are done. □

Corollary 1.14. *Under the setting as in the proposition above, there is a natural bijection between the sets $X(\Omega)/G$ and $(X/G)(\Omega)$.*

Proof. It remains to show that the map $X(\Omega) \to (X/G)(\Omega), \overline{x} \mapsto p \circ \overline{x}$ is surjective. So let $\overline{z} \in (X/G)(\Omega)$ and $z \in X/G$ be its image. Then there exists by Proposition 1.4(ii) a point $x \in X$ such that $p(x) = z$. Since $\kappa(x)|\kappa(z)$ is an algebraic extension by Proposition 1.4(iii) and Ω is algebraically closed, the embedding $\overline{z}^\sharp : \kappa(z) \hookrightarrow \Omega$ can be extended to an embedding $\varphi : \kappa(x) \hookrightarrow \Omega$. Hence the geometric point $\overline{x} \in X(\Omega)$ given by x and φ satisfies the condition $p \circ \overline{x} = \overline{z}$ as desired. □

Remark 1.15. In the proof above, we used the fact that there is a one-to-one correspondence between the geometric points $\overline{x} \in X(\Omega)$ and a pair consisting of a point $x \in X$ with an embedding $\kappa(x) \hookrightarrow \Omega$. In fact, if $g \in G$ and $\overline{x} \in X(\Omega)$ is a geometric point with the underlying point $x \in X$ and $z := p(x) \in X/G$, then $\overline{x}g = \overline{x}$ if and only if the corresponding embeddings $(g\overline{x})^\sharp, \overline{x}^\sharp : \operatorname{Spec}\Omega \to X$ are the same, i.e. if and only if g lies in the kernel of the group homomorphism $G_x \to \operatorname{Aut}(\kappa(x)|\kappa(z))$ as in Proposition 1.4(iii).

Now we come to a special class of group actions on schemes, namely the group actions avoiding fixed points.

Definition 1.16. *We say that the group action of a finite group G on X* avoids fixed points *or G acts on X* without fixed points *if for each $g \in G \setminus \{1\}$ and each geometric point $\overline{x} \in X(\Omega)$, we have $\overline{x}g \neq \overline{x}$.*

Note that if G acts admissibly on X, then by the remark above, the action of G avoids fixed points if and only if for all $x \in X$ with its image $z \in X/G$ under the quotient map, the group homomorphism $G_x \to \operatorname{Aut}(\kappa(x)|\kappa(z))$ as in the Proposition 1.4(iii) is injective (in this case, this is even bijective because of loc.cit.).

A useful criterion to determine that a group action avoids fixed points is the following:

Proposition 1.17. *Let G be a finite group acting admissibly on a scheme X from the right via $\rho : G^{\mathrm{op}} \to \operatorname{Aut}(X)$, $\overline{x} : \operatorname{Spec}\Omega \to X$ be a geometric point and $x' \in X$ be a specialisation*[1] *of the point lying under $\overline{x}$. If $g \in G$ is an element such that $\rho_g(x') \neq x'$, then $\overline{x}g \neq \overline{x}$.*

Proof. Since the group G acts admissibly on the scheme X, we may assume without loss of generality, in virtue of Proposition 1.6, that $X = \operatorname{Spec} A$ is an affine scheme, for otherwise we can replace X by an open affine subscheme containing the G-orbit of x' which must contain the point lying under $\overline{x}$ for it is a generisation of the point x'.

Now let $\varphi : A \to \Omega$ be the ring homomorphism corresponding to $\overline{x} : \operatorname{Spec}\Omega \to X$. Then $\mathfrak{p} := \ker\varphi$ is the prime ideal corresponding to the point $x \in X$ lying under $\overline{x}$. The specialisation x' of x corresponds to the prime ideal $\mathfrak{p}' \trianglelefteq A$ containing $\mathfrak{p}$. It follows from $\rho_g(x') \neq x'$ that $g(\mathfrak{p}') \neq \mathfrak{p}'$, i.e. there is an element $a \in \mathfrak{p}'$ such that $g(a) \notin \mathfrak{p}'$. This implies

$$g(a) - a \notin \mathfrak{p}' \supseteq \mathfrak{p} = \ker\varphi, \quad \text{i.e. } \varphi(g(a) - a) \neq 0.$$

Therefore, we have $\varphi(g(a)) \neq g(a)$, i.e. $\varphi \circ g \neq \varphi$. Hence $\rho_g \circ \overline{x} \neq \overline{x}$ as desired. □

Corollary 1.18. *If X is a locally noetherian scheme and a finite group G acts admissibly on X from the right via $\rho : G^{\mathrm{op}} \to \operatorname{Aut}(X)$ in such a way that $\rho_g(x) \neq x$ for all closed points $x \in X$ and $g \in G \setminus \{1\}$, then the group G acts on the scheme X without fixed points.*

Proof. This follows from the fact that every point on the locally noetherian scheme has a specialisation which is a closed point, see [GW10, Ex.5.5]. □

It is also worthwhile to mention that the group action on a scheme without fixed points induces the one on a base extension avoiding fixed points.

[1] i.e. x' is a point in the topological closure of the set of the given point. In this case we also say that the point lying under $\overline{x}$ is a generisation of x', compare [GW10, Def.2.8].

Proposition 1.19. *Let S be a scheme, $\varphi : X \to S$ be a scheme over S, and G be a finite group acting on X from the right as a scheme on S via $\rho : G^{\mathrm{op}} \to \mathrm{Aut}(X|S)$. For a base extension $\psi : S' \to S$, let $X' := X \times_S S'$. Then the following holds:*

(1) *If the action of G on X avoids fixed points, then so does the one of G on X'.*

(2) *The converse also holds if $S' \to S$ is surjective. In particular this holds for $S = \mathrm{Spec}\, k$ and $S' = \mathrm{Spec}\, k'$ if $k'|k$ is a field extension.*

Proof. In what follows, the canonical projection of the fibre product $X' = X \times_S S'$ to X will be denoted by $\mathrm{pr}_1 : X' \to X$, and the group homomorphism corresponding to the group action of G on X' over S' induced by the one of G on X over S will be denoted by $\rho' : G^{\mathrm{op}} \to \mathrm{Aut}(X'|S')$. Note that for each $g \in G$, we have $\mathrm{pr}_1 \circ \rho'_g = \rho_g \circ \mathrm{pr}_1$

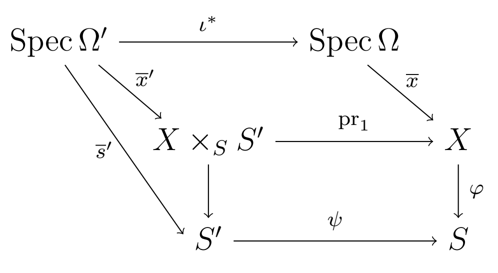
Suppose that G acts on X without fixed points and let $\overline{x}' : \mathrm{Spec}\, \Omega \to X'$ be an arbitrary geometric point of X'. We are going to show that if $g \in G$ is an element such that $\overline{x}'g = \overline{x}'$, then $g = 1$. So assume that $\overline{x}'g = \overline{x}'$, i.e. $\rho'_g \circ \overline{x}' = \overline{x}'$, and let $\overline{x} := \mathrm{pr}_1 \circ \overline{x}' : \mathrm{Spec}\, \Omega \to X$. Then we have

$$\overline{x}g = \rho_g \circ \overline{x} = \rho_g \circ \mathrm{pr}_1 \circ \overline{x}' = \mathrm{pr}_1 \circ \rho'_g \circ \overline{x}' = \mathrm{pr}_1 \circ \overline{x}' = \overline{x}.$$

But G acts on X without fixed points. So we have $g = 1$, i.e. G also acts on X' without fixed points.

Now suppose that $S' \to S$ is surjective and $\overline{x} \in X(\Omega)$ is a geometric point. Let $s \in S$ be the image of $\varphi \circ \overline{x} : \mathrm{Spec}\, \Omega \to S$. By the surjectivity of ψ, there exists a point $s' \in S'$ such that $\psi(s') = s$. Thus the field $\kappa(s)$ can be embedded in Ω using the morphism $\varphi \circ \overline{x}$ and into $\kappa(s')$ using ψ. Therefore there exists an algebraically closed field Ω' together with two embeddings $\iota : \Omega \hookrightarrow \Omega'$ and $\kappa(s') \hookrightarrow \Omega'$ over $\kappa(s)$. In fact, Ω and $\kappa(s')$ can be embedded in a field that is a factor ring of the tensor product $\Omega \otimes_{\kappa(s)} \kappa(s')$ modulo a maximal ideal, hence also in its algebraic closure.

Let $\overline{s}' : \mathrm{Spec}\, \Omega' \to S'$ be the geometric point corresponding to the point $s' \in S'$ with the embedding $\kappa(s') \hookrightarrow \Omega'$ as above. Then we have $\psi \circ \overline{s}' = (\iota^* \circ \overline{x}) \circ \varphi$ since both of them have the same image and their corresponding embeddings $\kappa(s) \hookrightarrow \Omega$ are the same. By the universal mapping property of $X \times_S S'$, we obtain the geometric point $\overline{x}' : \mathrm{Spec}\, \Omega \to X \otimes_k k'$ such that the diagram



is commutative. Now let $g \in G$ be such that $\overline{x}g = \overline{x}$, i.e. $\rho_g \circ \overline{x} = \overline{x}$. Using the notation as above, we see that $\mathrm{pr}_1 \circ (\overline{x}'g) = \mathrm{pr}_1 \circ \rho'_g \circ \overline{x}' = \rho_g \circ \mathrm{pr}_1 \circ \overline{x}' = \rho_g \circ \overline{x} \circ \iota^* = \overline{x} \circ \iota^* = \mathrm{pr}_1 \circ \overline{x}'$. Thus $\overline{x}'g = \overline{x}'$ by the universal mapping property of $X \times_S S'$. But since G acts on X' without fixed points, we get $g = 1$. Hence G also acts on X without fixed points as desired. □

The reason why we are interested in admissible group actions on a scheme avoiding fixed points is the fact that in this case, the quotient map is a finite étale covering under some additional conditions. This will be discussed in more detail in the next chapter.

1.4 Semilinear actions on schemes

Consider for example a scheme X over $\mathbb{C}$. It is sometimes useful to consider not only a morphism of X as scheme over $\mathbb{C}$, but also over $\mathbb{R}$. This, however, may be somewhat too weak to make any useful assertions about it. A somewhat stronger condition which will still have something to do with the structure of X as a scheme over $\mathbb{C}$ is that this morphism is in some sense compatible with an automorphism of $\mathbb{C}$ such as the complex conjugation. In this section, we will discuss some facts about such morphisms which will be useful for the question about schemes defined over a field which is not algebraically closed.

Definition 1.20. *Let $k'|k$ be a Galois field extension and Y, Z be two schemes over k'.*

(1) *A morphism of schemes $\phi : Y \to Z$ is said to be* compatible with an automorphism $\sigma \in \mathrm{Gal}(k'|k)$ *or a* σ-morphism over k' *if the diagram*

$$\begin{array}{ccc} Y & \xrightarrow{\phi} & Z \\ \downarrow & & \downarrow \\ \operatorname{Spec} k' & \xrightarrow{\sigma^*} & \operatorname{Spec} k' \end{array}$$

is commutative, where $\sigma^* : \operatorname{Spec} k' \to \operatorname{Spec} k'$ denotes the morphism of $\operatorname{Spec} k'$ *induced by $\sigma : k' \to k'$.*

(2) *Let $\pi : E \to \mathrm{Gal}(k'|k)$ be a group homomorphism. A right group action*

$$\rho : E^{\mathrm{op}} \to \mathrm{Aut}(Y),\ g \mapsto \rho_g$$

is said to be semilinear with respect to π *or* π-semilinear *(or just* semilinear *if it is clear which group homomorphism $E \to \mathrm{Gal}(k'|k)$ is to be considered) if for each $g \in E$, the morphism $\rho_g : Y \to Y$ is compatible with $\pi(g) \in \mathrm{Gal}(k'|k)$.*

For example, if $k'|k$ is a Galois field extension, then the action of $\mathrm{Gal}(k'|k)$ on $\operatorname{Spec} k'$ induced by the canonical one of $\mathrm{Gal}(k'|k)$ on the field k' is a semilinear action (with respect to $\mathrm{id}_{\mathrm{Gal}(k'|k)}$).

Remark 1.21. Let $k'|k$ and Y, Z be as in the definition above.

(1) If a morphism $\phi : Y \to Z$ is compatible with $\sigma \in \mathrm{Gal}(k'|k)$, then ϕ is a morphism from Y to Z over $\operatorname{Spec} k$.

(2) If a group E acts semilinearly w.r.t. a group homomorphism $\pi : E \to \mathrm{Gal}(k'|k)$ on a scheme Y, then the kernel $\ker \pi \trianglelefteq E$ acts on Y as a scheme over k'.

We are going to treat some properties of the quotient scheme of a scheme under an admissible semilinear action. For this purpose, we begin with the following lemma:

Lemma 1.22. *Let $k'|k$ be a Galois field extension, X a scheme over k and Y a scheme over k'. Consider a right group action $\rho : E^{\mathrm{op}} \to \mathrm{Aut}(Y)$ of a group E which is semilinear with respect to a group homomorphism $\pi : E \to \mathrm{Gal}(k'|k)$. Then each morphism $\varphi : Y \to X$ of schemes over k such that $\varphi \circ \rho_g = \varphi$ for all $g \in E$ induces a morphism $\psi : Y \to X \otimes_k k'$ such that for each $g \in E$ and $\sigma := \pi(g) \in \mathrm{Gal}(k'|k)$, the diagram*

$$\begin{array}{ccc} Y & \xrightarrow{\rho_g} & Y \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi} \\ X \otimes_k k' & \xrightarrow{\sigma_X^*} & X \otimes_k k', \end{array}$$

where $\sigma_X^* := \mathrm{id}_X \times \sigma^* : X \otimes_k k' \to X \otimes_k k'$ *denotes the morphism induced by the morphisms* $\mathrm{id}_X : X \to X$ *and* $\sigma^* : \operatorname{Spec} k' \to \operatorname{Spec} k'$ *over* $\operatorname{Spec} k$*, is commutative.*

Proof. First of all, observe that by the universal mapping property of the fibre product $X \otimes_k k'$, there exists a unique morphism $\psi : Y \to X \otimes_k k'$ making the diagram

$$\begin{array}{ccccc} Y & & & & \\ & \searrow^{\psi} & & \searrow & \\ & & X \otimes_k k' & \xrightarrow{\mathrm{pr}_1} & X \\ & \searrow & \downarrow{\scriptstyle \mathrm{pr}_2} & & \downarrow \\ & & \operatorname{Spec} k' & \longrightarrow & \operatorname{Spec} k \end{array}$$

commutative. Here the morphisms $\mathrm{pr}_1 : X \otimes_k k' \to X$ and $\mathrm{pr}_2 : X \otimes_k k' \to \operatorname{Spec} k'$ are the canonical projection from the fibre product to its factors. In particular, the structure morphism $Y \to \operatorname{Spec} k'$ is a composition of ψ with pr_2. So let $g \in E$. We are going to show that

$$\psi \circ \rho_g = \sigma_X^* \circ \psi : Y \to X \otimes_k k'.$$

In fact, since the action of E on Y is π-semilinear, we have the following equation

$$\sigma^* \circ \mathrm{pr}_2 \circ \psi = \mathrm{pr}_2 \circ \psi \circ \rho_g.$$

Furthermore, by the definition of $\sigma_X^* : X \otimes_k k' \to X \otimes_k k'$, we have

$$\sigma^* \circ \mathrm{pr}_2 = \mathrm{pr}_2 \circ \sigma_X^*.$$

Composing both of the morphisms $\psi \circ \rho_g$ and $\sigma_X^* \circ \psi$ with the canonical projections pr_1 and pr_2 yields

$$\mathrm{pr}_2 \circ (\psi \circ \rho_g) = \sigma^* \circ \mathrm{pr}_2 \circ \psi = (\mathrm{pr}_2 \circ \sigma_X^*) \circ \psi = \mathrm{pr}_2 \circ (\sigma_X^* \circ \psi),$$

and, by using that $\mathrm{pr}_1 \circ \psi = \varphi : Y \to X$ and $\mathrm{pr}_1 \circ \sigma_X^* = \mathrm{id}_X \circ \mathrm{pr}_1 = \mathrm{pr}_1$,

$$\mathrm{pr}_1 \circ (\psi \circ \rho_g) = (\mathrm{pr}_1 \circ \psi) \circ \rho_g = \varphi \circ \rho_g = \varphi = \mathrm{pr}_1 \circ \psi = (\mathrm{pr}_1 \circ \sigma_X^*) \circ \psi = \mathrm{pr}_1 \circ (\sigma_X^* \circ \psi).$$

Hence it follows from the universal mapping property of the fibre product $X \otimes_k k'$ that the both morphisms $\psi \circ \rho_g$ and $\sigma_X^* \circ \psi$ must be the same, and we are done. □

Corollary 1.23. *Under the setting as above, if* $E = \mathrm{Gal}(k'|k)$*, then the morphism* ψ *is* $\mathrm{Gal}(k'|k)$*-equivariant.*

The other lemma which will also be useful later is the following:

Lemma 1.24. *Let* $k'|k$ *be a Galois extension,* E *be a finite group and* $\pi : E \to \mathrm{Gal}(k'|k)$ *be a group homomorphism. Furthermore, let* Y *be a scheme over* k' *on which the group* E *acts* π*-semilinearly. Then for each* $g \in E$ *and each geometric point* $\overline{y} : \operatorname{Spec} \Omega \to Y$*, we can have* $\overline{y} g = \overline{y}$ *only if* $g \in \ker \pi$*.*

Proof. Let $\overline{y} : \operatorname{Spec} \Omega \to Y$ be a geometric point of Y. Composing this with the structure morphism $Y \to \operatorname{Spec} k'$ yields a morphism $\operatorname{Spec} \Omega \to \operatorname{Spec} k'$ with a corresponding field homomorphism $\sigma : k' \to \Omega$. Now for each element $g \in E$, we also get a geometric point $\overline{y}g = \rho_g \circ \overline{y} : \operatorname{Spec} \Omega \to Y$, whose composition with the structure morphism $Y \to \operatorname{Spec} k'$ is the morphism $\operatorname{Spec} \Omega \to \operatorname{Spec} k'$ corresponding to $\sigma \circ \pi(g) : k' \to \Omega$ since E acts semilinearly on Y. This may be illustrated by the following diagram

$$\begin{array}{ccccc} \operatorname{Spec} \Omega & \longrightarrow & Y & \xrightarrow{\rho_g} & Y \\ & \searrow^{\sigma^*} & \downarrow & \overset{(\sigma \circ \pi(g))^*}{\dashrightarrow} & \downarrow \\ & & \operatorname{Spec} k' & \xrightarrow[\pi(g)^*]{} & \operatorname{Spec} k'. \end{array}$$

So if $\overline{y}g = \overline{y}$, then we have $\sigma = \sigma \circ \pi(g)$, but since every field homomorphism is injective, we have $\pi(g) = \mathrm{id}$, i.e. $g \in \ker \pi$ as desired. □

Now we come to the quotient scheme by an admissible semilinear action.

