Quaternionic Arithmetic Lattices of Rank 2 and a Fake Quadric in Characteristic 2

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Introduction

In this thesis we construct a torsion-free quaternionic lattice of rank 2 over $\mathbb{F}_2(z)$ which yields a square complex with four vertices as quotient under its action on the product of two trees. It turns out that 4 is the minimal possible number of vertices of a quotient square complex obtained by this way over a global field of characteristic 2. The lattice we constructed is a normal subgroup of index 4 of another lattice acting on the product of two trees simply transitively on the vertices. Moreover, it can be used in the construction of a fake quadric. The problem of constructing such a lattice can be viewed from the following different aspects.

We shall denote in what follows the tree with constant valency $n \in \mathbb{N}$ by T_n. In geometric group theory, one considers a cocompact lattice $\Gamma \leq \mathrm{Aut}(T_m) \times \mathrm{Aut}(T_n)$, i.e. a discrete subgroup such that the induced action on $T_m \times T_n$ yields a compact quotient. Conversely, one may start with a square complex with product of two trees as universal covering. The associated lattice is then determined by the fundamental group of the square complex. Such a lattice admits a finite presentation by Seifert-van Kampen theorem. However, it rarely has an arithmetic origin in the sense that Γ can be considered an arithmetic subgroup of a linear group.

In arithmetic theory of algebraic groups, the question of explicitly determining a presentation of an arithmetic group is an old one. Such a presentation for a lattice in a Lie group can be determined in general by finding first a Dirichlet fundamental domain. A necessary and sufficient condition on the global and local rank for an arithmetic group to be finitely generated resp. finitely presented has been given by Behr in [Beh98]. Several explicit results are known in the rank 1 case. For rank at least 2, however, very few (explicit) results are known so far, so that it is already interesting to study first those lattices with small fundamental domain.

In the case of quaternionic arithmetic lattices of rank 2, we may consider the following situation: Let K be a global field, Q be a quaternion division algebra over K and $S \subseteq \mathbb{P}_K$ consist of all ramified places and two finite unramified places $\mathfrak{p}, \mathfrak{p}'$. Then an S-arithmetic subgroup Γ of the projective linear group

$$G = \mathrm{PGL}_{1,Q} = \mathrm{GL}_{1,Q} / \mathbb{G}_m$$

acts as a cocompact lattice on the product of two trees $T_{q+1} \times T_{q'+1}$, where q and q' are the norms of $\mathfrak{p}$ and $\mathfrak{p}'$ respectively. Here the two unramified places $\mathfrak{p}, \mathfrak{p}'$ yield the rank 2 condition of Γ, and the action of Γ is given by the diagonal embedding

$$\Gamma \hookrightarrow G(K_{\mathfrak{p}}) \times G(K_{\mathfrak{p}'}) \cong \mathrm{PGL}_2(K_{\mathfrak{p}}) \times \mathrm{PGL}_2(K_{\mathfrak{p}'}),$$

where $\mathrm{PGL}_2(K_{\mathfrak{p}}) \times \mathrm{PGL}_2(K_{\mathfrak{p}'})$ acts component-wise on $T_{q+1} \times T_{q'+1}$ via the Bruhat-Tits action. Although there are plenty of such lattices, only few of them are torsion-free and yield a small quotient under their action on $T_{q+1} \times T_{q'+1}$. The best case would be when the action is simply transitive on the vertices. This leads us to the question to find such a lattice in a quaternion algebra over a global field, or even whether it is possible to do so.

The question from the previous paragraph has an affirmative answer in the following cases: For $K = \mathbb{Q}$, we consider two places defined by two odd primes $p \neq \ell$. It turns out that there exists a quaternionic lattice of $\mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_\ell)$ acting on $T_{p+1} \times T_{\ell+1}$ simply transitively on the vertices. Such a lattice was first found by Mozes in [Moz95] for $p \equiv \ell \equiv 1 \bmod 4$. The local permutation structure in the sense of [SV13, §5.1] of this lattice was used in the construction of Ramanujan graphs in [LPS88]. The result was later generalised to arbitrary odd primes $p \neq \ell$ by Rattaggi in [Rat04, Ch.3]. In fact, this lattice is a $\{p, \ell\}$-arithmetic subgroup of the rational Hamiltonian quaternion group.

The case of global function fields has been rarely established so far. If K is a global function field of a smooth curve over a finite field $\mathbb{F}_q$, we might restrict to the case $\mathfrak{p}$ and $\mathfrak{p}'$ rational over $\mathbb{F}_q$, so that Γ acts on the product $T_{q+1} \times T_{q+1}$ of two trees with the same valency. For odd prime powers q, such lattices were found by Stix and Vdovina in [SV13], while its local permutation structure in the sense of [SV13, §5.1] had already been used in the construction of Ramanujan graphs in [Mor94].

In the case of characteristic 2, however, it is not possible to find a torsion-free lattice with a simply transitive action on the vertices of the product of two trees as above. In fact, if N is the number of vertices of the quotient square complex $\Sigma_\Gamma := \Gamma\backslash T_{q+1} \times T_{q+1}$, then the condition of Σ_Γ to be complete implies that each vertex is attached to exactly $(q+1)^2$ square corners. Summing up together and observing that each square has four square corners, we see that the number of squares in this complex is

$$\#\mathbb{S}(\Sigma_\Gamma) = \frac{1}{4}N(q+1)^2.$$

Since $q+1$ is odd, N must be at least 4, i.e. the action of Γ on the vertices of $T_{q+1} \times T_{q+1}$ must have at least four orbits and hence cannot be transitive. Nevertheless this leads to another question to find a torsion-free quaternionic lattice in characteristic 2 which yields a quotient square complex with four vertices, i.e. with smallest possible number of vertices.

The question of finding a torsion-free quaternionic lattice in characteristic 2 yielding a quotient square complex with four vertices also arises from algebraic geometry, namely the construction of a fake quadric by means of non-archimedean uniformisation. Based on Mumford's construction in the 1-dimensional case in [Mum72], this technique was developed independently by Kurihara in [Kur78] and Mustafin in [Mus78]. Later, in [Mum79], Mumford employed this technique to construct a fake projective plane, a minimal surface of general type X with $c_1(X)^2 = 9$, $c_2(X) = 3$ and trivial Albanese variety. In the case of characteristic 0, these numerical conditions are equivalent to its Betti numbers coinciding with those of the projective plane. The construction is based on a torsion-free lattice acting simply transitively on the vertices of the Bruhat-Tits building of $\mathrm{PGL}_3(\mathbb{Q}_2)$ with a quotient complex of Euler characteristic 1.

The notion of fake quadrics is motivated by the notion of fake projective planes. A fake quadric is a minimal surface of general type X such that $c_1(X)^2 = 8$, $c_2(X) = 4$ and the Albanese variety is trivial. Again, these numerical conditions are equivalent to its Betti numbers coinciding with those of a quadric in the case of characteristic 0. Furthermore, the conditions on the Chern numbers imply that the Euler characteristic is

$$\chi(X) = \frac{1}{12}(c_1(X)^2 + c_2(X)) = 1.$$

Fake quadrics in characteristic 0 have been so far studied by Kuga and Shavel [Sha78], Bauer, Catanese and Grunewald [BCG08], Frapporti [Fra13], Džambić [Dža14] as well as Linowitz, Stover and Voight [LSV15].

Fake quadrics in positive characteristic have first been studied by Stix and Vdovina in [SV13]. The construction is explained in §7 of loc.cit. as well as in Chapter 5 of this thesis. It is based on a torsion-free lattice acting on the product of two copies of the Bruhat-Tits tree for $\mathrm{PGL}_2(\mathbb{F}_q((t)))$ for some prime power q. It turns out that if the quotient square complex under this action has N vertices, then the Euler characteristic of the resulting surface X_Γ is

$$\chi(X_\Gamma) = \frac{1}{4}N(q-1)^2.$$

It follows that $\chi(X_\Gamma) = 1$ if and only if $N = 1$, $q = 3$ or $N = 4$, $q = 2$. In particular, this construction can yield a fake quadric only for $q = 2, 3$. The case $q = 3$ has been established by Stix and Vdovina in [SV13], where a class of algebraic surfaces of general type with Chern ratio $c_1^2/c_2 = 2$, trivial Albanese variety and non-reduced Picard variety for any odd prime power q is constructed. This yields for $q = 3$ a fake quadric. The case $q = 2$ leads to the question of finding a torsion-free lattice which yields a quotient square complex with four vertices under the Bruhat-Tits action as discussed above.

One of the main results of this thesis is that there really exists a torsion-free arithmetic lattice Γ in a quaternion algebra of a global function field over $\mathbb{F}_2$ (i.e. $q = 2$) with the smallest possible number of orbits, i.e. that yields a quotient square complex with four vertices. The lattice we have found is contained in another arithmetic lattice Λ acting simply transitively on the vertices of $T_3 \times T_3$. In fact, Γ is a normal subgroup of Λ with the Klein four-group $V_4 = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ as quotient. This lattice is constructed as follows: We set $K := \mathbb{F}_2(z)$ and define the following quaternion algebra over K:

$$Q := \left[\frac{z, 1+z^3}{K}\right) = K\{I, J\}/\{I^2 + I = z,\ J^2 = 1 + z^3,\ JI = (I+1)J\}.$$

The basis $1, I, J, IJ$ then defines an integral structure on the algebraic group $G = \mathrm{PGL}_{1,Q}$ over the following rings:

$$R_0 := \mathbb{F}_2[z, \tfrac{1}{z}], \quad R_1 := \mathbb{F}_2[z, \tfrac{1}{z(1+z)}] \quad \text{and} \quad R := \mathbb{F}_2[z, \tfrac{1}{z(1+z^3)}].$$

It turns out that $G(R)$ has a subgroup $\Lambda := \langle b_1, b_2, c_1, c_2 \rangle$ with the presentation

$$\Lambda = \left\langle b_1, b_2, c_1, c_2 \;\middle|\; c_1^2,\ c_2^2,\ c_1c_2 = c_2c_1,\ b_1b_2c_1b_2,\ b_1c_2b_1b_2^{-1} \right\rangle,$$

see Proposition 4.26. In fact, the generators have the following images under the embeddings ρ_y in $\mathrm{PGL}_2(\mathbb{F}_2((y)))$ from Lemma 1.33 and ρ_t in $\mathrm{PGL}_2(\mathbb{F}_2((t)))$ from Lemma 1.34:

	ρ_y	ρ_t
b_1	$\begin{pmatrix} (1+z)y & 1+z^3 \\ 1 & (1+z)(1+y) \end{pmatrix}$	$\begin{pmatrix} (1+u)(1+u+t) & u+u^4 \\ 1 & (1+u)(u+t) \end{pmatrix}$
b_2	$\begin{pmatrix} z+z^2+(1+z)y & (1+z^3)(1+y) \\ y & 1+z^2+(1+z)y \end{pmatrix}$	$\begin{pmatrix} (1+u)t & (1+t)(1+u^3) \\ t/u & (1+u)(1+t) \end{pmatrix}$
c_1	$\begin{pmatrix} 1+z^2 & (1+z^3)y \\ 1+y & 1+z^2 \end{pmatrix}$	$\begin{pmatrix} u+u^2 & (1+u^3)(1+u+t) \\ (u+t)/u & u+u^2 \end{pmatrix}$
c_2	$\begin{pmatrix} z+z^2 & (1+z^3)y \\ 1+y & z+z^2 \end{pmatrix}$	$\begin{pmatrix} 1+u^2 & (1+u^3)(1+u+t) \\ (u+t)/u & 1+u^2 \end{pmatrix}$

Here $u = z^{-1}$ and y, t are such that $y^2 + y = z$ and $t^2 + t = u$. Now let w denote the standard vertex of the product of the Bruhat-Tits trees for $\mathrm{PGL}_2(\mathbb{F}_2((y)))$ and $\mathrm{PGL}_2(\mathbb{F}_2((t)))$, i.e. its components are given by the lattices $\mathbb{F}_2[\![y]\!]^2 \subseteq \mathbb{F}_2(\!(y)\!)^2$ and $\mathbb{F}_2[\![t]\!]^2 \subseteq \mathbb{F}_2(\!(t)\!)^2$ respectively. Then we see that under the Bruhat-Tits action, w is sent by b_1, b_1^{-1}, c_1 to its vertical and by b_2, b_2^{-1}, c_2 to its horizontal neighbours. This leads to a generalisation of a VH-structure in the sense of [SV13, §2.2, Def.4]. We call this a **V_4-equivariant vertical-horizontal structure**, in short **V_4-structure**.

In analogy to [SV13, §2.2.1], a V_4-structure gives rise to a square complex Σ with four vertices and a V_4-action. A consequence is that Λ is isomorphic to the orbital fundamental group of Σ by V_4, so that Λ has a presentation as above. Furthermore, we can show that $G(R_1) = \Lambda$ and the maximal arithmetic lattice $G(R)$ has the following presentation:

$$G(R) = \left\langle b_1, b_2, c_1, c_2, d \;\middle|\; \begin{array}{c} c_1^2,\ c_2^2,\ d^2,\ c_1c_2 = c_2c_1,\ c_1d = dc_1,\ c_2d = dc_2, \\ b_1b_2c_1b_2,\ b_1c_2b_1b_2^{-1},\ db_1db_1,\ db_2db_2 \end{array} \right\rangle,$$

see Theorem 4.27. The subgroup

$$\Gamma := \ker(\Lambda \cong \pi_1^{\mathrm{orb}}(\Sigma; V_4) \twoheadrightarrow V_4)$$

is then a normal subgroup of index 4 in Λ corresponding to the fundamental group $\pi_1(\Sigma)$. The quotient $\Gamma\backslash(T_3 \times T_3)$ is then a square complex with four vertices as desired.

A consequence is that we can use this Γ to construct an algebraic surface X_Γ of general type over $\mathbb{F}_2(\!(t)\!)$ by means of non-archimedean uniformisation as mentioned before. The fact that the resulting quotient square complex has four vertices implies that this algebraic surface has the Chern numbers

$$c_1(X_\Gamma)^2 = 8 \quad \text{and} \quad c_2(X_\Gamma) = 4.$$

Its Albanese variety can be computed via Kummer étale cohomology using the local permutation groups in the sense of §3.6, which is a modification of the local permutation groups in the sense of [SV13, §5.1]. In the end, the surface we have constructed is indeed a fake quadric over $\mathbb{F}_2(\!(t)\!)$ as will be proved in Theorem 5.19. However, it still remains an open question whether this fake quadric has reduced Picard scheme or not.

Outline

The basic facts about quaternion algebras, especially in characteristic 2, are treated in the first chapter. The introduced notion will be illustrated in the last section via the example of the quaternion algebra $Q = \left[\frac{z, 1+z^3}{\mathbb{F}_2(z)}\right)$, whose arithmetic lattices will be established later.

Chapter 2 aims to give basic facts about the fundamental group of a global orbispace, i.e. an orbispace arising from a group action on a topological space. The orbital fibre of a covering space is introduced as a generalisation of a fibre of a covering space. Further properties of a covering space and its deck transformation group are generalised to the global orbispace context.

In Chapter 3, we introduce the notion of a V_4-structure and construct an associated square complex with four vertices. A consequence is that its orbital fundamental group by V_4 can be described easily from the given V_4-structure. Furthermore, we prove the Comparison Theorem 3.17, which will later serve to identify the orbital fundamental group $\pi_1^{\mathrm{orb}}(\Sigma_{\mathcal{A},\mathcal{B}})$ as a subgroup of $G(R)$. The last section is concerned with the local permutation groups from a given V_4-structure.

Chapter 4 is devoted to determining presentations of arithmetic lattices by means of the Bruhat-Tits action. The main idea is to determine the stabiliser of a vertex and use the subgroup obtained from the Comparison Theorem in the previous chapter. We also give topological interpretations for the lattices $G(R_0)$, $G(R_1)$ and $G(R)$ as orbital fundamental group of $\Sigma_{\mathcal{A},\mathcal{B}}$ by appropriate group actions and identify the fundamental group $\pi_1(\Sigma_{\mathcal{A},\mathcal{B}})$ as a subgroup of $G(R_0)$ of finite index, hence a lattice in $\mathrm{PGL}_2(\mathbb{F}_2(\!(y)\!)) \times \mathrm{PGL}_2(\mathbb{F}_2(\!(t)\!))$.

Beginning with the definition and some historical remarks of fake quadrics, we discuss the construction of a fake quadric by means of non-archimedean uniformisation in Chapter 5. For this we construct a formal scheme over $\mathrm{Spf}(\mathbb{F}_2[\![t]\!])$ whose generic fibre is the product of two copies of the Drinfeld upper half plane over $\mathbb{F}_2(\!(t)\!)$ and whose special fibre has the dual complex isomorphic to the product of two Bruhat-Tits trees. The quotient scheme of this formal scheme by the lattice from Chapter 4 corresponding to $\pi_1(\Sigma_{\mathcal{A},\mathcal{B}})$ can then be algebraised to a projective scheme over $\mathbb{F}_2[\![t]\!]$. In this way we obtain a smooth projective surface over $\mathbb{F}_2(\!(t)\!)$. Numerical invariants are then computed explicitly to confirm that this surface is indeed a fake quadric.

This thesis could have been complete in its own by these five chapters. However, one might ask how to find such a quaternion algebra. This is in fact not easy and will be discussed heuristically in Appendix A. There we will begin with finding a suitable square complex Σ, then a quaternion algebra Q over a global function field K over $\mathbb{F}_2$ as well as an embedding $\Sigma \hookrightarrow Q^\times/K^\times$. By this heuristic we obtain candidates for a square complex and a quaternion algebra we are looking for. This is then to be verified for example by the method introduced in the main part of the thesis.

Notation and Terminology

By a **lattice** in a locally compact group G, we mean a discrete subgroup Γ such that the quotient $\Gamma\backslash G$ has finite volume with respect to the Haar measure induced by G.

For a field K, we denote its separable closure by K_s, its absolute Galois group by $G_K = \mathrm{Gal}(K_s/K)$ and its algebraic closure by K^{alg}. If K is a global field, the set of its places will be denoted by $\mathbb{P}_K$. For $S \subseteq \mathbb{P}_K$, we denote the ring of S-integers by $O_{K,S}$. The normalised valuation on K with respect to the place $\mathfrak{p} \in \mathbb{P}_K$ is denoted by $\nu_\mathfrak{p}$

In special case $K = \mathbb{F}_2(z)$, we denote the completion of K at the place $\{z = x\}$, where $x \in \mathbb{P}^1_{\mathbb{F}_2}$, by K_x, and its normalised valuation by ν_x. The Bruhat-Tits tree for $\mathrm{PGL}_2(K_x)$ will be denoted by T_x and its standard vertex by w_x. Note that for $n \in \mathbb{N}$, we also denote the infinite tree with constant valency n by T_n, but it should be clear from the context whether the index (x or n) stands for a point in $\mathbb{P}^1_{\mathbb{F}_2}$ or a natural number.

For the explicit computation with the quaternion algebra $Q = \big[\frac{z,1+z^3}{\mathbb{F}_2(z)}\big)$, we use the following notation:

K	the global function field $\mathbb{F}_2(z)$ of the projective curve $\mathbb{P}^1_{\mathbb{F}_2}$
Q	the quaternion algebra $\big[\frac{z,1+z^3}{\mathbb{F}_2(z)}\big)$
I, J	generators of Q with $I^2 = I + z$, $J^2 = 1 + z^3$ and $JI = (I+1)J$
G	the projective linear group $\mathrm{PGL}_{1,Q} = \mathrm{GL}_{1,Q}/\mathbb{G}_m$ over K
R_0, R_1, R	the rings $\mathbb{F}_2[z, \frac{1}{z}]$, $\mathbb{F}_2[z, \frac{1}{z+z^2}]$ and $\mathbb{F}_2[z, \frac{1}{z+z^4}]$ respectively
$\zeta \in \mathbb{F}_2^{\mathrm{alg}}$	a primitive third root of unity
ρ_y, ρ_t	the embeddings from Lemma 1.33 and 1.34 respectively
Z_2	the group $\mathbb{Z}/2\mathbb{Z}$
γ_v, γ_h	commuting generators of $V_4 \cong Z_2 \times Z_2$ of order 2
$\mathcal{A}, \mathcal{B}$	subsets of $G(R_1)$ consisting of b_1, b_1^{-1}, c_1 resp. b_2, b_2^{-1}, c_2, compare Def. 4.18
d	explicit element in $G(R)$ from Proposition 4.14

A **square complex** is a two-dimensional combinatorial cell complex such that each 2-cell is attached to a path of length 4 beginning and ending at the same point. If Σ is a square complex, we shall denote the set of its vertices, edges and squares by $\mathbb{V}(\Sigma)$, $\mathbb{E}(\Sigma)$ and $\mathbb{S}(\Sigma)$ respectively.

A square complex Σ is said to have a **vertical-horizontal structure**, in short **VH-structure**, if there is a partition $\mathbb{E}(\Sigma) = \mathbb{E}(\Sigma)_v \sqcup \mathbb{E}(\Sigma)_h$ of the edges into the vertical and horizontal ones, such that each path attached to a square alternates between vertical and horizontal edges.

Furthermore, for a given V_4-structure $(\mathcal{A}, \mathcal{B})$, we use the following notation:

$\Sigma_{\mathcal{A},\mathcal{B}}$	the square complex with four vertices associated to $(\mathcal{A}, \mathcal{B})$
$\mathrm{Sym}(X)$	the permutation group of the set X
$G \wr \mathrm{Sym}(I)$	the wreath product $G^I \rtimes \mathrm{Sym}(I)$
$\widehat{\cdot}$	the transposition between 0 and 1
$t^j_{(a,i)}, t^i_{(b,j)}$	the bijection between the squares attached to a vertical/horizontal edge and the horizontal/vertical edges attached to a vertex, see Def. 3.21
$P^j_{\mathcal{A}}, P^i_{\mathcal{B}}$	the local permutation groups associated to $(\mathcal{A}, \mathcal{B})$.
$\tau^{\mathcal{A}}, \tau^{\mathcal{B}}$	the maps sending each $a \in \mathcal{A}$ resp. $b \in \mathcal{B}$ to its inverse

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Chapter 1

Quaternion algebras in characteristic 2

In this chapter, we shall review basic facts about quaternion algebras over fields of characteristic 2 and their arithmetic, provided that the field we are working on is a global or local field. A concrete example, which will play a prominent role in this thesis, will be introduced in the last section.

1.1 Generalities

We shall begin this section by recalling the general definition of a quaternion algebra. Throughout this section, K will denote a field, K_s its separable closure, and $G_K := \mathrm{Gal}(K_s/K)$ its absolute Galois group.

Definition 1.1. A **quaternion algebra** over a field K is a central simple K-algebra Q which has dimension 4 as a vector space over K.

Note that by Wedderburn's theorem, every quaternion algebra over K is either a division algebra or isomorphic to the matrix algebra $\mathrm{M}_2(K)$.

Lemma 1.2. *Let L/K be a Galois extension of degree 2 contained in K_s and $\bar{\cdot} \in \mathrm{Gal}(L/K)$ the non-trivial K-automorphism of L. Define the character $\psi = \psi_L \in H^1(G_K, \mathbb{Q}/\mathbb{Z})$ by*

$$\psi_L : G_K \to \mathbb{Q}/\mathbb{Z},\ s \mapsto \begin{cases} 0 & \text{if } s|_L = \mathrm{id}_L, \\ 1/2 & \text{if } s|_L = \bar{\cdot}. \end{cases}$$

Consider the connecting homomorphism $H^1(G_K, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\delta} H^2(G_K, \mathbb{Z})$ obtained by the exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$. Then for each $b \in K^\times = H^0(G_K, K_s^\times)$, the element $\delta\psi \cup b \in H^2(G_K, K_s^\times)$ can be represented by the quaternion algebra (L, b) defined by

$$(L, b) := L \oplus Lu \ \text{ as } K\text{-vector space}, \ \ ux = \bar{x}u \ \text{ for all } x \in L, \text{ and } \ u^2 = b.$$

Proof. Let $G := \mathrm{Gal}(L/K)$. Since this can be considered as a factor group of G_K by $\mathrm{Gal}(K_s/L)$, by the definition of ψ, it is the inflation of $\tilde{\psi} \in H^1(G, \mathbb{Q}/\mathbb{Z})$ given by $\tilde{\psi} : G \to \mathbb{Q}/\mathbb{Z},\ \bar{\cdot} \mapsto 1/2$. Then under the connecting homomorphism $H^1(G, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\delta} H^2(G, \mathbb{Z})$, it is mapped to

$$\delta\tilde{\psi} : G \times G \to \mathbb{Z},\ (\tau, \rho) \mapsto \begin{cases} 1 & \text{if } \tau = \rho = \bar{\cdot}, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the cup product $\delta\tilde{\psi} \cup b \in H^2(G, L^\times)$ maps $(\tau, \rho) \in G \times G$ to $b \in K^\times$ if $\tau = \rho = \bar{\cdot}$ and 1 otherwise. This can then be represented by the quaternion algebra (L, b) which has $L \oplus Lu$ as the underlying K-vector space and whose multiplication rule is given by $ux = \overline{x}u$ for all $x \in L$ and $u^2 = b$. Hence we are done since $\delta\tilde{\psi} \cup b \in H^2(G, L^\times)$ inflates to $\delta\psi \cup b \in H^2(G_K, K_s^\times)$. □

The quaternion algebra (L, b) obtained from the lemma above can also be embedded in the matrix algebra over L as follows:

Lemma 1.3. *Let L/K be a Galois extension of degree 2 with the non-trivial K-automorphism $\bar{\cdot} \in \mathrm{Gal}(L/K)$ and $b \in K^\times$, then (L, b) has the following injective 2-dimensional representation over L:*

$$\rho : (L, b) \longrightarrow \mathrm{M}_2(L),\ x + yu \longmapsto \begin{pmatrix} x & by \\ \overline{y} & \overline{x} \end{pmatrix}.$$

Proof. It is easy to see that ρ is an injective K-linear mapping. To show that ρ is multiplicative, consider the following calculation for $x_1, x_2, y_1, y_2 \in L$:

$$\begin{aligned} \rho(x_1 + y_1 u)\rho(x_2 + y_2 u) &= \begin{pmatrix} x_1 & by_1 \\ \overline{y_1} & \overline{x_1} \end{pmatrix} \begin{pmatrix} x_2 & by_2 \\ \overline{y_2} & \overline{x_2} \end{pmatrix} = \begin{pmatrix} x_1x_2 + by_1\overline{y_2} & b(x_1y_2 + \overline{x_2}y_1) \\ \overline{x_1y_2 + x_2\overline{y_1}} & \overline{x_1x_2 + b\overline{y_1}y_2} \end{pmatrix} \\ &= \rho((x_1x_2 + by_1\overline{y_2}) + (x_1y_2 + x_2\overline{y_1})u) = \rho((x_1 + y_1u)(x_2 + y_2u)) \end{aligned}$$

This shows that ρ is an injective representation of (L, b) in $\mathrm{M}_2(L)$ as desired. □

From now on to the end of this section, we shall assume that K has characteristic 2, and introduce the following notations:

Notation 1.4. Let $\wp : K_s \to K_s$ be the additive homomorphism on $(K_s, +)$ given by

$$\wp(x) := x^2 + x \quad \text{for all} \quad x \in K_s.$$

Furthermore, for each $a \in K$, define the character $\psi_a \in H^1(G_K, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(G_K, \mathbb{Q}/\mathbb{Z})$ by

$$\psi_a : G_K \to \mathbb{Q}/\mathbb{Z},\ s \mapsto \frac{1}{2}(s(\alpha) - \alpha), \quad \text{where} \quad \alpha \in \wp^{-1}(a).$$

Note that this is well-defined since for $\alpha, \alpha' \in K_s$, we have $\wp(\alpha) = \wp(\alpha')$ if and only if $\alpha = \alpha'$ or $\alpha = \alpha' + 1$. Furthermore ψ_a is bilinear since $K \to H^1(G_K, \mathbb{Q}/\mathbb{Z}),\ a \to \psi_a$ is a homomorphism.

Using these notations, we can describe the structure of a quaternion algebra over K more precisely as follows:

Definition 1.5. We define the pairing $[\cdot, \cdot) = [\cdot, \cdot)_K : K \times K^\times \to \mathrm{Br}_K$ by the following composition:

$$\begin{array}{ccccccc} K \times K^\times & \xrightarrow{\psi_{\cdot} \times \mathrm{id}} & H^1(G_K, \mathbb{Q}/\mathbb{Z}) \times K^\times & \xrightarrow{\delta \times \mathrm{id}} & H^2(G_K, \mathbb{Z}) \times K^\times & \xrightarrow{- \cup -} & H^2(G_K, K^\times) = \mathrm{Br}_K, \\ (a, b) & \longmapsto & (\psi_a, b) & \longmapsto & (\delta\psi_a, b) & \longmapsto & \delta\psi_a \cup b. \end{array}$$

Proposition 1.6. *Suppose that $a \in K$ and $b \in K^\times$. Then $[a, b) \in H^2(G_K, K_s^\times)$ defined above can be represented by*

$$\left[\frac{a, b}{K}\right) := K\{I, J\}/(I^2 + I = a,\ J^2 = b,\ IJ = J(I + 1)).$$

Proof. We first fix an element $\alpha \in \wp^{-1}(a)$ and define the following K-algebra homomorphism:

$$\varphi : \left[\frac{a,b}{K}\right) \to \mathrm{M}_2(K(\alpha)), \quad \varphi(I) := \begin{pmatrix} \alpha & 0 \\ 0 & \alpha+1 \end{pmatrix} \text{ and } \varphi(J) := \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}.$$

This is injective since $1, I, J, IJ$ generate $\left[\frac{a,b}{K}\right)$ and their images in $\mathrm{M}_2(K(\alpha))$ are linearly independent. Depending on whether $a \in \wp(K)$, we proceed the proof to the end as follows:

- If $a \in \wp(K)$, then $K(\alpha) = K$ and φ is an injective linear map between vector spaces of dimension 4 over K. Hence φ is an isomorphism of K-algebras, i.e. $\left[\frac{a,b}{K}\right) \cong \mathrm{M}_2(K)$. On the other hand, $\psi_a : G_K \to \mathbb{Q}/\mathbb{Z}$ is also trivial, which implies that $[a,b) = 0 \in \mathrm{Br}_K$. Hence $[a,b)$ is indeed representable by the matrix algebra $\mathrm{M}_2(K)$.
- If $a \notin \wp(K)$, then $L_\alpha := K(\alpha)$ is a Galois extension of degree 2, and $\psi_a : G_K \to \mathbb{Q}/\mathbb{Z}$ does exactly the same as ψ in Lemma 1.2, so that $[a,b)$ is represented by the quaternion algebra (L_α, b) defined there. On the other hand, we have an injective representation $\rho : (L_\alpha, b) \to \mathrm{M}_2(L_\alpha)$ from Lemma 1.3, which has the same image as φ. This implies that $\left[\frac{a,b}{K}\right)$ and (L_α, b) are isomorphic.

Hence in the both cases, the K-algebra $\left[\frac{a,b}{K}\right)$ belongs to the class $[a,b) \in \mathrm{Br}_K$. □

Remark 1.7. For $a \in K$ and $b \in K^\times$, we also see from the proof above that $1, I, J, IJ$ form a basis of $\left[\frac{a,b}{K}\right)$ as K-vector space, and we have the following multiplication table:

	I	J	IJ
I	$a+I$	IJ	$aJ+IJ$
J	$J+IJ$	b	$b+bI$
IJ	aJ	bI	ab

Corollary 1.8. *For $a \in K$ and $b \in K^\times$, $\left[\frac{a,b}{K}\right)$ is a quaternion algebra.*

Now we consider specifically the case of a local field, i.e. $K = k(\!(t)\!)$, where k is a finite field with 2^n elements for some $n \in \mathbb{N}$. In this case we know that $\mathrm{Br}_K \cong \mathbb{Q}/\mathbb{Z}$. Since the class of a quaternion algebra in Br_K is a 2-torsion, a quaternion algebra over a local field is either the matrix algebra or the uniquely determined division algebra having dimension 4 as a vector space over K.

In order to determine whether the quaternion algebra $[a,b)$ is matrix algebra or divison algebra, recall first that if $\omega = f\,dt$ is a differential form of K, the coefficient of t^{-1} in f is called the **residue** of ω, which will be denoted by $\mathrm{res}(\omega)$. It can be shown that this is independent of the choice of the uniformiser t, see e.g. [Sti09, Prop.4.2.9].

Proposition 1.9. *Let $a \in K$ and $b \in K^\times$. Furthermore, let $c := \mathrm{res}\,\frac{a\,db}{b}$. Then*

(1) $[a,b) = [c,t)$ *in* Br_K.

(2) $[a,b) = 0$ *in* Br_K *if and only if* $\mathrm{tr}_{k/\mathbb{F}_2}(c) = 0$.

Proof. [Ser79, Ch.XIV, Prop.12,13,15] □

The following corollary will be useful later when we want to determine the ramified places of a quaternion algebra over a global field.

Corollary 1.10. *If $a \in k[\![t]\!]$ and $b \in k[\![t]\!]^\times$, then $[a,b) = 0$.*

Proof. By writing $db = b'\,dt$, we see that $ab' \in k[\![t]\!]$, then also $\frac{ab'}{b}$. Hence the coefficient of t^{-1} in $\frac{ab'}{b}$ is zero, which implies that $[a,b) = 0$. □

1.2 Orders and their discriminant

The goal of this section is to introduce the notion of an order over a Dedekind ring in a quaternion algebra over the fraction field. To compute the discriminant, one need the trace form, so that the notion of trace is needed. We shall begin this section with the definition of the conjugation, reduced trace and reduced norm.

Lemma–Definition 1.11. *Let Q be a quaternion algebra over a field K. Then there exists a uniquely determined K-antiautomorphism $\bar{\cdot} : Q \to Q$ with the property that $q\bar{q} \in K$ for all $q \in Q$.*

This antiautomorphism is called the ***conjugation*** *of Q and satisfies $q + \bar{q} \in K$ for all $q \in Q$.*

Proof. The existence and uniqueness of the conjugation is shown in [Sch85, Ch.8 Thm.11.2]. Furthermore, for $q \in Q$, we have

$$(q+1)\overline{(q+1)} = (q+1)(\bar{q}+1) = q\bar{q} + q + \bar{q} + 1 \in K.$$

Since $q\bar{q} \in K$, we also have $q + \bar{q} \in K$. □

Definition 1.12. Let Q be a quaternion algebra over a field K. Then we define the **reduced norm** $\mathrm{n} : Q \to K$ and **reduced trace** $\mathrm{tr} : Q \to K$ by

$$\mathrm{n}(q) := q\bar{q} \quad \text{and} \quad \mathrm{tr}(q) := q + \bar{q}$$

respectively for all $q \in Q$.

Remark 1.13. The following hold for all $q \in Q$:

(1) Since $q + \bar{q} = \mathrm{tr}(q)$ lies in the center of Q, we have

$$q\bar{q} = q\bar{q} + q^2 - q^2 = q(\bar{q} + q) - q^2 = (\bar{q} + q)q - q^2 = \bar{q}q + q^2 - q^2 = \bar{q}q.$$

This implies that $n(q) = n(\bar{q})$.

(2) The polynomial $X^2 - \mathrm{tr}(q)X + \mathrm{n}(q) \in K[X]$ annihilates q as this can be verified easily. In particular, if $q \notin K$, then $X^2 - \mathrm{tr}(q)X + \mathrm{n}(q)$ is the minimal polynomial of q over K.

The following proposition tells us how to compute the conjugation, the norm and the trace explicitly.

Proposition 1.14. *Suppose that $Q = (L, b)$ for a Galois extension L/K of degree 2 with the non-trivial automorphism $\bar{\cdot} \in \mathrm{Gal}(L/K)$ and $b \in K^\times$. Then the following statements hold for all $q = x + yu \in Q = L \oplus Lu$:*

(1) *$\bar{q} = \bar{x} - yu$, $\mathrm{n}(q) = x^2 - by^2$ and $\mathrm{tr}(q) = x + \bar{x}$.*

(2) *Under the embedding $\rho : Q \to \mathrm{M}_2(L)$ from Lemma 1.3, we have*

$$\rho(\bar{q}) = \mathrm{adj}(\rho(q)), \quad \mathrm{n}(q) = \det(\rho(q)) \quad \textit{and} \quad \mathrm{tr}(q) = \mathrm{tr}(\rho(q)).$$

Proof. The first statement follows from [Sch85, Ch.8 Lemma 11.3]. Indeed, we have $\bar{u} = -u$ since its minimal polynomial is $X^2 - b$ and hence $\mathrm{tr}(u) = 0$, so that

$$\bar{q} = \bar{x} + \bar{u}\bar{y} = \bar{x} - u\bar{y} = \bar{x} - yu.$$

Consequently, we have the formulas for $\mathrm{n}(q)$ and $\mathrm{tr}(q)$. For the second statement, we have

$$\rho(\bar{q}) = \rho(\bar{x} - yu) = \begin{pmatrix} \bar{x} & -by \\ -\bar{y} & x \end{pmatrix} = \mathrm{adj}(\rho(q)),$$

so that $\rho(\mathrm{n}(q)) = \rho(q)\rho(\bar{q}) = \det(\rho(q))\mathbf{1}_2 = \rho(\det(\rho(q)))$, hence by injectivity of ρ, $\mathrm{n}(q) = \det(\rho(q))$. The statement about $\mathrm{tr}(q)$ can be verified directly. □

And in the case of a field of characteristic 2, we even have another explicit formula.

Proposition 1.15. *Suppose that K is a field of characteristic 2. Then for $a \in K$ and $b \in K^\times$, the conjugation on $\left[\frac{a,b}{K}\right)$ is given on the basis $\{1, I, J, IJ\}$ by*

$$\bar{1} = 1,\ \bar{I} = I + 1,\ \bar{J} = J \text{ and } \overline{IJ} = IJ.$$

Hence for $x, y, z, w \in K$, we have

$$\mathrm{n}(x + yI + zJ + wIJ) = (x^2 + xy + ay^2) + b(z^2 + zw + aw^2) \quad \text{and}$$
$$\mathrm{tr}(x + yI + zJ + wIJ) = y.$$

Proof. Since the minimal polynomial of I is $X^2 + \mathrm{tr}(I)X + \mathrm{n}(I) = X^2 + X + a$, we have $\bar{I} = \mathrm{tr}(I) - I = 1 + I$. The same argumentation applies for J with the minimal polynomial $X^2 + b$, implying that $\overline{IJ} = \bar{J}\bar{I} = J\bar{I} = IJ$. The formulas for the reduced norm and trace can be verified by a direct computation. □

Definition 1.16. The **trace form** on a quaternion algebra Q over a field K is defined by

$$(\cdot,\cdot) : Q \times Q \to K, \quad (x, y) := \mathrm{tr}(x \cdot y) \quad \text{for } x, y \in Q.$$

It can be shown that this is a symmetric, non-degenerate bilinear form.

Lemma 1.17. *Let K be field of characteristic 2, $a \in K$ and $b \in K^\times$. Then the trace form on $\left[\frac{a,b}{K}\right)$ can be represented with respect to the basis $\{1, I, J, IJ\}$ by the matrix*

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & b & 0 \end{pmatrix}$$

Proof. This follows from the multiplication table in remark 1.7 and proposition 1.15 □

Now we come to the definition of an order and a maximal order. In what follows, let R be a Dedekind ring, K its fraction field and Q a quaternion algebra over K.

Definition 1.18. (1) An R-**ideal** of Q is a finitely generated R-submodule $I \subseteq Q$ such that $KI = Q$.

(2) An R-**order** of Q is a subring $\mathfrak{O} \subseteq Q$ which is also an R-ideal.

(3) An R-order is **maximal** if it is not properly contained in another R-order in Q.

We shall see in the next section that there is a criterion for an order over a ring of integers of a global field to be maximal. This criterion is based on the discriminant of the order. In order to give an intrinsic definition of the discriminant, we need the following notions:

Definition 1.19. Let $I \subseteq Q$ be an ideal. The **inverse** of I is given by

$$I^{-1} := \{h \in Q \mid IhI \subseteq I\}.$$

The **reduced norm** of I, denoted by $\mathrm{n}(I)$, is the fractional ideal of R generated by the reduced norms of elements of I.

Now we can define the discriminant of an R-order.

Definition 1.20. Let $\mathfrak{O} \subseteq Q$ be an R-order. We define the **dual order** of $\mathfrak{O}$ by

$$\mathfrak{O}^* := \{x \in Q \mid \mathrm{tr}(x\mathfrak{O}) \subseteq R\}.$$

The **different** of $\mathfrak{O}$ is defined as the inverse of the dual order of $\mathfrak{O}$ and its norm is called **reduced discriminant**, denoted by $d(\mathfrak{O})$, i.e.

$$d(\mathfrak{O}) := \mathrm{n}((\mathfrak{O}^*)^{-1}).$$

In certain cases, the discriminant of an order can be computed easily:

Proposition 1.21. *Let $\mathfrak{O} \subseteq Q$ be an order which is a free R-module with a basis $(u_i)_i$. Then*

$$d(\mathfrak{O})^2 = R(\det(\mathrm{tr}(u_i u_j))_{i,j}).$$

Proof. see [Vig80, Lemme I.4.7]. □

1.3 Quaternion algebras over global fields

Throughout this section, suppose that K is a global field with the set of places denoted by $\mathbb{P}_K$. To determine whether a quaternion algebra Q over K is a division algebra, we can consider the base extension to the completion of K with respect to each place of K, i.e. we can transfer the problem to the case of local fields, for which we have a criterion from the first section. We shall begin with the following definition:

Definition 1.22. We say that a quaternion algebra Q over K **ramifies** at $\mathfrak{p} \in \mathbb{P}_K$ if and only if $Q_{K_\mathfrak{p}} := Q \otimes_K K_\mathfrak{p}$ is a division algebra. Otherwise we say that Q **splits** at $\mathfrak{p}$.

The set of ramified places of a quaternion algebra Q over K will be denoted by $\mathrm{Ram}(Q)$. The next Proposition gives us a fact about the possible number of ramification places of a quaternion algebra.

Proposition 1.23. *If Q is a quaternion algebra over K, then the cardinality of $\mathrm{Ram}(Q)$ is finite and even. Conversely, given a finite subset $S \subseteq \mathbb{P}_K$ of even cardinality, there exists a quaternion algebra Q, which is uniquely determined up to isomorphism, such that $\mathrm{Ram}(Q) = S$.*

Proof. [Vig80, Lemme 3.1, Ch. III Thm.3.1] □

In the case of characteristic 2, we have the following proposition which tells us at which places a given quaternion algebra can at most ramify.

Proposition 1.24. *Suppose that* $\operatorname{char} K = 2$, $a \in K$ *and* $b \in K^\times$. *Then the quaternion algebra* $Q := \left[\frac{a,b}{K}\right)$ *ramifies at most in the places* $\mathfrak{p}$ *where* $\nu_\mathfrak{p}(a) < 0$ *or* $\nu_\mathfrak{p}(b) \neq 0$. *Here* $\nu_\mathfrak{p}$ *denotes the normalised valuation with respect to the place* $\mathfrak{p}$.

Proof. Notice first that for each place $\mathfrak{p}$, we have $Q_{K_\mathfrak{p}} = \left[\frac{a,b}{K_\mathfrak{p}}\right)$. Then the statement follows from Corollary 1.10 since being in the ring of integers resp. its unit group is equivalent to that the valuation is non-negative resp. zero. □

Remark 1.25. There is also a similar criterion for the case $\operatorname{char} K \neq 2$, which is not to be used in this thesis. This Proposition has also a consequence that the set of ramification places must be finite since every element in $K \setminus \{0\}$ has a zero valuation at almost all places. To show that its cardinality is even is, however, more complicated.

In a global function field K, we can also define for a non-empty subset $S \subseteq \mathbb{P}_K$ of places the ring of S-integers

$$O_{K,S} := \{x \in K \mid \forall \mathfrak{p} \in \mathbb{P}_K \backslash S : \nu_\mathfrak{p}(x) \geq 0\}.$$

This is a Dedekind ring having K as fraction field. This allows us to determine whether an $O_{K,S}$-order in a quaternion algebra over K is maximal by considering its discriminant.

Proposition 1.26. *Let* Q *be a quaternion algebra over* K, $\emptyset \neq S \subseteq \mathbb{P}_K$ *and* $R := O_{K,S}$. *Then an* R*-order* $\mathfrak{O}$ *in* Q *is maximal if and only if*

$$d(\mathfrak{O}) = \prod_{\substack{\mathfrak{p} \in \mathrm{Ram}(Q) \\ \mathfrak{p} \notin S}} \mathfrak{p}$$

Proof. [Vig80, Ch. III, Cor.5.3] □

1.4 An example

Having recalled the necessary notions for quaternion algebras, we are going to establish an example of such quaternion algebras which will have an important role in this thesis. Throughout this section, let $K := \mathbb{F}_2(z)$. Consider the following quaternion algebra

$$Q := \left[\frac{z, 1+z^3}{K}\right) = K\{I, J\}/\{I^2 + I = z,\ J^2 = 1 + z^3,\ IJ = J(I+1)\}.$$

Lemma 1.27. *Q ramifies exactly in $z = 1$ and $z = \zeta$, where $\zeta \in \bar{\mathbb{F}}_2$ denotes a primitive third root of unity. In particular, Q is a division algebra.*

Here by the place $z = \zeta$, we mean the place of $\mathbb{F}_2(z)$ obtained by the evaluation map $z \mapsto \zeta$, i.e. the valuation ring consists of those elements in $\mathbb{F}_2(z)$ for which the evaluation map is well-defined. This doesn't depend on the choice of ζ since any Galois conjugation of ζ would define the same valuation ring. A uniformiser is given by its minimal polynomial over $\mathbb{F}_2$.

Proof. By Proposition 1.24, the algebra Q ramifies at most in the places $\mathfrak{p}$ where $\nu_\mathfrak{p}(z) < 0$ or $\nu_\mathfrak{p}(1+z^3) \neq 0$. If $\nu_\mathfrak{p}(z) < 0$, then $\mathfrak{p}$ is the place $\{z = \infty\}$, i.e. $u := z^{-1}$ is a uniformiser and we have

$$\frac{z\,d(1+z^3)}{1+z^3} = \frac{u^{-1}\,d(1+u^{-3})}{1+u^{-3}} = \frac{u^2 \cdot u^{-4}\,du}{1+u^3} = (u^{-2} + u + u^4 + u^7 + \cdots)\,du$$

So the residue of this differential form vanishes, which implies that $[z, 1+z^3)_{K_\mathfrak{p}} = 0$ by Proposition 1.9, i.e. Q splits at this place.

If $\nu_\mathfrak{p}(1+z^3) \neq 0$, then $\mathfrak{p}$ is the place $\{z = 1\}$ or $\{z = \zeta\}$. In either case, $1+z^3$ is a uniformiser and $\nu_\mathfrak{p}(z) = 0$, which implies that the residue of the differential form $\frac{z\,d(1+z^3)}{1+z^3}$ does not vanish. In particular, $[z, 1+z^3)_{K_\mathfrak{p}} \neq 0$ for $\mathfrak{p} = \{z = 1\}$, i.e. Q ramifies at this place. Since the number of ramified places is even, Q must also ramify at ζ. □

In this thesis we are going to deal with several arithmetic lattices for the algebraic group

$$G := \mathrm{PGL}_{1,Q} = \mathrm{GL}_{1,Q} / \mathbb{G}_m$$

over K of projective linear group of rank 1 of Q.

Lemma 1.28. *G is connected, reductive, semisimple and anisotropic over K.*

Proof. Since G is a twisted form of PGL_2 and PGL_2 is connected, reductive and semisimple, the same holds for G. To show that G is anisotropic over K, observe first that $\pi : \mathrm{GL}_{1,Q} \to G$ is a central surjective K-morphism. On the other hand, since Q is a central division algebra over K, the maximal K-split torus in $\mathrm{GL}_{1,Q}$ is exactly its centre, which is also the kernel of π, cf. [Bor91, 23.1]. Hence by [Bor91, Thm.22.7], the maximal K-split torus in G must be trivial, i.e. G is anisotropic over K. □

The arithmetic lattices with which we shall be concerned are defined over the following rings:

$$R_0 := \mathbb{F}_2[z, \frac{1}{z}], \quad R_1 := \mathbb{F}_2[z, \frac{1}{z(1+z)}] \quad \text{and} \quad R := \mathbb{F}_2[z, \frac{1}{z(1+z^3)}].$$

We see that $R_0 \subseteq R_1 \subseteq R$ and that $R_0 = O_{K,S_0}$, $R_1 = O_{K,S_1}$ and $R = O_{K,S}$, where S_0 is the set of places at $0, \infty$; S_1 the set of places at $0, 1, \infty$; and S the set of places at $0, 1, \zeta, \infty$ respectively.

Lemma 1.29. *The S-rank of G is 2.*

Proof. By definition, the S-rank of G is

$$\sum_{\mathfrak{p} \in S} \mathrm{rk}_{K_\mathfrak{p}} G(K_\mathfrak{p}).$$

For $\mathfrak{p} = \{z = 1\}$ or $\{z = \zeta\}$, we know from Lemma 1.27 that $Q_{K_\mathfrak{p}}$ is a division algebra. Hence the group G is anisotropic over $K_\mathfrak{p}$ by the same reasoning as in Lemma 1.28, so that the corresponding local ranks are zero. For $\mathfrak{p} = \{z = 0\}$ or $\mathfrak{p} = \{z = \infty\}$, the quaternion algebra Q splits over $\mathfrak{p}$. This implies that $G(K_\mathfrak{p}) \cong \mathrm{PGL}_2(K_\mathfrak{p})$ has the rank 1 over $K_\mathfrak{p}$. Summing up all local ranks together, we obtain 2 as the S-rank of G. □

Now we are going to give G an integral structure. To make the corresponding arithmetic subgroup as large as possible, we find first an R-maximal order of Q:

Lemma 1.30. *The R_0-order $\mathfrak{O}_0 := R_0 \oplus R_0 \cdot I \oplus R_0 \cdot J \oplus R_0 \cdot IJ$ is maximal.*

Proof. From Lemma 1.17, we see that the trace form on the basis $1, I, J, IJ$ has the discriminant $(1+z^3)^2$, which implies that the reduced discriminant $d(\mathfrak{O}_0)$ is generated by $1+z^3$, which is exactly the product of $1+z$ and $1+z+z^2$, i.e. of prime elements for the places at 1 and ζ. Since these places are exactly the ramified places of Q, the order is maximal by Proposition 1.26. □

Corollary 1.31. *The R_1-order $\mathfrak{O}_1 := \mathfrak{O}_0 \otimes_{R_0} R_1$ and the R-order $\mathfrak{O} := \mathfrak{O}_0 \otimes_{R_0} R$ are maximal over its corresponding rings and $\mathfrak{O}$ is an Azumaya algebra over R.*

Proof. This follows from the Lemma above. Furthermore, since $1+z^3$ is invertible in R, $\mathfrak{O}$ is even an Azumaya algebra over R. □

For the integral structure on G with respect to the basis $1, I, J, IJ$. The elements in $G(\tilde{R})$ for $\tilde{R} \in \{R_0, R_1, R\}$ can be described explicitly as in the follows:

Lemma 1.32. *For $\tilde{R} \in \{R_0, R_1, R\}$, we have $G(\tilde{R}) = (\mathfrak{O}_0 \otimes_{\mathrm{R}_0} \tilde{R})^\times / \tilde{R}^\times$.*

Proof. Consider the exact sequence

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathrm{GL}_{1,Q} \longrightarrow G \longrightarrow 1.$$

This yields for $\tilde{R} \in \{R_0, R_1, R\}$ the following long exact cohomology sequence:

$$1 \longrightarrow \tilde{R}^\times \longrightarrow (\mathfrak{O}_0 \otimes_{\mathrm{R}_0} \tilde{R})^\times \longrightarrow G(\tilde{R}) \longrightarrow H^1(\tilde{R}, \mathbb{G}_m) \longrightarrow \dots .$$

Since all $\tilde{R} \in \{R_0, R_1, R\}$ are principal ideal domains, we have $H^1(\tilde{R}, \mathbb{G}_m) = \mathrm{Pic}(\tilde{R}) = 1$, which implies that $G(\tilde{R}) = (\mathfrak{O}_0 \otimes_{\mathrm{R}_0} \tilde{R})^\times / \tilde{R}^\times$ as desired. □

From the lemma we have the following arithmetic subgroups of $G(K)$:

$$\mathfrak{O}_0^\times / R_0^\times = G(R_0) \subseteq \mathfrak{O}_1^\times / R_1^\times = G(R_1) \subseteq \mathfrak{O}^\times / R^\times = G(R) \subseteq Q^\times / K^\times = G(K)$$

We are going to find presentations for the arithmetic subgroups $G(R_0)$, $G(R_1)$ and $G(R)$ in the next chapters. What we are going to do is, roughly speaking, consider their action on the product of two copies of Bruhat-Tits-Tree for $\mathrm{PGL}_2(\mathbb{F}_2(\!(\pi)\!))$. Here the action on each component is induced by its embeddings $G(K) \hookrightarrow G(\mathbb{F}_2(\!(\pi)\!)) = \mathrm{PGL}_2(\mathbb{F}_2(\!(\pi)\!))$, where $K \hookrightarrow \mathbb{F}_2(\!(\pi)\!)$ denotes the completion with respect to splitting places with residue class field $\mathbb{F}_2$, i.e. $\pi = z$ for $\mathfrak{p} = \{z = 0\}$ and $\pi = u = z^{-1}$ for $\mathfrak{p} = \{z = \infty\}$.

In order to describe the embeddings above more explicitly, we look first for representations of Q over its splitting fields, for which the place $\{z = 0\}$ resp. $\{z = \infty\}$ splits. For the first splitting, let $\mathbb{F}_2(y)$ be the quadratic extension of K defined by

$$y^2 + y = z.$$

This extension has a non-trivial Galois automorphism given by $y \mapsto 1 + y$. Moreover, the place $\{z = 0\}$ splits under this extension since the above equation reduces modulo z to $y(y+1) = y^2 + y = 0$ over $\mathbb{F}_2$.

Lemma 1.33. *The quaternion algebra Q splits over $\mathbb{F}_2(y)$, i.e., Q has a 2-dimensional representation $\rho_y : Q \to M_2(\mathbb{F}_2(y))$ over $\mathbb{F}_2(y)$ given by*

$$I \mapsto \rho_y(I) := \begin{pmatrix} y & 0 \\ 0 & 1+y \end{pmatrix} \quad \text{and} \quad J \mapsto \rho_y(J) := \begin{pmatrix} 0 & 1+z^3 \\ 1 & 0 \end{pmatrix}.$$

Proof. This follows directly from Lemma 1.3. □

For the second splitting, denote $u = z^{-1} \in K$ as before, and let $\mathbb{F}_2(t)$ be the quadratic extension of K defined by

$$t^2 + t = u.$$

This extension has a non-trivial Galois automorphism given by $t \mapsto 1 + t$, and we can show by using the same argument as before that the place $\{u = 0\} = \{z = \infty\}$ splits under this extension.

Lemma 1.34. *The quaternion algebra Q splits over $\mathbb{F}_2(t)$, i.e., Q has a 2-dimensional representation $\rho_t : Q \to M_2(\mathbb{F}_2(t))$ over $\mathbb{F}_2(t)$ given by*

$$I \mapsto \rho_t(I) := \begin{pmatrix} 1+u+t & 1+u^3 \\ u^{-1} & u+t \end{pmatrix} \quad \textit{and} \quad J \mapsto \rho_t(J) := \begin{pmatrix} 0 & u^{-1}+u^2 \\ u^{-2} & 0 \end{pmatrix}.$$

Proof. According to the definition, we have

$$\rho_t(I)\rho_t(J) = u^{-2} \begin{pmatrix} 1+u^3 & (1+u+t)(u+u^4) \\ u+t & 1+u^3 \end{pmatrix},$$

$$\rho_t(J)\rho_t(I) = u^{-2} \begin{pmatrix} 1+u^3 & (u+t)(u+u^4) \\ 1+u+t & 1+u^3 \end{pmatrix}, \quad \text{and}$$

$$\rho_t(I)^2 = \begin{pmatrix} u^{-1}+1+u+t & 1+u^3 \\ u^{-1} & u^{-1}+u+t \end{pmatrix}.$$

So we can easily verify that $\rho_t(I)^2 + \rho_t(I) = u^{-1} = z$, $\rho_t(J)^2 = u^{-3} + 1 = 1 + z^3$, and $\rho_t(I)\rho_t(J) = \rho_t(J)\rho_t(I) + \rho_t(J) = \rho_t(J)(\rho_t(I) + 1)$, which implies that ρ_t is well-defined. □

Chapter 2

The fundamental groups of global orbispaces

Having introduced a quaternion algebra $Q = \left[\frac{z,1+z^3}{\mathbb{F}_2(z)}\right)$ and its order $\mathfrak{O}$ over $R = \mathbb{F}_2[z, \frac{1}{z}]$ in the previous chapter, we are going to find a presentation of the associated arithmetic subgroup $\mathfrak{O}^\times/R^\times$ of $Q^\times/K^\times$. As we shall see later, this group occurs indeed as the fundamental group of the quotient of a square complex by V_4-action. In this chapter, we shall introduce some basic facts about such fundamental groups which are mostly similar to the known facts for fundamental groups of topological spaces.

2.1 Definitions and basic properties

In this section, we are going to introduce the concept of the orbital fundamental group, i.e. the fundamental group of an orbispace. In general, an orbispace is a topological space obtained by an open covering of quotient spaces by a continuous action of a topological group on a locally connected space. Here we shall restrict to **global orbispaces**, i.e. those orbispaces which are global quotients by discrete group actions. This leads to the following definition:

Definition 2.1 (The category of global orbispace structures)**.**

(1) A **global orbispace structure** is a pair (S, G) consisting of a topological space S and a discrete group G with a continuous left group action $G \times S \to S,\ (g, s) \mapsto gs$, i.e. a group homomorphism $G \to \mathrm{Aut}(S)$ from G to the group of the homeomorphisms on S.

If $\mathbf{P}$ is a property of a topological space, we shall say that a global orbispace structure (S, G) has property $\mathbf{P}$ if the underlying topological space S has this property.

(2) By a **morphism** between two global orbispace structures (S, G) and (S', G'), we mean a pair $f = (f, \varphi)$ consisting of a continuous map $f : S \to S'$ and a group homomorphism $\varphi : G \to G'$ such that $f(gs) = \varphi(g)f(s)$ for all $s \in S$ and $g \in G$.

Definition 2.2. A **global orbispace** is a triple $(S_0, (S, G), q)$ consisting of a topological space S_0, a global orbispace structure (S, G) and a continuous mapping $q : S \to S_0$ which is G-invariant and induces a homeomorphism between the quotient space $G\backslash S$ and S_0.