Proposition 1.25. *Let $k'|k$ be a finite Galois extension and*

$$1 \longrightarrow G \longrightarrow E \xrightarrow{\pi} \operatorname{Gal}(k'|k) \longrightarrow 1 \qquad (*)$$

be an exact sequence of finite groups. Furthermore, let $Y \to \operatorname{Spec} k'$ be a scheme over k' with an admissible action of the group E which is semilinear w.r.t. $\pi : E \to \operatorname{Gal}(k'|k)$. Let $X := Y/E$ be the quotient scheme. Then the following holds:

(1) *$Y \to \operatorname{Spec} k' \to \operatorname{Spec} k$ induces a morphism $X \to \operatorname{Spec} k$, i.e. X has a structure as a scheme over k.*

(2) *Letting G act on Y as a normal subgroup of E, the quotient scheme Y/G also exists and is a scheme over k', on which the action of $\operatorname{Gal}(k'|k)$ induced by the one of E on Y is a well-defined semilinear action without fixed points.*

(3) *The morphisms $Y/G \to X$ and $Y/G \to \operatorname{Spec} k'$ induce a morphism $Y/G \to X \otimes_k k'$ which is $\operatorname{Gal}(k'|k)$-equivariant.*

Proof. Applying the universal mapping property of the quotient scheme to the morphism $Y \to \operatorname{Spec} k' \to \operatorname{Spec} k$, the first statement follows from the assumption of the group E and the observation that E acts on Y as a scheme over k by the remark above.

For the second statement, the quotient scheme Y/G exists by Corollary 1.7. By the exactness of the sequence $(*)$, the group G acts on Y as a scheme over k', so there exists a unique morphism $\overline{\psi} : Y/G \to \operatorname{Spec} k'$ induced by the structure morphism $\psi : Y \to \operatorname{Spec} k'$ such that $\overline{\psi} \circ p_G = \psi$, where $p_G : X \to X/G$ denotes the quotient morphism of X by G. Furthermore, the group action of E on Y induces a well-defined admissible action of $\operatorname{Gal}(k'|k) \cong E/G$ on Y/G via $\overline{\rho} : (E/G)^{\mathrm{op}} \to \operatorname{Aut}(Y/G)$ by Proposition 1.8.

To see that this group action is semilinear, consider an element $g \in E$ with its image $\sigma := \pi(g) \in \operatorname{Gal}(k'|k)$ and the following diagram

$$\begin{CD}
Y @>{\rho_g}>> Y \\
@V{p_G}VV @VV{p_G}V \\
Y/G @>{\overline{\rho}_g}>> Y/G \\
@V{\overline{\psi}}VV @VV{\overline{\psi}}V \\
\operatorname{Spec} k' @>{\sigma^*}>> \operatorname{Spec} k'.
\end{CD}$$

We have to show that the lower square is commutative. As in the proof of Proposition 1.8, we see that the upper square is commutative, i.e. $p_G \circ \rho_g = \overline{\rho}_g \circ p_G$. On the other hand, since the group action of E on Y is semilinear, the big square is also commutative, i.e. $\sigma^* \circ \psi = \psi \circ \rho_g$. Hence we have

$$(\overline{\psi} \circ \overline{\rho}_g) \circ p_G = \overline{\psi} \circ (\overline{\rho}_g \circ p_G) = \overline{\psi} \circ (p_G \circ \rho_g) = (\sigma^* \circ \overline{\psi}) \circ p_G.$$

Therefore, we have $\overline{\psi} \circ \overline{\rho}_g = \sigma^* \circ \overline{\psi}$ for all $g \in E$, i.e. the action of $\mathrm{Gal}(k'|k)$ on E is semilinear. The fact that this group action avoids fixed points follows from Lemma 1.24.

Summarising the results up to now, we get the following commutative diagram

$$\begin{array}{ccccc}
Y & \longrightarrow & Y/G & \longrightarrow & X \\
 & \searrow & \downarrow & & \downarrow \\
 & & \operatorname{Spec} k' & \longrightarrow & \operatorname{Spec} k.
\end{array}$$

We see that $Y/G \to X$ is a morphism of schemes over k, which is invariant under the action of $\mathrm{Gal}(k'|k)$ on Y/G since it is actually the quotient map of Y/G by $\mathrm{Gal}(k'|k)$. So the condition in Lemma 1.22, replacing there Y by Y/G, is satisfied and the last statement follows from Corollary 1.23. □

We conclude this section with the remark that if Y is of finite type over k', then the morphism $Y/G \to X \otimes_k k'$ is even an isomorphism. A proof using properties of finite étale coverings will be demonstrated in the next chapter.

Chapter 2

Étale Fundamental groups of schemes

In this chapter we will recall first the definition and some basic properties of étale fundamental groups of connected schemes as in [SGA1, Exp.V]. Afterwards we will study properties of fundamental groups of quotient schemes under admissible group actions, especially those which are also semilinear on schemes defined over a finite Galois field extension.

2.1 Definitions and basic properties

Definition 2.1. *Let X be a connected scheme.*

(1) *The category $\mathbf{Fet}_X$ is defined as the full subcategory of the category of schemes over X whose objects are the finite étale coverings of X, i.e. those finite étale morphisms whose target is the scheme X.*

(2) *If $\overline{x} : \operatorname{Spec}\Omega \to X$ is a geometric point of a connected scheme X, then we define the* fibre functor *at the geometric point $\overline{x}$ as*

$$F_{\overline{x}} : \mathbf{Fet}_X \longrightarrow \mathbf{sets}, \quad F_{\overline{x}}(Y) := \operatorname{Hom}_X(\operatorname{Spec}\Omega, Y).$$

(3) *A finite étale covering $Y \to X$, such that $Y \neq \emptyset$ is a connected scheme, is said to be a* Galois covering *if the group action of $\operatorname{Aut}(Y|X)$ on $F_{\overline{x}}(Y)$ obtained by the one on Y is transitive.*

Remark 2.2. In many books, the authors define the fibre functor of a finite étale covering $Y \to X$ to be $\operatorname{sp}(Y \times_X \operatorname{Spec}\Omega)$ (the set of points of the scheme $Y \times_X \operatorname{Spec}\Omega$). This also yields the same result as the above definition since in this case, $Y \times_X \operatorname{Spec}\Omega$ is the spectrum of a finite product of copies of Ω, i.e. each point in $\operatorname{sp}(Y \times_X \operatorname{Spec}\Omega)$ corresponds to a morphism $\operatorname{Spec}\Omega \to Y \times_X \operatorname{Spec}\Omega$ over $\operatorname{Spec}\Omega$. So there exists a canonical bijection between $\operatorname{sp}(Y \times_X \operatorname{Spec}\Omega)$ and $\operatorname{Hom}_{\operatorname{Spec}\Omega}(\operatorname{Spec}\Omega, Y \times_X \operatorname{Spec}\Omega)$, and also $\operatorname{Hom}_X(\operatorname{Spec}\Omega, Y)$ by the universal mapping property of fibre product.

Proposition 2.3. *Let X be a connected scheme and $\overline{x} : \operatorname{Spec}\Omega \to X$ be a geometric point. Then the category $\mathbf{Fet}_X$, together with the fibre functor $F_{\overline{x}}$, is a* Galois category, *i.e. it satisfies the following conditions:*

(G1) *$\mathbf{Fet}_X$ has a terminal object, namely the identity $\operatorname{id}_X : X \to X$, and if $Y \to W$ and $Z \to W$ are morphisms in $\mathbf{Fet}_X$, then $Y \times_W Z \to X$ is a finite étale covering of X.*

(G2) *If* $Y_j \to X$ *is a finite étale covering for each* $j = 1, \dots, n$*, then so is the direct sum* $\coprod_{j=1}^{n} Y_j \to X$*. In particular,* $\emptyset \to X$ *is the initial object. Furthermore, if* $Y \to X$ *is a finite étale covering and* $G \leq \mathrm{Aut}(Y|X)$ *is a finite subgroup of the automorphisms, then* Y/G *also exists and* $Y/G \to X$ *is also a finite étale covering.*

(G3) *Every morphism* u *in* $\mathbf{Fet}_X$ *can be written as* $u = u'u''$*, where* u'' *is a strict epimorphism and* u' *is a momomorphism in* $\mathbf{Fet}_X$*. Furthermore, if* $u : Y \to Z$ *is a momomorphism, then* Z *can be written as* $Z = Z_1 \coprod Z_2$ *with* $Z_1, Z_2 \in \mathbf{Fet}_X$*, so that* $Y \cong Z_1$*.*

(G4) $F_{\overline{x}}(X)$ *is a one-pointed set, and if* $Y \to W$ *and* $Z \to W$ *are morphisms in* $\mathbf{Fet}_X$*, then* $F_{\overline{x}}(Y \times_W Z) = F_{\overline{x}}(Y) \times_{F_{\overline{x}}(W)} F_{\overline{x}}(Z)$*.*

(G5) *If* $Y_j \in \mathbf{Fet}_X$ *for* $j = 1, \dots, n$*, then* $F_{\overline{x}}(\coprod_{j=1}^{n} Y_j) = \coprod_{j=1}^{n} F_{\overline{x}}(Y_j)$*. Furthermore, if* u *is a strict epimorphism in* $\mathbf{Fet}_X$*, then* $F(X)$ *is an epimorphism, and if* $Y \to X$ *is a finite étale covering and* $G \leq \mathrm{Aut}(Y|X)$ *is finite, then* $F(Y/G) = F(Y)/G$*.*

(G6) *If* u *is a morphism in* $\mathbf{Fet}_X$ *such that* $F_{\overline{x}}(u)$ *is an isomorphism, then so is* u*.*

Proof. [SGA1, V §7] □

Remark 2.4. If X is a connected scheme, then (G3) implies that every finite étale morphism $\phi : Y \to X$ with $Y \neq \emptyset$ is surjective. In fact, one can show more generally that every morphism $Y \to Z$ in $\mathbf{Fet}_X$ such that $Y \neq \emptyset$ and Z is a connected scheme is a strict epimorphism, see [SGA1, V §4e].

Definition 2.5. *Given a connected scheme* X *and a geometric point* $\overline{x} : \mathrm{Spec}\,\Omega \to X$*, we define the* (étale) fundamental group $\pi_1(X, \overline{x})$ *to be the group of automorphisms of the fibre functor* $F_{\overline{x}}$ *on* $\mathbf{Fet}_X$*.*

By the definition above, the étale fundamental group of a scheme is defined as the automorphism group of the fibre functor. On the other hand, it can be written as the projective limit of the projective system consisting of the opposite groups of automorphism groups of Galois coverings as follows:

Definition 2.6. *The partial order on the collection of Galois coverings of* X *is defined as follows: If* $P_\alpha \to X$ *and* $P_\beta \to X$ *are two Galois coverings of* X*, then* $P_\alpha \leq P_\beta$ *if and only if there exists a morphism* $\phi : P_\beta \to P_\alpha$ *of schemes over* X*.*

Proposition 2.7. *Let* X *be a connected scheme and* $\overline{x} : \mathrm{Spec}\,\Omega \to X$ *be a geometric point.*

(1) *If* $P_\alpha \to X$ *and* $P_\beta \to X$ *are two Galois coverings of* X *such that* $P_\alpha \leq P_\beta$*, then for each* $\overline{y}_\alpha \in F_{\overline{x}}(P_\alpha)$ *and* $\overline{y}_\beta \in F_{\overline{x}}(P_\beta)$*, there exists a unique morphism* $\phi_{\alpha\beta} : P_\beta \to P_\alpha$ *of schemes over* X *such that* $F_{\overline{x}}(\phi_{\alpha\beta})(\overline{y}_\beta) = \overline{y}_\alpha$*. Furthermore, this defines a natural surjective group homomorphism*

$$\mathrm{Aut}(P_\beta|X) \to \mathrm{Aut}(P_\alpha|X), \quad \sigma \mapsto \tau \quad \text{if } \tau \circ \phi_{\alpha\beta} = \phi_{\alpha\beta} \circ \tau.$$

(2) *For each Galois covering* $P_\alpha \to X$*, fix an element* $\overline{y}_\alpha \in F_{\overline{x}}(P_\alpha)$*, and for each two Galois coverings* $P_\alpha \leq P_\beta$*, let* $\phi_{\alpha\beta} : P_\beta \to P_\alpha$ *be as in the first part. Then the automorphism groups* $\mathrm{Aut}(P_\alpha|X)$ *indexed over the Galois coverings of* X*, together with the group*

homomorphisms $\mathrm{Aut}(P_\beta|X) \to \mathrm{Aut}(P_\alpha|X)$ *as defined above, form a projective system and there is an isomorphism*

$$\pi_1(X, \overline{x}) \cong \varprojlim_{P_\alpha} \mathrm{Aut}(P_\alpha|X)^{\mathrm{op}}.$$

Furthermore, if $P_\alpha \to X$ *is a Galois covering of* X*, then the group homomorphism* $\pi_1(X, \overline{x}) \to \mathrm{Aut}(P_\alpha|X)^{\mathrm{op}}$ *induced by the isomorphism above is surjective.*

Proof. A direct proof can be found in [Sza09, Prop.5.4.6, Cor.5.4.7 and 5.4.8]. For a proof using properties of a Galois category, let $P_\alpha \to X, P_\beta \to X$ be two Galois coverings such that $P_\alpha \leq P_\beta$ with a morphism $\phi : P_\beta \to P_\alpha$ over X. For a fixed $\overline{y}_\alpha \in F_{\overline{x}}(P_\alpha)$ and $\overline{y}_\beta \in F_{\overline{x}}(P_\beta)$, there exists an automorphism $\sigma \in \mathrm{Aut}(P_\alpha|X)$ such that $F_{\overline{x}}(\sigma) \circ F_{\overline{x}}(\phi)(\overline{y}_\beta) = \overline{y}_\alpha$ since $\mathrm{Aut}(P_\alpha|X)$ acts on $F_{\overline{x}}(P_\alpha)$ transitively. Thus $\phi_{\alpha\beta} := \sigma \circ \phi : P_\beta \to P_\alpha$ satisfies $F_{\overline{x}}(\phi_{\alpha\beta})(\overline{y}_\beta) = \overline{y}_\alpha$. Its uniqueness follows from the injectivity of the map

$$\mathrm{Hom}_X(P_\beta, P_\alpha) \longrightarrow F_{\overline{x}}(P_\alpha), \quad \phi \mapsto F_{\overline{x}}(\phi)(\overline{y}_\beta)$$

since P_β is a connected object in $\mathbf{Fet}_X$, compare [Len08, 3.13].

Now by [SGA1, V §4f], there are bijections between $F_{\overline{x}}(P_\alpha)$ and $\mathrm{Aut}(P_\alpha|X)$, and between $F_{\overline{x}}(P_\beta)$ and $\mathrm{Aut}(P_\beta|X)$. These bijections are so defined that the group homomorphism $\mathrm{Aut}(P_\beta|X) \to \mathrm{Aut}(P_\alpha|X)$ as above and the map $F_{\overline{x}}(\phi_{\alpha\beta}) : F_{\overline{x}}(P_\alpha) \to F_{\overline{x}}(P_\beta)$ are compatible. Since $\phi_{\alpha\beta}$ is a strict epimorphism by the remark above, $F_{\overline{x}}(\phi_{\alpha\beta})$ is surjective. Hence $\mathrm{Aut}(P_\beta|X) \to \mathrm{Aut}(P_\alpha|X)$ is a surjective group homomorphism.

For the second part of the proposition, the collection of Galois coverings of X with the defined partial order is a directed set as shown in [Len08, 3.13 and 3.15]. The fact that $\pi_1(X, \overline{x}) \cong \varprojlim \mathrm{Aut}(P_\alpha|X)^{\mathrm{op}}$ follows from [SGA1, V §4h], and the last statement always holds for any surjective projective system of groups. □

Proposition 2.8. *Let* X *be a connected scheme and* $\overline{x} : \mathrm{Spec}\,\Omega \to X$ *and* $\overline{x}' : \mathrm{Spec}\,\Omega' \to X$ *be two geometric points. Then the fibre functors* $F_{\overline{x}}$ *and* $F_{\overline{x}'}$ *are isomorphic. In particular, there exists a continuous isomorphism of profinite groups* $\pi_1(X, \overline{x}) \xrightarrow{\sim} \pi_1(X, \overline{x}')$*.*

Therefore we will sometimes write $\pi_1(X)$ instead of $\pi_1(X, \overline{x})$.

Proof. This follows from [SGA1, V Thm.4.1] since both $F_{\overline{x}}$ and $F_{\overline{x}'}$ define functors of the Galois category $\mathbf{Fet}_X$. For a proof without the language of Galois categories, see [Sza09, Prop.5.5.1 and Cor.5.5.2]. □

Let $\phi : X' \to X$ be a morphism of connected schemes, $\overline{x} \in X(\Omega)$ and $\overline{x}' \in X'(\Omega)$ be geometric points on X and X' respectively such that $\overline{x} = \phi \circ \overline{x}'$. Then one has a well-defined base change functor

$$BC_{X,X'} : \mathbf{Fet}_X \to \mathbf{Fet}_{X'}, Y \mapsto Y \times_X X'.$$

Furthermore, for each $Y \in \mathbf{Fet}_X$, we have $F_{\overline{x}'} \circ BC_{X,X'}(Y) = \mathrm{Hom}_{X'}(\mathrm{Spec}\,\Omega, Y \times_X X')$, which is in a bijection to $\mathrm{Hom}_X(\mathrm{Spec}\,\Omega, Y)$ defined by the projection $\mathrm{pr}_1 : Y \times_X X' \to Y$ and this bijection is functorial in Y by the universal mapping property of fibre products. Hence we have an isomorphism of functors $F_{\overline{x}'} \circ BC_{X,X'} \cong F_{\overline{x}}$, and we obtain a group homomorphism

$$\phi_* : \pi_1(X', \overline{x}') \longrightarrow \pi_1(X, \overline{x}).$$

If moreover $\phi : X' \to X$ is a finite étale covering, then we have the following statement:

Proposition 2.9. *Let $\phi : Y \to X$ be a finite étale connected covering of a connected scheme and let $\overline{x} \in X(\Omega)$ and $\overline{y} \in Y(\Omega)$ be geometric points such that $\overline{x} = \phi \circ \overline{y}$. Then the group homomorphism $\phi_* : \pi_1(Y, \overline{y}) \to \pi_1(X, \overline{x})$ induces an isomorphism between $\pi_1(Y, \overline{y})$ and the open subgroup of $\pi_1(X, \overline{x})$ containing those automorphisms stabilising $\overline{y} \in F_{\overline{x}}(Y)$, i.e.*

$$\operatorname{im} \phi_* = \{\alpha \in \pi_1(X, \overline{x}) \mid \alpha_Y(\overline{y}) = \overline{y}\}.$$

Proof. This is the interpretation of [SGA1, V Prop.6.13] in the category of finite étale coverings of the scheme X. □

2.2 Fundamental groups of quotient schemes

As mentioned before, the Godeaux-Serre variety was constructed as the quotient variety of a complete intersection of a projective space. The fact about the fundamental group of the Godeaux-Serre variety follows from a relation between the fundamental group of a scheme and that of its quotient. In this section, we will introduce this relation and some more properties to be used in the main result.

Proposition 2.10. *Let G be a finite group acting admissibly on a scheme X from the right. Suppose that X/G is locally noetherian and the quotient morphism $X \to X/G$ is finite. If $\overline{x} \in X(\Omega)$ is a geometric point such that $\overline{x}g \neq \overline{x}$ for all $g \in G \setminus \{1\}$ and $x \in X$ is the underlying point of $\overline{x}$, then the quotient morphism $p : X \to X/G$ is étale in x, i.e. the local ring homomorphism $p_x^\sharp : \mathcal{O}_{X/G,z} \to \mathcal{O}_{X,x}$ is étale, where $z := p(x) \in X/G$ denotes the image of x under the quotient map.*

Proof. This follows from [SGA1, V Prop.2.2], using that each $g \in G$ lies in the kernel of the homomorphism $G_x \to \operatorname{Aut}(\kappa(x)|\kappa(z))$ as in Proposition 1.4(iii) if and only if $\overline{x}g = \overline{x}$. □

Proposition 2.11. *Let G and X be defined as in the proposition above. Suppose in addition that the action of G on X avoids fixed points. Then the following holds:*

(1) *The quotient morphism $p : X \to X/G$ is a finite étale Galois covering.*

(2) *Let $\overline{x} \in X(\Omega)$ and $\overline{z}$ be its image on X/G under p. Then the mapping*

$$\Phi : \pi_1(X/G, \overline{z}) \to G, \quad \alpha \mapsto g_\alpha \quad \text{if } \alpha_X(\overline{x}) = \overline{x}g_\alpha \in F_{\overline{z}}(X)$$

is a well-defined surjective group homomorphism.

(3) *The sequence*

$$1 \longrightarrow \pi_1(X, \overline{x}) \longrightarrow \pi_1(X/G, \overline{z}) \xrightarrow{\Phi} G \longrightarrow 1 \tag{2.1}$$

is exact.

Proof. By the previous proposition, the quotient morphism $p : X \to X/G$ is a finite étale covering. To show that this is a Galois covering, notice first that

$$F_{\overline{z}}(X) = \operatorname{Hom}_{X/G}(\operatorname{Spec} \Omega, X) = \{y \in X(\Omega) \mid p \circ \overline{y} = \overline{z}\}.$$

Since $\overline{x} \in F_{\overline{z}}(X)$, it follows from Proposition 1.13 that the fibre of $\overline{z}$ is the G-orbit of $\overline{x} \in X(\Omega)$, i.e. for each $\overline{y} \in F_{\overline{z}}(X)$, there exists an element $g \in G$ such that $\overline{x}g = \overline{y}$, i.e. G acts on the fibre $F_{\overline{z}}(X)$ transitively. On the other hand, since each element $g \in G$ also defines an automorphism ρ_g of X over X/G, the group $\operatorname{Aut}(X|(X/G))$ also acts on $F_{\overline{z}}(X)$ transitively, hence the first statement follows.

We come to the second statement. Since the group G acts on X without fixed points, the group homomorphism $G \to \mathrm{Aut}(X|(X/G))^{\mathrm{op}}, g \mapsto \rho_g$ is injective. On the other hand, there is a bijection between $\mathrm{Aut}(X|(X/G))$ and $F_{\overline{z}}(X)$ by [SGA1, V §4f]. So the group $\mathrm{Aut}(X|(X/G))$ has as many elements as $F_{\overline{z}}(X)$ which also has as many elements as G since it is a G-orbit of $\overline{x}$ and $\overline{x}g \neq \overline{x}$ for all $g \in G\backslash\{1\}$. This implies that the group homomorphism $G \to \mathrm{Aut}(X|(X/G))^{\mathrm{op}}$ as defined above is an isomorphism. Its inverse is the map sending $\sigma \in \mathrm{Aut}(X|(X/G))$ to $g \in G$ if $\overline{x}g = \sigma \circ \overline{x}$. So the map $\Phi : \pi_1(X/G, \overline{z}) \to G$ is induced by

$$\pi_1(X/G, \overline{z}) \cong \varprojlim \mathrm{Aut}(Y|(X/G))^{\mathrm{op}} \longrightarrow \mathrm{Aut}(X|(X/G))^{\mathrm{op}}.$$

Therefore Φ is a well-defined surjective group homomorphism by Proposition 2.7.

For the last statement, we have to show the exactness of (2.1) at $\pi_1(X, \overline{x})$ and $\pi_1(X/G, \overline{z})$. The injectivity of the map $\pi_1(X, \overline{x}) \to \pi_1(X/G, \overline{z})$ follows from Proposition 2.9. Moreover, the same proposition shows that the image of $\pi_1(X, \overline{x}) \to \pi_1(X/G, \overline{z})$ is just the subgroup of those fibre functor automorphisms $\alpha \in \pi_1(X/G, \overline{z})$ such that $\alpha_X(\overline{x}) = \overline{x} = \overline{x}1$, i.e. the kernel of Φ. Therefore the whole sequence is exact and we are done. □

Corollary 2.12. *Let $k'|k$ be a finite Galois extension and X be a geometrically connected scheme over k. Then $(X \otimes_k k')/\mathrm{Gal}(k'|k) \cong X$ and the sequence*

$$1 \longrightarrow \pi_1(X \otimes_k k', \overline{x}') \longrightarrow \pi_1(X, \overline{x}) \longrightarrow \mathrm{Gal}(k'|k) \longrightarrow 1, \tag{2.2}$$

where $\overline{x} \in X(\Omega)$ denotes the image of $\overline{x}' \in (X \otimes_k k')(\Omega)$ under the canonical projection $X \otimes_k k' \to X$, is exact. The action of $\mathrm{Gal}(k'|k)$ on $X \otimes_k k'$ is defined by the one on $\mathrm{Spec}\, k'$.

Proof. Since the structure morphism $X \to \mathrm{Spec}\, k$ is flat, it follows from Proposition 1.9 that $(X \otimes_k k')/\mathrm{Gal}(k'|k) \cong X \times_k (\mathrm{Spec}\, k'/\mathrm{Gal}(k'|k)) \cong X \times_k \mathrm{Spec}\, k \cong X$. And the claim follows from the previous proposition. □

By the proposition above, we obtain a group homomorphism $\Phi : \pi_1(X/G, \overline{z}) \to G$ if the finite group G acts admissibly on the scheme X without fixed points. In this case, if $H \leq G$ is a subgroup, then H also acts admissibly on X without fixed points and we also obtain the group homomorphism $\Phi_H : \pi_1(X/H, \overline{y}) \to H$, where $\overline{y}$ is the image of $\overline{x}$ under the quotient morphism $p_H : X \to X/H$. The following proposition states that Φ and Φ_H are compatible with the canonical map $X/H \to X/G$:

Proposition 2.13. *Under the setting as above, the diagram*

$$\begin{array}{ccc} \pi_1(X/H, \overline{y}) & \xrightarrow{\overline{p}_{G,*}} & \pi_1(X/G, \overline{z}) \\ \Big\downarrow{\scriptstyle \Phi_H} & & \Big\downarrow{\scriptstyle \Phi} \\ H & \hookrightarrow & G \end{array}$$

is commutative, where $\overline{p}_{G,*} : \pi_1(X/H, \overline{y}) \to \pi_1(X/G, \overline{z})$ denotes the group homomorphism corresponding to $\overline{p}_G : X/H \to X/G$, which is obtained by the quotient map $p_G : X \to X/G$, compare the proof of Proposition 1.8 (note that we didn't use that H is a "normal" subgroup of G to obtain the morphism $\overline{p}_G : X/H \to X/G$).

Proof. Let $\alpha \in \pi_1(X/H, \overline{y})$. By the definition of $\overline{p}_{G,*}$, the map $(\overline{p}_{G,*}\alpha)_X : F_{\overline{z}}(X) \to F_{\overline{z}}(X)$ is obtained by $\alpha_{X'} : F_{\overline{y}}(X') \to F_{\overline{y}}(X')$, where $X' := X \times_{(X/G)} (X/H)$, and using that $F_{\overline{y}}(X') \to F_{\overline{z}}(X), \overline{x}' \mapsto \mathrm{pr}_1 \circ \overline{x}'$, where $\mathrm{pr}_1 : X' \to X$ denotes the canonical projection to the first component, is bijective. In other words, $(\overline{p}_{G,*}\alpha)_X : F_{\overline{z}}(X) \to F_{\overline{z}}(X)$ is defined so that for all $\overline{x}' \in F_{\overline{y}}(X')$, we have

$$(\overline{p}_{G,*}\alpha)_X(\mathrm{pr}_1 \circ \overline{x}') = \mathrm{pr}_1 \circ (\alpha_{X'}(\overline{x}')).$$

Up to now, the fibre $F_{\overline{y}}(X)$ hasn't played any role yet. In order to give it a role, note that we have a relation $p_G \circ \mathrm{id}_X = \overline{p}_G \circ p_H$, which implies by the universal mapping property of $X' = X \times_{(X/G)} (X/H)$ that there exists a unique morphism $\delta : X \to X'$ such that $\mathrm{pr}_1 \circ \delta = \mathrm{id}_X$ and $\mathrm{pr}_2 \circ \delta = p_H$, where $\mathrm{pr}_2 : X' \to X/H$ denotes the canonical projection to the second component. So the morphism δ is a morphism of schemes over X/H, and the following diagram is commutative

$$\begin{array}{ccc} F_{\overline{y}}(X) & \xrightarrow{\alpha_X} & F_{\overline{y}}(X) \\ \big\downarrow{\scriptstyle \delta \circ -} & & \big\downarrow{\scriptstyle \delta \circ -} \\ F_{\overline{y}}(X') & \xrightarrow{\alpha_{X'}} & F_{\overline{y}}(X') \\ {\scriptstyle \wr}\big\downarrow{\scriptstyle \mathrm{pr}_1 \circ -} & & {\scriptstyle \wr}\big\downarrow{\scriptstyle \mathrm{pr}_1 \circ -} \\ F_{\overline{z}}(X) & \xrightarrow[(\overline{p}_{G,*}\alpha)_X]{} & F_{\overline{z}}(X). \end{array}$$

It follows from $\mathrm{pr}_1 \circ \delta = \mathrm{id}_X$ that $\alpha_X(\overline{w}) = \overline{p}_{G,*}(\overline{w})$ for all $\overline{w} \in F_{\overline{y}}(X)$ (note that $F_{\overline{y}}(X) \subseteq F_{\overline{z}}(X)$ because of the definition of the fibres $F_{\overline{y}}(X)$ and $F_{\overline{z}}(X)$ and $\overline{p}_G \circ \overline{y} = \overline{z}$). This holds especially for $\overline{x}$. So if $h = \Phi_H(\alpha)$, then we have

$$\overline{x}h = \alpha_X(\overline{x}) = (\overline{p}_{G,*}\alpha)_X(\overline{w}),$$

which implies that $\Phi(\overline{p}_{G,*}\alpha) = h = \Phi_H(\alpha)$, and we are done. □

Since by Proposition 1.19, the property that a finite group acts on a scheme without fixed points remains unchanged under a base extension, we may do an analogue to the previous proposition for a base extension.

Proposition 2.14. *Let X be a scheme of finite type over a locally noetherian scheme S and G be a group acting admissibly on X from the right without fixed points as a scheme over S. Let $S' \to S$ be a flat base extension and $\mathrm{pr}_1 : X' \to X$ be the canonical projection from the fibre product $X' := X \times_S S'$ to the first component. Furthermore, let $p'_G : X' \to X'/G$ be the quotient morphism of X' by G.*

Consider a geometric point $\overline{x}' \in X'(\Omega)$ with its images $\overline{x} := \mathrm{pr}_1 \circ \overline{x}' \in X(\Omega)$, $\overline{z}' := p'_G \circ \overline{x}' \in (X'/G)(\Omega)$ and $\overline{z} := p_G \circ \overline{x} \in (X/G)(\Omega)$, then the diagram

$$\begin{array}{ccc} \pi_1(X', \overline{x}') & \xrightarrow{\mathrm{pr}_{1,*}} & \pi_1(X, \overline{x}) \\ \big\downarrow & & \big\downarrow \\ \pi_1(X'/G, \overline{z}') & \xrightarrow{\overline{\mathrm{pr}}_{1,*}} & \pi_1(X/G, \overline{z}) \\ & {\scriptstyle \Phi'}\searrow \quad \swarrow{\scriptstyle \Phi} & \\ & G & \end{array}$$

is commutative, where $\overline{\mathrm{pr}}_1 : X'/G \to X/G$ *denotes the morphism induced by* $p_G \circ \mathrm{pr}_1 : X' \to X \to X/G$*, so that* $\overline{\mathrm{pr}}_1 \circ p'_G = p_G \circ \mathrm{pr}_1$*, and* $\Phi' : \pi_1(X'/G, \overline{z}') \to G$ *and* $\Phi : \pi_1(X/G, \overline{z})$ *denote the group homomorphisms obtained by Proposition 2.11.*

Proof. The assumption that X is of finite type over S (hence X' is also of finite type over S') assures that the quotient maps $X \to X/G$ and $X' \to X'/G$ are finite étale, compare Propositions 1.4 and 2.10. It follows from $\overline{\mathrm{pr}}_1 \circ p'_G = p_G \circ \mathrm{pr}_1$ immediately that the upper square commutes. So it remains to show that the lower triangle also commutes.

For this purpose, let $\alpha \in \pi_1(X'/G, \overline{z}')$. By the definition of $\overline{\mathrm{pr}}_{1,*}$ and the same argument as in the proof of the previous proposition, we see that the map $(\overline{\mathrm{pr}}_{1,*}\alpha)_X : F_{\overline{z}}(X) \to F_{\overline{z}}(X)$ satisfies the property

$$(\overline{\mathrm{pr}}_{1,*}\alpha)_X(\mathrm{pr}_1 \circ \overline{w}') = \mathrm{pr}_1 \circ (\alpha_{X'}(\overline{w}')) \quad \text{for all } \overline{w}' \in F_{\overline{z}'}(X')$$

Here we used that $X'/G \cong (X/G) \times_S S'$ and that the isomorphism is defined so that it is compatible with $\overline{\mathrm{pr}}_1 : X'/G \to X/G$ and the canonical projection $(X/G) \times_S S'$, compare Proposition 1.9, to get $X \times_{(X/G)} (X'/G) = X \times_{(X/G)} ((X/G) \times_S S') = X \times_S S' = X'$. This property holds especially for $\overline{x}' \in F_{\overline{z}'}(X')$. So we see that if $g = \Phi'(\alpha)$, then $\overline{x}'g = \alpha_{X'}(\overline{x}')$, and therefore

$$(\overline{\mathrm{pr}}_{1,*}\alpha)_X(\overline{x}) = (\overline{\mathrm{pr}}_{1,*}\alpha)_X(\mathrm{pr}_1 \circ \overline{x}') = \mathrm{pr}_1 \circ (\alpha_{X'}(\overline{x}')) = \mathrm{pr}_1 \circ (\overline{x}'g) = \overline{x}g.$$

Hence $\Phi(\overline{\mathrm{pr}}_{1,*}\alpha) = g = \Phi'(\alpha)$ for all $\alpha \in \pi_1(X'/G, \overline{z}')$ as desired. □

2.3 Semilinear actions and fundamental groups

Having introduced the notion of étale fundamental groups and derived some facts about fundamental groups of quotient schemes, we can now establish a connection between étale fundamental groups and semilinear actions on schemes. As promised in the previous chapter, we will first prove that the morphism $Y/G \to X \otimes_k k'$ from Proposition 1.25 is a $\mathrm{Gal}(k'|k)$-equivariant isomorphism.