Remark 2.3. Every global orbispace structure (S, G) yields a global orbispace

$$S//G = (G\backslash S, (G, S), q_{S,G}),$$

where $q_{S,G} : S \to G\backslash S$ denotes the canonical quotient map, the **global orbispace associated to** (S, G). The image of $s \in S$ under $q_{S,G}$ will be denoted by $[s] := q_{S,G}(s) = Gs \in G\backslash S$

Remark 2.4. It is also possible to define a morphism for global orbispaces in such a way that each morphism of global orbispace structures yields a morphism of the associated global orbispaces, and if G is a group acting on a space S freely, then $S//G$ and $(G\backslash S)//1$ are isomorphic as global orbispaces. However, we are not going to discuss the proper definition here.

Before we come to the definition of the fundamental group of a global orbispace, let us fix the following notation: A **path** on a topological space S is a continuous map $\alpha : [0,1] \to S$. If $\alpha, \beta : [0,1] \to S$ are two paths with $\alpha(1) = \beta(0)$, then the composition $\alpha * \beta : [0,1] \to S$ is defined by

$$(\alpha * \beta)(t) := \begin{cases} \alpha(2t) & \text{for } 0 \le t \le 1/2, \\ \beta(2t-1) & \text{for } 1/2 \le t \le 1. \end{cases}$$

Two paths $\alpha, \alpha' : [0,1] \to S$ are said to be **homotopic** if $\alpha(0) = \alpha'(0)$, $\alpha(1) = \alpha'(1)$ and there exists a continuous map $H : [0,1] \times [0,1] \to S$ such that $H(0,-) = \alpha$ and $H(1,-) = \alpha'$. Note that the homotopy of paths is an equivalence relation, and the composition of paths yields a well-defined composition of homotopy classes of paths. Furthermore, we shall denote the reverse path of the path α on S by $\bar{\alpha}$, i.e. $\bar{\alpha}(t) = \alpha(1-t)$ for $t \in [0,1]$. Note that $\bar{\bar{\alpha}} = \alpha$ and $\alpha * \bar{\alpha}$ is homotopic to the constant path at the point $\alpha(0)$.

We now come to the definition of the fundamental group of a global orbispace. The following definition, which will be sufficient for this thesis, comes from [Rho66].

Definition 2.5. Let (S, G) be a global orbispace structure and s be a point in S.

(1) A **loop of order** $g \in G$ with basepoint s is a path from s to gs.

(2) A homotopy class of a loop α of order g will be denoted by $[\alpha; g]$. Note that in this case, every path homotopic to α is also a loop of order g as its endpoint is $\alpha(1) = gs$.

(3) The **orbital fundamental group** of S by G with basepoint s, denoted by $\pi_1^{\mathrm{orb}}(S, G, s)$, is defined as the set of homotopy classes of loops of any order with basepoint s, together with the composition law
$$[\alpha; g] * [\beta; h] = [\alpha * (g\beta); gh].$$

Remark 2.6. Let (S, G) and s be as above.

(1) A loop of order 1 is simply a loop in the topological space S in the normal sense. Hence if $G = \{1\}$, we get $\pi_1^{\mathrm{orb}}(S, G, s) = \pi_1(S, s)$, i.e. the topological fundamental group is a special case of this orbital fundamental group.

(2) It is easy to show that $\pi_1^{\mathrm{orb}}(S, G, s)$ with the given composition law is in fact a group. The associativity is done by an easy calculation. The neutral element is $[\bullet, 1]$, where $\bullet$ denotes the constant path at the point s. And the inverse element of $[\alpha; g]$ is $[g^{-1}\bar{\alpha}; g^{-1}]$.

(3) If $s' \in S$ is another point lying on the same path component as s, then one can show, as in the case of the fundamental group of a topological space, that there is a non-canonical isomorphism between $\pi_1^{\mathrm{orb}}(S, G, s)$ and $\pi_1^{\mathrm{orb}}(S, G, s')$.

Proposition 2.7 (functoriality of $\pi_1^{\mathrm{orb}}(S, G, s_0)$)**.** *Let (S, G) and (S', G') be two global orbispace structures, and $f = (f, \varphi) : (S, G) \to (S', G')$ be a morphism between them. Furthermore, let $s_0 \in S$ be a base point and $s_0' := f(s_0) \in S'$. Then we have a group homomorphism*

$$f_\# : \pi_1^{\mathrm{orb}}(S, G, s_0) \longrightarrow \pi_1^{\mathrm{orb}}(S', G', s_0'), \quad [\alpha; g] \longmapsto [f \circ \alpha; \varphi(g)].$$

Proof. First of all, note that $f_\#$ is well-defined since if α is a path in S beginning at s_0 and ending at gs_0, then $f\circ\alpha$ begins at $f(s_0)$ and ends at $f(gs_0)=\varphi(g)f(s_0)$. Furthermore, if α and β are homotopic loops of order g with a homotopy H, then $f\circ H$ is a homotopy between $f\circ\alpha$ and $f\circ\beta$, whence $[f\circ\alpha;\varphi(g)]=[f\circ\beta;\varphi(g)]$. Now for $[\alpha;g],[\beta;h]\in\pi_1^{\mathrm{orb}}(S,G,s_p)$, we have

$$\begin{aligned} f_\#([\alpha;g]*[\beta;h]) &= f_\#([\alpha*(g\beta);gh]) = [f\circ(\alpha*(g\beta));\varphi(gh)] \\ &= [(f\circ\alpha)*(f\circ(g\beta));\varphi(gh)] = [(f\circ\alpha)*(\varphi(g)(f\circ\beta)),\varphi(g)\varphi(h)] \\ &= [f\circ\alpha;\varphi(g)]*[f\circ\beta;\varphi(h)] = f_\#([\alpha;g])*f_\#([\beta;h]). \end{aligned}$$

Hence $f_\#$ as a group homomorphism as desired. □

As a special case of the proposition above, setting $S=S'$, $f:=\mathrm{id}_S$ and replacing $\varphi:G\to G'$ by an embedding $H\hookrightarrow G$, we get the canonical homomorphism

$$\pi_1^{\mathrm{orb}}(S,H,s)\longrightarrow\pi_1^{\mathrm{orb}}(S,G,s),\quad [\alpha]\longmapsto[\alpha;1],$$

which is injective as can be shown easily. If moreover H is a normal subgroup in G, we can combine the map above with the projection map

$$\pi_1^{\mathrm{orb}}(S,G,s)\to G\twoheadrightarrow G/H,\quad [\alpha;g]\longmapsto gH,$$

and obtain the following exact sequence:

Proposition 2.8. *Let (S,G) be a global orbispace structure and $s\in S$ be a point such that the orbit Gs lies in the same path component of S. Then there is an exact sequence*

$$1\longrightarrow\pi_1^{\mathrm{orb}}(S,H,s)\longrightarrow\pi_1^{\mathrm{orb}}(S,G,s)\longrightarrow G/H\longrightarrow 1.$$

Proof. The assumption implies that there exists a path from s to gs for each $g\in G$, in particular a loop or order g with basepoint s. This implies that the projection map $\pi_1^{\mathrm{orb}}(S,G,s)\to G$ as above is surjective, hence also $\pi_1^{\mathrm{orb}}(S,G,s)\to G/H$. Furthermore, $[\alpha;g]\in\pi_1^{\mathrm{orb}}(S,G,s)$ belongs to its kernel if and only if $g\in H$, i.e. $\alpha\in\pi_1^{\mathrm{orb}}(S,H,s)$. □

Corollary 2.9. *Under the same condition for (S,G) and $s\in S$ as in Proposition 2.8, there is an exact sequence*

$$1\longrightarrow\pi_1(S,s)\longrightarrow\pi_1^{\mathrm{orb}}(S,G,s)\longrightarrow G\longrightarrow 1.$$

Proof. This follows immediately from Proposition 2.8 by setting $H=1$. □

2.2 Orbital fibre of a covering space over a global orbispace

In this section, we are going to introduce the orbital fibre of a covering space over a quotient space, which is an analogue to a fibre of a covering space over a topological space. In general the orbital fibre contains more information than the preimage of a point in the quotient space under the covering map, so that we can define an action of the orbital fundamental group on the orbital fibre in a natural way. However, we shall see that the orbital fibre is in a canonical bijection with the preimage under the covering map if the stabiliser of the base point is trivial.

Throughout this section, let (S,G) be a path connected global orbispace structure with the quotient map $q_{S,G}:S\to G\backslash S$. Furthermore, let $p:Y\to S$ be a covering space and $p_0:=p\circ q_{S,G}$.

Definition 2.10. The **orbital fibre** of the covering space $p : Y \to S$ with respect to the group action of G on S over a point $s \in S$ is defined by

$$p^{-1}(s)_G := \{(y, g) \in Y \times G \mid p(y) = gs\}.$$

Lemma 2.11. *If the stabiliser of s is trivial, then the canonical projection*

$$p^{-1}(s)_G \longrightarrow p_0^{-1}([s]),\ (y, g) \longmapsto y,$$

is a bijection

Proof. It is easy to see that this mapping is well-defined. To show the surjectivity, consider an element $y \in p_0^{-1}([s])$. Since $q_{S,G}(p(y)) = p_0(y) = [s] = q_{S,G}(s)$, there is a $g \in G$ such that $p(y) = gs$, so that $(y, g) \in p^{-1}(s)_G$ with y as image under the projection.

Now we come to the injectivity. if $(y, g), (y', g') \in p^{-1}(s)_G$ have the same image under the projection, then $y = y'$ and therefore $gs = p(y) = p(y') = g's$, i.e. $g^{-1}g'$ stabilises s. Since the stabiliser of s is trivial, we also have $g = g'$, i.e. this mapping is injective, hence bijective. □

Remark 2.12. Even if the stabiliser of s is non-trivial, we can still say that there is a non-canonical bijection between $p^{-1}(s)_G$ and $p_0^{-1}([s]) \times G_s$, where $G_s \leq G$ denotes the stabiliser group of $s \in S$ under G.

An advantage of the orbital fibre is that we can define an action of the orbital fundamental group in a natural way, even if the stabiliser of the base point is not trivial.

Proposition 2.13. *The group $\pi_1^{\mathrm{orb}}(S, G, s)$ acts on the orbital fibre $p^{-1}(s)_G$ from the right by the rule*

$$(y, g) \cdot [\alpha; g'] := (y.(g\alpha), gg').$$

Here $y.(g\alpha)$ denotes the endpoint of the path in Y which is the lifting of the path $g.\alpha$ in S and begins at y. Furthermore, this group action is transitive if Y is path-connected.

Proof. We show first that this is a group action. Since $p(y) = gs$ and $g\alpha$ also begins at gs, it is possible to lift $g\alpha$ to a path beginning at y. The end of this lifting path then has $gg's$ as the image under the covering map, so that $(y.(g\alpha), gg') \in p^{-1}(s)_G$. Note that this is also independent of the choice of α from its homotopy class since the whole homotopy between two such paths can be lifted to a homotopy of two paths in Y with the same ending point. Hence the given rule is well-defined. To show that this is defines a group action, let $(y, g) \in p^{-1}(s)_G$ and $[\alpha; h], [\alpha'; h'] \in \pi_1^{\mathrm{orb}}(S, G, s)$. Then

$$\begin{aligned}((y, g) \cdot [\alpha; h]) \cdot [\alpha'; h'] &= (y.(g\alpha), gh) \cdot [\alpha'; h'] = ((y.(g\alpha)).(gh\alpha'), (gh)h') \\ = (y.(g(\alpha * (h\alpha'))), g(hh')) &= (y, g) \cdot [\alpha * (h\alpha'); hh'] = (y, g) \cdot ([\alpha; h] * [\alpha'; h'])\end{aligned}$$

This shows that the given rule is a group action.

Now assume that Y is path-connected. To show that the action is transitive, fix a preimage $y_0 \in Y$ of s, i.e. $(y_0, 1) \in p^{-1}(s)_G$, and consider any element $(y, g) \in p^{-1}(s)_G$. Since Y is path-connected, we can find a path γ from y_0 to y. Then $\alpha := p \circ \gamma$ is a loop of order g in S since it begins at s and ends at $p(y) = gs$. This implies that $(y, g) = (y_0, 1) \cdot [\alpha; g]$, i.e. the group action is transitive. □

Proposition 2.14. *If $p : Y \to S$ is the universal covering, then the action of $\pi_1^{\mathrm{orb}}(S, G, s)$ on $p^{-1}(s)_G$ is simply transitive.*

Proof. As in the proof of the previous proposition, fix a preimage $y_0 \in Y$ of $s \in S$ under p, i.e. $(y_0, 1) \in p^{-1}(s)_G$. Since this group action is transitive, it suffices to show that if we have $[\alpha; g], [\beta; h] \in \pi_1^{\mathrm{orb}}(S, G, s)$ such that $(y_0, 1) \cdot [\alpha; g] = (y_0, 1) \cdot [\beta; h]$, then $[\alpha; g] = [\beta; h]$. But in this case we have

$$(y_0.\alpha, g) = (y_0.\beta, h), \quad \text{i.e.} \quad y_0.\alpha = y_0.\beta \text{ and } g = h.$$

Let $\tilde{\alpha}$ and $\tilde{\beta}$ be the lifting of α resp. β beginning at y_0 respectively. These paths end at the same point, namely at $y_0.\alpha = y_0.\beta$. So by the simply connectedness of Y, there is a homotopy $\tilde{H}$ between $\tilde{\alpha}$ and $\tilde{\beta}$. Hence $p \circ \tilde{H}$ is a homotopy between α and β. This implies that $[\alpha; g] = [\beta; h]$ as desired. □

Corollary 2.15. *If the stabiliser of s is trivial and $p : Y \to S$ is the universal covering, then the group action from Proposition 2.14 induces the simply transitive group action of $\pi_1^{\mathrm{orb}}(S, G, s)$ on the fibre $p_0^{-1}([s])$*

Proof. This follows from Proposition 2.14 and Lemma 2.11. □

2.3 Deck transformations over global orbispaces

The goal of this section is to extend the theory of deck transformations of covering spaces over a topological space to those over a global orbispace. We shall, however, continue to restrict to those covering spaces obtained from covering spaces over its underlying space of its global orbispace structure.

Throughout this section, let (S, G) be a path-connected and locally path-connected global orbispace structure and $G_0 := \ker(G \to \mathrm{Aut}(S))$. Furthermore, let $p : Y \to S$ be a path-connected covering space over S.

Definition 2.16. By a **deck transformation** of a covering space $p : Y \to S$ over a global orbispace structure (S, G), we mean a homeomorphism $\phi : Y \to Y$ such that the mappings $p, p \circ \phi : Y \to S$ differ only by an action of a group element of G on S. The set of such deck transformations will be denoted by $\mathrm{Aut}(Y/(S, G))$, i.e.

$$\mathrm{Aut}(Y/(S, G)) = \{\phi \in \mathrm{Aut}(Y) \mid \exists\, g_\phi \in G : p \circ \phi = g_\phi p\}.$$

Proposition 2.17. *The composition defines a group structure on* $\mathrm{Aut}(Y/(S, G))$ *and there is the following group homomorphism*

$$\mathrm{Aut}(Y/(S, G)) \longrightarrow G/G_0, \quad \phi \longmapsto (g_\phi \in G \textit{ such that } p \circ \phi = g_\phi p).$$

Proof. First of all, we are going to show that for any deck transformations $\phi, \psi \in \mathrm{Aut}(Y/(S, G))$, we have $\phi \circ \psi, \phi^{-1} \in \mathrm{Aut}(Y/(S, G))$. Indeed, there exist $g_\phi, g_\psi \in G$ such that $p \circ \phi = g_\phi p$ and $p \circ \psi = g_\psi p$. This implies that

$$p \circ (\phi \circ \psi) = (p \circ \phi) \circ \psi = (g_\phi p) \circ \psi = g_\phi (p \circ \psi) = g_\phi (g_\psi p) = (g_\phi g_\psi) p, \quad \text{and}$$
$$p \circ \phi^{-1} = g_\phi^{-1} (g_\phi p) \circ \phi^{-1} = g_\phi^{-1} (p \circ \phi) \circ \phi^{-1} = g_\phi^{-1} p \circ (\phi \circ \phi^{-1}) = g_\phi^{-1} p.$$

Since id_Y is also a deck transformation, we conclude that $\mathrm{Aut}(Y/(S, G))$ is a subgroup of $\mathrm{Aut}(Y)$. Now suppose for a given $\phi \in \mathrm{Aut}(Y/(S, G))$ that we have $g_\phi, g'_\phi \in G$ with $p \circ \phi = g_\phi p$ and $p \circ \phi = g'_\phi p$. Then we have $g_\phi p = g'_\phi p$ which is equivalent to $g_\phi^{-1} g'_\phi p = p$. Since p is surjective, this implies that the homeomorphism on S defined by $g_\phi^{-1} g'_\phi \in G$ is trivial, i.e. $g_\phi^{-1} g'_\phi \in G_0$. Hence the given mapping is well-defined and is a group homomorphism by the calculation above. □

Now we are going to establish the connection between the deck transformation group and the orbital fundamental group. For this we need the following lifting criterion:

Proposition 2.18. *Suppose that $y \in Y$ is a preimage of $s \in S$ under the covering $p : Y \to S$ and $f : Z \to S$ is a continuous map from a path-connected and locally path-connected space Z with a point $z \in Z$ such that $f(z) = s$. Then a lift $\tilde{f} : Z \to Y$ such that $\tilde{f}(z) = y$ exists iff $f_\#(\pi_1(Z, z)) \subseteq p_\#(\pi_1(Y, y))$. In this case, such a lift $\tilde{f}$ is uniquely determined.*

Proof. [Hat01, Prop.1.33-1.34] □

The following proposition is an extension of a part of [Hat01, Prop.1.39].

Proposition 2.19. *Let $y \in Y$ be a preimage of $s \in S$ under the covering $p : Y \to S$ and H be the subgroup $p_\#(\pi_1(Y, y)) \leq \pi_1(S, s) \trianglelefteq \pi_1^{\mathrm{orb}}(S, G, s)$. Then there exists a surjective group homomorphism*

$$N_{\pi_1^{\mathrm{orb}}(S,G,s)}(H) \longrightarrow \mathrm{Aut}(Y/(S, G)), \quad [\alpha; g] \longmapsto \phi_{[\alpha;g]},$$

where $N_{\pi_1^{\mathrm{orb}}(S,G,s)}(H)$ denotes the normaliser of H in $\pi_1^{\mathrm{orb}}(S, G, s)$. Its kernel is an internal direct product of H and $G_0 := \ker(G \to \mathrm{Aut}(S))$.

Proof. We consider first an arbitrary element $[\alpha; g] \in \pi_1^{\mathrm{orb}}(S, G, s)$ and try to construct a deck transformation $\phi_{[\alpha;g]}$ such that $\phi_{[\alpha;g]}(y) = y'$, where y' denotes the endpoint of $\alpha_Y : [0, 1] \to Y$, the lifting of α in Y starting at y. If such a deck transformation exists, it must fit into the following diagram:

$$\begin{array}{ccc} (Y, y) & \overset{\phi_{[\alpha;g]}}{\dashrightarrow} & (Y, y') \\ \downarrow p & & \downarrow p \\ (S, s) & \overset{g}{\longrightarrow} & (S, gs). \end{array} \tag{2.1}$$

By Proposition 2.18, such a map exists if and only if $(gp)_\#(\pi_1(Y, y)) \subseteq p_\#(\pi_1(Y, y'))$, i.e. if and only if for all $\gamma \in \pi_1(Y, y)$, there exists a $\delta \in \pi_1(Y, y')$ such that $g(p \circ \gamma) = p \circ \delta$. By writing $\delta = \bar{\alpha}_Y * \delta' * \alpha_Y$ for a uniquely determined $\delta' \in \pi_1(Y, y)$, the last equation is equivalent to $g(p \circ \gamma) = \bar{\alpha} * (p \circ \delta') * \alpha$ or $\alpha * (g(p \circ \gamma)) * \bar{\alpha} = p \circ \delta'$. Now consider the following calculation:

$$[\alpha; g] * (p \circ \gamma) * [\alpha; g]^{-1} = [\alpha * (g(p \circ \gamma)); g] * [g^{-1}\bar{\alpha}; g^{-1}] = \alpha * (g(p \circ \gamma)) * \bar{\alpha},$$

where we consider every element $\beta \in \pi_1(S, s)$ as an element $[\beta; 1] \in \pi_1^{\mathrm{orb}}(S, G, s)$. This shows that the map $\phi_{[\alpha;g]}$ we want to construct exists if and only if for all $\gamma \in \pi_1(Y, y)$, there exists a $\delta' \in \pi_1(Y, y)$ such that $[\alpha; g] * (p \circ \gamma) * [\alpha; g]^{-1} = p \circ \delta' \in H$, i.e. if and only if $[\alpha; g]H[\alpha; g]^{-1} \subseteq H$. This is the case especially for $[\alpha; g] \in N_{\pi_1^{\mathrm{orb}}(S,G,s)}(H)$.

Next, we are going to show that $\phi_{[\alpha;g]} \circ \phi_{[\beta;h]} = \phi_{[\alpha*(g\beta);gh]}$ for all $[\alpha; g], [\beta; h] \in N_{\pi_1^{\mathrm{orb}}(S,G,s)}(H)$. To this end let α_Y and β_Y be the liftings of α resp. β in Y beginning at y. Then $\phi_{[\beta;h]}(y) = \beta_Y(1)$, which implies that $(\phi_{[\alpha;g]} \circ \phi_{[\beta;h]})(y) = \phi_{[\alpha;g]}(\beta_Y(1))$. On the other hand, by the property of $\phi_{[\alpha;g]}$ we have

$$p \circ \phi_{[\alpha;g]} \circ \beta_Y = gp \circ \beta_Y = g\beta.$$

This implies that $\phi_{[\alpha;g]} \circ \beta_Y$ is the lifting of $g\beta$ beginning at $\phi_{[\alpha;g]}(\beta_Y(0)) = \phi_{[\alpha;g]}(y) = y.\alpha$. Hence the lifting of $\alpha * (g\beta)$ beginning at y ends at $\phi_{[\alpha;g]}(\beta_Y(1)) = (\phi_{[\alpha;g]} \circ \phi_{[\beta;h]})(y)$. On the other hand, $\phi_{[\alpha*(g\beta);gh]}(y)$ is also the end of the lifting of $\alpha * (g\beta)$ beginning at y, meaning that the images of y under $\phi_{[\alpha;g]} \circ \phi_{[\beta;h]}$ and $\phi_{[\alpha*(g\beta);gh]}$ are the same. By the uniqueness of the lifting from Proposition 2.18, we have $\phi_{[\alpha;g]} \circ \phi_{[\beta;h]} = \phi_{[\alpha*(g\beta);gh]}$ as desired.

Consequently, since $\phi_{[\bullet,1]} = \mathrm{id}_Y$, the map $\phi_{[\alpha;g]}$ is a homeomorphism on Y for every $[\alpha;g] \in N_{\pi_1^{\mathrm{orb}}(S,G,s)}(H)$. Hence we get a well-defined group homomorphism

$$N_{\pi_1^{\mathrm{orb}}(S,G,s)}(H) \longrightarrow \mathrm{Aut}(Y/(S,G)), \quad [\alpha;g] \longmapsto \phi_{[\alpha;g]}.$$

To show that this is surjective, consider a deck transformation $\phi \in \mathrm{Aut}(Y/(S,G))$. By definition there exists a $g \in G$ such that $gp = p \circ \phi$. Furthermore, let γ be a path from y to $\phi(y)$ in Y and $\alpha := p \circ \gamma$. Then $\alpha(1) = gs$ and $\phi_{[\alpha,g]}$ fits into the diagram (2.1). This is also the case for ϕ, which implies that $\phi = \phi_{[\alpha;g]}$. It remains to show that $[\alpha;g] \in N_{\pi_1^{\mathrm{orb}}(S,G,s)}(H)$, but from the argumentation at the beginning of the proof, we already have $[\alpha;g]H[\alpha;g]^{-1} \subseteq H$. So we are going to show that the reverse inclusion also holds as follows:

Since $\phi \in \mathrm{Aut}(Y/(G,S))$ with $p \circ \phi = gp$, we also have $\phi^{-1} \in \mathrm{Aut}(Y/(G,S))$ with $p \circ \phi^{-1} = g^{-1}p$. Hence the path $\phi^{-1} \circ \gamma$ projects to $p \circ \phi^{-1} \circ \gamma = g^{-1}\alpha$. Hence $\phi^{-1} \circ \bar{\gamma}$ is a lifting of $g^{-1}\bar{\alpha}$. But this path ends at $\phi^{-1}(\gamma(0)) = \phi^{-1}(y)$, so that ϕ^{-1} does exactly the same as $\phi_{[g^{-1}\bar{\alpha};g^{-1}]}$, i.e. they are the same map. Since $[g^{-1}\bar{\alpha};g^{-1}] = [\alpha;g]^{-1}$. We have $[\alpha;g]^{-1}H([\alpha;g]^{-1})^{-1} \subseteq H$, or equivalently, $H \subseteq [\alpha;g]H[\alpha;g]^{-1}$. This implies that $[\alpha;g] \in N_{\pi_1^{\mathrm{orb}}(S,G,s)}(H)$ as desired.

The last thing to do here is to determine the kernel. So let $[\alpha;g] \in N_{\pi_1^{\mathrm{orb}}(S,G,s)}(H)$ be such that $\phi_{[\alpha;g]} = \mathrm{id}_Y$. Then we have $\phi_{[\alpha;g]}(y) = y$, meaning that α lifts to a loop $\gamma : [0,1] \to Y$ beginning at y, i.e. $\alpha = p \circ \gamma \in H$. Furthermore, the diagram (2.1) implies that $gp = p$, which implies by the surjectivity of p that $g \in G_0$. Hence $[\alpha;g]$ lies in direct product of H and G_0. Note also that H and G_0 build a direct product in $\pi_1^{\mathrm{orb}}(G,S,s)$ since G_0 acts on the elements of H trivially. Conversely, it is easy to see that each element $[\alpha;g]$ with $\alpha \in H$ and $g \in G_0$ yields the trivial deck transformation on Y, so that the kernel of $N_{\pi_1^{\mathrm{orb}}(S,G,s)}(H) \longrightarrow \mathrm{Aut}(Y/(S,G))$ is the direct product of H and G_0 as desired. □

Corollary 2.20. *If $\tilde{S} \to S$ is the universal covering, then we have an exact sequence*

$$1 \longrightarrow G_0 \longrightarrow \pi_1^{\mathrm{orb}}(S,G,s) \longrightarrow \mathrm{Aut}(\tilde{S}/(S,G)) \longrightarrow 1,$$

where $s \in S$ and $G_0 := \ker(G \to \mathrm{Aut}(S))$. In particular, if G acts on S effectively, i.e. G_0 is trivial, then we have an isomorphism between $\pi_1^{\mathrm{orb}}(S,G,s)$ and $\mathrm{Aut}(\tilde{S}/(S,G))$

Proof. Let $\tilde{s}$ be a preimage of s under this covering. Then $\pi_1(\tilde{S},\tilde{s})$ is trivial. With the notation from Proposition 2.19, it means that $H = 1$, and consequently, its normaliser is the whole orbital fundamental group. Hence the claim follows immediately from loc.cit. □

For the universal covering, we can also prove the isomorphism between the orbital fundamental group and the deck transformation group by comparing their actions on the orbital fibre as follows:

Theorem 2.21. *Let $p : \tilde{S} \to S$ be the universal covering and $s \in S$. Furthermore, suppose that the group G acts on S effectively. Then there exists a group action of $\mathrm{Aut}(\tilde{S}/(S,G))$ on the orbital fibre $p^{-1}(s)_G$ from left defined by*

$$\phi(y,h) := (\phi(y), g_\phi h) \quad \text{for all } \phi \in \mathrm{Aut}(\tilde{S}/(S,G)) \text{ and } (y,h) \in p^{-1}(s)_G. \tag{2.2}$$

This action is compatible with the action of $\pi_1^{\mathrm{orb}}(S,G,s)$ from Proposition 2.15. Moreover, this induces the group isomorphism which is the same as the one in Corollary 2.20.

Proof. We show first that (2.2) is a group action. To do this, let $\phi, \psi \in \mathrm{Aut}(\tilde{S}/(S,G))$ and $(y,h) \in p^{-1}(s)_G$. Then we have $p(\phi(y)) = g_\phi p(y) = (g_\phi h)s$, so that the rule given in (2.2) is well-defined. Furthermore,

$$\psi(\phi(y,h)) = \psi(\phi(y), g_\phi h) = ((\psi \circ \phi)(y), g_\psi g_\phi h) = ((\psi \circ \phi)(y), g_{\psi \circ \phi} h) = (\psi \circ \phi)(y,h),$$

so that this is in fact a group action. To show that this is simply transitive, suppose that $(x,g),(y,h) \in p^{-1}(s)_G$. Then $p(x) = gs$ and $p(y) = hs = (hg^{-1})(gs)$, so that we can apply the lifting criterion from Proposition 2.18, using the simply connectedness of $\tilde{S}$, to get a uniquely determined $\phi \in \mathrm{Aut}(\tilde{S}/(S,G))$ such that $p \circ \phi = (hg^{-1})p$ and $\phi(x) = y$, i.e. $\phi(x,g) = (y,h)$. Notice that ϕ is in fact invertible since its inverse is the uniquely determined deck transformation sending (y,h) to (x,g) by lifting criterion. This shows that the action is simply transitive.

The next thing to show is that the action is compatible with the one of $\pi_1^{\mathrm{orb}}(S,G,s)$. So suppose that $\phi \in \mathrm{Aut}(\tilde{S}/(S,G))$ and $[\alpha; g] \in \pi_1^{\mathrm{orb}}(S,G,s)$. Then for $(y,h) \in p^{-1}(s)_G$, we have $\phi(y,h) = (\phi(y), g_\phi h)$ and $(y,h) \cdot [\alpha; g] = (y.h\alpha, hg)$. So we need to show that the end of the lifting of $g_\phi h\alpha$ in Y beginning at $\phi(y)$ is the same as $\phi(y.(h\alpha))$. For this purpose, let $\tilde{\beta}$ be the lifting of $h\alpha$ in Y beginning at y (such a lifting exists uniquely since by definition, $p(y) = hs$). Then $\phi \circ \tilde{\beta}$ begins at $\phi(y)$ and

$$p \circ (\phi \circ \tilde{\beta}) = (p \circ \phi) \circ \tilde{\beta} = (g_\phi p) \circ \tilde{\beta} = g_\phi (p \circ \tilde{\beta}) = g_\phi h\alpha.$$

This shows that $\phi \circ \tilde{\beta}$ is the lifting of $g_\phi h\alpha$ in Y beginning at $\phi(y)$. Since this ends at $\phi(\tilde{\beta}(1)) = \phi(y.(h\alpha))$, we have $(\phi(y,h)) \cdot [\alpha; g] = (\phi(y).g_\phi h\alpha, g_\phi hg) = (\phi(y.(h\alpha)), g_\phi hg) = \phi((y,h) \cdot [\alpha; g])$ as desired, i.e. the right action of $\pi_1^{\mathrm{orb}}(S,G,s)$ is compatible with the left one of $\mathrm{Aut}(\tilde{S}/(S,G))$.

This pair of compatible actions induce a well-defined isomorphism between $\pi_1^{\mathrm{orb}}(S,G,s)$ and $\mathrm{Aut}(\tilde{S}/(S,G))$ as follows: Fix a preimage $\tilde{s} \in \tilde{S}$ of s. Then we send $[\alpha; g] \in \pi_1^{\mathrm{orb}}(S,G,s)$ to $\phi_{[\alpha;g]} \in \mathrm{Aut}(\tilde{S}/(S,G))$ such that $(\tilde{s},1) \cdot [\alpha; g] = \phi_{[\alpha;g]}(\tilde{s},1)$, i.e. $\phi_{[\alpha;g]}(s) = \tilde{s}.\alpha$ and $p \circ \phi_{[\alpha;g]} = gp$, but this is exactly the same as in Corollary 2.20. The isomorphism property can be shown here using the simply transivity of the both actions, and we are done. □

Chapter 3

Square complexes with V_4-actions

In this chapter, we are going to study a special class of square complexes with four vertices. The construction of such a complex is based on a V_4-equivariant vertical-horizontal structure (in short, a V_4-structure) of an abstract group as will be defined in the first section. Furthermore, there is an action of the Klein four-group V_4 on such a complex, so that its orbital fundamental group has a nice presentation and is isomorphic to the group generated by the corresponding V_4-structure under a certain condition. This will play a prominent role later as we shall see that there is a square complex from this class occurring as the quotient of $T_3 \times T_3$ by an arithmetic subgroup of $\mathrm{PGL}_2(\mathbb{F}_2(\!(y)\!)) \times \mathrm{PGL}_2(\mathbb{F}_2(\!(t)\!))$.

3.1 Construction

We begin with the definition of a V_4-equivariant vertical-horizontal structure.

Definition 3.1. A **V_4-equivariant vertical-horizontal structure**, in short **V_4-structure**, of a group Λ is an ordered pair $(\mathcal{A}, \mathcal{B})$ of finite subsets $\mathcal{A}, \mathcal{B} \subseteq \Lambda$ satisfying the following properties:

(1) $\mathcal{A} \cup \mathcal{B}$ generates Λ.

(2) $\mathcal{A}$ and $\mathcal{B}$ are closed under taking inverse.

(3) $\mathcal{AB} = \mathcal{BA}$ and has so many elements as the cartesian product $\mathcal{A} \times \mathcal{B}$, in other words, the maps

$$\mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{AB},\ (a, b) \longmapsto ab \text{ resp. } (a, b) \longmapsto ba$$

are well-defined bijections.

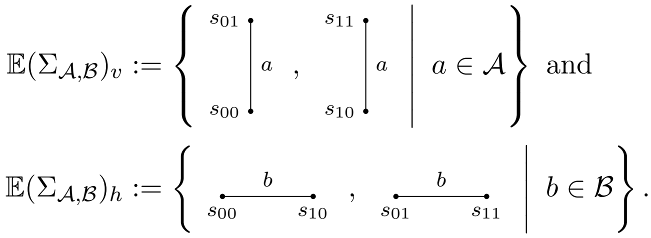
Note that in contrast to a VH-structure in [SV13, §2], we also allow 2-torsions to be in the sets $\mathcal{A}$ and $\mathcal{B}$.

Having introduced a V_4-equivariant vertical-horizontal structure, we can construct a square complex as follows:

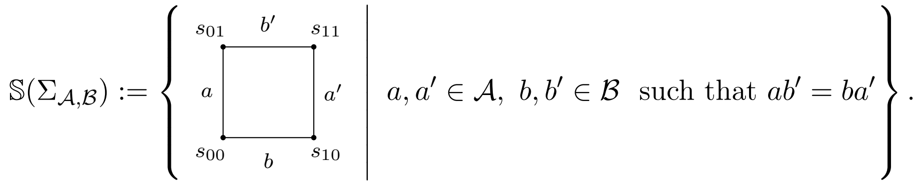
Definition 3.2. The **square complex associated to a V_4-structure** $(\mathcal{A}, \mathcal{B})$ of a group Λ, denoted by $\Sigma_{\mathcal{A},\mathcal{B}}$, is a square complex with the following data: The vertex set is given by $\mathbb{V}(\Sigma_{\mathcal{A},\mathcal{B}}) := \{s_{00}, s_{01}, s_{10}, s_{11}\}$. The set of edges is given by

$$\mathbb{E}(\Sigma_{\mathcal{A},\mathcal{B}}) := \mathbb{E}(\Sigma_{\mathcal{A},\mathcal{B}})_v \sqcup \mathbb{E}(\Sigma_{\mathcal{A},\mathcal{B}})_h,$$

where $\mathbb{E}(\Sigma_{\mathcal{A},\mathcal{B}})_v$ and $\mathbb{E}(\Sigma_{\mathcal{A},\mathcal{B}})_h$ are the sets of vertical and horizontal edges respectively defined by



Finally, the set of squares in $\Sigma_{\mathcal{A},\mathcal{B}}$ is given by



Note that $\#\mathbb{S}(\Sigma_{\mathcal{A},\mathcal{B}}) = (\#\mathcal{A})(\#\mathcal{B})$ by the definition of a V_4-structure and the fact that each square is attached to all four vertices.

Notation 3.3. For future references, we shall let $(a, i) \in \mathcal{A} \times \{0, 1\}$ denote the vertical edge connecting the vertices s_{i0} and s_{i1} labelled by a, and $(b, j) \in \mathcal{B} \times \{0, 1\}$ the horizontal edge connecting the vertices s_{0j} and s_{1j} labelled by b. Furthermore, we shall write $\mathcal{A}_i := \mathcal{A} \times \{i\}$ for $i \in \{0, 1\}$ and $\mathcal{B}_j := \mathcal{B} \times \{j\}$ for $j \in \{0, 1\}$, i.e.

$$\mathbb{E}(\Sigma_{\mathcal{A},\mathcal{B}})_v = \mathcal{A} \times \{0, 1\} = \mathcal{A}_0 \sqcup \mathcal{A}_1 \quad \text{and} \quad \mathbb{E}(\Sigma_{\mathcal{A},\mathcal{B}})_h = \mathcal{B} \times \{0, 1\} = \mathcal{B}_0 \sqcup \mathcal{B}_1.$$

For $a, a' \in \mathcal{A}$ and $b, b' \in \mathcal{B}$ such that $ab' = ba'$, the square with the edges $(a, 0)$, $(b', 1)$, $(b, 0)$ and $(a', 1)$ will be denoted by

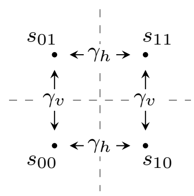
$$[a, b'; b, a'].$$

The name V_4-equivariant vertical-horizontal structure comes from the fact that we can define a V_4-action on the square complex $\Sigma_{\mathcal{A},\mathcal{B}}$. We shall use the following notation for the group V_4:

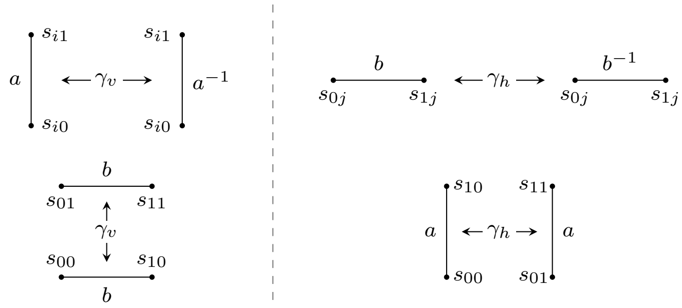
$$V_4 = \langle \gamma_v, \gamma_h \mid \gamma_v^2 = \gamma_h^2 = 1,\ \gamma_v\gamma_h = \gamma_h\gamma_v \rangle \cong Z_2^2,$$

where $Z_2 := \mathbb{Z}/2\mathbb{Z}$ denotes the cyclic group of order 2.

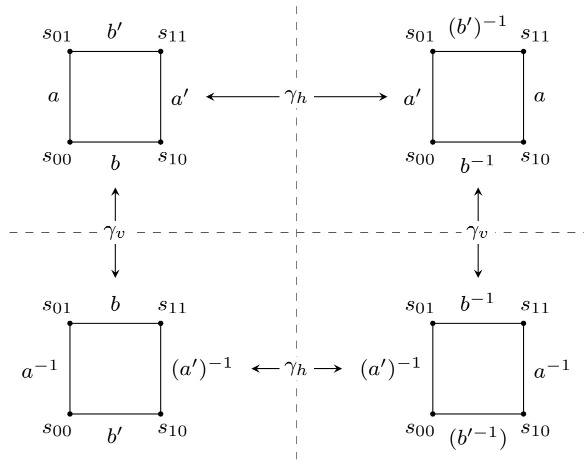
Definition 3.4. We define the action of the group V_4 on the square complex $\Sigma = \Sigma_{\mathcal{A},\mathcal{B}}$ as follows: On the set $\mathbb{V}(\Sigma)$ of vertices, γ_v interchanges s_{ij} with $s_{i(1-j)}$, while γ_h interchanges s_{ij} with $s_{(1-i)j}$ $(i, j \in \{0, 1\})$. This can be visualised by the diagram below.



To define the action of V_4 on $\mathbb{E}(\Sigma)$, we let γ_v interchange the edge $(a, i) \in \mathcal{A} \times \{0, 1\}$ with (a^{-1}, i), and the edge $(b, 0)$ with $(b, 1)$ for $b \in \mathcal{B}$. Furthermore, we let γ_h interchange the edge $(a, 0)$ with $(a, 1)$ for $a \in \mathcal{A}$, and the edge $(b, j) \in \mathcal{B} \times \{0, 1\}$ with (b^{-1}, j).



The action of V_4 on the vertices and edges of Σ forces the action on the square of Σ to be as shown in the diagram below.



Remark 3.5. From the definition, we can assert the following statements:

(1) Since the relation $ab' = ba'$ is equivalent to the relations $a'(b')^{-1} = b^{-1}a$, $a^{-1}b = b'(a')^{-1}$ and $(a')^{-1}b^{-1} = (b')^{-1}a^{-1}$, the V_4-action as above is well-defined.

(2) This V_4-action apparently respects the vertical-horizontal structure of the complex.

3.2 Topological properties of $\Sigma_{\mathcal{A},\mathcal{B}}$

In this section, we shall establish some basic topological properties of the square complex $\Sigma_{\mathcal{A},\mathcal{B}}$ we have just constructed in the previous section. The first property deals with its universal covering space.

Lemma 3.6. *For each point $s \in \mathbb{V}(\Sigma_{\mathcal{A},\mathcal{B}})$, the link* Lk_s *is a complete bipartite graph with vertical vertices labelled by A and horizontal vertices labelled by B.*

Recall that for a square complex Σ and a vertex $s \in \mathbb{V}(\Sigma)$, the **link** $\mathrm{Lk}_s = \mathrm{Lk}_s(\Sigma)$ is the (undirected multi-)graph whose set of vertices is given by the end of the edges in Σ attached to s (in particular, a loop at s corresponds to two vertices in Lk_s), and whose set of edges joining two vertices $a, b \in \mathrm{Lk}_s$ is given by the square corners attached to a and b.

Proof. We shall prove this first for $s = s_{01}$, namely that for each $a \in \mathcal{A}$ and $b \in \mathcal{B}$, there exists a unique square having an edge from s_{01} to s_{00} labelled by a and an edge from s_{01} to s_{11} labelled by b. Since the squares of $\Sigma_{\mathcal{A},\mathcal{B}}$ are of the form $[a, b\,; b', a']$ with $a, a' \in \mathcal{A}$ and $b, b' \in \mathcal{B}$ such that $ab = b'a'$, we have to show that for each $a \in \mathcal{A}$ and $b \in \mathcal{B}$, there are unique $a' \in \mathcal{A}$ and $b' \in \mathcal{B}$ such that $ab = b'a'$. But this is contained in the definition of a V_4-structure. Thus the link of s_{01} is a complete bipartite graph since each square attached to s_{01} has exactly one corner at s_{01}.

The claim for the other vertices follows by considering the V_4-action on $\Sigma_{\mathcal{A},\mathcal{B}}$ defined above, which is transitive on the set of the vertices of $\Sigma_{\mathcal{A},\mathcal{B}}$, and the fact that the action of each element in V_4 yields an isomorphism between the links of the corresponding vertices. □

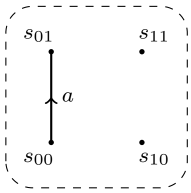
Corollary 3.7. *If $m = \#\mathcal{A}$ and $n = \#\mathcal{B}$, then the universal covering space of Σ is a product of trees $T_m \times T_n$.*

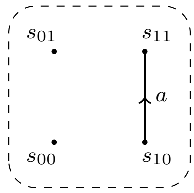
Proof. This follows from Lemma 3.6 and the fact that the universal covering space of a square complex is a product of trees if and only if the links of its vertices are all complete bipartite graphs, see [Wis07, Thm.3.8] □

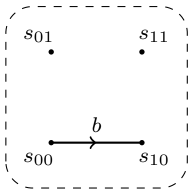
Now we come to the computation of the fundamental group $\pi_1(\Sigma, s_{00})$.

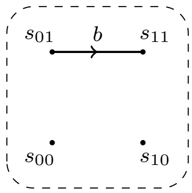
Definition 3.8. In the square complex $\Sigma = \Sigma_{\mathcal{A},\mathcal{B}}$, we define the following paths:

- For each $a \in \mathcal{A}$ and $i \in \{0, 1\}$, $\delta_{i,a}$ is the path from s_{i0} to s_{i1} along the edge a.
- For each $b \in \mathcal{B}$ and $i \in \{0, 1\}$, $\varepsilon_{i,b}$ is the path from s_{0i} to s_{1i} along the edge b.

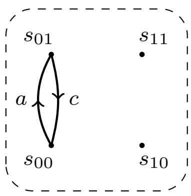
$\delta_{0a} =$ 

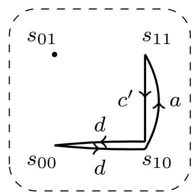
$\delta_{1a} =$ 

$\varepsilon_{0b} =$ 

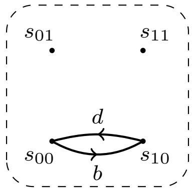
$\varepsilon_{1b} =$ 

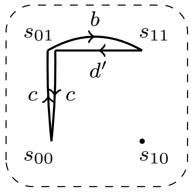
Now choose a square $[c, d'; d, c']$ of the square complex $\Sigma = \Sigma_{\mathcal{A},\mathcal{B}}$. For each $a \in \mathcal{A}$ and $b \in \mathcal{B}$, we define the following loops:

$x_{0a} = \delta_{0,a} * \overline{\delta_{0,c}} =$ 

, $x_{1a} = \varepsilon_{0,d} * \delta_{1,a} * \overline{\delta_{1,c'}} * \overline{\varepsilon_{0,d}} =$ 

,

$y_{0b} = \varepsilon_{0,b} * \overline{\varepsilon_{0,d}} =$ 

, $y_{1b} = \delta_{0,c} * \varepsilon_{1,b} * \overline{\varepsilon_{1,d'}} * \overline{\delta_{0,c}} =$ 

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Using the Seifert-van Kampen theorem, we obtain the presentation of the fundamental group $\pi_1(\Sigma, s_{00})$ as in the following lemma.

Lemma 3.9. *Under the notations as above, the fundamental group $\pi_1(\Sigma, s_{00})$ has the following presentation:*

$$\pi_1(\Sigma, s_{00}) = \left\langle \begin{array}{c} x_{0a}, x_{1a}, y_{0b}, y_{1b} \\ \text{for } a \in \mathcal{A},\ b \in \mathcal{B} \end{array} \middle| \begin{array}{c} x_{0a}y_{1b'} = y_{0b}x_{1a'} \\ \text{for } a, a' \in \mathcal{A},\ b, b' \in \mathcal{B} \text{ s.t. } ab' = ba' \text{ in } \Lambda \\ x_{0c} = x_{1c'} = y_{0d} = y_{1d'} = 1 \end{array} \right\rangle \tag{3.1}$$

Proof. Let $X \subseteq \Sigma$ be the 1-skeleton of the square complex $\Sigma_{\mathcal{A},\mathcal{B}}$. Then by the Seifert-van Kampen theorem, the fundamental group $\pi_1(X, s_{00})$ is a free group generated by the loops x_{0a} for $a \in \mathcal{A} \setminus \{c\}$, y_{0b} for $b \in \mathcal{B} \setminus \{d\}$, x_{1a} for $a \in \mathcal{A} \setminus \{c'\}$, y_{1b} for $b \in \mathcal{B} \setminus \{d'\}$ as well as the loop $z := \delta_{0,c} * \varepsilon_{1,d'} * \overline{\delta_{1,c'}} * \overline{\varepsilon_{0,d}}$, i.e.

$$\pi_1(X, s_{00}) = \left\langle x_{0a}, x_{1a}, y_{0b}, y_{1b}, z \;\middle|\; a \in \mathcal{A},\ b \in \mathcal{B},\ x_{0c} = x_{1c'} = y_{0d} = y_{1d'} = 1 \right\rangle.$$

By [Hat01, Prop.1.26], $\pi_1(\Sigma, s_{00})$ is the quotient of $\pi_1(X, s_{00})$ by its normal subgroup N generated by the homotopy classes of those loops around the squares attached to Σ. These are exactly the loops of the form $\delta_{0,a} * \varepsilon_{1,b'} * \overline{\delta_{1,a'}} * \overline{\varepsilon_{1,b}}$ for $a, a' \in \mathcal{A}$ and $b, b' \in \mathcal{B}$ such that $ab' = ba'$. And as homotopy class of paths in X, we have

$$\delta_{0,a} * \varepsilon_{1,b'} * \overline{\delta_{1,a'}} * \overline{\varepsilon_{1,b}} = x_{0a}y_{1b'}z(y_{0b}x_{1a'})^{-1}.$$

This is especially the case for $(a, b', b, a') = (c, d', d, c')$, where the homotopy classes of x_{0c}, $x_{1c'}$, y_{0d} and $y_{1d'}$ vanish in $\pi_1(X, s_{00})$, so that z also lies in N. Consequently, the generator z of $\pi_1(X, s_{00})$ vanishes from the presentation of $\pi_1(\Sigma, s_{00})$, Furthermore we can replace the generators $\delta_{0,a} * \varepsilon_{1,b'} * \overline{\delta_{1,a'}} * \overline{\varepsilon_{1,b}}$ of N by $x_{0a}y_{1b'}(y_{0b}x_{1a'})^{-1}$, which can be replaced by the relations $x_{0a}y_{1b'} = y_{0b}x_{1a'}$ as in (3.1), and we are done. □

In the next step, we are going to extend this group to the orbital fundamental group $\pi_1^{\mathrm{orb}}(\Sigma_{\mathcal{A},\mathcal{B}}, V_4, s_{00})$. In order to find its presentation, we are going to study presentations of group extensions in details first as to be done in the next section.

3.3 Presentations of group extensions

Now that we have a presentation for the fundamental group $\pi_1(\Sigma_{\mathcal{A},\mathcal{B}}, s_{00})$ and know from the last chapter that this is a normal subgroup the orbital fundamental group $\pi_1^{\mathrm{orb}}(\Sigma_{\mathcal{A},\mathcal{B}}, V_4, s_{00})$ with V_4 as quotient, we are going to find a presentation for $\pi_1^{\mathrm{orb}}(\Sigma_{\mathcal{A},\mathcal{B}}, V_4, s_{00})$ based on this fact. This leads to studying presentations of a group occurring as a group extension of two other groups with known presentations as to be demonstrated in this section.

To fix a notation, by a group extension of a group G by a group Γ, we shall mean a group E fitting into an exact sequence

$$1 \longrightarrow \Gamma \longrightarrow E \longrightarrow G \longrightarrow 1.$$

We shall begin with a construction of such group extensions.

Definition 3.10. An **extension datum** of G by Γ is a pair $(\{\sigma_g\}_{g\in G}, x)$ consisting of a collection of automorphisms

$$\sigma_g : \Gamma \longrightarrow \Gamma,\ \alpha \longmapsto \sigma_g(\alpha)$$

for each $g \in G$ and a mapping

$$x : G \times G \longrightarrow \Gamma,\ (g,h) \longmapsto x(g,h),$$

such that the following properties hold for all $g, h, k \in G$:

(E0) $\sigma_1 = \mathrm{id}_\Gamma$ and $x(g,h) = 1$ whenever $g = 1$ or $h = 1$,

(E1) $\sigma_g \circ \sigma_h = \iota_{x(g,h)} \circ \sigma_{gh}$, where $\iota_\alpha \in \mathrm{Aut}(\Gamma)$ denotes the conjugation by $\alpha \in \Gamma$,

(E2) $x(g,h) \cdot x(gh,k) = \sigma_g(x(h,k)) \cdot x(g,hk)$.

Note that if Γ is an abelian group, then the condition (E1) says that we have a group homomorphism $G \to \mathrm{Aut}(\Gamma), g \mapsto \sigma_g$, i.e. Γ has a structure as a G-module. The condition (E2) is nothing other than the cocycle condition, i.e. an extension datum is simply a 2-cocycle of Γ as a G-module.

To justify the name extension datum, we are going to prove in the next proposition that every extension datum yields a group extension, and that every group extension comes form such a datum.

Proposition 3.11. *Let Γ and G be two groups.*

(1) *Each extension datum $(\{\sigma_g\}_{g\in G}, x)$ defines a group extension E of G by Γ as follows: The underlying set of E is the cartesian product $\Gamma \times G$ and the group law is defined by*

$$(\alpha, g)(\beta, h) := (\alpha \cdot \sigma_g(\beta) \cdot x(g,h), gh)$$

for each $\alpha, \beta \in \Gamma$ and $g, h \in G$.

(2) *Each group extension corresponds to an extension datum, i.e., given a group extension*

$$1 \longrightarrow \Gamma \longrightarrow E \longrightarrow G \longrightarrow 1,$$

there is an extension datum such that the group extension obtained by the construction above is isomorphic to the given one.

Proof. For the first statement, the group axioms can be showed by a direct calculation. The neutral element is $(1,1)$ and the inverse element of $(\alpha, g) \in E$ is $(\sigma_{g^{-1}}(\alpha \cdot x(g, g^{-1}))^{-1}, g^{-1})$.

For the second statement, consider a group extension

$$1 \longrightarrow \Gamma \longrightarrow E \xrightarrow{\ p\ } G \longrightarrow 1,$$

Choose a lift $\widehat{g} \in E$ for each $g \in G$ with $\widehat{1} := 1$ and define the extension datum $(\{\sigma_g\}_g, x)$ as follows:

$$\sigma_g : \Gamma \to \Gamma,\ \alpha \mapsto \widehat{g}\alpha\widehat{g}^{-1} \quad \text{for each } g \in G, \text{ and}$$
$$x(g,h) := \widehat{g}\widehat{h}\widehat{gh}^{-1} \quad \text{for each } g, h \in G.$$

Note that $\widehat{g}\alpha\widehat{g}^{-1} \in \Gamma$ since $p(\widehat{g}\alpha\widehat{g}^{-1}) = gg^{-1} = 1$ and $x(g,h) \in \Gamma$ since $p(x(g,h)) = gh(gh)^{-1} = 1$. One can see by a direct calculation that this satisfies the axioms of an extension datum. Let

$\tilde{E}$ be the group obtained by this extension datum. Then there are isomorphisms between E and $\tilde{E}$ given by

$$E \longrightarrow \tilde{E}, \ \gamma \longmapsto (\gamma \cdot \widehat{p(\gamma)}^{-1}, p(\gamma))$$

and

$$\tilde{E} \longrightarrow E, \ (\alpha, g) \longmapsto \alpha\widehat{g}.$$

One can see easily that this is also compatible with the embeddings of Γ in E resp. $\tilde{E}$ and their projections onto G. This implies that the both group extensions are isomorphic to each other. $\square$

Now we are going to give a presentation of a group extension. In virtue of the previous Proposition, we shall begin with a group extension obtained by an extension datum.

Proposition 3.12. *Let Γ be a group with the following presentation:*

$$\Gamma = \langle\, u_1, \ldots, u_n \mid R_1(u_1, \ldots, u_n) = \ldots = R_m(u_1, \ldots, u_n) = 1 \,\rangle\,.$$

Let G be another group, $(\{\sigma_g\}_{g\in G}, x)$ an extension datum of G by Γ, and E the group extension obtained by this datum. Then E has the following presentation:

$$E = \left\langle \begin{array}{c} u_1, \ldots, u_n, \\ \gamma_g \text{ for } g \in G \end{array} \middle| \begin{array}{c} R_1(u_1, \ldots, u_n) = \ldots = R_m(u_1, \ldots u_n) = \gamma_1 = 1 \\ \gamma_g u_i \gamma_g^{-1} = \sigma_g(u_i) \text{ for } g \in G \text{ and } i = 1, \ldots, n \\ \gamma_g\gamma_h\gamma_{gh}^{-1} = x(g,h) \text{ for } g, h \in G \end{array} \right\rangle.$$

Proof. Let $\tilde{E}$ be the group with the presentation as above. The isomorphisms between E and $\tilde{E}$ are given by

$$E \longrightarrow \tilde{E}, \ (\alpha, g) \longmapsto W_\alpha(u_1, \ldots, u_n)\gamma_g,$$

where $W_\alpha(u_1, \ldots, u_n)$ denotes a fixed word in $u_i^{\pm 1}$ representing $\alpha \in \Gamma$, and

$$\tilde{E} \longrightarrow E, \ u_i \longmapsto (u_i, 1) \text{ and } \gamma_g \longmapsto (1, g).$$

Note that the latter one is compatible with the given relations in $\tilde{E}$, i.e. it is a well-defined homomorphism. Furthermore, both are inverse to each other. In particular, both maps are in fact isomorphisms. Hence we get a presentation of E as desired. $\square$

Corollary 3.13. *Let Γ be a group with a presentation as in Proposition 3.12, G be another group, and*

$$1 \longrightarrow \Gamma \longrightarrow E \xrightarrow{p} G \longrightarrow 1$$

be a group extension. Choosing for each $g \in G$ a lifting $\hat{g} \in E$, we have the following presentation:

$$E = \left\langle \begin{array}{c} u_1, \ldots, u_n, \\ \gamma_g \text{ for } g \in G \end{array} \middle| \begin{array}{c} R_1(u_1, \ldots, u_n) = \ldots = R_m(u_1, \ldots u_n) = \gamma_1 = 1 \\ \gamma_g u_i \gamma_g^{-1} = \widehat{g} u_i \widehat{g}^{-1} \text{ for } g \in G \text{ and } i = 1, \ldots, n \\ \gamma_g\gamma_h\gamma_{gh}^{-1} = \widehat{g}\widehat{h}\widehat{gh}^{-1} \text{ for } g, h \in G \end{array} \right\rangle.$$

Proof. This follows immediately from Propositions 3.11 and 3.12 $\square$

And as the last result of this section, we can simplify the presentation from Corollary 3.13 if the extension has a section as follows:

Corollary 3.14. *Under the same condition as above, suppose that the sequence has a section $s : G \to E$ and G has a presentation*

$$G = \langle\, g_1, \dots, g_l \mid S_1(g_1, \dots, g_l) = \dots = S_k(g_1, \dots, g_l) = 1 \,\rangle\,.$$

Then E has the following presentation:

$$E = \left\langle \begin{array}{c} u_1, \dots, u_n, \\ g_1, \dots, g_l \end{array} \left| \begin{array}{c} R_1(u_1, \dots, u_n) = \dots = R_m(u_1, \dots u_n) = 1 \\ S_1(g_1, \dots, g_l) = \dots = S_k(g_1, \dots, g_l) = 1 \\ g_j u_i g_j^{-1} = s(g_j) u_i s(g_j)^{-1} \textit{ for } i = 1, \dots, n;\ j = 1, \dots, l \end{array} \right. \right\rangle. \tag{3.2}$$

Proof. From Corollary 3.13, choosing $s(g)$ as a lift of each $g \in G$, so that $s(g)s(h) = s(gh)$ for each $g, h \in G$, we obtain the following presentation:

$$E = \left\langle \begin{array}{c} u_1, \dots, u_n, \\ \gamma_g \text{ for } g \in G \end{array} \left| \begin{array}{c} R_1(u_1, \dots, u_n) = \dots = R_m(u_1, \dots u_n) = \gamma_1 = 1 \\ \gamma_g u_i \gamma_g^{-1} = s(g) u_i s(g)^{-1} \text{ for } g \in G \text{ and } i = 1, \dots, n \\ \gamma_g \gamma_h = \gamma_{gh} \text{ for } g, h \in G \qquad (*) \end{array} \right. \right\rangle \tag{3.3}$$

Notice first that the relation $(*)$ implies that the assignment $G \to E,\ g \mapsto \gamma_g$ is a group homomorphism. Hence we have $\gamma_1 = 1$, $\gamma_{g^{-1}} = \gamma_g^{-1}$ for all $g \in G$ and the relations

$$S_1(\gamma_{g_1}, \dots, \gamma_{g_l}) = \dots = S_k(\gamma_{g_1}, \dots, \gamma_{g_l}) = 1$$

in E, so that we can add this relations to the above presentation of E.