Proposition 2.15. *Let* $k'|k$ *be a finite Galois extension and*

$$1 \longrightarrow G \longrightarrow E \xrightarrow{\pi} \mathrm{Gal}(k'|k) \longrightarrow 1 \tag{2.3}$$

be an exact sequence of finite groups. Furthermore, let $Y \to \operatorname{Spec} k'$ *be a scheme of finite type over* k' *with an admissible semilinear action of the group* E *with respect to* $\pi : E \to \mathrm{Gal}(k'|k)$*. Let* $X := Y/E$ *be the quotient scheme. Then the morphisms* $Y/G \to X$ *and* $Y/G \to \operatorname{Spec} k'$ *induce a* $\mathrm{Gal}(k'|k)$*-equivariant canonical isomorphism*

$$Y/G \cong X \otimes_k k'.$$

Proof. In order to use the properties of étale coverings, let us first show that both $Y/G \to X$ and $X \otimes_k k' \to X$ are finite étale coverings. In fact, since Y is of finite type over k' and hence also over k, so is the scheme Y/G. By Proposition 1.8, $X = Y/E$ is the quotient scheme of Y/G by $E/G \cong \mathrm{Gal}(k'|k)$. Now by Lemma 1.24, we see that $\mathrm{Gal}(k'|k)$ acts on Y/G without fixed points. Therefore, the quotient map $Y/G \to (Y/G)/\mathrm{Gal}(k'|k) = X$ is a finite étale covering by Proposition 2.11. On the other hand, $X \otimes_k k' \to X$ is obtained by base extension from $\operatorname{Spec} k' \to \operatorname{Spec} k$, therefore also a finite étale covering.

As we saw in Proposition 1.25, the morphism $\psi : Y \to X \otimes_k k'$ obtained by the morphisms $Y/G \to X$ and $Y/G \to \operatorname{Spec} k'$ is $\operatorname{Gal}(k'|k)$-equivariant. To show that ψ is an isomorphism, consider a geometric point $\overline{x}$ of X and the fibre functor $F_{\overline{x}} : \mathbf{Fet}_X \to \mathbf{sets}$. Since ψ is $\operatorname{Gal}(k'|k)$-equivariant, it is also $F_{\overline{x}}(\psi) : F_{\overline{x}}(Y/G) \to F_{\overline{x}}(X \otimes_k k')$, i.e. the image of a $\operatorname{Gal}(k'|k)$-orbit of $F_{\overline{x}}(Y/G)$ is a $\operatorname{Gal}(k'|k)$-orbit of $F_{\overline{x}}(X \otimes_k k')$. But the group $\operatorname{Gal}(k'|k)$ acts on the fibre $F_{\overline{x}}(X \otimes_k k')$ transitively since this fibre is in a canonical bijection with $\operatorname{Hom}_k(\operatorname{Spec} \Omega, \operatorname{Spec} k') \cong \operatorname{Gal}(k'|k)$, so the map $F_{\overline{x}}(\psi)$ is surjective.

To show that $F_{\overline{x}}(\psi)$ is even bijective, observe that the fibre $F_{\overline{x}}(Y/G)$ is just a $\operatorname{Gal}(k'|k)$-orbit of geometric points of Y/G lying over $\overline{x}$ by the same argument as in Proposition 2.11 and has therefore the same cardinality as $\operatorname{Gal}(K|k)$. Furthermore, we have seen above that there is a bijection between the fibre $F_{\overline{x}}(X \otimes_k k')$ and $\operatorname{Gal}(k'|k)$. Hence the both fibres have the same finite cardinality. Therefore, $F_{\overline{x}}(\psi)$ is bijective. But $F_{\overline{x}}$ is a Galois functor of the Galois category $\mathbf{Fet}_X$, hence $\psi : Y/G \to X \otimes_k k'$ is an isomorphism as desired. □

Corollary 2.16. *Let $k'|k$ be a finite Galois extension and A be a finitely generated k'-algebra. Suppose that G and E are finite groups fitting into the exact sequence* (2.3) *and E acts on A as a ring in such a way that for each $g \in E$, the automorphism $g : A \to A$ is $\pi(g)$-linear. Then A^E is a finitely generated k-algebra and one has a canonical isomorphism*

$$A^E \otimes_k k' \cong A^G.$$

Proof. Since the action of E on A is linear over k, the invariant subring A^E is a k-algebra which is finitely generated since this holds for A. Now consider the scheme $Y := \operatorname{Spec} A$. Then the group action on Y induced by the one on A is admissible by Proposition 1.4 and π-semilinear. Now by loc.cit., we see that $Y/E \cong \operatorname{Spec} A^E$ and $Y/G \cong \operatorname{Spec} A^G$, and by the previous proposition, we have

$$Y/G \cong (Y/E) \otimes_k k' \cong \operatorname{Spec} A^E \otimes_k k'.$$

So we get a canonical isomorphism $\operatorname{Spec} A^E \otimes_k k' \cong \operatorname{Spec} A^G$ and we are done. □

As we saw in the previous section, if a π-semilinear action of E on Y avoids fixed points and $X = Y/E$, we obtain a group homomorphism $\Phi : \pi_1(X, \overline{x}) \to E$. On the other hand, the structure morphism $X \to \operatorname{Spec} k$ induces a group homomorphism $\pi_1(X, \overline{x}) \to \operatorname{Gal}_k$. Since the group $\operatorname{Gal}(k'|k)$ can be considered a quotient group of E as well as of Gal_k, we may ask about the compatibility of both homomorphisms.

Proposition 2.17. *Let k be a field, $\bar{k}$ its algebraic closure, and $k'|k$ be a finite Galois extension contained in $\bar{k}$. Let G, E, π and Y be as in the proposition above. Suppose that E acts on Y without fixed points. Fix a geometric point $\overline{y} \in Y(\Omega)$ with its image $\overline{x} \in X(\Omega)$ under the quotient map $Y \to X = Y/E$. Then the diagram*

$$\begin{array}{ccc} \pi_1(X, \overline{x}) & \xrightarrow{\psi_*} & \operatorname{Gal}_k \\ \Big\downarrow{\scriptstyle \Phi} & & \Big\downarrow \\ E & \xrightarrow{\pi} & \operatorname{Gal}(k'|k) \end{array}$$

is commutative, where $\psi_* : \pi_1(X, \overline{x}) \to \mathrm{Gal}_k = \pi_1(\mathrm{Spec}\, k)$ *denotes the group homomorphism induced by the structure morphism* $\psi : X \to \mathrm{Spec}\, k$*, and* $\Phi : \pi_1(X, \overline{x}) \to E$ *denotes the group homomorphism as in Proposition 2.11.*

Proof. First of all, observe that $\pi_1(\mathrm{Spec}\, k) = \mathrm{Gal}_k = \varprojlim_L \mathrm{Gal}(L|k)$, where L runs over all finite Galois extensions over k contained in $\bar{k}$. So the image of $\gamma \in \pi_1(\mathrm{Spec}\, k)$ under the map $\mathrm{Gal}_k = \pi_1(\mathrm{Spec}\, k) \to \mathrm{Gal}(k'|k)$ is the automorphism $\sigma \in \mathrm{Gal}(k'|k)$ corresponding to

$$\gamma_{\mathrm{Spec}\, k'} : \mathrm{Hom}_k(\mathrm{Spec}\, \Omega, \mathrm{Spec}\, k') \to \mathrm{Hom}_k(\mathrm{Spec}\, \Omega, \mathrm{Spec}\, k').$$

Furthermore, by the universal mapping property of the fibre product $X \otimes_k k'$, there is a canonical bijection between $\mathrm{Hom}_k(\mathrm{Spec}\, \Omega, \mathrm{Spec}\, k')$ and $\mathrm{Hom}_X(\mathrm{Spec}\, \Omega, X \otimes_k k')$. So the image of $\alpha \in \pi_1(X, \overline{x})$ under the composition $\pi_1(X, \overline{x}) \to \mathrm{Gal}_k \to \mathrm{Gal}(k'|k)$ is the automorphism $\sigma \in \mathrm{Gal}(k'|k)$ corresponding to $\alpha_{X \otimes_k k'} : \mathrm{Hom}_X(\mathrm{Spec}\, \Omega, X \otimes_k k') \to \mathrm{Hom}_X(\mathrm{Spec}\, \Omega, X \otimes_k k')$, or in other words: the automorphism $\sigma \in \mathrm{Gal}(k'|k)$ making the diagram

$$\begin{array}{ccc} \mathrm{Hom}_X(\mathrm{Spec}\, \Omega, X \otimes_k k') & \xrightarrow{\alpha_{X \otimes_k k'}} & \mathrm{Hom}_X(\mathrm{Spec}\, \Omega, X \otimes_k k') \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{Hom}_k(\mathrm{Spec}\, \Omega, \mathrm{Spec}\, k') & \xrightarrow{\sigma^* \circ -} & \mathrm{Hom}_k(\mathrm{Spec}\, \Omega, \mathrm{Spec}\, k') \end{array}$$

commutative. Now let $\varphi : Y \to X \otimes_k k'$ denote the morphism obtained by $Y \to X$ and $Y \to \mathrm{Spec}\, k'$. Then $\overline{x}' := \varphi \circ \overline{y} : \mathrm{Spec}\, \Omega \to X \otimes_k k'$ is also a geometric point of $X \otimes_k k'$ lying over $\overline{x}$. Furthermore, the bijection $\mathrm{Hom}_X(\mathrm{Spec}\, \Omega, X \otimes_k k') \to \mathrm{Hom}_k(\mathrm{Spec}\, \Omega, \mathrm{Spec}\, k')$ is obtained by composing $\overline{z} \in \mathrm{Hom}_X(\mathrm{Spec}\, \Omega, X \otimes_k k')$ with $\mathrm{pr}_2 : X \otimes_k k' \to \mathrm{Spec}\, k'$. Since the composition $\pi_1(X, \overline{x}) \to \mathrm{Gal}_k \to \mathrm{Gal}(k'|k)$ is a well-defined map, we obtain by the diagram above that $\sigma \in \mathrm{Gal}(k'|k)$ is the image of $\alpha \in \pi_1(X, \overline{x})$ if and only if

$$\mathrm{pr}_2 \circ \sigma_X^* \circ \overline{x}' = \sigma^* \circ \mathrm{pr}_2 \circ \overline{x}' = \mathrm{pr}_2 \circ \alpha_{X \otimes_k k'}(\overline{x}'),$$

where $\sigma_X^* : X \otimes_k k' \to X \otimes_k k'$ denotes the morphism as in Lemma 1.22. But $\mathrm{pr}_1 \circ \sigma_X^* = \mathrm{pr}_1 : X \otimes_k k' \to X$, so we also have

$$\mathrm{pr}_1 \circ \sigma_X^* \circ \overline{x}' = \mathrm{pr}_1 \circ \overline{x}' = \overline{x} = \mathrm{pr}_1 \circ \alpha_{X \otimes_k k'}(\overline{x}').$$

Here the last equality follows form $\alpha_{X \otimes_k k'}(\overline{x}') \in \mathrm{Hom}_X(\mathrm{Spec}\, \Omega, X \otimes_k k')$. So by the universal mapping property, the image of $\sigma \in \mathrm{Gal}(k'|k)$ is the image of $\alpha \in \pi_1(X, \overline{x})$ under the composition $\pi_1(X, \overline{x}) \to \mathrm{Gal}_k \to \mathrm{Gal}(k'|k)$ if and only if

$$\sigma_X^* \circ \overline{x}' = \alpha_{X \otimes_k k'}(\overline{x}'). \tag{2.4}$$

We turn to the composition $\pi_1(X, \overline{x}) \to E \to \mathrm{Gal}(k'|k)$. Let $g := \Phi(\alpha) \in E$. Then by the construction of Φ, we have $\alpha_Y(\overline{y}) = \overline{y}g = \rho_g \circ \overline{y}$. Using Lemma 1.22, which can be used since the map $Y \to X$ is simply the quotient map of Y by E and therefore invariant under the action of E on Y, we see that if $\sigma = \pi(g) \in \mathrm{Gal}(k'|k)$, then

$$\sigma_X^* \circ \overline{x}' = \sigma_X^* \circ \varphi \circ \overline{y} = \varphi \circ \rho_g \circ \overline{y} = \varphi \circ \alpha_Y(\overline{y}) = \alpha_{X \otimes_k k'}(\varphi \circ \overline{y}) = \alpha_{X \otimes_k k'}(\overline{x}').$$

Hence σ satisfies the equation (2.4). Therefore the diagram is commutative as desired. □

By the proposition above and Corollary 2.12, we see that the group homomorphism $\pi_1(X, \overline{x}) \to \mathrm{Gal}_k$ induced by $X \to \operatorname{Spec} k$ is compatible with $\pi_1(X, \overline{x}) \to \mathrm{Gal}(k'|k)$. On the other hand, the groups $\mathrm{Gal}(k'|k)$, where k' runs over the finite Galois extension of k contained in a fixed algebraic closure $\bar{k}$, form a projective system with Gal_k as the projective limit. So it remains to show that the groups $\pi_1(X \otimes_k k')$ also form a projective system with $\pi_1(X \otimes_k \bar{k})$ as the projective limit. Indeed, one can prove the following statement:

Theorem 2.18. *Let k be a field and $\bar{k}$ its algebraic closure. Let X be a quasi-compact geometrically connected scheme over k. Put $\overline{X} := X \otimes_k \bar{k}$ and let $\overline{x}' \in \overline{X}(\Omega)$ be a geometric point in $\overline{X}$ with its image $\overline{x}$ in X. Then there is a canonical exact sequence*

$$1 \longrightarrow \pi_1(\overline{X}, \overline{x}') \longrightarrow \pi_1(X, \overline{x}) \longrightarrow \mathrm{Gal}_k \longrightarrow 1.$$

Proof. [SGA1, IX Thm.6.1] □

Chapter 3

Regularity conditions

In this chapter, we will review some properties of regular local rings and regular schemes and derive some consequences that will be used in the main result.

3.1 Regular local rings

In this section, we will review some facts about regular local rings to be used in the construction of generalised Godeaux-Serre varieties. The proof can be found in several books in commutative algebra such as in [Mat89]. Recall that a Noetherian local ring $(A, \mathfrak{m})$ is *regular* if $\mathfrak{m}$ can be generated by $r := \dim A$ elements. In this case, the set of elements $x_1, \ldots, x_r \in \mathfrak{m}$ generating $\mathfrak{m}$ is called a *regular system of parameters.*

Proposition 3.1. *Let $(R, \mathfrak{m})$ be an n-dimensional regular local ring and $x_1, \ldots, x_i$ elements of $\mathfrak{m}$. Then the following conditions are equivalent:*

(1) *$x_1, \ldots, x_i$ is a subset of a regular system of parameters of R;*

(2) *the images of $x_1, \ldots, x_i$ in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent over $R/\mathfrak{m}$;*

(3) *$R/(x_1, \ldots, x_i)$ is an $(n-i)$-dimensional regular local ring.*

Proof. [Mat89, Thm.14.2] □

Proposition 3.2. *A regular local ring is an integral domain.*

Proof. [Mat89, Thm.14.3] □

Proposition 3.3. *Let $(A, \mathfrak{m})$ be a regular local ring and $a \in \mathfrak{m}$ such that $a \neq 0$. Then* $\dim(A/(a)) = \dim A - 1$.

Proof. We first show that $\dim(A/(a)) \geq \dim A - 1$. In fact, since $a \in \mathfrak{m}$ and $\mathfrak{m}/(a) \trianglelefteq A/(a)$ is the maximal ideal, it follows from [Mat89, Thm.13.6 (ii)] that

$$\dim A/(a) = \operatorname{ht} \mathfrak{m}/(a) \geq \operatorname{ht} \mathfrak{m} - 1 = \dim A - 1.$$

On the other hand, since A is a regular local ring, it is also an integral domain. So every chain of prime ideals in A corresponding to a chain in $A/(a)$ can be extended by the zero ideal, i.e. $\dim A \geq \dim A/(a) + 1$, hence the equality as desired. □

Proposition 3.4. *Let $\phi : (A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)$ be an étale homomorphism of regular local rings. If the elements $x_1, \ldots, x_k \in \mathfrak{m}_A$ form a subset of a regular system of parameters of A, then $\phi(x_1), \ldots, \phi(x_k) \in \mathfrak{m}_B$ form a subset of a regular system of parameters of B.*

Proof. Let $x_1, \ldots, x_k \in \mathfrak{m}_A$ be a part of a regular system of parameters of A. Let $\bar{x}_j$ denote the image of x_j in $\mathfrak{m}_A/\mathfrak{m}_A^2$ for each $j = 1, \ldots, k$. Then by Proposition 3.1, $\bar{x}_1, \ldots, \bar{x}_k$ are linearly independent over $A/\mathfrak{m}_A$. Now consider the mapping

$$f : (A/\mathfrak{m}_A)^k \to \mathfrak{m}_A/\mathfrak{m}_A^2, (a_1, \ldots, a_k) \mapsto a_1\bar{x}_1 + \cdots + a_k\bar{x}_k \in \mathfrak{m}_A/\mathfrak{m}_A^2.$$

This defines a linear mapping between vector spaces over $A/\mathfrak{m}_A$, which is injective since $\bar{x}_1, \ldots, \bar{x}_k \in \mathfrak{m}_A/\mathfrak{m}_A^2$ are linearly independent, hence also an injective A-module homomorphism. Since B is a flat A-module, we have, by identifying $A \otimes_A B$ with B, $\mathfrak{m}_A \otimes_A B = \mathfrak{m}_A B = \mathfrak{m}_B$ (the last equality holds since $A \to B$ is unramified), and by tensoring the map f with B over A, we get the injective B-module homomorphism

$$f \otimes \mathrm{id}_B : (B/\mathfrak{m}_B)^k \to \mathfrak{m}_B/\mathfrak{m}_B^2, (b_1, \ldots, b_k) \mapsto b_1\overline{\phi(x_1)} + \cdots + a_k\overline{\phi(x_k)} \in \mathfrak{m}_B/\mathfrak{m}_B^2,$$

where $\overline{\phi(x_1)}, \ldots, \overline{\phi(x_k)}$ denote the images of $\phi(x_1), \ldots, \phi(x_k)$ in $\mathfrak{m}_B/\mathfrak{m}_B^2$. Here we used the fact that $A/\mathfrak{m}_A \otimes_A B = B/\mathfrak{m}_A B = B/\mathfrak{m}_B$. Hence $\overline{\phi(x_1)}, \ldots, \overline{\phi(x_k)}$ are also linearly independent over $B/\mathfrak{m}_B$. So again by Proposition 3.1, $\phi(x_1), \ldots, \phi(x_k) \in \mathfrak{m}_B$ form a subset of a regular system of parameters of B. □

Proposition 3.5. *Let A, B be Noetherian local rings and $\phi : A \to B$ be a local homomorphism such that B is flat over A. If B is regular, then so is A.*

Proof. [Mat89, Thm.23.7(i)] □

Corollary 3.6. *Let $f : X \to Y$ be a morphism of schemes*

(1) *If a point $x \in X$ is regular and f is flat in x, then $f(x) \in Y$ is also regular.*

(2) *If X is regular and f is flat and surjective, then Y is also regular.*

Proof. Since a morphism $f : X \to Y$ is flat in $x \in X$ if and only if $f_x^\sharp : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat, the claim follows immediately from the proposition above. □

3.2 Bertini's theorems

In the construction of Godeaux-Serre varieties, it appeared in one step that one should consider a complete intersection of hyperplanes in a projective space which is regular. Its existence is assured by Bertini's theorems as presented in this section. We will first deal with the cases of infinite fields and finite fields separately and then discuss some consequences.

3.2.1 Bertini's theorem for infinite fields

Throughout this subsection, let k denote an infinite field. The set k^n will be considered with the *Zariski topology*, i.e. the closed subsets of k^n will be of the form

$$N(\mathfrak{a}) := \{\alpha = (\alpha_1, \ldots, \alpha_n) \in k^n \mid \forall f \in \mathfrak{a} : f(\alpha_1, \ldots, \alpha_n) = 0\}$$

for an ideal $\mathfrak{a} \trianglelefteq k[T_1, \ldots, T_n]$. Note that $N(\mathfrak{a}) = k^n$ if and only if $\mathfrak{a} = 0$. We will treat Bertini's theorem for infinite fields as in [FOV99, §3.4] and begin with the following definition.

Definition 3.7. *We will say that a property holds for* generic $\alpha \in k^n$ *if it holds for all* α *in an open dense subset of* k^n

Let X be a scheme of finite type over k and $\mathcal{L}$ be a line bundle on X. Furthermore, let $s_1, \dots, s_n \in \Gamma(X, \mathcal{L})$ be sections over X having no common zero (i.e. for each $x \in X$, there exists a $j \in \{1, \dots, n\}$ such that $s_j(x) \neq 0 \in \mathcal{L}_x/\mathfrak{m}_x\mathcal{L}_x$), and $\mathfrak{X} \subseteq X \times_k \mathbb{A}_k^n$ be the closed subschemes defined by the global section $s := \sum_{j=1}^n s_j U_j$, where $\mathbb{A}_k^n = \operatorname{Spec} k[U_1, \dots, U_n]$. Now consider the morphism

$$q : \mathfrak{X} \hookrightarrow X \times_k \mathbb{A}_k^n \xrightarrow{\mathrm{pr}_2} \mathbb{A}_k^n,$$

where $\mathrm{pr}_2 : X \times_k \mathbb{A}_k^n \to \mathbb{A}_k^n$ denotes the canonical projection onto $\mathbb{A}_k^n$. Then for each $\alpha := (\alpha_1, \dots, \alpha_n) \in k^n$, considered as a closed point in $\mathbb{A}_k^n$, the fibre $X_\alpha := \mathfrak{X} \times_{\mathbb{A}_k^n} \operatorname{Spec} k$, where the morphism $\operatorname{Spec} k \to \mathbb{A}_k^n$ is induced by the ring homomorphism

$$k[U_1, \dots, U_n] \to k, U_j \mapsto \alpha_j \quad \text{for each } j = 1, \dots, n,$$

is just a closed subscheme of X defined by the global section $\sum_{j=1}^n s_j \alpha_j$. In fact, considering the scheme X locally, so that $X = \operatorname{Spec} A$ for a finitely generated k-algebra A and $\mathcal{L} \cong \mathcal{O}_X$, we get $\mathfrak{X} = \operatorname{Spec} A[U_1, \dots, U_n]/(\sum_{j=1}^n f_j U_j)$, where $f_1, \dots, f_n \in A = \Gamma(X, \mathcal{O}_X)$ are elements corresponding to $s_1, \dots, s_n$. From this it follows that

$$X_\alpha = \mathfrak{X} \times_{\mathbb{A}_k^n} \operatorname{Spec} k = \operatorname{Spec}\Big(A[U_1, \dots, U_n]\Big/\big(\sum_{j=1}^n f_j U_j\big)\Big) \otimes_{k[U_1,\dots,U_n]} k = \operatorname{Spec}(A/(f)),$$

where $f := \sum_{j=1}^n f_j \alpha_j$. Having prepared the notations, we can formulate Bertini's theorem for regular schemes as follows:

Theorem 3.8 (Bertini for regular schemes over an infinite field)**.** *Let* X *be a regular scheme of finite type over an infinite field* k *and* $\mathcal{L}, s_1, \dots, s_n$ *be as above. Assume that for every* $x \in X$*, the differentials* $d(\frac{s_1}{s_i}), \dots, d(\frac{s_n}{s_i})$ *generate* $\Omega^1_{X|k,x}$ *for* $i = 1, \dots, n$ *such that* $s_i(x) \neq 0 \in \mathcal{L}_x/\mathfrak{m}_x\mathcal{L}_x$*. Then for generic* $\alpha \in k^n$*, the subscheme* X_α *is also regular.*

Proof. [FOV99, Thm.3.4.13] □

Now consider the projective space $\mathbb{P}_k^n = \operatorname{Proj} k[T_0, \dots, T_n]$ and a subscheme $X \subseteq \mathbb{P}_k^n$. We know that $\Gamma(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(1))$ is the vector space over k generated by $T_0, \dots, T_n \in k[T_0, \dots, T_n]$, i.e. $T_0, \dots, T_n$ are sections of the line bundle $\mathcal{O}_{\mathbb{P}_k^n}(1)$, and these also define global sections in $\Gamma(X, \mathcal{O}_X(1))$, which we will also denote by $T_0, \dots, T_n$. According to the discussion above, we see that for each $\alpha = (\alpha_0, \dots, \alpha_n) \in k^{n+1} \setminus \{0\}$, the subscheme X_α is simply the (schematic) intersection of $H_\alpha := V(\sum_{i=0}^n \alpha_j T_j)$ with X. Using Bertini's theorem as stated above, we obtain the following corollary:

Corollary 3.9. *Suppose that* $X \subseteq \mathbb{P}_k^n$ *is a regular subscheme. Then for generic* $\alpha \in k^{n+1}$*, the intersection* $H_\alpha \cap X$ *is also regular. In this case, we have* $\dim_x H_\alpha \cap X = \dim_x X - 1$ *for each closed point* $x \in H_\alpha \cap X$*.*

Proof. Only the last statement is to be checked. So let $\alpha \in K^{n+1}$ be such that $H_\alpha \cap X$ is regular and consider the scheme $H_\alpha \cap X \subseteq X$ locally at x. Furthermore, let $s_x \in \mathcal{O}_{X,x}$ be an element obtained by $\sum_{i=0}^n \alpha_j T_j$, so that $\mathcal{O}_{H_\alpha \cap X,x} \cong \mathcal{O}_{X,x}/(s_x)$. Then $s_x \in \mathfrak{m}_{X,x}$ since $s_x \notin \mathcal{O}_{X,x}^\times$. Thus the claim follows from Proposition 3.3 since $\dim \mathcal{O}_{X,x} = \dim_x X$ and $\dim_x H_\alpha \cap X = \dim \mathcal{O}_{H_\alpha \cap X,x}$. □

Using the fact that the intersection of two open dense subsets of a topological space is again a dense open subset, we obtain a somewhat stronger result as follows:

Corollary 3.10. *Suppose that $X \subseteq \mathbb{P}^n_k$ is a regular subscheme and $Z \subseteq \mathbb{P}^n_k$ is a finite subscheme. Then for generic $\alpha \in k^{n+1}$, the intersection $H_\alpha \cap X$ is regular and $H_\alpha \cap Z = \emptyset$.*

Proof. Let $\mathrm{sp}(Z) = \{y_1, \ldots, y_m\}$, i.e. $y_1 \ldots, y_m$ are the closed points of Z, and define $W_j := \{\alpha \in k^{n+1} \mid y_j \notin H_\alpha\}$ for each $j = 1, \ldots, m$. Then the sets $W_1, \ldots, W_m$ are open dense subsets of k^{n+1}. Now the set $W := \{\alpha \in k^{n+1} \mid H_\alpha \cap X \text{ is regular}\}$ is also an open dense subset as in the corollary above. Hence the set

$$W \cap W_1 \cap \cdots \cap W_m = \{\alpha \in k^{n+1} \mid H_\alpha \cap X \text{ is regular and } y_j \notin H_\alpha \text{ for all } j = 1, \ldots, m\}$$

is also a dense open subset of k^{n+1}. Furthermore, it is easy to see that every $\alpha \in W$ satisfies the properties in the statement of the corollary, and we are done. □

Remark 3.11. All statements above also hold if we replace "regular" by "smooth". Indeed, recall that a scheme $X \to \operatorname{Spec} k$ over a field k is smooth if and only if it is geometrically regular, i.e. $\bar{X} := X \otimes_k \bar{k}$ is a regular scheme. So we may replace X by $\bar{X}$ in all proofs above and obtain an open dense subset $U \subseteq \bar{k}^{n+1}$ such that each $\alpha \in U$ satisfies the desired properties. Now the set $U \cap k^{n+1}$ is an open subset of k^{n+1} which is not empty since k is infinite, i.e. a polynomial $f \in \bar{k}[T_0, \ldots, T_n]$ satisfies $f(\alpha) = 0$ for all $\alpha \in k^{n+1}$ if and only if $f = 0$. Hence $U \cap k^{n+1}$ is also dense in k^{n+1}, and for each $\alpha \in U$, the intersection $(H_\alpha \cap X) \otimes_k \bar{k} = (H_\alpha \otimes_k \bar{k}) \cap \bar{X}$ is regular, i.e. $H_\alpha \cap X$ is a smooth scheme over k.

3.2.2 Bertini's theorem for finite fields

Now we come to the case when we are working over a finite field. In the previous section we have seen, for example, that if $X \subseteq \mathbb{P}^n_k$ is a regular subscheme over an infinite field k, then there exists an open dense subset $U \subseteq k^{n+1}$ such that each $\alpha \in k^{n+1}$ gives a linear hyperplane in $\mathbb{P}^n_k$ whose intersection with X is a regular subscheme. This open dense subset U is the complement of the zero set of an ideal $\mathfrak{a} \trianglelefteq k[T_0, \ldots, T_n]$ and is not empty if $\mathfrak{a} \neq 0$.

If we try to transfer the results to the case of finite fields, say $k = \mathbb{F}_q$, we encounter a problem that there exists a nonzero polynomial over a finite fields having all elements of $\mathbb{F}_q^{n+1}$ as its roots. So we have to do something other than just consider $\mathbb{F}_q^{n+1}$ as the space of parameters. In this section, we will follow the paper [Poo04] and begin with the following preparations:

Notation 3.12. Let $k = \mathbb{F}_q$ be a finite field with q elements and $S := \mathbb{F}_q[T_0, \ldots, T_n]$ be the polynomial ring over $\mathbb{F}_q$ in variables $T_0, \ldots, T_n$.

- For $d \geq 0$, let $S_d \subseteq S$ be the set of homogeneous polynomials of degree d.
- $S_{\text{homog}} := \bigcup_{d=0}^{\infty} S_d$ is defined as the set of homogeneous polynomials.
- For $f \in S_{\text{homog}}$, let H_f be the subscheme $\operatorname{Proj}(S/(f)) \subseteq \mathbb{P}^n_k$.
- If $\mathcal{P} \subseteq S_{\text{homog}}$, define the *density* of $\mathcal{P}$ by $\mu(\mathcal{P}) := \lim\limits_{d \to \infty} \dfrac{\#(\mathcal{P} \cap S_d)}{\# S_d}$.

- For a scheme X of finite type over $\mathbb{F}_q$, define the zeta function ζ_X by

$$\zeta_X(s) := \prod_{x \in |X|} \frac{1}{1 - q^{-s \deg x}} = \exp\left(\sum_{r=1}^{\infty} \frac{\#X(\mathbb{F}_{q^r})}{r} q^{-rs}\right),$$

where $|X|$ denotes the set of closed points in X.

Now we can state Bertini's Theorem over a finite field as follows:

Theorem 3.13 (Bertini over a finite field)**.** *Let $k = \mathbb{F}_q$ be a finite field with q elements, $X \subseteq \mathbb{P}^n_k$ be a quasiprojective subscheme of $\mathbb{P}^n_k$ and $Z \subseteq \mathbb{P}^n_k$ be a finite subscheme. Assume that $U := X \setminus (X \cap Z)$ is smooth of dimension $m \geq 0$. For each $f \in S_d$, let $f|_Z$ be the element of $H^0(Z, \mathcal{O}_Z)$ that on each connected component Z_i equals the restriction of $T_j^{-d} f$ to Z_i, where $j = j(i)$ is the smallest $j \in \{0, 1, \dots, n\}$ such that the coordinate T_j is invertible on Z_i. Fix a subset $T \subset H^0(Z, \mathcal{O}_Z)$ and define*

$$\mathcal{P} := \{f \in S_{\mathrm{homog}} \mid H_f \cap U \text{ is smooth of dimension } m-1 \text{ and } f|_Z \in T\}.$$

Then

$$\mu(\mathcal{P}) = \frac{\#T}{\#H^0(Z, \mathcal{O}_Z)} \zeta_U(m+1)^{-1}.$$

Proof. [Poo04, Thm.1.2] □

Corollary 3.14. *Let $k = \mathbb{F}_q$ and $X, Z, U \subseteq \mathbb{P}^n_k$ be as in the theorem above. Then there exists a homogenous polynomial $f \in S_{\mathrm{homog}}$ such that $H_f \cap U$ is smooth of dimension $m-1$ and $H_f \cap Z = \emptyset$.*

Proof. Consider the set

$$T := \prod_{Z_i} (H^0(Z_i, \mathcal{O}_{Z_i}) \setminus \{0\}) \subseteq \prod_{Z_i} H^0(Z_i, \mathcal{O}_{Z_i}) = H^0(Z, \mathcal{O}_Z),$$

where Z_i in the product runs over the connected components of Z. Since $Z_i \cap H \neq \emptyset$ if and only if $f|_{Z_i} = 0 \in H^0(Z_i, \mathcal{O}_{Z_i})$ for each connected component $Z_i \subseteq Z$, a hypersurface H_f defined by a homogeneous polynomial $f \in S_{\mathrm{homog}}$ does not meet Z if and only if $f|_Z \in T$. Hence a homogeneous f lies in $\mathcal{P}$ if and only if $H_f \cap U$ is smooth of dimension $m-1$ and $H_f \cap Z = \emptyset$. Now by the theorem above, there exists a $d \in \mathbb{N}$ such that $\frac{\#(\mathcal{P} \cap S_d)}{\#S_d} > 0$, i.e. there exists a homogeneous polynomial of degree d lying in $\mathcal{P}$ and we are done. □

3.2.3 Consequences

Having introduced Bertini's Theorems over an infinite and a finite field, we can now discuss its applications in the both cases. The first statement is a consequence of Corollaries 3.10 and 3.14 together with Remark 3.11:

Corollary 3.15. *Let k be an arbitrary field, $X \subseteq \mathbb{P}^n_k$ be a regular subscheme and $Y \subseteq \mathbb{P}^n_k$ a closed subscheme. Then there exists a hypersurface $H_f \subseteq \mathbb{P}^n_k$ defined by a homogeneous polynomial $f \in k[T_0, \dots, T_n]$ such that $H_f \cap X$ is regular and $\dim H_f \cap Y \leq \dim Y - 1$. The same holds if one replaces "regular" by "smooth".*

Proof. Since $Y \subseteq \mathbb{P}^n_k$ is a closed subscheme, we can write Y as a union of irreducible components $Y = Y_1 \cup \cdots \cup Y_m$ (using the primary decomposition of the ideal of $k[T_0, \ldots, T_N]$ corresponding to Y). Choose any closed point $y_j \in Y_j$ for each $j = 1, \ldots, m$ and let $Z \subseteq \mathbb{P}^n_k$ be the finite subscheme consisting of $y_1, \ldots, y_m$. Then there exists a hypersurface $H_f \subseteq \mathbb{P}^n_k$ defined by a homogeneous polynomial $f \in k[T_0, \ldots, T_n]$ such that $H_f \cap X$ is regular and $H_f \cap Z = \emptyset$ (note that if k is infinite, then we can choose f to have degree 1 and H_f is simply a linear hyperplane in $\mathbb{P}^n_k$). For this hyperplane H_f, we have

$$\dim H_f \cap Y = \sup_{j=1,\ldots,m} \dim H_f \cap Y_j \leq \sup_{j=1,\ldots,m} \dim Y_j - 1 = \dim Y - 1,$$

i.e. H_f satisfies the desired property. □

Applying the corollary above inductively, we also obtain the following statement.