Now we are going to eliminate the generators γ_g for $g \in G \setminus \{g_1, \dots, g_l\}$. Let $W_g(g_1, \dots, g_l)$ be a fixed word in $g_i^{\pm 1}$ representing g. The relation $(*)$ implies that $W_g(\gamma_{g_1}, \dots, \gamma_{g_l}) = \gamma_g$, so that all γ_g's for $g \in G \setminus \{g_1, \dots, g_l\}$ can be eliminated, but then we obtain instead the following relations:

$$W_g(\gamma_{g_1}, \dots, \gamma_{g_l}) W_h(\gamma_{g_1}, \dots, \gamma_{g_l}) W_{gh}(\gamma_{g_1}, \dots, \gamma_{g_l})^{-1} = 1 \qquad \text{for all } g, h \in G. \tag{3.4}$$

Since this also holds when replacing γ_{g_i} by g_i in G, the words $W_g W_h W_{gh}^{-1}$ must be products of some conjugations of the words $S_1, \dots, S_k$ and their inverses. Since we have the relations $S_1(\gamma_{g_1}, \dots, \gamma_{g_l}) = \dots = S_k(\gamma_{g_1}, \dots, \gamma_{g_l}) = 1$, all relations from (3.4) can be eliminated. By replacing all γ_{g_i} in the presentation we have up to now by g_i for $i = 1, \dots, l$, we get the presentation as in (3.2). Hence we are done. ☐

3.4 The orbital fundamental group of $\Sigma_{\mathcal{A},\mathcal{B}}$ by V_4

Having discussed how to determine a presentation for a group extension, we now come to finding a presentation for the orbital fundamental group $\pi_1^{\mathrm{orb}}(\Sigma_{\mathcal{A},\mathcal{B}}, V_4, s_{00})$ as a group extension of V_4 by $\pi_1(\Sigma_{\mathcal{A},\mathcal{B}}, s_{00})$. In fact, the presentation we obtained from the theory in the previous section can be modified to get a nice one as follows:

Proposition 3.15. *Let $(\mathcal{A}, \mathcal{B})$ be a V_4-structure of Λ, and let $\Sigma = \Sigma_{\mathcal{A},\mathcal{B}}$ be the square complex with the V_4-action as in Definitions 3.2 and 3.4. Furthermore, define*

$$\alpha_a := [\delta_{0,a}, \gamma_v],\ \beta_b := [\varepsilon_{0,b}, \gamma_h] \in \pi_1^{\mathrm{orb}}(\Sigma, V_4, s_{00}) \quad \textit{for each } a \in \mathcal{A},\ b \in \mathcal{B}.$$

Then the orbital fundamental group $\pi_1^{\mathrm{orb}}(\Sigma, V_4, s_{00})$ has the following presentation:

$$\pi_1^{\mathrm{orb}}(\Sigma, V_4, s_{00}) = \left\langle \begin{array}{c} \alpha_a, \beta_b \\ \textit{for } a \in \mathcal{A},\ b \in \mathcal{B} \end{array} \left| \begin{array}{c} \alpha_a \beta_{b'} = \beta_b \alpha_{a'} \textit{ for } a, a' \in \mathcal{A},\ b, b' \in \mathcal{B} \\ \textit{s.t. } ab' = ba' \textit{ in } \Lambda, \\ \alpha_a \alpha_{a^{-1}} = \beta_b \beta_{b^{-1}} = 1 \textit{ for } a \in \mathcal{A},\ b \in \mathcal{B} \end{array} \right. \right\rangle. \tag{3.5}$$

Proof. Recall first that from Lemma 3.9, we have the following presentation for the fundamental group of the square complex Σ:

$$\pi_1(\Sigma, s_{00}) = \left\langle \begin{array}{c} x_{0a}, x_{1a}, y_{0b}, y_{1b} \\ \text{for } a \in \mathcal{A},\ b \in \mathcal{B} \end{array} \middle| \begin{array}{c} x_{0a}y_{1b'} = y_{0b}x_{0a'} \\ \text{for } a, a' \in \mathcal{A},\ b, b' \in \mathcal{B} \text{ s.t. } ab' = ba' \text{ in } \Lambda \\ x_{0c} = x_{1c'} = y_{0d} = y_{1d'} = 1 \end{array} \right\rangle.$$

We shall compute the orbital fundamental group using the exact sequence from Proposition 2.9 and the presentation above. In fact, α_c is a lifting of γ_v and β_d a lifting of γ_h. And we have the following relations for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$:

$$\alpha_c x_{0a} \alpha_c^{-1} = x_{0,a^{-1}}^{-1} x_{0,c^{-1}} \tag{3.6a}$$
$$\alpha_c x_{1a} \alpha_c^{-1} = y_{1d} x_{1,a^{-1}}^{-1} x_{1,(c')^{-1}} y_{1d}^{-1} \tag{3.6b}$$
$$\alpha_c y_{0b} \alpha_c^{-1} = y_{1b} y_{1d}^{-1} \tag{3.6c}$$
$$\alpha_c y_{1b} \alpha_c^{-1} = x_{0,c^{-1}}^{-1} y_{0b} y_{0d'}^{-1} x_{0,c^{-1}} \tag{3.6d}$$
$$\beta_d x_{0a} \beta_d^{-1} = x_{1a} x_{1c}^{-1} \tag{3.6e}$$
$$\beta_d x_{1a} \beta_d^{-1} = y_{0,d^{-1}} x_{0a} x_{0c'}^{-1} y_{0,d^{-1}} \tag{3.6f}$$
$$\beta_d y_{0b} \beta_d^{-1} = y_{0,b^{-1}}^{-1} y_{0,d^{-1}} \tag{3.6g}$$
$$\beta_d y_{1b} \beta_d^{-1} = x_{1c} y_{1,b^{-1}}^{-1} y_{1,(d')^{-1}} x_{1c}^{-1} \tag{3.6h}$$

Furthermore, choosing $\alpha_c\beta_d$ as a lifting of γ_r, we get the following relations:

$$\begin{aligned}
x(\gamma_v, \gamma_v) &= \alpha_c^2 = x_{0,c^{-1}}^{-1} \\
x(\gamma_v, \gamma_h) &= 1 \\
x(\gamma_v, \gamma_r) &= \alpha_c(\alpha_c\beta_d)\beta_d^{-1} = \alpha_c^2 = x(\gamma_v, \gamma_v) \\
x(\gamma_h, \gamma_v) &= \beta_d\alpha_c(\alpha_c\beta_d)^{-1} = x_{1c} y_{1d}^{-1} \\
x(\gamma_h, \gamma_h) &= \beta_d^2 = y_{0,d^{-1}}^{-1} \\
x(\gamma_h, \gamma_r) &= \beta_d(\alpha_c\beta_d)\alpha_c^{-1} = \beta_d x(\gamma_h, \gamma_v)^{-1} x(\gamma_h, \gamma_h)\beta_d^{-1} \\
x(\gamma_r, \gamma_v) &= \alpha_c\beta_d\alpha_c\beta_d^{-1} = \alpha_c x(\gamma_h, \gamma_v) x(\gamma_v, \gamma_v)\alpha_c^{-1} \\
x(\gamma_r, \gamma_h) &= (\alpha_c\beta_d)\beta_d\alpha_c^{-1} = \alpha_c x(\gamma_h, \gamma_h)\alpha_c^{-1} \\
x(\gamma_r, \gamma_r) &= \alpha_c\beta_d\alpha_c\beta_d = x(\gamma_r, \gamma_v) x(\gamma_h, \gamma_h)
\end{aligned}$$

Hence the group $\pi_1^{\text{orb}}(\Sigma, V_4, s_{00})$ has the following presentation:

$$\pi_1^{\text{orb}}(\Sigma, V_4, s_{00}) = \left\langle \begin{array}{c} x_{0a}, x_{1a}, y_{0b}, y_{1b} \\ \text{for } a \in \mathcal{A},\ b \in \mathcal{B} \\ \alpha_c, \beta_d \end{array} \middle| \begin{array}{c} x_{0a}y_{1b'} = y_{0b}x_{0a'} \text{ for } a, a' \in \mathcal{A},\ b, b' \in \mathcal{B} \text{ s.t. } ab' = ba' \\ x_{0c} = x_{1c'} = y_{0d} = y_{1d'} = 1,\ \beta_d\alpha_c\beta_d^{-1}\alpha_c^{-1} = x_{1c}y_{1d}^{-1} \\ \alpha_c^2 = x_{0,c^{-1}}^{-1},\ \beta_d^2 = y_{0,d^{-1}}^{-1} \text{ and (3.6a) – (3.6h) hold} \end{array} \right\rangle.$$

To express a presentation of $\pi_1^{\text{orb}}(\Sigma, V_4, s_{00})$ in terms of α_a's and β_b's for $a \in \mathcal{A}$ and $b \in \mathcal{B}$, observe first that its elements x_{0a}, x_{1a} for $a \in \mathcal{A}$ and y_{0b}, y_{1b} for $b \in \mathcal{B}$ can be expressed in terms of α_a's and β_b's as follows:

$$\begin{aligned}
x_{0a} &= \alpha_a\alpha_c^{-1}, \\
x_{1a} &= \beta_d\alpha_a\alpha_{c'}^{-1}\beta_d^{-1}, \\
y_{0b} &= \beta_b\beta_d^{-1}, \qquad \text{and} \\
y_{1b} &= \alpha_c\beta_b\beta_{d'}^{-1}\alpha_c^{-1},
\end{aligned}$$

so that α_a's and β_b's for $a \in \mathcal{A}$ and $b \in \mathcal{B}$ generate the group $\pi_1^{\text{orb}}(\Sigma, V_4, s_{00})$. One sees immediately from these equations that $x_{0c} = x_{1c'} = y_{0d} = y_{1d'} = 1$. Furthermore, the given relations for $\pi_1^{\text{orb}}(\Sigma, V_4, s_{00})$ can be expressed in terms of α_a's and β_b's as follows:

- $\beta_d\alpha_c\beta_d^{-1}\alpha_c^{-1} = x_{1c}y_{1d}^{-1} = \beta_d\alpha_c\alpha_{c'}^{-1}\beta_d^{-1}\alpha_c\beta_{d'}\beta_d^{-1}\alpha_c^{-1}$ is equivalent to $\alpha_{c'}^{-1}\beta_d^{-1}\alpha_c\beta_{d'} = 1$, which is equivalent to $\alpha_c\beta_{d'} = \beta_d\alpha_{c'}$.
- $\alpha_c^2 = x_{0,c^{-1}}^{-1} = \alpha_c\alpha_{c^{-1}}^{-1}$ is equivalent to $\alpha_c\alpha_{c^{-1}} = 1$.
- $\beta_d^2 = y_{0,d^{-1}}^{-1} = \beta_d\beta_{d^{-1}}^{-1}$ is equivalent to $\beta_d\beta_{d^{-1}} = 1$.
- $x_{0a}y_{1b'} = y_{0b}x_{0a'}$ is equivalent to $\alpha_a\beta_{b'}\beta_{d'}^{-1}\alpha_c^{-1} = \beta_b\alpha_{a'}\alpha_{c'}^{-1}\beta_d^{-1}$. Using that $\alpha_c\beta_{d'} = \beta_d\alpha_{c'}$, this is equivalent to $\alpha_a\beta_{b'} = \beta_b\alpha_{a'}$.
- (3.6a) is equivalent to $\alpha_c\alpha_a\alpha_c^{-2} = \alpha_c\alpha_{a^{-1}}^{-1}\alpha_{c^{-1}}\alpha_c^{-1}$. Using that $\alpha_c\alpha_{c^{-1}} = 1$, this is equivalent to $\alpha_a\alpha_{a^{-1}} = 1$.
- (3.6g) is equivalent to $\beta_d\beta_b\beta_d^{-2} = \beta_d\beta_{b^{-1}}^{-1}\beta_{d^{-1}}\beta_d^{-1}$. Using that $\beta_d\beta_{d^{-1}} = 1$, this is equivalent to $\beta_b\beta_{b^{-1}} = 1$.
- The other relations from (3.6a) to (3.6h) are equivalent to the products of the relations above with elements from $\pi_1^{\text{orb}}(\Sigma, V_4, s_{00})$

Hence we obtain the presentation for $\pi_1^{\text{orb}}(\Sigma, V_4, s_{00})$ as in (3.5), and we are done. □

Remark 3.16. The presentation in (3.5) can be reduced as follows: Let $\bar{\mathbb{S}}(\Sigma_{\mathcal{A},\mathcal{B}})$ be a set of representatives of the V_4-orbits of squares in $\Sigma_{\mathcal{A},\mathcal{B}}$. Then for $[a, b'; b, a'] \in \bar{\mathbb{S}}(\Sigma_{\mathcal{A},\mathcal{B}})$, we have the relation

$$\alpha_a\beta_{b'} = \beta_b\alpha_{a'}.$$

Using that $\alpha_{a^{-1}} = \alpha_a^{-1}$, $\beta_{b^{-1}} = \beta_b^{-1}$ etc., this is equivalent to the relations

$$\alpha_{a^{-1}}\beta_b = \beta_{b'}\alpha_{(a')^{-1}},\ \alpha_{a'}\beta_{(b')^{-1}} = \beta_b\alpha_{a^{-1}} \text{ and } \alpha_{(a')^{-1}}\beta_{b^{-1}} = \beta_{(b')^{-1}}\alpha_{a^{-1}}.$$

These three relations correspond to the squares

$$[a^{-1}, b; b', (a')^{-1}],\ [a', (b')^{-1}; b, a^{-1}] \text{ and } [(a')^{-1}, b^{-1}; (b')^{-1}, a^{-1}],$$

which are exactly those in the V_4-orbit of $[a, b'; b, a']$. This show that it suffices to take only one relation associated to a square from each V_4-orbit, i.e.

$$\pi_1^{\text{orb}}(\Sigma, V_4, s_{00}) = \left\langle \begin{array}{c} \alpha_a, \beta_b \\ \text{for } a \in \mathcal{A},\ b \in \mathcal{B} \end{array} \middle| \begin{array}{c} \alpha_a\beta_{b'} = \beta_b\alpha_{a'} \text{ for } [a, b'; b, a'] \in \bar{\mathbb{S}}(\Sigma_{\mathcal{A},\mathcal{B}}), \\ \alpha_a\alpha_{a^{-1}} = \beta_b\beta_{b^{-1}} = 1 \text{ for } a \in \mathcal{A},\ b \in \mathcal{B} \end{array} \right\rangle.$$

Furthermore, we can reduce the number of generators by taking only one of α_a and $\alpha_{a^{-1}}$ for $a \in \mathcal{A}$ such that $a \neq a^{-1}$ and one of β_b and $\beta_{b^{-1}}$ for $b \in B$ such that $b \neq b^{-1}$ and replacing $\alpha_{a^{-1}}$ by α_a^{-1} and $\beta_{b^{-1}}$ by β_b^{-1} in the relations obtained from the squares in $\bar{\mathbb{S}}(\Sigma_{\mathcal{A},\mathcal{B}})$ if necessary.

3.5 Comparison theorem

Having found a nice presentation of $\pi_1^{\text{orb}}(\Sigma_{\mathcal{A},\mathcal{B}}, V_4, s_{00})$, we can compare its structure with the structure of Λ by means of their actions on products of trees as follows:

Theorem 3.17. *Let Λ be a group, $(\mathcal{A},\mathcal{B})$ be a V_4-structure of Λ, $m := \#\mathcal{A}$ and $n := \#\mathcal{B}$. Suppose that Λ acts on the product $Y := T_m \times T_n$ in such a way that the group action respects the cellular structure and for a distinguished vertex $v \in Y$, the orbits $\mathcal{A}.v$ and $\mathcal{B}.v$ agree with the set of vertical and horizontal neighbours of v respectively. Then the following holds:*

(1) *Λ acts simply transitively on the vertices of Y.*

(2) *The group homomorphism*

$$\varphi : \pi_1^{\mathrm{orb}}(\Sigma_{\mathcal{A},\mathcal{B}}, V_4, s_{00}) \longrightarrow \Lambda,\ \alpha_a \longmapsto a \text{ and } \beta_b \longmapsto b,$$

is an isomorphism.

(3) *There exists a unique group homomorphism $\Lambda \to V_4$ sending every $a \in \mathcal{A}$ to γ_v and every $b \in \mathcal{B}$ to γ_h. This is surjective and its kernel Γ is isomorphic under φ to the fundamental group $\pi_1(\Sigma_{AB}, s_{00})$.*

(4) *For the normal subgroup $\Gamma \trianglelefteq \Lambda$ as above, we have a canonical isomorphism of topological spaces $\Gamma\backslash Y \cong \Sigma_{\mathcal{A},\mathcal{B}}$*

Proof. We are going to show first that the group action of Λ on the vertices of Y is transitive. Note first that Y as product of trees is connected. Hence it is possible to finish the issue by induction on distance to v. The case of distance 1 is clear from assumption. For the inductive step, if $w \in Y$ is a vertex and w' its neighbour with $d(w', v) = d(w, v) - 1$ (this exists by considering a path of minimal length from v to w), then $w' = g.v$ for some $g \in \Lambda$ and $d(g^{-1}.w, v) = d(w, w') = 1$, i.e. $g^{-1}.w = a.v$ for some $a \in \mathcal{A}$ or $b.v$ for some $b \in \mathcal{B}$. This impies that $w = ga.v$ resp. $gb.v$, i.e. $w \in \Lambda.v$ as desired.

In what follows, let us denote $\Lambda' := \pi_1^{\mathrm{orb}}(\Sigma_{\mathcal{A},\mathcal{B}}, V_4, s_{00})$. The universal covering $\widetilde{\Sigma}$ of $\Sigma_{\mathcal{A},\mathcal{B}}$ is then isomorphic to $T_m \times T_n$ by Corollary 3.7. By choice of a distinguished vertex $\tilde{s} \in \widetilde{\Sigma}$ in the fibre of s_{00}, we obtain an action of Λ' on $\widetilde{\Sigma}$ under the isomorphism $\Lambda' \to \mathrm{Aut}(\widetilde{\Sigma}/(\Sigma_{\mathcal{A},\mathcal{B}}, V_4))$ from Theorem 2.21, which is simply transitive on the vertices of $\widetilde{\Sigma}$.

On the other hand, the homomorphism φ as in the statement of the theorem is well-defined as can be easily seen from the presentation of Λ' in (3.5). Based on this fact, we can construct a φ-equivariant mapping $f : \widetilde{\Sigma} \to Y$ sending $\tilde{s}$ to v as follows:

- On the 0-skeleton, f is uniquely determined since the action of Λ' on the vertices of $\widetilde{\Sigma}$ is simply transitive. More precisely, if $t \in \widetilde{\Sigma}$ is a vertex, then there exists exactly one $\delta \in \Lambda'$ such that $t = \delta\tilde{s}$, so that we have exactly one possibility to define $f(t)$, namely $f(t) := \varphi(\delta).v$.

- On the 1-skeleton, we are going to show that if t, t' are vertices of $\widetilde{\Sigma}$ with distance 1, then so are $f(t), f(t') \in Y$. By translation invariance and transitivity of Λ' on the vertices of $\widetilde{\Sigma}$, we may assume that $t' = \tilde{s}$, but then $t = \tilde{s} \cdot \alpha_a$ for some $a \in \mathcal{A}$ or $t = \tilde{s} \cdot \beta_b$ for some $b \in \mathcal{B}$, so that $f(t) = a.v$ or $f(t) = b.v$ respectively. This implies that $f(t)$ and $v = f(\tilde{s})$ have the distance 1 by the assumption on $\mathcal{A}.v$ and $\mathcal{B}.v$.

- On the 2-skeleton, we are going to show that the images of the vertices of a square in $\widetilde{\Sigma}$ under f are the vertices of a square in Y. As before we can translate a given square in $\widetilde{\Sigma}$ by a deck transformation over $(\Sigma_{\mathcal{A},\mathcal{B}}, V_4)$ so that one of its vertices is $\tilde{s}$. Then two of the other three vertices must be a vertical and a horizontal neighbour of $\tilde{s}$, say $\tilde{s} \cdot \alpha_a$ and $\tilde{s} \cdot \beta_b$ for some $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Hence the last point must be of the form $(\tilde{s} \cdot \alpha_a) \cdot \beta_{b'} = (\tilde{s} \cdot \beta_b) \cdot \alpha_{a'}$ for uniquely determined $a' \in \mathcal{A}$ and $b' \in \mathcal{B}$ such that $ab' = ba'$ since this must project to the square of type $[a, b'; b, a']$ in $\Sigma_{\mathcal{A},\mathcal{B}}$.

Mapping this square to Y, we see that the four vertices of this square will be mapped to the vertices $v, a.v, b.v$ and $ab'.v = ba'.v$ in Y. These build indeed a square with the vertical edges $[v, a.v]$, $[b.v, ba'.v]$ and horizontal edges $[v, b.v]$, $[a.v, ab'.v]$. Using φ-equivariance, the mapping between 1-skeletons of $\widetilde{\Sigma}$ and Y can be extended uniquely to a mapping between square complexes sending a square in $\widetilde{\Sigma}$ to a square in Y.

We are going to show next that this is a covering map by considering first the distinguished points $\tilde{s} \in \widetilde{\Sigma}$ and $v \in Y$ as well as their neighbours. We see that there are one-to-one correspondences between the sets of vertical neighbours of $\tilde{s}$ and v, namely $\{\tilde{s} \cdot \alpha_a \mid a \in \mathcal{A}\}$ and $\mathcal{A}.v$, as well as between the sets of horizontal ones, namely $\{\tilde{s} \cdot \beta_b \mid b \in \mathcal{B}\}$ and $\mathcal{B}.v$. These induces one-to-one correspondences between the vertical resp. horizontal edges from $\tilde{s}$ and from v, as well as between the squares adjacent to $\tilde{s}$ and v. This implies that f is a local homeomorphism between the neighbourhood of $\tilde{s}$ and the one of v. Since the action of Λ' on $\widetilde{\Sigma}$ and the one of Λ on Y are transitive on their vertices and f is φ-equivariant, we obtain that f is a covering map.

On the other hand, Y as a product of two trees is simply connected. Therefore, the covering map $f : \widetilde{\Sigma} \to Y$ is even a homeomorphism, meaning that the action of $\varphi(\Lambda')$ on the vertices of Y is also simply transitive and the map φ in assertion (2) is injective. Since $\varphi(\Lambda')$ contains $\mathcal{A} \cup \mathcal{B}$ which generates Λ, we have $\varphi(\Lambda') = \Lambda$ and φ is an isomorphism. This, in combination with the canonical mapping $\Lambda' \to V_4$, also induces the homomorphism $\Lambda \to V_4$ as in assertion (3) with the kernel Γ isomorphic under φ to the kernel of $\Lambda' \to V_4$, namely $\pi_1(\Sigma_{\mathcal{A},\mathcal{B}}, s_{00})$.

Since $\pi_1(\Sigma_{\mathcal{A},\mathcal{B}}, s_{00})$ acts on $\widetilde{\Sigma}$ properly discontinuously as known from theory of covering space, so does the group Γ on Y with the same quotient as the quotient of $\widetilde{\Sigma}$ by $\pi_1(\Sigma_{\mathcal{A},\mathcal{B}}, s_{00})$. This implies that $\Gamma \backslash Y \cong \Sigma_{\mathcal{A},\mathcal{B}}$ as desired. □

3.6 The local permutation groups

This section aims to give an analogue to the local permutation structure in the sense of [SV13, §5.1]. Since we are dealing with a square complex with four vertices and each square is attached to all these four vertices, a modification is needed. We begin by fixing the notation of the following special case of wreath product.

Definition 3.18. Let G be a group and I be a set. The **wreath product** $G \wr \mathrm{Sym}(I)$ is defined as the semidirect product $G^I \rtimes \mathrm{Sym}(I)$ with $G^I = \prod_I G$ as normal subgroup and the action of $\mathrm{Sym}(I)$ on G^I defined by

$$\sigma((g_i)_{i\in I}) := (g_{\sigma^{-1}(i)})_{i\in I} \qquad \text{for} \quad (g_i)_{i\in I} \in G^I,\ \sigma \in \mathrm{Sym}(I).$$

Remark 3.19. If $G = \mathrm{Sym}(X)$ is the permutation group of another set X, then the wreath product $\mathrm{Sym}(X) \wr \mathrm{Sym}(I)$ can be embedded in $\mathrm{Sym}(X \times I)$ by defining

$$\sigma(x, i) = (\sigma_{\bar{\sigma}(i)}(x), \bar{\sigma}(i))$$

for each $(x, i) \in X \times I$ and $\sigma = ((\sigma_i)_i, \bar{\sigma}) \in \mathrm{Sym}(X) \wr \mathrm{Sym}(I)$.

From now on, let $(\mathcal{A}, \mathcal{B})$ be a V_4-structure of a group Λ. To define the local permutation groups associated to the V_4-structure $(\mathcal{A}, \mathcal{B})$, we begin with the following notation:

Notation 3.20. Throughout this section, set $I := \{0,1\}$. For $i, j \in I$, we shall write $\widehat{i} = 1 - i$ and $\widehat{j} = 1 - j$, i.e. $\widehat{0} = 1$ and $\widehat{1} = 0$.

Definition 3.21. For $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $i, j \in I$, let $\mathbb{S}_{(a,i)}, \mathbb{S}_{(b,j)} \subseteq \mathbb{S}(\Sigma_{\mathcal{A},\mathcal{B}})$ be the set of the squares attached to the edge (a,i) and (b,j) respectively. Furthermore, the mappings

$$t^j_{(a,i)} : \mathbb{S}_{(a,i)} \longrightarrow \mathcal{B}_j \quad \text{and} \quad t^i_{(b,j)} : \mathbb{S}_{(b,j)} \longrightarrow \mathcal{A}_i$$

are defined by sending each square in $\mathbb{S}_{(a,i)}$ and $\mathbb{S}_{(b,j)}$ to its horizontal and vertical edge attached to the vertex s_{ij} respectively.

Note that $t^j_{(a,i)}$ and $t^i_{(b,j)}$ are well-defined by the construction of $\Sigma_{\mathcal{A},\mathcal{B}}$. These are bijective since the link $\mathrm{Lk}_{s_{ij}}$ is a complete bipartite graph. Hence it is possible to consider the maps

$$t^j_{(a,i)} \circ (t^{\widehat{j}}_{(a,i)})^{-1} : \mathcal{B}_{\widehat{j}} \longrightarrow \mathcal{B}_j \quad \text{and} \quad t^i_{(b,j)} \circ (t^{\widehat{i}}_{(b,j)})^{-1} : \mathcal{A}_{\widehat{i}} \longrightarrow \mathcal{A}_i.$$

These are all bijective, so that $t^0_{(a,i)} \circ (t^1_{(a,i)})^{-1} \sqcup t^1_{(a,i)} \circ (t^0_{(a,i)})^{-1}$ is a bijection from $\mathcal{B}_0 \sqcup \mathcal{B}_1 = \mathcal{B} \times I$ to $\mathcal{B}_1 \sqcup \mathcal{B}_0 = \mathcal{B} \times I$. We can do the similar things replacing (a,i) by (b,j) and $\mathcal{B}$ by $\mathcal{A}$ and obtain the following definitions.

Definition 3.22. We defines the following mappings

$$\begin{aligned} \mathcal{A} \times I \longrightarrow \mathrm{Sym}(\mathcal{B} \times I), \quad (a,i) \longmapsto \sigma^{\mathcal{B}}_{(a,i)} &:= t^0_{(a,i)} \circ (t^1_{(a,i)})^{-1} \sqcup t^1_{(a,i)} \circ (t^0_{(a,i)})^{-1} \quad \text{and} \\ \mathcal{B} \times I \longrightarrow \mathrm{Sym}(\mathcal{A} \times I), \quad (b,j) \longmapsto \sigma^{\mathcal{A}}_{(b,j)} &:= t^0_{(b,j)} \circ (t^1_{(b,j)})^{-1} \sqcup t^1_{(b,j)} \circ (t^0_{(b,j)})^{-1}. \end{aligned}$$

The **local permutation groups associated to the V_4-structure** $(\mathcal{A}, \mathcal{B})$ are then defined by

$$\begin{aligned} P^j_{\mathcal{A}} &:= \langle \sigma^{\mathcal{A}}_{(b,j)} \mid b \in \mathcal{B} \rangle \leq \mathrm{Sym}(\mathcal{A} \times I) \quad \text{and} \\ P^i_{\mathcal{B}} &:= \langle \sigma^{\mathcal{B}}_{(a,i)} \mid a \in \mathcal{A} \rangle \leq \mathrm{Sym}(\mathcal{B} \times I) \end{aligned}$$

for each $i, j \in I$.