Corollary 3.16. *Under the condition as in the corollary above with $d \leq n$, there exists a closed subscheme $L \subseteq \mathbb{P}^n_k$ defined by d homogeneous polynomials in $k[T_0, \ldots, T_n]$ such that $L \cap X$ is regular and $\dim (L \cap Y) \leq \dim Y - d$. In particular, if $\dim Y < d$, then we can choose a closed subscheme $L \subseteq \mathbb{P}^N_k$ with these properties such that $L \cap Y = \emptyset$.*

Furthermore, if $\dim X = \dim \mathcal{O}_{X,x}$ for all closed points $x \in X$ (for example, if X is an integral subscheme), then $\dim L \cap X = \dim X - d$.

The same holds if one replaces "regular" by "smooth". Moreover, if k is an infinite field, we can choose such a closed subscheme $L \subseteq \mathbb{P}^n_k$ as a linear subspace of codimension d.

Chapter 4

Godeaux-Serre varieties and their k-forms

In this chapter we will prove the main result as stated in the introduction. First of all, we will give a construction for an analogous problem due to a finite Galois extension, which will be used in the main result. One of its consequences is that we obtain the main result for the special case $k = \mathbb{R}$. In order to transfer this result to the general case, we will need a tool from profinite group theory as discussed in Section 4.3.

4.1 A construction

In this section we will construct a variety over a finite Galois field extension which will play an important role for our main result. The idea is to find a projective variety Y which is a complete intersection in an appropriate projective space $\mathbb{P}^n_{k'}$ such that for a given extension of finite groups

$$1 \longrightarrow G \xrightarrow{\iota} \widetilde{E} \xrightarrow{\pi} \mathrm{Gal}(k'|k) \longrightarrow 1, \qquad (4.1)$$

one can find a π-semilinear action of $\tilde{E}$ on Y avoiding fixed points. The following construction is inspired from [Ser58, Prop.15]:

Proposition 4.1. *Let $k'|k$ be a finite Galois extension in a fixed algebraic closure $\bar{k}$ and an extension of finite groups* (4.1) *be given. Then there exists for each $r \geq 2$ a smooth and geometrically connected projective variety of dimension r which is a complete intersection in $\mathbb{P}^n_{k'}$ for an appropriate $n \in \mathbb{N}$, on which the group $\tilde{E}$ acts π-semilinearly and admissibly without fixed points.*

Recall that a *complete intersection* in $\mathbb{P}^n_{k'}$ is a closed subscheme $Y \subseteq \mathbb{P}^n_{k'}$ such that the corresponding homogeneous ideal can be generated by $n - r$ elements, where $r = \dim Y$.

Proof. Throughout the proof, if $K|k$ is a field extension of the field k and $X \to \operatorname{Spec} k$ is a scheme over k, then let $X_K := X \otimes_k K$. We will proceed in several steps.

Step 1. *A semilinear action of $\tilde{E}$ on the homogeneous coordinate ring over k'*

Since the group $\widetilde{E}$ is finite, there exists a faithful linear representation of $\widetilde{E}^{\mathrm{op}}$ over k given by an injective group homomorphism $\rho : \widetilde{E}^{\mathrm{op}} \to \mathrm{GL}_{n+1}(k)$ such that $\rho(g)$ is not a multiple of the identity matrix for all $g \in \widetilde{E} \setminus \{1\}$ (for example, the regular representation). Using this representation, we can define the semilinear action of $\widetilde{E}$ on the homogeneous coordinate ring $k'[T_0,\dots,T_n]$ as follows:

$$\begin{array}{rcl}\widetilde{E}\times k'[T_0,\dots,T_n] & \to & k'[T_0,\dots,T_n]\\ (g,f(T_0,\dots,T_n)) & \mapsto & (gf)(T_0,\dots,T_n):=\pi(g)(f((T_0,\dots,T_n)\rho(g)))\end{array}$$

or more precisely, if $\rho(g):=(a_{ij})_{0\le i,j\le n}$ and $f=\sum r_{k_0,\dots,k_n}T_0^{k_0}\cdots T_n^{k_n}\in k[T_0,\dots,T_n]$, then

$$(gf)(T_0,\dots,T_n):=\sum\sigma(r_{k_0,\dots,k_n})S_0^{k_0}\cdots S_n^{k_n},$$

where $\sigma:=\pi(g)\in\mathrm{Gal}(k'|k)$ and $S_j:=a_{0j}T_0+\cdots+a_{nj}T_n$ for each $j=0,\dots,n$. Also note that this actually defines a left action of $\widetilde{E}$ on $k'[T_0,\dots,T_n]$ as a graded ring automorphism.

STEP 2. *The right action of $\widetilde{E}$ on $\mathbb{P}^n_{k'}$ induced by the left one on $k'[T_0,\dots,T_n]$*

By the first step, we obtain the right action of $\widetilde{E}$ on $\mathbb{P}^n_{k'}=\mathrm{Proj}\,k'[T_0,\dots,T_n]$ from the left one on $k'[T_0,\dots,T_n]$, which is clearly π-semilinear. By Proposition 1.12, we see that E acts admissibly on $\mathbb{P}^n_{k'}$ and

$$\mathbb{P}^n_{k'}/\widetilde{E}\cong\mathrm{Proj}\,k'[T_0,\dots,T_n]^{\widetilde{E}}.$$

So let us consider the invariant ring $A:=k'[T_0,\dots,T_n]^{\widetilde{E}}$. We see that $A_0=(k')^{\widetilde{E}}=k$. Thus A is finitely generated as a k-algebra since it is an invariant subalgebra of $k'[T_0,\dots,T_n]$ which is a finitely generated algebra over k' and hence also over k, compare [Bou89, V §1.9 Thm.2]. Furthermore, by [Bou89, III §1.3 Prop.3], there exists a natural number $d\in\mathbb{N}$ such that the subring $A^{(d)}$ as defined in Proposition 1.10 is generated by A_d, i.e.

$$A^{(d)}:=\bigoplus_{l\ge 0}A_{dl}=A_0[A_d]=k[A_d].$$

So the k-Algebra $A^{(d)}$ is (finitely) generated by homogeneous polynomials over k' of degree d invariant under $\widetilde{E}$, say $A^{(d)}=k[f_0,\dots,f_s]$ for some $f_0,\dots,f_s\in A_d$, and therefore isomorphic to $k[U_0,\dots,U_s]/\mathfrak{a}$ for some homogeneous ideal $\mathfrak{a}\trianglelefteq k[U_0,\dots,U_s]$. Also note that $\mathfrak{a}$ is actually a prime ideal since the factor ring $k[U_0,\dots,U_s]/\mathfrak{a}$ is isomorphic to $A^{(d)}$ which is free from zero divisors.

On the other hand, Proposition 1.10 shows that $\mathrm{Proj}\,A\cong\mathrm{Proj}\,A^{(d)}$, i.e. we have $\mathbb{P}^n_{k'}/\widetilde{E}\cong\mathrm{Proj}\,A^{(d)}$. So we may consider the quotient subscheme $\mathbb{P}^n_{k'}/\widetilde{E}$ as a projective variety $Z\subseteq\mathbb{P}^s_k$ defined by the ideal $\mathfrak{a}$ as mentioned above with the quotient map $p:\mathbb{P}^n_{k'}\to Z$.

STEP 3. *The closed subscheme of "bad points" in $\mathbb{P}^n_{k'}$*

Since we are going to find a subvariety of $\mathbb{P}^n_{k'}$ on which the group $\widetilde{E}$ acts without fixed points, we consider in this step the closed subscheme of $\mathbb{P}^n_{k'}$ whose geometric points are invariant under the action of some elements of $\widetilde{E}$. At this point, we should remark that it suffices to consider the action of G on $\mathbb{P}^n_{k'}$ as a subgroup of $\widetilde{E}$ since for each $g\in\widetilde{E}$ with $\pi(g)\ne\mathrm{id}_{k'}\in\mathrm{Gal}(k'|k)$ and geometric point $\overline{x}\in\mathbb{P}^n_{k'}(\Omega)$, we have $\overline{x}g\ne\overline{x}$ by Lemma 1.24.

In order to deal with the closed points more easily, we will work in the algebraic closure $\bar{k}$. The set of closed points of $\mathbb{P}^n_{\bar{k}}$ may be identified with the set $\mathbb{P}^n(\bar{k}):=(\bar{k}^{n+1}\setminus\{0\})/\sim$, where $\sim$ is the equivalence relation on $\bar{k}^{n+1}\setminus\{0\}$ defined by

$$(x_0,\dots,x_n)\sim(y_0,\dots,y_n)\ :\Leftrightarrow\ \exists\lambda\in\bar{k}^\times:(x_0,\dots,x_n)=(\lambda y_0,\dots,\lambda y_n).$$

Furthermore, the group G acts on $\mathbb{P}^n_{k'}$ as a scheme over k', thus also on $\mathbb{P}^n_{\bar{k}} = \mathbb{P}^n_{k'} \otimes_{k'} \bar{k}$ as a scheme over $\bar{k}$. Now for each $g \in G \setminus \{1\}$, consider the set

$$\tilde{Q}_g := \{x \in \mathbb{P}^n(\bar{k}) \mid \rho_g(x) = x\} \subseteq \mathbb{P}^n(\bar{k}).$$

Since $\rho_g \neq \mathrm{id}$, $\tilde{Q}_g$ is the union of the linear subspaces corresponding to the eigenspaces of ρ_g as a linear mapping on $\bar{k}^{n+1}$, none of which is the whole space $\bar{k}^{n+1}$ since ρ_g is not a multiple of the identity matrix. So the corresponding closed subscheme $Q_g \subseteq \mathbb{P}^n_{\bar{k}}$ has the dimension $\dim Q_g = \max\{\dim V_{g,\lambda} \mid \lambda \in \bar{k}\} - 1 \leq n-1$, where $V_{g,\lambda} \subseteq \bar{k}^{n+1}$ denotes the eigenspace of ρ_g corresponding to $\lambda \in \bar{k}$.

Furthermore, Q_g is defined by a homogeneous ideal of $\bar{k}[T_0, \dots, T_n]$ generated by polynomials with coefficients in k since ρ_g is given by a linear mapping with coefficients in k. Therefore, if $Q_g \subseteq \mathbb{P}^n_{\bar{k}}$ is the closed subscheme corresponding to $\tilde{Q}_g$, there exists a (proper) closed subscheme $Q'_g \subseteq \mathbb{P}^n_{k'}$ such that $Q'_g \otimes_{k'} \bar{k} = Q_g$, namely the one defined by the ideal of $k'[T_0, \dots, T_n]$ generated by those polynomials generating the ideal which defines Q_g.

By this construction, $Q' := \bigcup_{g \in G\setminus\{1\}} Q'_g \subseteq \mathbb{P}^n_{k'}$ is actually a closed subscheme whose base extension under $k' \hookrightarrow \bar{k}$ is the subscheme $Q := \bigcup_{g \in G\setminus\{1\}} Q_g \subseteq \mathbb{P}^n_{\bar{k}}$, i.e. the subscheme of $\mathbb{P}^n_{\bar{k}}$ whose closed points are invariant under the action of some elements of G.

STEP 4. *Smoothness condition*

Consider again the closed subscheme $Q' \subseteq \mathbb{P}^n_{k'}$. For each $g \in \widetilde{E}$, $g(Q')$ is also a closed subscheme of $\mathbb{P}^n_{k'}$ (and it is exactly Q' if $g \in G$). So the finite union $\hat{Q} := \bigcup_{g \in \widetilde{E}} g(Q)$ is a closed subscheme of $\mathbb{P}^n_{k'}$ invariant under $\widetilde{E}$ which is not the whole scheme $\mathbb{P}^n_{k'}$ because of its dimension and the fact that $\mathbb{P}^n_{k'}$ is irreducible. Hence $p(\hat{Q}) \subseteq Z (\cong \mathbb{P}^n_{k'}/\widetilde{E})$ is a closed subscheme of the same dimension as $\hat{Q}$.

We are going to show that $Z' := Z \setminus p(\hat{Q})$ is smooth over k by showing that $Z'_{\bar{k}}$ is regular. In fact, $Z' \subseteq Z$ is an open subscheme. So by Corollary 1.5, E also acts admissibly on $p^{-1}(Z')$ and its quotient is Z'. Thus the group G also acts admissibly on $p^{-1}(Z')$ and its quotient scheme is $Z'_{k'}$ by Proposition 2.15. Since G acts on $p^{-1}(Z')$ as a scheme over k', it also acts admissibly on $W := p^{-1}(Z') \otimes_{k'} \bar{k}$ with the quotient $Z'_{k'} \otimes_{k'} \bar{k} = Z'_{\bar{k}}$ by Proposition 1.9.

On the other hand, $p^{-1}(Z') \cap Q' = \emptyset$ by the construction of Z'. Thus also $W \cap Q = \emptyset$, i.e. G acts on the set of closed points of W without fixed points, which implies that G on the scheme W without fixed points by Corollary 1.18. Therefore the quotient morphism $W \to Z'_{\bar{k}}$ is étale by Proposition 2.11. Since $W \subseteq \mathbb{P}^n_{\bar{k}}$ is regular, so is $Z'_{\bar{k}}$ by Corollary 3.6. So Z' is smooth over k as desired.

STEP 5. *Using Bertini*

Observe that we can assume without loss of generality that $r < n - \dim Q$, since otherwise we can consider a representation $\tilde{\rho} : E^{\mathrm{op}} \to \mathrm{GL}_{\tilde{n}+1}(k')$, where $\tilde{n} := m(n+1) - 1$, obtained by m copies of ρ for some $m \in \mathbb{N}$. Indeed, let Q_ρ, $Q_{\tilde{\rho}}$ denote the subscheme $Q \subseteq \mathbb{P}^n_{\bar{k}}$ obtained from the construction in Step 3 by using the representation ρ resp. $\tilde{\rho}$ and similarly for $Q_{\rho,g}$, $Q_{\tilde{\rho},g}$, etc. Then for each $g \in G \setminus \{1\}$ and $\lambda \in \bar{k}$, we have $\dim V_{\tilde{\rho},g,\lambda} = m \dim V_{\rho,g,\lambda}$ and consequently $\dim Q_{\tilde{\rho},g} = m(\dim Q_{\rho,g} + 1) - 1 \leq mn - 1 = \tilde{n} - m$. Thus for $m > r$, we have $\dim Q_{\tilde{\rho}} = \max\{\dim Q_{\tilde{\rho},g} \mid g \in G \setminus \{1\}\} \leq \tilde{n} - m < \tilde{n} - r$, i.e. $r < \tilde{n} - \dim Q_{\tilde{\rho}}$ as desired.

Applying Bertini's theorem, see Corollary 3.16, we obtain a closed subvariety $L \subseteq \mathbb{P}^s_k$, say $L = V(h_1, \dots, h_{n-r})$ for some homogeneous polynomials $h_1, \dots, h_{n-r} \in k[U_0, \dots, U_s]$, such that $L \cap p(\hat{Q}) = \emptyset$ and $L \cap Z = L \cap (Z \setminus p(\hat{Q}))$ is smooth over k and has dimension $n-(n-r) = r$. Note that if k is an infinite field, we can choose L as a linear subspace and $h_1, \dots, h_{n-r}$ as linear polynomials.

Now let $g_j := h_j(f_0, \dots, f_s) \in k'[T_0, \dots, T_n]$ for $j = 1, \dots, n-r$. Then $L \cap Z$ as a closed subscheme of $Z \cong \operatorname{Proj}(A)$ is isomorphic to $\operatorname{Proj}(A/(g_1, \dots, g_{n-r}))$. Since by Corollary 2.16, $A \otimes_k k' \cong B := k'[T_0, \dots, T_n]^G$, we have $(L \cap Z)_{k'} \cong \operatorname{Proj}(B/(g_1, \dots, g_{n-r}))$, which can be considered a closed subscheme of $\operatorname{Proj} B = \mathbb{P}^n_{k'}/G$ in a natural way. Hence we also have $(L \cap Z)_{\bar{k}} \cong \operatorname{Proj}((B \otimes_{k'} \bar{k})/(g_1, \dots, g_{n-r}))$ as a closed subscheme of $\operatorname{Proj}(B \otimes_{k'} \bar{k}) = \mathbb{P}^n_{\bar{k}}/G$.

Step 6. *Using regularity*

From the previous step, we see that $\overline{Y} := V(g_1, \dots, g_{n-r}) \subseteq \mathbb{P}^n_{\bar{k}}$ is the preimage of $(L \cap Z)_{\bar{k}}$ under the quotient map $p_{\bar{k}} : \mathbb{P}^n_{\bar{k}} \to Z_{\bar{k}} \cong \mathbb{P}^n_{\bar{k}}/G$. Since $L \cap Z \subseteq Z'$, the local ring $\mathcal{O}_{Z_{\bar{k}},z}$ is regular for all $z \in (L \cap Z)_{\bar{k}} \subseteq Z_{\bar{k}}$ as shown in Step 4.

We are going to prove that $\overline{Y}$ is a regular scheme of dimension r by showing that $\mathcal{O}_{\overline{Y},y}$ is regular for each closed point $y \in \overline{Y}$. In this case, let $z := p(y) \in (L \cap Z)_{\bar{k}} \subseteq Z_{\bar{k}}$, then the ring $\mathcal{O}_{(L\cap Z)_{\bar{k}},z} = \mathcal{O}_{Z_{\bar{k}},z}/(h_1, \dots, h_{n-r})$ is a regular local ring of dimension $r = n-(n-r)$. It follows from Proposition 3.1 that $\{h_1, \dots, h_{n-r}\}$ is a subset of a regular parameter system of $\mathcal{O}_{Z_{\bar{k}},z}$. Since $\mathcal{O}_{Z_{\bar{k}},z} \to \mathcal{O}_{\mathbb{P}^n_{\bar{k}},y}$ is étale, $\{g_1, \dots, g_{n-r}\}$ is also such a subset of $\mathcal{O}_{\mathbb{P}^n_{\bar{k}},y}$ by Proposition 3.4. Hence $\mathcal{O}_{\overline{Y},y} = \mathcal{O}_{\mathbb{P}^n_{\bar{k}},y}/(g_1, \dots, g_{n-r})$ is also a regular local ring of dimension $n-(n-r) = r$. Therefore $\overline{Y}$ is regular and has dimension r.

Step 7. *The projective variety Y*

Let $Y := V(\mathfrak{b}) \subseteq \mathbb{P}^n_{k'}$, where $\mathfrak{b} := (g_1, \dots, g_{n-r}) \trianglelefteq k'[T_0, \dots, T_n]$. We are going to show that Y is geometrically integral by showing that $\overline{Y} = Y \otimes_{k'} \bar{k}$ is integral, i.e. reduced and irreducible. In fact, for each $y \in \overline{Y}$, the local ring $\mathcal{O}_{\overline{Y},y}$ is integral since it is regular by the previous step, see Proposition 3.2, hence also reduced. Therefore $\overline{Y}$ is reduced.

On the other hand, since the closed subscheme $\overline{Y} \subseteq \mathbb{P}^n_{\bar{k}}$ has dimension $r > 1$ and its corresponding ideal $(g_1, \dots, g_{n-r}) \trianglelefteq \bar{k}[T_0, \dots, T_n]$ is generated by $n-r$ polynomials, it is a complete intersection in $\mathbb{P}^n_{\bar{k}}$, hence connected, see [Har77, III Ex.5.5]. Since $\overline{Y}$ is connected and regular, it is also irreducible, which implies that $\overline{Y}$ is integral.

Since $\overline{Y} = Y \otimes_{k'} \bar{k}$ is regular and integral of dimension r, we see that $Y = V(\mathfrak{b}) \subseteq \mathbb{P}^n_{k'}$ is a smooth and geometrically connected projective variety over k' of dimension r. Its corresponding ideal is generated by $n-r$ polynomials, i.e. it is a complete intersection in $\mathbb{P}^n_{k'}$. The action of $\widetilde{E}$ on Y as a subscheme of $\mathbb{P}^n_{k'}$ is induced by the action on $k'[T_0, \dots, T_n]/\mathfrak{b}$ as a factor ring of $k'[T_0, \dots, T_n]$, which is well-defined since $\mathfrak{b}$ is invariant under $\widetilde{E}$. Thus $\widetilde{E}$ acts admissibly and π-semilinearly on Y. Furthermore, we see from Steps 3 – 4 that this group action avoids fixed points, i.e. the variety Y satisfies the desired properties. □

4.2 A consequence – The case $k = \mathbb{R}$

As mentioned before, the construction from the previous section can be used to obtain our main result for the case $k = \mathbb{R}$. To this end, we begin first with the following statement which will also be used in the general case.

Proposition 4.2. *Let $k'|k$ be a finite Galois extension in a fixed algebraic closure $\bar{k}$, let G be a finite group and $\widetilde{E}$ be another one fitting into the exact sequence* (4.1)*. Let Y be a geometrically connected scheme of finite type over k' with an admissible π-semilinear action of the group $\widetilde{E}$ avoiding fixed points. If $X := Y/\widetilde{E}$ is the quotient scheme of Y by $\widetilde{E}$, then X is also geometrically connected, $X \otimes_k \bar{k} = (Y \otimes_{k'} \bar{k})/G$ and the diagram*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(X \otimes_k \bar{k}) & \longrightarrow & \pi_1(X) & \longrightarrow & \mathrm{Gal}_k & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \Phi_{\bar{k}}} & & \downarrow{\scriptstyle \Phi} & & \downarrow & & \\ 1 & \longrightarrow & G & \xrightarrow{\iota} & \widetilde{E} & \xrightarrow{\pi} & \mathrm{Gal}(k'|k) & \longrightarrow & 1 \end{array}$$

is commutative, where the group homomorphisms $\Phi_{\bar{k}} : \pi_1(X \otimes_k \bar{k}) \to G$ and $\Phi : \pi_1(X) \to \widetilde{E}$ are those obtained by Proposition 2.11.

Note that the action of G on $Y \otimes_{k'} \bar{k}$ also avoids fixed points by Proposition 1.19, which implies that the homomorphism $\pi_1(X \otimes_k \bar{k}) \to G$ is well-defined since $X \otimes_k \bar{k} \cong (Y \otimes_{k'} \bar{k})/G$.

Proof. The commutativity of the right square has been shown in Proposition 2.17. We are going to show that X is also geometrically connected and $X \otimes_k \bar{k} \cong (Y \otimes_{k'} \bar{k})/G$. For this purpose, notice first that $X \otimes_k k' \cong Y/G$ by Proposition 2.15. Since $\operatorname{Spec} \bar{k} \to \operatorname{Spec} k'$ is a flat base extension, it follows from Proposition 1.9 that

$$X \otimes_k \bar{k} = (X \otimes_k k') \otimes_{k'} \bar{k} \cong (Y/G) \otimes_{k'} \bar{k} \cong (Y \otimes_{k'} \bar{k})/G.$$

Since $Y \otimes_{k'} \bar{k}$ is connected, so is its quotient by G, i.e. X is geometrically connected.

It remains to show that the left square of the diagram above is also commutative. For this purpose, observe that the isomorphism $Y/G \xrightarrow{\cong} X \otimes_k k'$ from Proposition 2.15 is a morphism of schemes over X. Using that $X \otimes_k \bar{k} = (X \otimes_k k') \otimes_{k'} \bar{k}$, we see that the diagram

$$\begin{array}{ccc} (Y/G) \otimes_{k'} \bar{k} & \xrightarrow{\cong} & X \otimes_k \bar{k} \\ \downarrow & & \downarrow \\ Y/G & \xrightarrow{\cong} & X \otimes_k k' \\ & \searrow \quad X \quad \swarrow & \end{array}$$

is commutative. It follows that the morphisms $X \otimes_k \bar{k} \xrightarrow{\cong} (Y/G) \otimes_{k'} \bar{k} \to Y/G$ are compatible with the quotient map $Y/G \to X$ and the canonical projection $X \otimes_k \bar{k} \to X$. Now consider the following diagram

$$\begin{array}{ccc} \pi_1(X \otimes_k \bar{k}) & & \\ & \searrow & \searrow \\ & \pi_1(Y/G) & \longrightarrow \ \pi_1(X) \\ & \downarrow & \downarrow \\ \searrow & G & \longrightarrow \ \widetilde{E}. \end{array}$$

We see that the upper triangle is commutative by the assertion above, the left one by Proposition 2.14, and the right square by Proposition 2.13. Therefore the whole diagram is commutative, which completes the proof. $\square$

Using the Lefschetz hyperplane theorem, which implies that if X is a regular projective variety over an algebraically closed field and $Y \subseteq X$ is a subscheme of dimension ≥ 2 that is a complete intersection in X, then $\pi_1(Y) \cong \pi_1(X)$ (see [Har70, IV Cor.2.2]), we can prove the main result for the special case $k = \mathbb{R}$ with $\mathrm{Gal}_{\mathbb{R}} = \mathrm{Gal}(\mathbb{C}|\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$ as follows:

Corollary 4.3. *Given $r \geq 2$ and an extension of finite groups*

$$1 \longrightarrow G \longrightarrow E \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1,$$

there exists a real smooth geometrically connected projective variety X of dimension r such that its fundamental group is isomorphic to E, and the exact sequence

$$1 \longrightarrow \pi_1(X \otimes_{\mathbb{R}} \mathbb{C}) \longrightarrow \pi_1(X) \longrightarrow \mathrm{Gal}_{\mathbb{R}} \longrightarrow 1$$

is compatible with the one given above.

Proof. By Proposition 4.1, there exists a complex smooth geometrically connected projective variety Y of dimension r which is a complete intersection in $\mathbb{P}^n_{\mathbb{C}}$ for an appropriate $n \in \mathbb{N}$, on which the group action of E on Y is semilinear and avoids fixed points. Then the previous proposition shows that $X := Y/E$ is a real projective variety which is geometrically connected, and we have the following commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(X \otimes_{\mathbb{R}} \mathbb{C}) & \longrightarrow & \pi_1(X) & \longrightarrow & \mathrm{Gal}_{\mathbb{R}} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow \cong & & \\ 1 & \longrightarrow & G & \longrightarrow & E & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 1. \end{array}$$

It remains to show that the maps $\pi_1(X \otimes_{\mathbb{R}} \mathbb{C}) \to G$ and $\pi_1(X) \to E$ are isomorphisms. But since $\pi_1(\mathbb{P}^n_{\mathbb{C}}) = 1$ by [SGA1, XI Prop.1.1], it follows from the Lefschetz hyperplane theorem that $\pi_1(Y) = 1$. Using that $X \otimes_{\mathbb{R}} \mathbb{C} \cong Y/G$, we see that both maps mentioned before are isomorphisms by the exact sequence in Proposition 2.11. □

4.3 A partial-lifting in an extension of profinite groups

In this section, we will prove a tool from the theory of profinite groups to be used in the constuction of Godeaux-Serre varieties with prescribed arithmetic fundamental groups, namely the existence of a "partial lifting" in an extension of a profinite group by a finite group, i.e. the existence of an open normal subgroup of the factor group that can be lifted to an open normal subgroup of the group extension. For this we need the following lemma, which will also be used later to show that the constructed variety has the desired properties.

Lemma 4.4. *Let $E, \widetilde{E}, \Gamma, \widetilde{\Gamma}$ be profinite groups, $\pi : E \to \Gamma$, $\widetilde{\pi} : \widetilde{E} \to \widetilde{\Gamma}$, $p : E \to \widetilde{E}$ and $q : \Gamma \to \widetilde{\Gamma}$ be continuous surjective group homomorphisms such that $\widetilde{\pi} \circ p = q \circ \pi$, i.e. the diagram*

$$\begin{array}{ccc} E & \xrightarrow{\pi} & \Gamma \\ \downarrow p & & \downarrow q \\ \widetilde{E} & \xrightarrow{\widetilde{\pi}} & \widetilde{\Gamma} \end{array}$$

is commutative. Then the restriction $p|_{\ker \pi} : \ker \pi \to \ker \widetilde{\pi}$ is an isomorphism if and only if the same holds for $\pi|_{\ker p} : \ker p \to \ker q$. In this case the following holds:

(1) *Two elements* $g, h \in E$ *coincide if and only if* $p(g) = p(h)$ *and* $\pi(g) = \pi(h)$. *In particular,* $\ker p \cap \ker \pi$ *contains only the neutral element.*

(2) (E, p, π) *is isomorphic to the fibre product* $\widetilde{E} \times_{\widetilde{\Gamma}} \Gamma$.

Proof. First of all, notice that we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \ker \pi & \hookrightarrow & E & \xrightarrow{\pi} & \Gamma & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} & & \\ 1 & \longrightarrow & \ker \widetilde{\pi} & \hookrightarrow & \widetilde{E} & \xrightarrow{\widetilde{\pi}} & \widetilde{\Gamma} & \longrightarrow & 1. \end{array}$$

So if $p|_{\ker \pi} : \ker \pi \to \ker \widetilde{\pi}$ is an isomorphism, then the cokernel of $p|_{\ker \pi}$ exists and is trivial. By the same argumentation as in the Snake Lemma we see that $\pi|_{\ker p} : \ker p \to \ker q$ is also an isomorphism. The converse can be shown with the same argument by the symmetry along the diagonal from E to $\widetilde{\Gamma}$.

Assume from now on that $p|_{\ker \pi} : \ker \pi \to \ker \widetilde{\pi}$ is an isomorphism and $g \in \ker \pi \cap \ker p$. Since $g \in \ker p$, we have $p(g) = 1$, and since $g \in \ker \pi$, the term $(p|_{\ker \pi})(g)$ makes sense and we have $(p|_{\ker \pi})(g) = p(g) = 1$, i.e. $g \in \ker(p|_{\ker \pi})$. But $p|_{\ker \pi}$ is an isomorphism, so $g = 1$, i.e. $\ker \pi \cap \ker p = \{1\}$. Furthermore, if $g, h \in E$ are such that $p(g) = p(h)$ and $\pi(g) = \pi(h)$, then $g^{-1}h \in \ker p \cap \ker \pi$, which implies that $g^{-1}h = 1$, i.e. $g = h$.

To show that (E, p, π) is the fibre product $\widetilde{E} \times_{\widetilde{\Gamma}} \Gamma$, observe first that since $\widetilde{\pi} \circ p = q \circ \pi$, there exists a unique homomorphism of profinite groups $(p, \pi) : E \to \widetilde{E} \times_{\widetilde{\Gamma}} \Gamma$ such that $\mathrm{pr}_1 \circ (p, \pi) = p$ and $\mathrm{pr}_2 \circ (p, \pi) = \pi$, where $\mathrm{pr}_1, \mathrm{pr}_2$ denote the canonical projections of $\widetilde{E} \times_{\widetilde{\Gamma}} \Gamma$ onto $\widetilde{E}$ and Γ respectively. So we have to show that (p, π) is indeed an isomorphism. For this purpose, consider the following diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \ker \pi & \hookrightarrow & E & \xrightarrow{\pi} & \Gamma & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle (p,\pi)} & & \| & & \\ 1 & \longrightarrow & \ker \widetilde{\pi} & \xrightarrow[i]{} & \widetilde{E} \times_{\widetilde{\Gamma}} \Gamma & \xrightarrow[\mathrm{pr}_2]{} & \Gamma & \longrightarrow & 1, \end{array}$$

where $i : \ker \widetilde{\pi} \to \widetilde{E} \times_{\widetilde{\Gamma}} \Gamma$ is given by $i(a) := (a, 1)$ for all $a \in \ker \widetilde{\pi}$. Note that this is well-defined since $\widetilde{\pi}(a) = 1 = q(1)$ and obviously an injective group homomorphism. The left square is commutative since the isomorphism $\ker \pi \to \ker \widetilde{\pi}$ is the restriction of $p : E \to \widetilde{E}$. The right square is commutative since $\mathrm{pr}_2 \circ (p, \pi) = \pi$ as we have seen before. Hence the whole diagram is commutative.

To see that the second row of the diagram is exact, observe that the surjectivity of pr_2 follows from the surjectivity of $\widetilde{\pi}$. Furthermore, we see that an element $(a, \sigma) \in \widetilde{E} \times_{\widetilde{\Gamma}} \Gamma$ lies in the kernel of pr_2 if and only if $\sigma = 1$, which implies that $\widetilde{\pi}(a) = q(\sigma) = 1$, i.e. $a \in \ker \widetilde{\pi}$. Since the map $\ker \pi \to \ker \widetilde{\pi}$ in the diagram is an isomorphism, the (not necessarily commutative) five lemma can be applied to deduce that $(p, \pi) : E \to \widetilde{E} \times_{\widetilde{\Gamma}} \Gamma$, i.e. E is isomorphic to the fibre product $\widetilde{E} \times_{\widetilde{\Gamma}} \Gamma$ as desired. □

As mentioned before, we now come to the existence of a partial lifting in an extension of a profinite group by a finite group:

Theorem 4.5. *Let G be a finite group and*

$$1 \longrightarrow G \xrightarrow{\iota} E \xrightarrow{\pi} \Gamma \longrightarrow 1$$

be an extension of profinite groups. Then there exists an open normal subgroup $H \trianglelefteq \Gamma$ which can be lifted to an open normal subgroup of E and we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G & \xrightarrow{\iota} & E & \xrightarrow{\pi} & \Gamma & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & G & \longrightarrow & E/H & \longrightarrow & \Gamma/H & \longrightarrow & 1. \end{array}$$

If G is an abelian group, we obtain the statement above from the fact that if A is a discrete continuous Γ-module, then the cohomology group $H^2(\Gamma, A)$ is in a bijection to the equivalence classes of group extensions of Γ by A and that

$$H^2(\Gamma, A) \cong \varinjlim_{\substack{H \trianglelefteq \Gamma \\ \text{open}}} H^2(\Gamma/H, A^H),$$

compare [NSW08, Thm.1.2.4 and Prop.1.2.5]. In the case that G is not an abelian group, one can also construct a group extension of Γ by G from a given non-abelian 2-cocycle, see [EM47]. Here we will give a direct proof based on this idea.