Remark 3.23. It can be easily seen that for each $(a,i) \in \mathcal{A} \times I$, the permutation $\sigma^{\mathcal{B}}_{(a,i)}$ lies in the wreath product $\mathrm{Sym}(\mathcal{B}) \wr \mathrm{Sym}(I)$ as a subgroup of $\mathrm{Sym}(\mathcal{B} \times I)$ with the non-trivial image under the projection $\mathrm{Sym}(\mathcal{B}) \wr \mathrm{Sym}(I) \to \mathrm{Sym}(I)$. Hence there are $\sigma^{\mathcal{B}}_{(a,i),0}, \sigma^{\mathcal{B}}_{(a,i),1} \in \mathrm{Sym}(\mathcal{B})$ such that

$$\sigma^{\mathcal{B}}_{(a,i)} = ((\sigma^{\mathcal{B}}_{(a,i),0}, \sigma^{\mathcal{B}}_{(a,i),1}), \widehat{\cdot}). \tag{3.7}$$

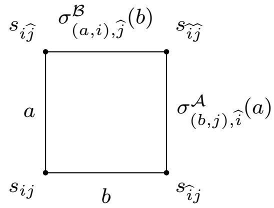
Note that by this notation, we have a commutative diagramm

$$\begin{array}{ccc} \mathcal{B}_0 & \overset{t^1_{(a,i)} \circ (t^0_{(a,i)})^{-1}}{\underset{t^0_{(a,i)} \circ (t^1_{(a,i)})^{-1}}{\rightleftarrows}} & \mathcal{B}_1 \\ {\scriptstyle \mathrm{pr}}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}} \\ \mathcal{B} & \overset{\sigma_{(a,i),1}}{\underset{\sigma_{(a,i),0}}{\rightleftarrows}} & \mathcal{B}, \end{array}$$

i.e. $\sigma^{\mathcal{B}}_{(a,i),0}$ and $\sigma^{\mathcal{B}}_{(a,i),1}$ are, up to identification of $\mathcal{B}$ with $\mathcal{B}_0$ or $\mathcal{B}_1$, the same as $t^0_{(a,i)} \circ (t^1_{(a,i)})^{-1}$ and $t^1_{(a,i)} \circ (t^0_{(a,i)})^{-1}$ respectively. In particular, $\sigma^{\mathcal{B}}_{(a,i),0}$ and $\sigma^{\mathcal{B}}_{(a,i),1}$ are inverse to each other. Similarly, there are $\sigma^{\mathcal{A}}_{(b,j),0}, \sigma^{\mathcal{A}}_{(b,j),1} \in \mathrm{Sym}(\mathcal{A})$ such that, by considering $\sigma^{\mathcal{A}}_{(b,j)}$ as element in $\mathrm{Sym}(\mathcal{A}) \wr \mathrm{Sym}(I)$, we have

$$\sigma^{\mathcal{A}}_{(b,j)} = ((\sigma^{\mathcal{A}}_{(b,j),0}, \sigma^{\mathcal{A}}_{(b,j),1}), \widehat{\cdot}). \tag{3.8}$$

The pair $(\sigma^{\mathcal{A}}_{(b,j),0}, \sigma^{\mathcal{A}}_{(b,j),1})$ has similar properties as $(\sigma^{\mathcal{B}}_{(a,i),0}, \sigma^{\mathcal{B}}_{(a,i),1})$. Alternatively, we can define $\sigma^{\mathcal{A}}_{(b,j)} \in \mathrm{Sym}(\mathcal{A}) \wr \mathrm{Sym}(I)$ and $\sigma^{\mathcal{B}}_{(a,i)} \in \mathrm{Sym}(\mathcal{B}) \wr \mathrm{Sym}(I)$ by (3.7) and (3.8), where $\sigma^{\mathcal{A}}_{(b,j),i} \in \mathrm{Sym}(\mathcal{A})$ and $\sigma^{\mathcal{B}}_{(a,i),j} \in \mathrm{Sym}(\mathcal{B})$ are defined in such a way that the square



(3.9)

is a square in $\Sigma_{\mathcal{A},\mathcal{B}}$ for all $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $i, j \in I$.

We conclude this section by deriving some properties of the local permutation groups which will be useful in computing the Albanese variety in Section 5.4.

Proposition 3.24. *For $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $i, j \in I$, let $\sigma^{\mathcal{A}}_{(b,j),i} \in \mathrm{Sym}(\mathcal{A})$, $\sigma^{\mathcal{B}}_{(a,i),j} \in \mathrm{Sym}(\mathcal{B})$ be as in Remark 3.23. Furthermore, define*

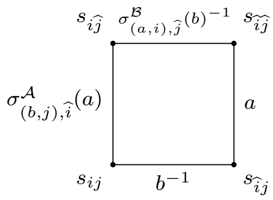
$$\iota^{\mathcal{A}} := ((\tau^{\mathcal{A}}, \tau^{\mathcal{A}}), \mathrm{id}) \in \mathrm{Sym}(\mathcal{A}) \wr \mathrm{Sym}(I) \quad \textit{and} \quad \iota^{\mathcal{B}} := ((\tau^{\mathcal{B}}, \tau^{\mathcal{B}}), \mathrm{id}) \in \mathrm{Sym}(\mathcal{B}) \wr \mathrm{Sym}(I),$$

where $\tau^{\mathcal{A}} := (\cdot)^{-1} \in \mathrm{Sym}(\mathcal{A})$ and $\tau^{\mathcal{B}} := (\cdot)^{-1} \in \mathrm{Sym}(\mathcal{B})$. Then the following holds:

(1) $\sigma^{\mathcal{A}}_{(b^{-1},j),i} = \sigma^{\mathcal{A}}_{(b,j),\widehat{i}}$ *and* $\sigma^{\mathcal{B}}_{(a^{-1},i),j} = \sigma^{\mathcal{B}}_{(a,i),\widehat{j}}$.

(2) $\sigma^{\mathcal{A}}_{(b,\widehat{j})} = \iota^{\mathcal{A}} \circ \sigma^{\mathcal{A}}_{(b,j)} \circ \iota^{\mathcal{A}}$ *and* $\sigma^{\mathcal{B}}_{(a,\widehat{i})} = \iota^{\mathcal{B}} \circ \sigma^{\mathcal{B}}_{(a,i)} \circ \iota^{\mathcal{B}}$. *In particular, $P^{\mathcal{A}}_0, P^{\mathcal{A}}_1 \le \mathrm{Sym}(\mathcal{A}) \wr \mathrm{Sym}(I)$ differ by conjugation with $\iota^{\mathcal{A}}$ and $P^{\mathcal{B}}_0, P^{\mathcal{B}}_1 \le \mathrm{Sym}(\mathcal{B}) \wr \mathrm{Sym}(I)$ differ by conjugation with $\iota^{\mathcal{B}}$.*

Proof. Applying the horizontal reflection γ_h to the square in (3.9), we obtain the square



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This implies that

$$\sigma^{\mathcal{A}}_{(b^{-1},j),i}(a) = \sigma^{\mathcal{A}}_{(b,j),\widehat{i}}(a) \quad \text{and} \quad \sigma^{\mathcal{B}}_{(a,i),\widehat{j}}(b)^{-1} = \sigma^{\mathcal{B}}_{(a,\widehat{i}),\widehat{j}}(b^{-1}).$$

The first equation implies that $\sigma^{\mathcal{A}}_{(b,j),\widehat{i}} = \sigma^{\mathcal{A}}_{(b^{-1},j),i}$. Furthermore, replacing i by $\widehat{i}$ in the second equation, we get

$$\begin{aligned} \iota^{\mathcal{B}} \circ \sigma^{\mathcal{B}}_{(a,i)} \circ \iota^{\mathcal{B}}(b,j) &= \iota^{\mathcal{B}} \circ \sigma^{\mathcal{B}}_{(a,i)}(b^{-1},j) = \iota^{\mathcal{B}}(\sigma^{\mathcal{B}}_{(a,i),\widehat{j}}(b^{-1}), \widehat{j}) \\ &= \iota^{\mathcal{B}}(\sigma^{\mathcal{B}}_{(a,\widehat{i}),\widehat{j}}(b)^{-1}, \widehat{j}) = (\sigma^{\mathcal{B}}_{(a,\widehat{i}),\widehat{j}}(b), \widehat{j}) = \sigma^{\mathcal{B}}_{(a,\widehat{i})}(b,j) \end{aligned}$$

for all $(b,j) \in \mathcal{B} \times I$, i.e. $\sigma^{\mathcal{B}}_{(a,\widehat{i})} = \iota^{\mathcal{B}} \circ \sigma^{\mathcal{B}}_{(a,i)} \circ \iota^{\mathcal{B}}$. This implies that $P^{\mathcal{B}}_0, P^{\mathcal{B}}_1 \le \mathrm{Sym}(\mathcal{B}) \wr \mathrm{Sym}(I)$ differ by conjugation with $\iota^{\mathcal{B}}$ since $\iota^{\mathcal{B}}$ has order two. Similarly, we can apply γ_v to the square in (3.9) to get the other statements of the proposition. □

Chapter 4

Presentations of quaternionic arithmetic lattices

The aim of this chapter is to find presentations of quaternionic arithmetic lattices of $Q = \left[\frac{z, 1+z^3}{\mathbb{F}_2(z)}\right)$ over $R = \mathbb{F}_2\left[z, \frac{1}{z+z^4}\right]$. We shall begin by recapitulating some important facts about Bruhat-Tits trees for the group $\mathrm{PGL}_2(K)$ over a local field K, and then explain how the lattice mentioned before defines an action on the product of Bruhat-Tits trees. After computing the stabiliser of a vertex on such a product and determining the V_4-structure on a subgroup of this lattice, we come to the main result in Section 4.5. The last section aims to give a computation of the fundamental group of the square complex defined by the V_4-structure found before, as well as the interpretation of the arithmetic subgroups occuring during the computation as (orbital) fundamental groups.

4.1 The Bruhat-Tits action

In what follows, let K be a local field with the normalised valuation ν, the ring of integers O_K, a uniformiser π and the residue field $\kappa = O_K/(\pi)$. We begin this section by recalling the definition of the Bruhat-Tits tree of $\mathrm{PGL}_2(K)$.

Definition 4.1. The **Bruhat-Tits tree** of $\mathrm{PGL}_2(K)$, denoted by $T_K = \Delta(\mathrm{PGL}_2(K))$, is a simple graph consisting of the following data:

- The vertex set $\mathbb{V}(T_K)$ is defined as the set of the equivalence classes of O_K-lattices in K^2 (i.e. O_K-submodules of rank $2 = \dim K^2$). Here two O_K-lattices $M_1, M_2 \subseteq K^2$ are said to be equivalent if there exists a $\lambda \in K^\times$ such that $M_1 = \lambda M_2$.

- The edge set is defined by

$$\mathbb{E}(T_K) := \left\{\{[M_1],[M_2]\} \subseteq \mathbb{V}(T_K) \;\middle|\; \exists\,\lambda\in K^\times : \pi M_1 \subsetneq \lambda M_2 \subsetneq M_1\right\}.$$

  Note that this definition doesn't depend on the choice of the lattices from the equivalence classes and is symmetric in $[M_1],[M_2]$ since $\pi M_1 \subsetneq \lambda M_2 \subsetneq M_1$ iff $\pi M_2 \subsetneq \frac{\pi}{\lambda} M_1 \subsetneq M_2$.

The action of $\mathrm{PGL}_2(K)$ on the vertices of $\mathbb{V}(T_K)$ is defined by $(a,[M]) \mapsto [a \cdot M]$. This induces a simplicial action on T_K.

Notation 4.2. If $M \subseteq K^2$ is an O_K-lattice with a basis $\{m_1, m_2\}$, we shall also denote the corresponding lattice $[M]$ by $[m_1, m_2]$. The **standard vertex** is the vertex $w_K := [e_1, e_2]$, where $\{e_1, e_2\}$ denotes the standard basis of K^2.

Remark 4.3. If $m_1, m_2 \in K^2$ form a basis of an O_K-lattice, the neighbours of the vertex $[m_1, m_2]$ are $[m_1, \pi m_2]$ and $[\pi m_1, \lambda m_1 + m_2]$, where $\lambda \in O_K$ runs through all liftings of elements of κ. In fact, denoting $M := \langle m_1, m_2 \rangle_{O_K}$, the neighbours of $[M]$ can be represented by those lattices M' such that $\pi M \subsetneq M' \subsetneq M$, or equivalently, $M'/\pi M$ is a non-trivial κ-subspace of $M/\pi M$, which has $\overline{m_1}, \overline{m_2}$ as a basis. Hence $M'/\pi M$ is generated by $\overline{m_1}$ or $\lambda \overline{m_1} + \overline{m_2}$ for some $\lambda \in \kappa$. The claim then follows by lifting such κ-subspaces to lattices contained in M. In particular, every vertex of T_K has exactly $\#\kappa + 1$ neighbours.

Lemma 4.4. *There exists a function $d : \mathbb{V}(T_K) \times \mathbb{V}(T_K) \to \mathbb{N}_0$ which is defined as follows: For $[M_1], [M_2] \in \mathbb{V}(T_K)$ with corresponding lattices M_1, M_2, there exist $a, b \in \mathbb{Z}$ and a basis $\{m_1, m_2\}$ of M_1 such that $\{\pi^a m_1, \pi^b m_2\}$ is a basis of M_2. In this case we define*

$$d([M_1], [M_2]) := |a - b|.$$

This is well-defined and symmetric in $[M_1], [M_2]$.

Proof. Let M_1 and M_2 be two lattices of K^2. By rescaling M_2 we may consider first the case $M_2 \subseteq M_1$. By the structure theorem of modules over principal ideal domains and the fact that π is up to associatedness the unique prime element in O_K, there exists a basis $\{m_1, m_2\}$ of M_1 such that $\{\pi^a m_1, \pi^b m_2\}$ is a basis of M_2, where $a, b \in \mathbb{N}_0$ with $a \leq b$ are uniquely determined from the structure of M_1/M_2 (in particular independent of the possible choice of such a basis $\{m_1, m_2\}$). This implies that such $a, b \in \mathbb{Z}$ as in the statement of the theorem exist.

To prove that $d([M_1], [M_2])$ is well-defined, suppose that $M_1' = \lambda M_1$ and $M_2' = \mu M_2$ with $\lambda, \mu \in K^\times$. Then $\{\lambda m_1, \lambda m_2\}$ is a basis of M_1' and, by writing $\frac{\mu}{\lambda} = \epsilon \pi^c$ with $\epsilon \in O_K^\times, c \in \mathbb{Z}$, $\{\pi^{a+c} \lambda m_1, \pi^{b+c} \lambda m_2\}$ is a basis of M_2, and consequently

$$d([M_1'], [M_2']) = |(a + c) - (b + c)| = |a - b| = d([M_1], [M_2]).$$

as desired. To show the symmetry, observe that under the condition as above, a basis of M_1 is $\{\pi^{-a}(\pi^a m_1), \pi^{-1}(\pi^b m_2)\}$, which implies that

$$d([M_2], [M_1]) = |(-a) - (-b)| = |b - a| = |a - b| = d([M_1], [M_2]).$$

Hence d is a symmetric function. □

Now we can justify the name Bruhat-Tits-tree by showing that it is indeed a tree, as well as the notation d which should stand for "distance" between two vertices.

Proposition 4.5. *The Bruhat-Tits tree T_K as defined above is in fact a tree, i.e. a connected simple graph without cycle. Moreover, if $[M_0], [M_1], \ldots, [M_n]$ is a sequence of vertices in a path without backtracking in T_K, then $n = d([M_0], [M_n])$. In particular, $d([M_0], [M_1]) = 1$ if and only if $[M_0]$ and $[M_1]$ are adjacent. Moreover, the tree T_K has the constant valency $\#\kappa + 1$.*

Proof. The main statement is proven in [Ser03, Ch.II, Thm.1]. The fact about the valency follows from Remark 4.3. □

The next proposition gives us information about the distance between the standard vertex and its image under the action of an element of $\mathrm{PGL}_2(K)$.

Proposition 4.6. *Suppose that $A \in \mathrm{GL}_2(K)$ with coprime entries in O_K. Then we have*

$$d(w_K, Aw_K) = \nu(\det(A)).$$

Proof. Observe first that the vertex Aw_K is represented by the lattice M_A generated by the column vectors of A. So by the elementary divisor theorem, there exist a basis $\{m_1, m_2\}$ of O_K^2 as well as $a, b \in \mathbb{N}_0$ with $a \neq b$ such that $\{\pi^a m_1, \pi^b m_2\}$ is a basis of M_A and $\det(A)$ is associated to π^{a+b}. Since the entries or A are coprime, we must have $a = 0$. Hence it follows from Lemma 4.4 that

$$d(w_K, Aw_K) = |0 - b| = b = \nu(\det(A))$$

as desired. □

This proposition has two consequences. The first one is easier but will be later useful in determining whether a matrix $A \in \mathrm{GL}_2(K)$ can stabilise the standard vertex w_K.

Corollary 4.7. *For $A \in \mathrm{GL}_2(K)$, we have*

$$d(w_K, Aw_K) \equiv \nu(\det(A)) \pmod 2.$$

Proof. For $A \in \mathrm{GL}_2(K)$, we can find a $\lambda \in K^\times$ such that λA has coprime entries in O_K (namely $\lambda = \pi^{-n}$, where n denotes the minimum valuation of entries in A). Then from Proposition 4.6, we have

$$\begin{aligned} d(w_K, Aw_K) &= d(w_K, (\lambda A)w_K) = \nu(\det(\lambda A)) = \nu(\lambda^2 \det(A)) \\ &= 2\nu(\lambda) + \nu(\det(A)) \equiv \nu(\det(A)) \pmod 2, \end{aligned}$$

and we are done. □

The second consequence is that we can describe the stabiliser of the standard vertex under the action of $\mathrm{PGL}_2(K)$ on T_K explicitly.

Corollary 4.8. *The stabiliser of the standard vertex w_K under the group action of $\mathrm{PGL}_2(K)$ on T_K is $\mathrm{PGL}_2(O_K)$, i.e. the group of elements in $\mathrm{PGL}_2(K)$ that can be lifted to $\mathrm{GL}_2(O_K)$.*

Proof. Every element in $\mathrm{PGL}_2(K)$ has a lifting $A \in \mathrm{GL}_2(K)$ with coprime entries in O_K. From Proposition 4.6, A stabilises w_K iff $\nu(\det(A)) = 0$, i.e. $\det(A) \in O_K^\times$. This implies that $A \in GL_2(O_K)$. Conversely, every element of $\mathrm{GL}_2(O_K)$ has coprime entries since their greatest common divisor must divide the determinant of the given matrix. Using the same argument as before, we can conclude that such matrices must stabilise the standard vertex. □

4.2 The lattice $G(R)$ and its action on $T_3 \times T_3$

We come back to what we have introduced in the first chapter. Recall the following notation:

$$\begin{aligned} K &= \mathbb{F}_2(z), \quad u = z^{-1}, \\ Q &= \left[\frac{z, 1+z^3}{K}\right) = K\{I, J\}/(I^2 + I = z, J^2 = 1 + z^3, IJ = JI + J), \\ G &= PGL_{1,Q} \quad \text{with integral structure defined by the basis } \{1, I, J, IJ\}, \\ R_0 &= \mathbb{F}_2[z, \frac{1}{z}] \ \subseteq \ R_1 = \mathbb{F}_2[z, \frac{1}{z(1+z)}] \ \subseteq \ R = \mathbb{F}_2[z, \frac{1}{z(1+z^3)}]. \end{aligned}$$

To define the action of $G(R)$ on the product of Bruhat-Tits trees $T_3 \times T_3$, we introduce first the following notation:

Notation 4.9. For each closed point $x \in \mathbb{P}^1_{\mathbb{F}_2}$, we denote by K_x the completion of K at the place $\{z = x\}$ with the normalised valuation ν_x. Furthermore, define $T_x := T_{K_x}$ as the Bruhat-Tits tree for $\mathrm{PGL}_2(K_x)$, and $w_x := w_{K_x} \in T_x$ as its standard vertex.

The group $G(R)$ has the following topological property:

Lemma 4.10. *$G(R)$ is a cocompact lattice under the diagonal embedding*

$$G(R) \longhookrightarrow G(K_0) \times G(K_\infty). \tag{4.1}$$

Proof. Since G is connected, reductive, semisimple and anisotropic over K, we can use [Mar91, Ch.I Thm.3.2.4-5] to conclude that the diagonal embedding

$$G(R) \longhookrightarrow G(K_0) \times G(K_1) \times G(K_\zeta) \times G(K_\infty)$$

makes $G(R)$ a cocompact lattice in $G(K_0) \times G(K_1) \times G(K_\zeta) \times G(K_\infty)$. Furthermore, Q ramifies at $\{z = 1\}$ and $\{z = \zeta\}$, i.e. $Q \otimes_K K_1$ and $Q \otimes_K K_\zeta$ are division algebras, so that the groups $G(K_1)$ and $G(K_\zeta)$ are compact. Therefore, the factors $G(K_1)$ and $G(K_\zeta)$ can be omitted to obtain a cocompact lattice under the embedding $G(R) \hookrightarrow G(K_0) \times G(K_\infty)$. □

Having embedded $G(R)$ as a subgroup of $G(K_0) \times G(K_\infty)$, we can define its action on the product of Bruhat-Tits trees $T_0 \times T_\infty$ as follows:

- The place $\{z = 0\}$ splits under the extension $\mathbb{F}_2(y)/\mathbb{F}_2(z)$ as in Lemma 1.33 with the prime elements y and $1 + y$ lying over z. Completion of K along $\{y = 0\}$ yields an isomorphism $K_0 \cong \mathbb{F}_2((y))$. The embedding ρ_y from Lemma 1.33 then induces an isomorphism $G(K_0) \cong \mathrm{PGL}_2(\mathbb{F}_2((y)))$, so that the embedding

  $$G(R) \longhookrightarrow G(K_0) \cong \mathrm{PGL}_2(\mathbb{F}_2((y)))$$

  yields the action of $G(R)$ on the horizontal component T_0 of $T_0 \times T_\infty$.

- The place $\{z = \infty\}$ splits under the extension $\mathbb{F}_2(t)/\mathbb{F}_2(z)$ as in Lemma 1.34 with the prime elements t and $1 + t$ lying over $u = z^{-1}$. Completion of K along $\{t = 0\}$ yields an isomorphism $K_\infty \cong \mathbb{F}_2((t))$. The embedding ρ_t from Lemma 1.34 then induces an isomorphism $G(K_\infty) \cong \mathrm{PGL}_2(\mathbb{F}_2((t)))$, so that the embedding

  $$G(R) \longhookrightarrow G(K_\infty) \cong \mathrm{PGL}_2(\mathbb{F}_2((t)))$$

  yields the action of $G(R)$ on the vertical component T_∞ of $T_0 \times T_\infty$.

Combining the horizontal and vertical action, we obtain the action of $G(R)$ on $T_0 \times T_\infty$, which is isomorphic to $T_3 \times T_3$ since both T_0 and T_∞ are trees with constant valency 3. This action will be studied in details in the next sections.

4.3 The stabiliser of the standard vertex

The goal of this section is determine the stabiliser group $G(R)_w \leq G(R)$ of the standard vertex $w = (w_0, w_\infty) \in T_0 \times T_\infty$. The main strategy is to show that each of its elements has the order at most 2. We shall begin with the following lemma:

Lemma 4.11. *$G(R)_w$ is a finite group.*

Proof. We use the embedding $\rho : G(R) \hookrightarrow \mathrm{PGL}_2(k(\!(y)\!)) \times \mathrm{PGL}_2(k(\!(t)\!))$, under which $G(R)$ becomes a discrete subgroup. From Corollary 4.8, we have

$$G(R)_w = \rho^{-1}(\mathrm{PGL}_2(k[\![y]\!]) \times \mathrm{PGL}_2(k[\![t]\!])) \subseteq G(R).$$

Hence we can embed $G(R)_w$ as a discrete subgroup in $\mathrm{PGL}_2(k[\![y]\!]) \times \mathrm{PGL}_2(k[\![t]\!])$, which is compact. This implies that $G(R)_w$ must be finite. □

A consequence of this lemma is that every stabiliser of v must have a finite order. As next step, we are going to show that this finite order can be at most 2. For this we shall consider first the group $\mathrm{PGL}_2(\mathbb{F}_2(\!(y)\!))$ which certainly contains $G(R)$ and for which the most important results are pretty well-known.

Lemma 4.12. *If $a \in \mathrm{PGL}_2(\mathbb{F}_2(\!(y)\!))$ has a finite order, then its order is 1, 2 or 3.*

Proof. Suppose that $a \in \mathrm{PGL}_2(\mathbb{F}_2(\!(y)\!)) \setminus \{1\}$ has a finite order $r \in \mathbb{N}$ and let $A \in \mathrm{GL}_2(\mathbb{F}_2(\!(y)\!))$ be its lifting. Then its minimal polynomial $m_A(X)$ has degree 2 and is a divisor of the polynomial $X^r - f$ for some $f \in \mathbb{F}_2(\!(y)\!)$. We shall consider the following cases:

- $r = 2^t$ for some $t \in \mathbb{N}$. Consider A as a matrix over $\mathbb{F}_2(\!(y)\!)^{\mathrm{alg}}$ and write $f = (\tilde{f})^{2^t}$ for some $\tilde{f} \in \mathbb{F}_2(\!(y)\!)^{\mathrm{alg}}$. Then $X^r - f = (X - \tilde{f})^{2^t}$, implying that $m_A(X) = (X - \tilde{f})^2 = X^2 - \tilde{f}^2$, i.e. $a^2 = 1$ in $\mathrm{PGL}_2(\mathbb{F}_2(\!(y)\!)^{\mathrm{alg}})$. Therefore r must be 2 in this case.

- r is an odd number. Then by [Bea10, Prop.1.1], there exists a primitive r-th root of unity $\xi \in \mathbb{F}_2(\!(y)\!)^{\mathrm{alg}}$ such that $\xi + \xi^{-1}$ lies in $\mathbb{F}_2(\!(y)\!)$. Since ξ is then algebraic over $\mathbb{F}_2$, we have $\xi + \xi^{-1} \in \mathbb{F}_2(\!(y)\!) \cap \mathbb{F}_2^{\mathrm{alg}} = \mathbb{F}_2$, i.e. $\xi + \xi^{-1}$ is 0 or 1. If $\xi + \xi^{-1} = 0$, then $\xi = \xi^{-1}$, i.e. $\xi^2 = 1$ or $\xi = 1$, contradicting our assumption on ξ. Hence we have $\xi + \xi^{-1} = 1$, which implies that $\xi^2 + \xi + 1 = 0$ and consequently $\xi^3 - 1 = (\xi - 1)(\xi^2 + \xi + 1) = 0$, i.e. $r = 3$.

In the general case, writing $r = 2^e n$ with $e, n \in \mathbb{N}_0$ and $n \nmid s$, we have from the argument above that $e \leq 1$ and $n \in \{1, 3\}$. Hence we still have to exclude the case $r = 6$. So let $\tilde{f} \in \mathbb{F}_2(\!(y)\!)^{\mathrm{alg}}$ be such that $\tilde{f}^6 = f$. Then $L := \mathbb{F}_2(\!(y)\!)(\tilde{f})$ is an extension of $\mathbb{F}_2(\!(y)\!)$ of separable degree at most 3 and inseparable degree at most 2, and we have the following prime factorisation in $L[X]$:

$$X^6 - f = (X^3 - \tilde{f}^3)^2 = (X - \tilde{f})^2(X^2 + \tilde{f}X + \tilde{f}^2).$$

Since $\deg m_A(X) = 2$, we have either $m_A(X) = (X - \tilde{f})^2 = X^2 - \tilde{f}^2$ which implies that a has order 2 in $\mathrm{PGL}_2(L)$, or $m_A(X) = X^2 + \tilde{f}X + \tilde{f}^2$ is a divisor of $X^3 - \tilde{f}^3$ which implies that a has order 3 in $\mathrm{PGL}_2(L)$. Both cases imply that a can't have order 6 in $\mathrm{PGL}_2(\mathbb{F}_2(\!(y)\!))$, and we are done. □

Lemma 4.13. *The group $G(K)$ has no element of order 3.*

Proof. Suppose that $G(K)$ had an element of order 3 with a lifting $A := p_1 + p_2 I + q_1 J + q_2 IJ \in Q$, where $p_1, p_2, q_1, q_2 \in \mathbb{F}_2[z]$ are coprime. Then the minimal polynomial of A over K, namely $X^2 - \mathrm{tr}(A)X + \mathrm{n}(A) \in K[X]$, must divide a polynomial of the form $X^3 - r$ for some $r \in K$. This implies that $\mathrm{n}(A) + \mathrm{tr}(A)^2 = 0$. Using the explicit formula from Proposition 1.15, we obtain the following equation:

$$p_1^2 + p_1 p_2 + (z+1)p_2^2 + (1+z^3)(q_1^2 + q_1 q_2 + z q_2^2) = 0. \tag{4.2}$$

Let $\zeta \in \bar{\mathbb{F}}_2$ be a primitive third root of unity. Then substituting ζ for z into above equation yields

$$p_1(\zeta)^2 + p_1(\zeta)p_2(\zeta) + (\zeta+1)p_2(\zeta)^2 = 0 \qquad \text{in } \mathbb{F}_4 = \mathbb{F}_2(\zeta).$$

If $p_2(\zeta) \neq 0$, then we could divide the above equation by $p_2(\zeta)^2$ to get $x^2 + x + (\zeta+1) = 0$ for $x = \frac{p_1(\zeta)}{p_2(\zeta)}$, which would be absurd since the polynomial $X^2 + X + (1+\zeta)$ has no roots in $\mathbb{F}_4 = \mathbb{F}_2(\zeta)$. Hence $p_2(\zeta) = 0$ and consequently $p_1(\zeta) = 0$, which implies that both p_1 and p_2 are divisible by the minimal polynomial of ζ over $\mathbb{F}_2$, namely $1+z+z^2$, say $p_1 = (1+z+z^2)\tilde{p}_1$ and $p_2 = (1+z+z^2)\tilde{p}_2$ for $\tilde{p}_1, \tilde{p}_2 \in \mathbb{F}_2[z]$. Substituting these into (4.2) and dividing the obtained equation by $(1+z+z^2)$, we obtain

$$(1+z+z^2)(\tilde{p}_1^2 + \tilde{p}_1\tilde{p}_2 + (z+1)\tilde{p}_2^2) + (1+z)(q_1^2 + q_1q_2 + zq_2^2) = 0. \tag{4.3}$$

We substitute again ζ for z but now into (4.3) to obtain

$$(1+\zeta)(q_1(\zeta)^2 + q_1(\zeta)q_2(\zeta) + \zeta q_2(\zeta)^2) = 0 \qquad \text{in } \mathbb{F}_4 = \mathbb{F}_2(\zeta).$$

Using the same argument as before, together with the fact that the polynomial $X^2 + X + \zeta$ has no roots in $\mathbb{F}_4$, we get $q_1(\zeta) = q_2(\zeta) = 0$, which implies that q_1 and q_2 are divisible by $(1+z+z^2)$. Hence $1+z+z^2$ is a common divisor of p_1, p_2, q_1, q_2, contradicting their coprimality assumed at the beginning. Therefore, the group $G(K)$ has no element of order 3. □

Proposition 4.14. *The stabiliser $G(R)_w$ of $w = (w_0, w_\infty)$ in $G(R)$ has only two elements, namely* 1 *and the image of $D := (1+z+z^2) + IJ$ under the projection $\mathfrak{O}^\times \to G(R)$.*

Notation 4.15. The image of $D = (1+z+z^2) + IJ$ under the projection $\mathfrak{O} \to G(R)$ will be denoted by $d \in G(R)$.

Proof. First of all, notice that $n(D) = 1+z+z^2 \in R^\times$, so that D is in fact invertible in $\mathfrak{O}$ and its image $d \in G(R)$ is well-defined. Now we are going to show that $d \in G(R)_w$. To do this, consider the embedding of D in $\mathrm{M}_2(\mathbb{F}_2(y))$. We have

$$\rho_y(D) = \begin{pmatrix} 1+z+z^2 & (1+z^3)y \\ 1+y & 1+z+z^2 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \pmod{y},$$

implying that $\rho_y(D) \in \mathrm{GL}_2(\mathbb{F}_2[\![y]\!])$. On the other hand,

$$\begin{aligned} u\rho_t(D) &= \begin{pmatrix} u^{-1}+1+u & 0 \\ 0 & u^{-1}+1+u \end{pmatrix} + u^{-1}\begin{pmatrix} 1+u^3 & (1+u+t)(u+u^4) \\ u+t & 1+u^3 \end{pmatrix} \\ &= \begin{pmatrix} 1+u+u^2 & (1+u+t)(1+u^3) \\ 1+t/u & 1+u+u^2 \end{pmatrix} \equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \pmod{t}, \end{aligned}$$

implying that $\rho_t(D) \in \mathrm{GL}_2(\mathbb{F}_2[\![t]\!])$. Hence we have $\rho(d) \in \mathrm{PGL}_2(\mathbb{F}_2[\![y]\!]) \times \mathrm{PGL}_2(\mathbb{F}_2[\![t]\!])$ and thus, by Corollary 4.8, $d \in G(R)_w$.

Next we have to show that $1, d$ are the only elements in $G(R)_w$. Since $G(R)_w$ can be considered a subgroup of $\mathrm{PGL}_2(\mathbb{F}_2(\!(y)\!))$, every non-trivial element of $G(R)_w$ must have order 2 or 3 by Lemma 4.12. However, the latter case is excluded by Lemma 4.13, so that $a^2 = 1$ for every $a \in G(R)_w$. This implies that $G(R)_w$ is a 2-elementary abelian group. Hence if $E_1, E_2 \in Q$ are two liftings of elements of $G(R)_w$, then E_1E_2 and E_2E_1 differ by a scalar multiple, say $E_1E_2 = \lambda E_2E_1$ for some $\lambda \in K^\times$. Taking the reduced norms of both side and dividing both sides by $\mathrm{n}(E_1)\,\mathrm{n}(E_2)$, we obtain $\lambda^2 = 1$, i.e. $\lambda = 1$. Therefore, the product of each two liftings of elements of $G(R)_v$ is commutative.

Let $L \subseteq Q$ be the K-subalgebra generated by the liftings of elements of $G(R)_w$. Then $G(R)_w$ becomes a subgroup of $L^\times/K^\times$. Furthermore, from the previous paragraph, we see that L is commutative, implying that L/K is a field extension of degree 2. Since $D \in L$ satisfies $D^2 = 1 + z + z^2$, we conclude that L/K is purely inseparable. Hence for all $x \in L$, we have $\mathrm{n}(x) = x^2$, i.e.

$$\mathrm{n}\,|_{L^\times} : L^\times \longrightarrow K^\times,\ x \longmapsto \mathrm{n}(x) = x^2$$

is injective. This also induces the injective map $L^\times/K^\times \hookrightarrow K^\times/(K^\times)^2$, which can be further restricted to the injection

$$G(R)_w \longrightarrow R^\times/(R^\times)^2 = (z)^{\mathbb{Z}/2\mathbb{Z}} \times (1+z)^{\mathbb{Z}/2\mathbb{Z}} \times (1+z+z^2)^{\mathbb{Z}/2\mathbb{Z}}.$$

Note that the target set can also be changed since the reduced norm of every element of $G(R)$ lies in $R^\times$ and $R^\times/(R^\times)^2$ can be considered a subgroup of $K^\times/(K^\times)^2$. But by Corollary 4.7, we must have $\nu_0(\mathrm{n}(E)) \equiv \nu_\infty(\mathrm{n}(E)) \equiv 0 \pmod 2$ for any lifting E of an element of $G(R)_w$. This implies that $G(R)_w$ can be embedded at most in the group $(1+z+z^2)^{\mathbb{Z}/2\mathbb{Z}}$, which has exactly two elements. And we already know these two elements explicitly, namely 1 and $d \in G(R)_w$, hence we are done. □

Remark 4.16. Here it is important to consider the subgroup $G(R)$ instead of the whole $G(K)$, since otherwise the stabiliser group could have infinitely many elements, or even worse, an element of infinite order. For instance, consider $E := 1 + zI + IJ \in Q$ with the image $e \in G(K)$. We see that its minimal polynomial is $X^2 + zX + (1 + z^3 + z^4)$, so that e cannot have the order 2, implying that its order is infinite by Lemma 4.12 and Lemma 4.13. However, we have

$$\rho_y(E) = \begin{pmatrix} 1+zy & (1+z^3)y \\ 1+y & 1+z(1+y) \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \pmod y,$$

implying that $\rho_y(e) \in \mathrm{PGL}_2(\mathbb{F}_2[\![y]\!])$, and

$$u^2\rho_t(E) = u^2\mathbf{1}_2 + u\rho(I) + u^2\rho(IJ) \equiv 0 + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \pmod t,$$

implying that $\rho_y(e) \in \mathrm{PGL}_2(\mathbb{F}_2[\![t]\!])$. Hence e stabilises the standard vertex s, although it has an infinite order.

Corollary 4.17. *The stabiliser $G(R_1)_w$ in $G(R_1)$ is trivial. In particular, if $a, b \in G(R_1)$ are such that $a.w = b.w$, then $a = b$.*

Proof. Considering $G(R_1)$ as a subgroup of $G(R)$, we have

$$G(R_1)_w = G(R_1) \cap G(R)_w \subseteq \{1, d\}.$$

Since a (and hence any) lifting of d in Q has an odd order at the place $\{z = \zeta\}$, it can't be invertible over R_1. Hence $d \notin G(R_1)$, implying that $G(R_1)_w$ is trivial. □

4.4 A V_4-structure

Having determined the stabiliser of the standard vertex, we are going to establish a subgroup of $G(R)$ acting on the vertices of $T_0 \times T_\infty$ simply transitively in this section. The strategy is to find a V_4-structure of such subgroup, so that we can refer to Theorem 3.17 to find its presentation.

Definition 4.18. In the quaternion algebra Q, we define the following elements:

$$\begin{aligned}
A_1 &:= zI + J, \\
A_2 &:= z + I, \\
C_1 &:= 1 + z^2 + IJ, \\
C_2 &:= z + z^2 + IJ, \\
B_1 &:= C_1 A_1 = (1+z)I + J, \\
B_2 &:= z^{-1} C_2 A_2 = z + z^2 + (1+z)I + J + IJ.
\end{aligned}$$

Lemma 4.19. *$A_1, A_2, C_1, C_2, B_1, B_2$ are all invertible in $\mathfrak{O}_1 = R_1 \oplus R_1 \cdot I \oplus R_1 \cdot J \oplus R_1 \cdot IJ \subseteq Q$*

Proof. We compute their reduced norms directly and obtain

$$\mathrm{n}(A_1) = 1, \ \ \mathrm{n}(A_2) = z^2, \ \ \mathrm{n}(C_1) = 1 + z \text{ and } \mathrm{n}(C_2) = z + z^2.$$

We see also that every element is invertible in $\mathfrak{O}_1$ since their norms are units in R_1. Since B_1, B_2 are their products, they are also invertible in $\mathfrak{O}_1$. $\square$

Remark 4.20. We also see from the proof that A_1 and A_2 are even invertible in $\mathfrak{O}_0$. For this reason, A_1 and A_2 are explicitly defined since they will be used in finding a presentation of $G(R_0)$.

Notation 4.21. The images of $A_1, A_2, C_1, C_2, B_1, B_2$ in $G(R_1) \subseteq G(K) = Q^\times / K^\times$ will be denoted by $a_1, a_2, c_1, c_2, b_1, b_2$ respectively. Furthermore, we define $\mathcal{A} := \{b_1, b_1^{-1}, c_1\}$ and $\mathcal{B} := \{b_2, b_2^{-1}, c_2\}$ as subsets of $G(R_1)$.

Lemma 4.22. *The sets $\mathcal{A}$ and $\mathcal{B}$ as above define a V_4-structure of the subgroup $\Lambda := \langle b_1, b_2, c_1, c_2 \rangle \leq G(R_1)$.*

Proof. Notice first that since $C_1^2 = 1 + z$ and $C_2^2 = z + z^2$, we have $c_1^2 = c_2^2 = 1$, implying that $\mathcal{A}$ and $\mathcal{B}$ are closed under taking inverse. By direct computation, we obtain the following equations:

$$\begin{aligned}
B_1 B_2 &= (1+z)(z + z^2 + zI + J + IJ) \\
B_1 B_2^{-1} &= z^{-1}(z + z^2 + I + IJ) \\
B_1 C_2 &= (1+z)(1 + z + z^2 + I + IJ) \\
B_1^{-1} B_2 &= I \\
B_1^{-1} B_2^{-1} &= (z + z^2)^{-1}(1 + z + zI + J) \\
B_1^{-1} C_2 &= 1 + I \\
C_1 B_2 &= (1+z)(z^2 + zI + J + IJ) \\
C_1 B_2^{-1} &= z^{-1}(1 + zI + J) \\
C_1 C_2 &= (1+z)(z^2 + IJ)
\end{aligned}$$

Since none of them is a scalar multiple of any others, the map $\mathcal{A} \times \mathcal{B} \to \mathcal{AB}$, $(g,h) \mapsto gh$ is indeed bijective. Furthermore, we have

- $B_2B_1 = (1+z)(1+zI+J)$, which implies the relation $b_2b_1 = c_1b_2^{-1}$, and consequently $b_2^{-1}c_1 = b_1b_2$, $b_2c_1 = b_1^{-1}b_2^{-1}$ and $b_2^{-1}b_1^{-1} = c_1b_2$.
- $C_2B_1 = (1+z)I$, which implies the relation $c_2b_1 = b_1^{-1}b_2$, and consequently $c_2b_1^{-1} = b_1b_2^{-1}$, $b_2^{-1}b_1 = b_1^{-1}c_2$ and $b_2b_1^{-1} = b_1c_2$.
- $C_2C_1 = (1+z)(z^2+IJ)$, i.e. $c_1c_2 = c_2c_1$.

Hence we get $\mathcal{BA} = \mathcal{AB}$, which implies that $\mathcal{A}$ and $\mathcal{B}$ define a V_4-structure of Λ as desired. □

Now that we have got a V_4-structure $(\mathcal{A}, \mathcal{B})$ of Λ, we are going to determine its presentation. For this we need the following lemma:

Lemma 4.23. *For the action of $G(R)$ on $T_0 \times T_\infty$ defined before, we have*

$$\mathcal{A}.w = \{(w_0, v) \mid v \in T_\infty \text{ is a neighbour of } w_\infty\} \text{ and}$$
$$\mathcal{B}.w = \{(v, w_\infty) \mid v \in T_0 \text{ is a neighbour of } w_0\}.$$

Proof. As computed before, we have $\mathrm{n}(B_1) = \mathrm{n}(C_1) = 1+z$. Since the images of B_1 and C_1 under ρ_y are matrices with entries from $\mathbb{F}_2[\![y]\!]$ and $1+z \in \mathbb{F}_2[\![y]\!]^\times$, we have $B_1.w_0 = C_1.w_0 = w_0$ by Corollary 4.8, hence also $B_1^{-1}.w_0 = w_0$. This implies that $\mathcal{A}.w_0 = \{w_0\}$.

Computing the images of B_1 and C_1 under ρ_t in $\mathrm{M}_2(\mathbb{F}_2(\!(t)\!))$, we obtain

$$u\,\rho_t(B_1) = \begin{pmatrix} (1+u)(1+u+t) & u+u^4 \\ 1 & (1+u)(u+t) \end{pmatrix} \text{ and}$$
$$u\,\rho_t(C_1) = \begin{pmatrix} u+u^2 & (1+u^3)(1+u+t) \\ (u+t)/u & u+u^2 \end{pmatrix}.$$

Since both matrices on the right hand side have coprime entries in $\mathbb{F}_2[\![t]\!]$ and determinant $u+u^2$, we have by Proposition 4.6 that

$$d(b_1.w_\infty, w_\infty) = d(c_1.w_\infty, w_\infty) = \nu_0(u+u^2) = 1$$

Since the action of $G(R)$ on T_∞ respects the distance between two vertices on the tree, we also have $d(b_1^{-1}.w_\infty, w_\infty) = d(w_\infty, b_1.w_\infty) = 1$, which implies that

$$\mathcal{A}.w \subseteq \{(w_0, v) \mid v \in T_\infty \text{ is a neighbour of } w_\infty\}.$$

Furthermore, we have by Corollary 4.17 that $b_1.w, b_1^{-1}, c_1.w$ are all different. Implying that $\mathcal{A}.w$ has exactly 3 different elements, i.e. so many elements as the vertical neighbours of w. Therefore the inclusion above is indeed an equality.

Now we come to the set $\mathcal{B}.w$. Since the images of B_2 and C_2 under ρ_y are matrices with coprime entries from $\mathbb{F}_2[\![y]\!]$ and have determinant equal to the reduced norm of B_2 and C_2, which is $z+z^2$, we have by Proposition 4.6 that

$$d(b_2.w_0, w_0) = d(c_2.w_0, w_0) = \nu_0(z+z^2) = 1$$

From this we can also conclude that $d(b_2^{-1}.w_0, w_0) = d(w_0, b_2.w_0) = 1$, i.e. $\mathcal{B}.w_0$ is contained in the subset of the neighbours of w_0

Now we consider the action of $\mathcal{B}$ on the vertical component. Under the representation ρ_t, we have

$$u\,\rho_t(B_2) = \begin{pmatrix} (1+u)t & (1+t)(1+u^3) \\ t/u & (1+u)(1+t) \end{pmatrix} \quad \text{and}$$
$$u\,\rho_t(C_2) = \begin{pmatrix} 1+u^2 & (1+u+t)(1+u^3) \\ (u+t)/u & 1+u^2 \end{pmatrix}.$$

Since $\mathrm{n}(B_2) = \mathrm{n}(C_2) = z + z^2 = u^{-2}(1+u)$, the matrices on the right hand side have the determinant $1+u$ and are therefore invertible over $\mathbb{F}_2[\![t]\!]$. This implies that $b_2.w_\infty = c_2.w_\infty = w_\infty$ by Corollary 4.8, hence also $b_2^{-1}.w_\infty = w_\infty$, i.e. $\mathcal{B}.w_\infty = \{w_\infty\}$. Hence we have

$$\mathcal{B}.w \subseteq \{(v, w_\infty) \mid v \in T_0 \text{ is a neighbour of } w_0\}.$$

The both sets coincide by the same argument applied to $\mathcal{A}.w$, and we are done. □

Corollary 4.24. *The group Λ acts on the set of vertices of $T_0 \times T_\infty$ simply transitively.*

Proof. From Lemmas 4.22 and 4.23, we see that the pair $(\mathcal{A}, \mathcal{B})$ satisfies the condition of Theorem 3.17, whence the claim follows. □

4.5 The presentations of $G(R_1)$ and $G(R)$

Having establish the stabiliser of $G(R)$ as well as its subgroup acting on the vertices of $T_0 \times T_\infty$ simply transitively, we can now find a presentation of $G(R)$. Let us recall the following lemma from the group theory:

Lemma 4.25. *Let G be a group acting on a set X, $G_x \leq G$ be the stabiliser of $x \in X$ and $H \leq G$ a subgroup acting on X transitively. Then*

$$G = HG_x.$$

Proof. Let $g \in G$. Since H acts on X transitively, there exists an $h \in H$ such that $gx = hx$. This implies that $h^{-1}gx = x$, i.e. $g' := h^{-1}g \in G_x$. Hence $g = hg' \in HG_x$ as desired. □

Before we come to $G(R)$, there is an easier result for its subgroup $G(R_1)$. Recall first that from the proof of Lemma 4.22, the squares in $\Sigma_{\mathcal{A},\mathcal{B}}$ can be divided in three V_4-orbits. The first one corresponds to the relation $b_1b_2c_1b_2 = 1$ and consists of the squares

$$[b_1, b_2; b_2^{-1}, c_1],\ [c_1, b_2^{-1}; b_2, b_1],\ [c_1, b_2; b_2^{-1}, b_1^{-1}] \text{ and } [b_1^{-1}, b_2^{-1}; b_2, c_1].$$

The second one corresponds to $b_1c_2b_1b_2^{-1} = 1$ and consists of the squares

$$[b_1, c_2; b_2, b_1^{-1}],\ [b_1^{-1}, c_2; b_2^{-1}, b_1],\ [b_1, b_2^{-1}; c_2, b_1^{-1}] \text{ and } [b_1^{-1}, b_2; c_2, b_1].$$

And the third one corresponds to $c_1c_2 = c_2c_1$ and consists of only the square $[c_1, c_2; c_2, c_1]$.

Proposition 4.26. *The group Λ coincides with $G(R_1)$ and has the presentation*

$$\Lambda = \left\langle b_1, b_2, c_1, c_2 \;\middle|\; c_1^2,\ c_2^2,\ c_1c_2 = c_2c_1,\ b_1b_2c_1b_2,\ b_1c_2b_1b_2^{-1} \right\rangle.$$

Proof. The proof continues from the proof of Corollary 4.24 in the previous section. By Theorem 3.17, Λ is isomorphic to the orbital fundamental group $\pi_1^{\mathrm{orb}}(\Sigma_{\mathcal{A},\mathcal{B}}, V_4, s_{00})$, the structure of which is known from Proposition 3.15 as well as Remark 3.16. Based on this fact, we can find a presentation for Λ as follows:

From the information about the V_4-orbits of the squares in the square complex $\Sigma_{\mathcal{A},\mathcal{B}}$ stated above, we can choose the following representatives:

$$[b_1, b_2; b_2^{-1}, c_1],\ [b_1, c_2; b_2, b_1^{-1}] \text{ and } [c_1, c_2; c_2, c_1].$$

Furthermore, we can use $\{\alpha_{b_1}, \alpha_{c_1}, \beta_{b_2}, \alpha_{c_2}\}$ as a generating system for $\pi_1^{\mathrm{orb}}(\Sigma_{\mathcal{A},\mathcal{B}}, V_4, s_{00})$ and obtain the following presentation:

$$\begin{aligned}\pi_1^{\mathrm{orb}}(\Sigma_{\mathcal{A},\mathcal{B}}, V_4, s_{00}) &= \left\langle \alpha_{b_1}, \alpha_{c_1}, \beta_{b_2}, \beta_{c_2} \;\middle|\; \begin{array}{c} \alpha_{b_1}\beta_{b_2} = \beta_{b_2}^{-1}\alpha_{c_1},\ \alpha_{b_1}\beta_{c_2} = \beta_{b_2}\alpha_{b_1}^{-1} \\ \alpha_{c_1}\beta_{c_2} = \beta_{c_2}\alpha_{c_1},\ \alpha_{c_1}^2 = \beta_{c_2}^2 = 1 \end{array} \right\rangle \\ &= \left\langle \alpha_{b_1}, \alpha_{c_1}, \beta_{b_2}, \beta_{c_2} \,\middle|\, \alpha_{c_1}^2,\ \beta_{c_2}^2,\ \alpha_{c_1}\beta_{c_2} = \beta_{c_2}\alpha_{c_1},\ \alpha_{b_1}\beta_{b_2}\alpha_{c_1}\beta_{b_2},\ \alpha_{b_1}\beta_{c_2}\alpha_{b_1}\beta_{b_2}^{-1} \right\rangle.\end{aligned}$$

The isomorphism from Theorem 3.17 then yields the presentation of Λ as in the statement of this proposition. Furthermore, since Λ acts on the vertices of $T_0 \times T_\infty$ transitively and $G(R_1)_w$ is trivial by Corollary 4.17, we have $G(R_1) = \Lambda$ as desired. □

Now we come to the main result, namely the presentation of the arithmetic lattice $G(R)$:

Theorem 4.27. *The quaternionic arithmetic lattice $G(R)$ has the following presentation:*

$$G(R) = \left\langle b_1, b_2, c_1, c_2, d \;\middle|\; \begin{array}{c} c_1^2,\ c_2^2,\ d^2,\ c_1c_2 = c_2c_1,\ c_1d = dc_1,\ c_2d = dc_2, \\ b_1b_2c_1b_2,\ b_1c_2b_1b_2^{-1},\ db_1db_1,\ db_2db_2 \end{array} \right\rangle.$$

Proof. By Lemma 4.25 and Corollary 4.24, we have

$$G(R) = \Lambda \cdot G(R)_w.$$

Since the action of Λ on the vertices of $T_0 \times T_\infty$ is simple, the intersection $\Lambda \cap G(R)_w$ is trivial. Furthermore, since $G(R)_w = \langle d \rangle$ has exactly two elements, Λ is a subgroup of $G(R)$ of index two and thus a normal subgroup. Hence by Corollary 3.14, we obtain a presentation of $G(R)$ by adding the relation $d^2 = 1$ from $G(R)_w$ and the relations obtained by conjugating the generators of Λ by d to the relations in the presentation of Λ as follows:

- Since C_1, C_2, D lie in the commutative subalgebra of Q generated by IJ, d lies in the centralisers of c_1 and c_2, i.e. the conjugations of c_1 and c_2 by d yield the same element. These are equivalent to $c_1d = dc_1$ and $c_2d = dc_2$.

- For b_1, we have $DB_1 = J$, implying that $(db_1)^2 = 1$ since $J^2 = 1 + z^3 \in K$. Consequently, we have $db_1d^{-1} = (db_1)^{-1}d^{-1} = b_1^{-1}d^{-2} = b_1^{-1}$, which is equivalent to $db_1db_1 = 1$.

- For b_2, we have $DB_2 = J + IJ$, implying that $(db_2)^2 = 1$ since $(J + IJ)^2 = z + z^4 \in K$. Hence we get $db_2d^{-1} = (db_2)^{-1}d^{-1} = b_2^{-1}d^{-2} = b_2^{-1}$, which is equivalent to $db_2db_2 = 1$.

Combining these relations to those in Λ obtained before, we get the presentation of $G(R)$ as desired. □

The group $G(R)$ also has a topological interpretation as the orbital fundamental group of $\Sigma_{\mathcal{A},\mathcal{B}}$. For this we need to define first the action of the group $Z_2^3 = V_4 \times Z_2$.

Definition 4.28. We define the action of the 2-elementary abelian group of order 8

$$Z_2^3 = \left\langle \gamma_v, \gamma_h, \delta \mid \gamma_v^2, \gamma_h^2, \delta^2 \right\rangle_{\text{ab}} = V_4 \times Z_2$$

on the square complex $\Sigma_{\mathcal{A},\mathcal{B}}$ as follows: The restriction to $V_4 = \langle \gamma_v, \gamma_h \rangle$ is the action defined in Chapter 3. The action of δ is trivial on the vertices, interchanges the vertical edges labelled by b_1 with b_1^{-1} and the horizontal ones labelled by b_2 with b_2^{-1}. The edges labelled by c_1 and c_2 remain unchanged under this action. The action on the squares is the induced by the action on the edges.

Lemma 4.29. *The action of Z_2^3 on $\Sigma_{\mathcal{A},\mathcal{B}}$ is well-defined.*

Proof. On the 1-skeleton of $\Sigma_{\mathcal{A},\mathcal{B}}$, we can see that the action of δ defines an involution, i.e. compatible to the condition $\delta^2 = 1$, and commutes with the actions of c_1 and c_2. Hence the action of Z_2^3 on the 1-skeleton is well-defined.

It remains to show that δ maps each square in $\Sigma_{\mathcal{A},\mathcal{B}}$ to a square in $\Sigma_{\mathcal{A},\mathcal{B}}$. Indeed, the action of Z_2^3 interchanges the square $[b_1, b_2; b_2^{-1}, c_1]$ with $[b_1^{-1}, b_2^{-1}; b_2, c_1]$, $[c_1, b_2^{-1}; b_2, b_1]$ with $[c_1, b_2; b_2^{-1}, b_1^{-1}]$, $[b_1, c_2; b_2, b_1^{-1}]$ with $[b_1^{-1}, c_2; b_2^{-1}, b_1]$, $[b_1^{-1}, b_2; c_2, b_1]$ with $[b_1, b_2^{-1}; c_2, b_1^{-1}]$ and fixes the square $[c_1, c_2; c_2, c_1]$. Hence the action of Z_2^3 is well-defined on the whole $\Sigma_{\mathcal{A},\mathcal{B}}$. □

Theorem 4.30. *The isomorphism $\varphi : \pi_1^{\text{orb}}(\Sigma_{\mathcal{A},\mathcal{B}}, V_4, s_{00}) \to G(R_1)$ from Theorem 3.17 can be extended to an isomorphism*

$$\tilde{\varphi} : \pi_1^{\text{orb}}(\Sigma_{\mathcal{A},\mathcal{B}}, Z_2^3, s_{00}) \longrightarrow G(R)$$

by mapping $[\bullet; \delta]$ to $d \in G(R)$, the element of order 2 from Proposition 4.14.