Proof. Consider an extension of a profinite group Γ by a finite group G as above and let $s : \Gamma \to E, \sigma \mapsto \hat{\sigma}$ be a continuous section of $\pi : E \to \Gamma$ such that $s(1) = 1$, which exists as shown in [Ser02, §1.2 Prop.1]. Define the map $x : \Gamma \times \Gamma \to G$ by

$$x(\sigma, \tau) := \hat{\sigma}\hat{\tau}\widehat{\sigma\tau}^{-1} \quad \text{for each} \quad \sigma, \tau \in \Gamma.$$

Note that this is well-defined since for each $\sigma, \tau \in \Gamma$, we have $\pi(\hat{\sigma}\hat{\tau}\widehat{\sigma\tau}^{-1}) = \sigma\tau(\sigma\tau)^{-1} = 1$, i.e. $\hat{\sigma}\hat{\tau}\widehat{\sigma\tau}^{-1}$ lies in the kernel of π, i.e. in the image of G under ι, which can be identified with the group G itself by the injectivity of ι. Furthermore, this map is continuous by the continuity of s. Hence for each $\gamma := (\sigma, \tau) \in \Gamma \times \Gamma$, the set $x^{-1}(x(\gamma))$ is an open subset of $\Gamma \times \Gamma$. So there exists an open normal subgroup $H_\gamma \trianglelefteq \Gamma$ such that $\sigma H_\gamma \times \tau H_\gamma \subseteq x^{-1}(x(\gamma))$, or in other words, $x|_{\sigma H_\gamma \times \tau H_\gamma}$ is constant. We obtain an open covering

$$\Gamma \times \Gamma = \bigcup_{\gamma \in \Gamma \times \Gamma} \sigma H_\gamma \times \tau H_\gamma.$$

By the compactness of Γ, there exist $\gamma_1, \ldots, \gamma_n \in \Gamma \times \Gamma$ such that

$$\Gamma \times \Gamma = \bigcup_{j=1}^{n} \sigma_j H_j \times \tau_j H_j, \tag{4.2}$$

where $\gamma_j = (\sigma_j, \tau_j)$ and $H_j := H_{\gamma_j}$ for all $j = 1, \ldots, n$.

We turn to consider the following mapping

$$\Gamma \times G \longrightarrow G, (\sigma, g) \mapsto {}^{\sigma}g := \hat{\sigma} g \hat{\sigma}^{-1}.$$

This mapping is also continuous and well-defined, since for each $\sigma \in \Gamma$ and $g \in G$, we have $\pi({}^{\sigma}g) = \sigma \cdot 1 \cdot \sigma^{-1} = 1$. Note that this need *not* be a group action of Γ on G since the group G is not abelian in general. However, it induces a continuous map

$$f : \Gamma \to \mathrm{Aut}(G), \sigma \mapsto \{f_\sigma : g \mapsto {}^{\sigma}g\},$$

where the set $\mathrm{Aut}(G)$ is considered with the discrete topology. Hence for each $\sigma \in \Gamma$, the set $f^{-1}(f_\sigma) \subseteq \Gamma$ is open, i.e. there exists an open normal subgroup $H'_\sigma \trianglelefteq \Gamma$ such that $\sigma H'_\sigma \subseteq f^{-1}(f_\sigma)$. We obtain an open covering

$$\Gamma = \bigcup_{\sigma \in \Gamma} \sigma H'_\sigma.$$

Again by the compactness of Γ, there exist $\sigma_1, \dots, \sigma_m \in \Gamma$ such that

$$\Gamma = \bigcup_{l=1}^{m} \sigma_l H'_l, \tag{4.3}$$

where $H'_l := H'_{\sigma_l}$ for all $l = 1, \dots, m$. It follows that $H := \left(\bigcap_{j=1}^{n} H_j\right) \cap \left(\bigcap_{l=1}^{m} H'_l\right)$, as a finite intersection of open normal subgroups, is an open normal subgroup of Γ.

We claim that this open normal subgroup H can be lifted to an open normal subgroup of E. For this purpose, consider the map $\overline{x} : \Gamma/H \times \Gamma/H \to G$ defined by

$$\overline{x}(\sigma H, \tau H) := x(\sigma, \tau) \quad \text{for each } \sigma, \tau \in \Gamma.$$

To see that $\overline{x}$ is well-defined, consider $\sigma, \tau \in \Gamma$ and $\rho_1, \rho_2 \in H$. By (4.2) there exists one $j \in \{1, \dots, n\}$ such that $(\sigma, \tau) \in \sigma_j H_j \times \tau_j H_j$, hence also $(\sigma\rho_1, \tau\rho_2)$ since $H \subseteq H_j$. But the map x is constant on $\sigma_j H_j \times \tau_j H_j$. Therefore $x(\sigma, \tau) = x(\sigma\rho_1, \tau\rho_2)$, i.e. $\overline{x}$ is well-defined.

Now we define the group $\widetilde{E}$ as the set $G \times (\Gamma/H)$ with the group law

$$(g, \sigma H)(h, \tau H) := (g\, {}^{\sigma}h \cdot \overline{x}(\sigma H, \tau H), \sigma\tau H) \quad \text{for each } g, h \in G \text{ and } \sigma, \tau \in \Gamma.$$

To see that this is well-defined, we only need to show that the term ${}^{\sigma}h$ is independent of the choice of the representative of σH. In fact, by (4.3) there exists an $l \in \{1, \dots, m\}$ such that $\sigma \in \sigma_l H'_l$. So if $\sigma' \in \sigma H$ is another representative, say $\sigma' = \sigma\rho$ for some $\rho \in H$, then σ' also lies in $\sigma_l H'_l$ since $H \subseteq H'_l$. But f is constant on $\sigma_l H'_l$, which implies that $f_\sigma = f_{\sigma'}$, i.e. ${}^{\sigma}h = {}^{\sigma'}h$. Therefore the operation on $\widetilde{E}$ as above is well-defined. Furthermore, by a direct computation, one sees that $\widetilde{E}$ with this operation is a group. Indeed, $(1, H) \in \widetilde{E}$ is a neutral element and an inverse element of $(g, \sigma H) \in \widetilde{E}$ is $(\hat{\sigma}^{-1} g^{-1} \widehat{\sigma^{-1}}^{-1}, \sigma^{-1} H)$.

We now construct a continuous surjective group homomorphism from E to $\widetilde{E}$ with a kernel that is isomorphic to H. This is given by

$$\varphi : E \to \widetilde{E}, \alpha \mapsto \varphi(\alpha) := (\alpha\hat{\sigma}^{-1}, \sigma H), \qquad \text{where } \sigma := \pi(\alpha) \in \Gamma.$$

Note that for each $\alpha \in E$ and $\sigma = \pi(\alpha)$, we have $\alpha\hat{\sigma}^{-1} \in G$ since $\pi(\alpha\hat{\sigma}^{-1}) = \pi(\alpha) \cdot \sigma^{-1} = 1$, i.e. $\alpha\hat{\sigma}^{-1} \in \ker \pi$. This map is a group homomorphism, since for each $\alpha, \beta \in E$ with $\sigma := \pi(\alpha)$ and $\tau := \pi(\beta)$, we have $\sigma\tau = \pi(\alpha\beta)$ and

$$\begin{aligned}\varphi(\alpha) \cdot \varphi(\beta) &= (\alpha\hat{\sigma}^{-1}, \sigma H) \cdot (\beta\hat{\tau}^{-1}, \tau H) = (\alpha\hat{\sigma}^{-1}\, {}^{\sigma}(\beta\hat{\tau}^{-1})\overline{x}(\sigma H, \tau H), \sigma\tau H) \\ &= (\alpha\hat{\sigma}^{-1}\hat{\sigma}\beta\hat{\tau}^{-1}\hat{\sigma}^{-1}\hat{\sigma}\hat{\tau}\widehat{\sigma\tau}^{-1}, \sigma\tau H) = (\alpha\beta\widehat{\sigma\tau}^{-1}, \sigma\tau H) = \varphi(\alpha\beta).\end{aligned}$$

To see that φ is surjective, let $\gamma \in \widetilde{E}$, say $\gamma = (g, \sigma H)$ for some $g \in G$ and $\sigma \in \Gamma$. Then we have $\pi(g\hat{\sigma}) = \pi(g)\pi(\hat{\sigma}) = \sigma$ and consequently

$$\varphi(g\hat{\sigma}) = (g\hat{\sigma}\hat{\sigma}^{-1}, \sigma H) = (g, \sigma H) = \gamma.$$

Furthermore, if $\tilde{\pi} : \widetilde{E} \to \Gamma/H, (g, \sigma H) \mapsto \sigma H$ denotes the projection of $\widetilde{E} = G \times (\Gamma/H)$ onto its second components (also note that this is actually a group homomorphism under the operation given above), then for each $\alpha \in E$ with $\sigma := \pi(\alpha) \in \Gamma$, we have

$$\tilde{\pi}(\varphi(\alpha)) = \tilde{\pi}(\alpha\hat{\sigma}^{-1}, \sigma H) = \sigma H.$$

It follows that the diagram

$$\begin{array}{ccccccccc}
 & & & & \ker\varphi & \dashrightarrow & H & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & G & \longrightarrow & E & \xrightarrow{\pi} & \Gamma & \longrightarrow & 1 \\
 & & \| & & \downarrow{\scriptstyle\varphi} & & \downarrow & & \\
1 & \longrightarrow & G & \underset{g \mapsto (g,H)}{\longrightarrow} & \widetilde{E} & \xrightarrow{\tilde{\pi}} & \Gamma/H & \longrightarrow & 1
\end{array}$$

is commutative. By the previous lemma, the restriction of π to $\ker\varphi$ induces an isomorphism from $\ker\varphi$ to H. Since φ is continuous and its image $\widetilde{E}$ is finite, its kernel is an open normal subgroup of E. This implies that H is isomorphic to an open normal subgroup of E and $\widetilde{E}$ is isomorphic to E/H as desired. □

4.4 The main result

Having derived some tools from profinite group theory in the previous section, we come now to our main result as follows:

Theorem 4.6. *Let k be a field and, G be a finite group, and let*

$$1 \longrightarrow G \xrightarrow{\iota} E \xrightarrow{\pi} \mathrm{Gal}_k \longrightarrow 1 \tag{4.4}$$

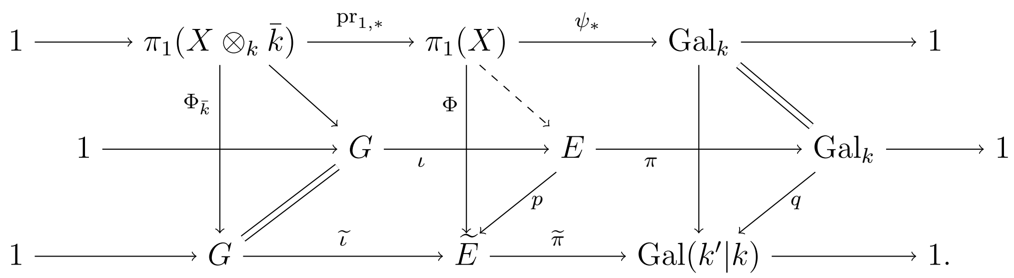
be an extension of profinite groups. Then there exists a geometrically connected smooth projective variety X over k such that $\pi_1(X) \cong E$ and the following diagram is commutative

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(X \otimes_k \bar{k}) & \longrightarrow & \pi_1(X) & \longrightarrow & \mathrm{Gal}_k & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \| & & \\
1 & \longrightarrow & G & \xrightarrow{\iota} & E & \xrightarrow{\pi} & \mathrm{Gal}_k & \longrightarrow & 1.
\end{array}$$

Proof. Using Theorem 4.5, we find an open normal subgroup $H \trianglelefteq \mathrm{Gal}_k$ which can be lifted to an open normal subgroup of E which will also be denoted by H. By the main theorem of Galois Theory for field extensions, there is a finite Galois extension $k'|k$ such that $H \cong \mathrm{Gal}_{k'}$ and $\Gamma/H \cong \mathrm{Gal}(k'|k)$. Furthermore, $\widetilde{E} := E/H$ is finite, and we obtain the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G & \xrightarrow{\iota} & E & \xrightarrow{\pi} & \mathrm{Gal}_k & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & G & \xrightarrow{\widetilde{\iota}} & \widetilde{E} & \xrightarrow{\widetilde{\pi}} & \mathrm{Gal}(k'|k) & \longrightarrow & 1. \end{array}$$

Consider the lower exact sequence. By Proposition 4.1, there exists a smooth and geometrically connected projective variety Y of dimension r which is a complete intersection in $\mathbb{P}^n_{k'}$, on which the group $\widetilde{E}$ acts admissibly, π-semilinearly and without fixed points, say $Y = V(g_1, \ldots, g_{n-r})$ for some $g_1, \ldots, g_{n-r} \in k'[T_0, \ldots, T_n]$. The scheme $X := Y/\widetilde{E}$ is the projective spectrum of a finitely generated k-algebra (namely $(k'[T_0, \ldots, T_n]/(g_1, \ldots, g_{n-r}))^{\widetilde{E}}$) and can thus be considered as a projective variety in $\mathbb{P}^s_k$ for some $s \in \mathbb{N}$. Now consider the following diagram



Here $\psi_* : \pi_1(X) \to \pi_1(\operatorname{Spec} k) = \mathrm{Gal}_k$ is the group homomorphism obtained by the structure morphism $\psi : X \to \operatorname{Spec} k$. By Proposition 4.2, $X \otimes_k \bar{k}$ is connected and the first and the third rows are compatible with the vertical arrows, which implies that the whole diagram up to the dashed arrow is commutative.

In order to obtain an isomorphism between the group $\pi_1(X)$ and E, we determine first a group homomorphism between them. Since $\widetilde{\pi} \circ \Phi = q \circ \psi_*$ and (E, p, π) is the fibre product of $\widetilde{E} \xrightarrow{\widetilde{\pi}} \mathrm{Gal}(k'|k)$ and $\mathrm{Gal}_k \xrightarrow{q} \mathrm{Gal}(k'|k)$ by Lemma 4.4, there exists a unique profinite group homomorphism $\varphi : \pi_1(X) \to E$ such that $\Phi = p \circ \varphi$ and $\psi_* = \pi \circ \varphi$. The latter implies that the upper right parallelogram is commutative, and we are going to show that the left one also commutes, i.e. that $\varphi \circ \mathrm{pr}_{1,*} = \iota \circ \Phi_{\bar{k}}$.

Since p and π are the canonical projection of E onto its factors $\widetilde{E}$ and $\mathrm{Gal}(k'|k)$, it suffices to show that the compositions of both homomorphisms above with p coincide and the same holds for π instead of p. This can be done as follows:

$$p \circ \varphi \circ \mathrm{pr}_{1,*} = \Phi \circ \mathrm{pr}_{1,*} = \widetilde{\iota} \circ \Phi_{\bar{k}} = p \circ \iota \circ \Phi_{\bar{k}}.$$

On the other hand, $\pi \circ \varphi \circ \mathrm{pr}_{1,*} = \psi_* \circ \mathrm{pr}_{1,*}$ is the trivial map as well as $\pi \circ \iota \circ \Phi_{\bar{k}}$ (since $\pi \circ \iota$ is trivial), i.e. $\pi \circ \varphi \circ \mathrm{pr}_{1,*} = \pi \circ \iota \circ \Phi_{\bar{k}}$. It follows that $\varphi \circ \mathrm{pr}_{1,*} = \iota \circ \Phi_{\bar{k}}$, i.e. the upper left parallelogram is commutative. Hence the first two rows of the diagram above are compatible with the vertical arrows. In other words, the whole diagram above is commutative.

Now consider the scheme $X \otimes_k \bar{k}$, which is isomorphic to $\overline{Y}/G$, where $\overline{Y} := Y \otimes_{k'} \bar{k}$, by Proposition 4.2. Since G acts on $\overline{Y}$ without fixed points, the quotient map $\overline{Y} \to X \otimes_k \bar{k}$ is étale, which implies that $X \otimes_k \bar{k}$ is also regular, compare Corollary 3.6, i.e. X is a smooth variety over k.

Moreover, since $\overline{Y}$ is a complete intersection in $\mathbb{P}^n_{\bar{k}}$ as shown in Proposition 4.1, we have $\pi_1(\overline{Y}) = 1$ and therefore $\Phi_{\bar{k}} : \pi_1(X \otimes_k \bar{k}) \to G$ is an isomorphism. Consider again the first two lines of the above diagram. We see that the left and right arrows are isomorphisms. Therefore $\varphi : \pi_1(X) \to E$ is also an isomorphism by the (not necessarily commutative) five lemma and we are done. □

From this theorem, we see that the question posed in the introduction is answered positively: There exists a geometrically integral and smooth projective variety over a field k such that the fundamental group is a group extension of the absolute Galois group Gal_k by a given finite group. Its base extension from k to the algebraic closure $\bar{k}$ is the Godeaux-Serre variety. Therefore this variety over k may be considered a k-form of a Godeaux-Serre variety with prescribed arithmetic fundamental group.

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Erklärung

Hiermit versichere ich, dass ich diese Arbeit selbstständig unter Anleitung verfasst habe, dass ich keine anderen als die angegebenen Quellen benutzt habe, und dass ich alle Stellen, die dem Wortlaut oder dem Sinne nach anderen Werken entlehnt sind, durch die Angabe der Quellen als Entlehnungen kenntlich gemacht habe.

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