Proof. Notice first that since $\delta.s_{00} = s_{00}$, the element $[\bullet, \delta] \in \pi_1^{\text{orb}}(\Sigma_{\mathcal{A},\mathcal{B}}, Z_2^3, s_{00})$ is well-defined. Since V_4 is a normal subgroup of Z_2^3 with the factor group Z_2, we have by Proposition 2.8 the exact sequence

$$1 \longrightarrow \pi_1^{\text{orb}}(\Sigma_{\mathcal{A},\mathcal{B}}, V_4, s_{00}) \longrightarrow \pi_1^{\text{orb}}(\Sigma_{\mathcal{A},\mathcal{B}}, Z_2^3, s_{00}) \longrightarrow Z_2 \longrightarrow 1$$

with a section $Z_2 \to \pi_1^{\text{orb}}(\Sigma_{\mathcal{A},\mathcal{B}}, Z_2^3, s_{00}), \delta \mapsto [\bullet; \delta]$. Hence in what follows we shall also use δ to denote the element $[\bullet, \delta] \in \pi_1^{\text{orb}}(\Sigma_{\mathcal{A},\mathcal{B}}, Z_2^3, s_{00})$ by abuse of notation. This implies that $\pi_1^{\text{orb}}(\Sigma_{\mathcal{A},\mathcal{B}}, Z_2^3, s_{00})$ is a semidirect product of $\pi_1^{\text{orb}}(\Sigma_{\mathcal{A},\mathcal{B}}, V_4, s_{00})$ and Z_2. By Corollary 3.14 and the presentation of $\pi_1^{\text{orb}}(\Sigma_{\mathcal{A},\mathcal{B}}, V_4, s_{00})$ obtained from the proof of Proposition 4.26, we have

$$\pi_1^{\text{orb}}(\Sigma_{\mathcal{A},\mathcal{B}}, Z_2^3, s_{00}) = \left\langle \begin{array}{c} \alpha_{b_1}, \beta_{b_2}, \\ \alpha_{c_1}, \beta_{c_2}, \delta \end{array} \middle| \begin{array}{c} \alpha_{c_1}^2,\ \beta_{c_2}^2,\ \alpha_{c_1}\beta_{c_2} = \beta_{c_2}\alpha_{c_1},\ \alpha_{b_1}\beta_{b_2}\alpha_{c_1}\beta_{b_2},\ \alpha_{b_1}\beta_{c_2}\alpha_{b_1}\beta_{b_2}^{-1} \\ \delta^2, \quad \delta\alpha_{c_1}\delta^{-1} = \alpha_{c_1},\ \delta\beta_{c_2}\delta^{-1} = \beta_{c_2} \\ \delta\alpha_{b_1}\delta^{-1} = \alpha_{b_1}^{-1},\ \delta\beta_{b_2}\delta^{-1} = \beta_{b_2}^{-1} \end{array} \right\rangle.$$

Comparing this with the presentation of $G(R)$ from Theorem 4.27, we can conclude that the homomorphism

$$\tilde{\varphi} : \pi_1^{\text{orb}}(\Sigma_{\mathcal{A},\mathcal{B}}, Z_2^3, s_{00}) \longrightarrow G(R),$$

given by $\tilde{\varphi}(\alpha_{b_1}) = b_1$, $\tilde{\varphi}(\alpha_{c_1}) = c_1$, $\tilde{\varphi}(\beta_{b_2}) = b_2$, $\tilde{\varphi}(\beta_{c_2}) = c_2$ and $\tilde{\varphi}(\delta) = d$, is a well-defined isomorphism which is an extension of the isomorphism $\varphi : \pi_1^{\text{orb}}(\Sigma_{\mathcal{A},\mathcal{B}}, V_4, s_{00}) \to G(R_1)$ from Theorem 3.17. □

Another consequence of the well-defined action of $\delta \in Z_2^3$ on $\Sigma_{\mathcal{A},\mathcal{B}}$ is the following nice result about the local permutation groups associated to the V_4-structure $(\mathcal{A},\mathcal{B})$.

Proposition 4.31. *Let $P_j^{\mathcal{A}}$ and $P_i^{\mathcal{B}}$ for $i,j \in I = \{0,1\}$ be the local permutation groups associated to the V_4-structure $(\mathcal{A},\mathcal{B})$ as introduced in Section 3.6. For $\mathcal{C} \in \{\mathcal{A},\mathcal{B}\}$, let $\tau^{\mathcal{C}} := (-)^{-1} \in \mathrm{Sym}(\mathcal{C})$ and*

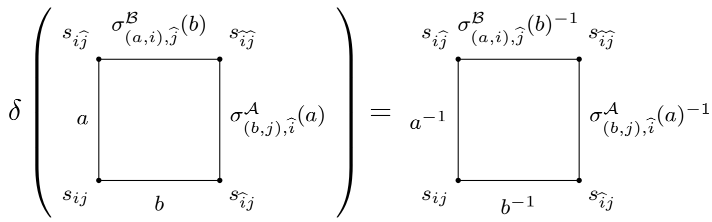
$$\psi_{\mathcal{C}} : \mathrm{Sym}(\mathcal{C}) \rtimes \{\pm 1\} \hookrightarrow \mathrm{Sym}(\mathcal{C}) \wr \mathrm{Sym}(I),\ (\sigma,\varepsilon) \longmapsto \begin{cases} ((\sigma, \tau^{\mathcal{C}}\sigma\tau^{\mathcal{C}}), \mathrm{id}_I), & \varepsilon = +1, \\ ((\sigma, \tau^{\mathcal{C}}\sigma\tau^{\mathcal{C}}), \widehat{\cdot}), & \varepsilon = -1, \end{cases}$$

where the semidirect product structure on $\mathrm{Sym}(\mathcal{C}) \rtimes \{\pm 1\}$ is given by sending $-1 \in \{\pm 1\}$ to the inner automorphism of $\mathrm{Sym}(\mathcal{C})$ given by conjugation with $\tau^{\mathcal{C}}$. Then

$$P_0^{\mathcal{C}} = P_1^{\mathcal{C}} = \mathrm{im}(\psi_{\mathcal{C}}) = \big\{((\sigma, \tau^{\mathcal{C}}\sigma\tau^{\mathcal{C}}), \varepsilon) \mid \sigma \in \mathrm{Sym}(\mathcal{C}), \varepsilon \in \mathrm{Sym}(I)\big\} \cong \mathrm{Sym}(\mathcal{C}) \rtimes \{\pm 1\}.$$

In particular, $P_0^{\mathcal{C}} = P_1^{\mathcal{C}}$ acts on $\mathcal{C} \times I$ as subgroup of $\mathrm{Sym}(\mathcal{C} \times I)$ transitively.

Proof. Observe first that $\psi_{\mathcal{A}}$ and $\psi_{\mathcal{B}}$ are apparently injective group homomorphisms. Now by applying δ to the square from (3.9), we get



for all $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $i,j \in I$. This implies that

$$\sigma^{\mathcal{A}}_{(b^{-1},j),\widehat{i}}(a^{-1}) = \sigma^{\mathcal{A}}_{(b,j),\widehat{i}}(a)^{-1} \quad \text{and} \quad \sigma^{\mathcal{B}}_{(a^{-1},i),\widehat{j}}(b^{-1}) = \sigma^{\mathcal{B}}_{(a,i),\widehat{j}}(b)^{-1}$$

Since $\sigma^{\mathcal{A}}_{(b^{-1},j),\widehat{i}}(a^{-1}) = \sigma^{\mathcal{A}}_{(b^{-1},\widehat{j}),\widehat{i}}(a)^{-1}$ by the proof of Proposition 3.24, we get $\sigma^{\mathcal{A}}_{(b^{-1},\widehat{j}),\widehat{i}} = \sigma^{\mathcal{A}}_{(b,j),\widehat{i}}$, implying that $\sigma^{\mathcal{A}}_{(b,j)} = \sigma^{\mathcal{A}}_{(b^{-1},\widehat{j})}$. Hence $P_0^{\mathcal{A}} = P_1^{\mathcal{A}}$ since both are generated by the same elements in $\mathrm{Sym}(\mathcal{A}) \wr \mathrm{Sym}(I)$. Furthermore, since $\sigma^{\mathcal{A}}_{(b^{-1},j),\widehat{i}} = \sigma^{\mathcal{A}}_{(b,j),i}$ by loc.cit., we get

$$\sigma^{\mathcal{A}}_{(b,j),i}\tau^{\mathcal{A}}(a) = \sigma^{\mathcal{A}}_{(b,j),i}(a^{-1}) = \sigma^{\mathcal{A}}_{(b^{-1},j),\widehat{i}}(a^{-1}) = \sigma^{\mathcal{A}}_{(b,j),\widehat{i}}(a)^{-1} = \tau^{\mathcal{A}}\sigma^{\mathcal{A}}_{(b,j),\widehat{i}}(a),$$

for all $a \in \mathcal{A}$ i.e. $\sigma^{\mathcal{A}}_{(b,j),i}\tau^{\mathcal{A}} = \tau^{\mathcal{A}}\sigma^{\mathcal{A}}_{(b,j),\widehat{i}}$, implying that $\sigma^{\mathcal{A}}_{(b,j),0}$ and $\sigma^{\mathcal{A}}_{(b,j),1}$ differ by conjugation with $\tau^{\mathcal{A}}$. Hence $P_0^{\mathcal{A}} = P_1^{\mathcal{A}}$ is contained in $\mathrm{im}(\psi_{\mathcal{A}})$. Similarly, we can show that $P_0^{\mathcal{B}}$ and $P_1^{\mathcal{B}}$ coincide and are both contained in $\mathrm{im}(\psi_{\mathcal{B}})$.

We are going to show that both equality hold by computing some elements of $P_0^{\mathcal{A}} = P_1^{\mathcal{A}}$ and $P_0^{\mathcal{B}} = P_1^{\mathcal{B}}$ as follows: Starting from $P^0_{\mathcal{A}}$, we have, by investigating the squares in $\Sigma_{\mathcal{A},\mathcal{B}}$ explicitly,

$$\sigma^{\mathcal{A}}_{(b_2,0)} = (((b_1\ c_1\ b_1^{-1}), (b_1^{-1}\ c_1\ b_1)), \widehat{\cdot}), \quad \sigma^{\mathcal{A}}_{(c_2,0)} = (((b_1\ b_1^{-1}), (b_1^{-1}\ b_1)), \widehat{\cdot}),$$

and by Proposition 3.24,

$$\sigma^{\mathcal{A}}_{(b_2^{-1},0)} = (((b_1^{-1}\ c_1\ b_1), (b_1\ c_1\ b_1^{-1})), \widehat{\cdot})$$

This implies that

$$\sigma^{\mathcal{A}}_{(b_2,0)}\sigma^{\mathcal{A}}_{(c_2,0)} = (((c_1\ b_1^{-1}),(c_1\ b_1)),\mathrm{id}_I) = \psi_{\mathcal{A}}((c_1\ b_1^{-1}),+1), \quad \text{and}$$
$$\sigma^{\mathcal{A}}_{(b_2^{-1},0)}\sigma^{\mathcal{A}}_{(c_2,0)} = (((c_1\ b_1),(c_1\ b_1^{-1})),\mathrm{id}_I) = \psi_{\mathcal{A}}((c_1\ b_1^{-1}),+1).$$

This means that the image of whole group $\mathrm{Sym}(\mathcal{A})$ as a subgroup of $\mathrm{Sym}(\mathcal{A}) \rtimes \{\pm 1\}$ under $\psi_{\mathcal{A}}$, being generated by two transpositions, is entirely contained in $P_0^{\mathcal{A}} = P_1^{\mathcal{A}}$. Since $P_0^{\mathcal{A}} = P_1^{\mathcal{A}}$ also contains at least one image of an element of $\mathrm{Sym}(\mathcal{A}) \rtimes \{\pm 1\}$ with non-trivial projection onto $\{\pm 1\}$, e.g. $\sigma^{\mathcal{A}}_{(c_2,0)} = ((b_1\ b_1^{-1}),-1)$, it must contain the whole image of $\mathrm{Sym}(\mathcal{A}) \rtimes \{\pm 1\}$ under $\psi_{\mathcal{A}}$, i.e. $P_0^{\mathcal{A}} = P_1^{\mathcal{A}} = \mathrm{im}(\psi_{\mathcal{A}})$. This acts apparently transitively on $\mathcal{A} \times I$ since an orbit must contain at least $\mathcal{A}_0$ or $\mathcal{A}_1$ and the element $\psi_{\mathcal{A}}(\mathrm{id},-1)$ interchanges $\mathcal{A}_0$ with $\mathcal{A}_1$.

Now for $P_1^{\mathcal{B}}$, we have

$$\sigma^{\mathcal{B}}_{(b_1,1)} = (((b_2\ c_2\ b_2^{-1}),(b_2^{-1}\ c_2\ b_2)),\widehat{\cdot}),\ \sigma^{\mathcal{B}}_{(c_1,0)} = (((b_2\ b_2^{-1}),(b_2^{-1}\ b_2)),\widehat{\cdot}) \ \text{ and}$$
$$\sigma^{\mathcal{B}}_{(b_1^{-1},1)} = (((b_2^{-1}\ c_2\ b_2),(b_2\ c_2\ b_2^{-1})),\widehat{\cdot}).$$

Hence the calculation of $P_0^{\mathcal{B}} = P_1^{\mathcal{B}}$ can be done in the similar way to $P_0^{\mathcal{A}}$, implying that $P_0^{\mathcal{B}} = P_1^{\mathcal{B}} = \mathrm{im}(\psi_{\mathcal{B}})$ and that this acts on $\mathcal{B} \times I$ transitively as desired. □

4.6 The group $G(R_0)$ and the fundamental group of $\Sigma_{\mathcal{A},\mathcal{B}}$

We have seen from the last section that the arithmetic lattice $G(R_1)$ concides with the orbital fundamental group $\pi_1^{\mathrm{orb}}(\Sigma_{\mathcal{A},\mathcal{B}}, V_4, s_{00})$. We shall use this to describe its subgroup $G(R_0)$. For this we begin with a normal subgroup of $G(R_1)$ corresponding to the fundamental group $\pi_1(\Sigma_{\mathcal{A},\mathcal{B}}, s_{00})$.

Proposition 4.32. $\Gamma := \langle a_1, a_2\rangle \leq G(R_1)$ *is a normal subgroup of index* 4 *and isomorphic to the fundamental group* $\pi_1(\Sigma_{\mathcal{A},\mathcal{B}}, s_{00})$ *under the isomorphism* $\varphi : \pi_1^{\mathrm{orb}}(\Sigma_{\mathcal{A},\mathcal{B}}, V_4, s_{00}) \xrightarrow{\cong} G(R_1)$ *from Theorem 3.17. Its presentation is given by*

$$\Gamma = \big\langle a_1, a_2 \ \big|\ a_1^2a_2^2a_1^{-1}a_2a_1a_2^{-1}a_1a_2^{-2}a_1^{-1}a_2^{-1},\ a_2a_1a_2^2a_1^{-1}a_2^2a_1^{-1}a_2^2a_1a_2a_1^{-1}\big\rangle. \tag{4.4}$$

Proof. Notice first that to show that Γ is isomorphic to $\pi_1(\Sigma_{\mathcal{A},\mathcal{B}}, s_{00})$ under φ, it suffices to show that the image of $\pi_1(\Sigma_{\mathcal{A},\mathcal{B}}, s_{00})$ as a subgroup of $\pi_1^{\mathrm{orb}}(\Sigma_{\mathcal{A},\mathcal{B}}, V_4, s_{00})$ under φ concides with Γ. For this purpose, we find first an appropriate presentation for $\pi_1(\Sigma_{\mathcal{A},\mathcal{B}}, s_{00})$ as follows:

Using the notations for loops in $\Sigma_{\mathcal{A},\mathcal{B}}$ from Definition 3.8 by fixing $[c_1, c_2; c_2, c_1]$ as the base square, i.e. $x_{0c_1} = x_{1c_1} = y_{0c_2} = y_{1c_2} = 1$, together with the presentation from Theorem 3.9, the group $\pi_1(\Sigma_{\mathcal{A},\mathcal{B}}, s_{00})$ can be generated by $\{x_{0b_1}, x_{0b_1^{-1}}, x_{1b_1}, x_{1b_1^{-1}}, y_{0b_2}, y_{0b_2^{-1}}, y_{1b_2}, y_{1b_2^{-1}}\}$ with the following relations:

$$y_{1b_2} = y_{0b_2^{-1}}x_{1b_1^{-1}} \tag{4.5a}$$
$$y_{1b_2^{-1}} = y_{0b_2}x_{1b_1} \tag{4.5b}$$
$$x_{0b_1^{-1}}y_{1b_2} = x_{1b_1} \tag{4.5c}$$
$$x_{0b_1}y_{1b_2^{-1}} = x_{1b_1^{-1}} \tag{4.5d}$$
$$x_{0b_1^{-1}}y_{1b_2^{-1}} = y_{0b_2} \tag{4.5e}$$
$$x_{0b_1}y_{1b_2} = y_{0b_2^{-1}} \tag{4.5f}$$
$$x_{0b_1} = y_{0b_2}x_{1b_1^{-1}} \tag{4.5g}$$
$$x_{0b_1^{-1}} = y_{0b_2^{-1}}x_{1b_1} \tag{4.5h}$$

Since $\varphi^{-1}(a_1) = \varphi^{-1}(c_1b_1) = x^{-1}_{0b_1^{-1}}$ and $\varphi^{-1}(a_2) = \varphi^{-1}(c_2b_2) = y^{-1}_{0b_2^{-1}}$, we are going to use the relations above to write the other generators in terms of $x_{0b_1^{-1}}$ and $y_{0b_2^{-1}}$ as follows: Using the equations (4.5e) – (4.5h), we can substitute

$$y_{1b_2^{-1}} = x^{-1}_{0b_1^{-1}}y_{0b_2},\ y_{1b_2} = x^{-1}_{0b_1}y_{0b_2^{-1}},\ x_{1b_1^{-1}} = y^{-1}_{0b_2}x_{0b_1} \text{ and } x_{1b_1} = y^{-1}_{0b_2^{-1}}x_{0b_1^{-1}}$$

into (4.5a) – (4.5d), so that $\pi_1(\Sigma_{\mathcal{A},\mathcal{B}}, s_{00})$ can be generated by $\{x_{0b_1}, x_{0b_1^{-1}}, y_{0b_2}, y_{0b_2^{-1}}\}$ with the following relations:

$$x^{-1}_{0b_1}y_{0b_2^{-1}} = y_{0b_2^{-1}}y^{-1}_{0b_2}x_{0b_1} \tag{4.6a}$$
$$x^{-1}_{0b_1^{-1}}y_{0b_2} = y_{0b_2}y^{-1}_{0b_2^{-1}}x_{0b_1^{-1}} \tag{4.6b}$$
$$x_{0b_1^{-1}}x^{-1}_{0b_1}y_{0b_2^{-1}} = y^{-1}_{0b_2^{-1}}x_{0b_1^{-1}} \tag{4.6c}$$
$$x_{0b_1}x^{-1}_{0b_1^{-1}}y_{0b_2} = y^{-1}_{0b_2}x_{0b_1} \tag{4.6d}$$

From (4.6c), we get $x_{0b_1} = y_{0b_2^{-1}}x^{-1}_{0b_1^{-1}}y_{0b_2^{-1}}x_{0b_1^{-1}}$. Substituting this into (4.6a), (4.6b) and (4.6d), we obtain

$$x^{-1}_{0b_1^{-1}}y^{-1}_{0b_2^{-1}}x_{0b_1^{-1}} = y_{0b_2^{-1}}y^{-1}_{0b_2}y_{0b_2^{-1}}x^{-1}_{0b_1^{-1}}y_{0b_2^{-1}}x_{0b_1^{-1}} \tag{4.7a}$$
$$x^{-1}_{0b_1^{-1}}y_{0b_2} = y_{0b_2}y^{-1}_{0b_2^{-1}}x_{0b_1^{-1}} \tag{4.7b}$$
$$y_{0b_2^{-1}}x^{-1}_{0b_1^{-1}}y_{0b_2^{-1}}y_{0b_2} = y^{-1}_{0b_2}y_{0b_2^{-1}}x^{-1}_{0b_1^{-1}}y_{0b_2^{-1}}x_{0b_1^{-1}} \tag{4.7c}$$

Now from (4.7a), we get $y_{0b_2} = y_{0b_2^{-1}}x^{-1}_{0b_1^{-1}}y^{2}_{0b_2^{-1}}x_{0b_1^{-1}}y_{0b_2^{-1}}$, which we can substitute in (4.7b) and (4.7c) to get

$$x^{-1}_{0b_1^{-1}}y_{0b_2^{-1}}x^{-1}_{0b_1^{-1}}y^{2}_{0b_2^{-1}}x_{0b_1^{-1}}y_{0b_2^{-1}} = y_{0b_2^{-1}}x^{-1}_{0b_1^{-1}}y^{2}_{0b_2^{-1}}x^{2}_{0b_1^{-1}} \tag{4.8a}$$
$$y_{0b_2^{-1}}x^{-1}_{0b_1^{-1}}y^{2}_{0b_2^{-1}}x^{-1}_{0b_1^{-1}}y^{2}_{0b_2^{-1}}x_{0b_1^{-1}}y_{0b_2^{-1}} = y^{-1}_{0b_2^{-1}}x^{-1}_{0b_1^{-1}}y^{-1}_{0b_2^{-1}}x_{0b_1^{-1}} \tag{4.8b}$$

This shows that $\pi_1(\Sigma_{\mathcal{A},\mathcal{B}}, s_{00})$ can be generated only by $\{x_{0b_1^{-1}}, y_{0b_2^{-1}}\}$. Its image is then generated by $\{\varphi(x_{0b_1^{-1}}), \varphi(y_{0b_2^{-1}})\} = \{a_1^{-1}, a_2^{-1}\}$, hence by $\{a_1, a_2\}$ as a subgroup of $G(R_1)$. This implies that Γ is exactly the image of $\pi_1(\Sigma_{\mathcal{A},\mathcal{B}}, s_{00})$ under φ and hence isomorphic to $\pi_1(\Sigma_{\mathcal{A},\mathcal{B}}, s_{00})$. Furthermore, the relations (4.8a) and (4.8b) are then, under the isomorphism φ, equivalent to

$$a_1^2a_2^2a_1^{-1}a_2a_1a_2^{-1}a_1a_2^{-2}a_1^{-1}a_2^{-1} = 1 \text{ and } a_2a_1a_2^2a_1^{-1}a_2^2a_1^{-1}a_2^2a_1a_2a_1^{-1} = 1$$

so that the group Γ has the presentation as in (4.4). □

Remark 4.33. This could also have been done without interpretation as the fundamental group. Indeed, one could use the Todd-Coxeter algorithm to show that Γ is a normal subgroup of index 4 in Λ, and use the Reidemeister-Schreier algorithm to find a presentation of Γ. But this would yield, before eliminating redundant generators, the presentation with eight generators and relations corresponding to the presentation obtained by computing the fundamental group of the square complex $\Sigma_{\mathcal{A},\mathcal{B}}$.

Corollary 4.34. *The abelianisation* Γ^{ab} *is isomorphic to* $\mathbb{Z}/15\mathbb{Z}$.

Proof. From the presentation in Proposition 4.32, we get

$$\Gamma^{\mathrm{ab}} = \big\langle a_1, a_2 \;\big|\; 2a_1 - a_2, 8a_2 - a_1 \big\rangle_{\mathrm{ab}}.$$

This means that Γ^{ab} is isomorphic to the cokernel of

$$\mathbb{Z}^2 \longrightarrow \mathbb{Z}^2,\; x \longmapsto Mx, \text{ where } M = \begin{pmatrix} 2 & -1 \\ -1 & 8 \end{pmatrix}.$$

Since the entries of M are relative prime and $\det(M) = 15$, the only elementary divisor of Γ^{ab} is then 15. Hence the claim follows. □

Proposition 4.35. *$G(R_0)$ is a normal subgroup of index 2 in $G(R_1)$ having $\{a_1, a_2, c\}$, where $c := c_1c_2$, as a generating system. Moreover, it is isomorphic under φ to the orbital fundamental group $\pi_1^{\mathrm{orb}}(\Sigma_{\mathcal{A},\mathcal{B}}, \langle\gamma\rangle, s_{00})$, where $\gamma := \gamma_v\gamma_h \in V_4$.*

Proof. Notice first that since $C := C_1C_2/(1+z) = z^2 + IJ$ has the norm z, it is invertible in $\mathfrak{O}_0$ and has therefore an image in $G(R_0)$, i.e. $c = c_1c_2 \in G(R_0)$. Its inverse image under φ is $\tilde{\gamma} := \alpha_{c_1}\beta_{c_2} \in \pi_1^{\mathrm{orb}}(\Sigma_{\mathcal{A},\mathcal{B}}, V_4, s_{00})$, which obviously doesn't lie in $\pi_1(\Sigma_{\mathcal{A},\mathcal{B}}, s_{00})$. Since $a_1, a_2 \in G(R_0)$ and Γ is generated by a_1, a_2, we see that Γ is a proper subgroup of $G(R_0)$, i.e. $(G(R_0) : \Gamma) \geq 2$.

On the other hand, since $\mathrm{n}(C_1) = 1 + z$, any lifting of $c_1 \in G(R_1)$ in $Q^\times$ has a norm with odd order at the place $\{z = 1\}$. This implies that c_1 can't lie in $G(R_0)$ and consequently $G(R_0)$ is a proper subgroup of $G(R_1)$, i.e. $(G(R_1) : G(R_0)) \geq 2$. But from the Proposition 4.32, we have

$$(G(R_1) : G(R_0)) \cdot (G(R_0) : \Gamma) = (G(R_1) : \Gamma) = 4.$$

This implies that $(G(R_1) : G(R_0)) = (G(R_0) : \Gamma) = 2$. In particular, $G(R_0)$ is a minimal subgroup that properly contains Γ. Since $\Gamma \lneq \langle a_1, a_2, c\rangle \leq G(R_0)$ from the consideration above, we have $G(R_0) = \langle a_1, a_2, c\rangle$. And its preimage under φ is then $\pi_1^{\mathrm{orb}}(\Sigma_{\mathcal{A},\mathcal{B}}, \langle\gamma\rangle, s_{00})$ since the latter one is an extension of $\langle\gamma\rangle$ by $\pi_1(\Sigma_{\mathcal{A},\mathcal{B}}, s_{00})$ and has therefore exactly $x_{0b_1^{-1}}, y_{0b_2^{-1}}, \tilde{\gamma}$ as generators, compare Proposition 3.12, whence the claim follows. □

To give an overview of all groups we have introduced and established in this chapter, we end up here with the following diagram of subgroups of $\pi_1^{\mathrm{orb}}(\Sigma_{\mathcal{A},\mathcal{B}}, Z_2^3, s_{00})$ and for each of them its isomorphic subgroup of $G(R)$ in the corresponding position in the other diagram. Each group is a subgroup of index 2, in particular a normal subgroup, of the group directly above it.

$$\begin{array}{ccc}
\pi_1^{\mathrm{orb}}(\Sigma_{\mathcal{A},\mathcal{B}}, Z_2^3, s_{00}) & \qquad & G(R) \\
| & & | \\
\pi_1^{\mathrm{orb}}(\Sigma_{\mathcal{A},\mathcal{B}}, V_4, s_{00}) & & G(R_1) = \Lambda \\
| & & | \\
\pi_1^{\mathrm{orb}}(\Sigma_{\mathcal{A},\mathcal{B}}, \langle\gamma\rangle, s_{00}) & & G(R_0) \\
| & & | \\
\pi_1(\Sigma_{\mathcal{A},\mathcal{B}}, s_{00}) & & \Gamma
\end{array}$$

Chapter 5

Construction of a fake quadric

In this chapter, we are going to discuss the construction of a fake quadric over a field of characteristic 2 by means of non-archimedean uniformisation. This is similar to the construction of such a surface in characteristic 3 given by Stix and Vdovina in [SV13], while the technique of non-archimedean uniformisation had already been employed by Mumford in his construction of a fake projective plane in [Mum79].

5.1 Generalities about fake quadrics

The original question which leads to the definition of a **fake quadric** asks for the existence of a complex Kähler manifold X with the same Betti numbers

$$b_1(X) = 0, \quad b_2(X) = 2 \tag{5.1}$$

as the quadric $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ but not isomorphic to it. Since $h^{1,1}(X) = h^1(X, \Omega_X) > 0$ and $b_2(X) = 2p_g(X) + h^{1,1}(X)$, where $p_g(X)$ denotes the geometric genus of X, the condition (5.1) implies that $p_g(X) = 0$ and $h^{1,1}(X) = 2$. It follows that $\chi(X, \mathscr{O}_X) = 1$. Furthermore, the Chern numbers in this case are $c_2(X) = c_2(X) = 2 - 2b_1 + b_2 = 4$, and by Noether's formula, $c_1(X)^2 = 12\chi(X, \mathscr{O}_X) - c_2(X) = 8$, i.e. X must satisfy the following conditions:

$$c_1(X)^2 = 8, \quad c_2(X) = 4 \quad \text{and} \quad \mathrm{Alb}_X = 0, \tag{5.2}$$

where Alb_X denotes the Albanese variety of X. Conversely, one can show that (5.2) implies (5.1), i.e. both conditions are equivalent in characteristic 0.

Now let X be a smooth projective surface over an arbitrary field K satisfying (5.2) and K_X its canonical divisor. Since $K_X^2 = c_1(X)^2 > 0$, the Kodaira dimenstion of X is either $-\infty$ or 2. In the first case, X is a Hirzebruch surface, i.e. of the form $\Sigma_n = \mathbb{P}(\mathscr{O} \oplus \mathscr{O}(n)) \to \mathbb{P}^1$ for some $n \in \mathbb{N}_0$. In the latter case, i.e. X is of general type, it is either the blow-up of a fake projective plane at a closed point or is a minimal surface. This leads to the following definition for a fake quadric over an arbitrary field:

Definition 5.1. Let K be an arbitrary field. A **fake quadric** over K is a smooth minimal projective surface of general type X over K satisfying the condition (5.2).

In characteristic 0, all fake quadrics known so far are uniformised over $\mathbb{C}$ by the product $\mathbb{H} \times \mathbb{H}$ of two copies of the upper half plane $\mathbb{H}$, i.e. of the form $X_\Gamma := \Gamma \backslash \mathbb{H} \times \mathbb{H}$ for some lattice Γ in the holomorphic isometry group $\mathrm{Isom}^h(\mathbb{H} \times \mathbb{H})$ acting on $\mathbb{H} \times \mathbb{H}$ freely. These can be divided into two following classes:

(1) *Reducible fake quadrics,* i.e. those fake quadrics X_Γ such that $\Gamma = \pi_1(X_\Gamma)$ is a reducible lattice in $\mathrm{Isom}^h(\mathbb{H}\times\mathbb{H})$. Here reducible means that Γ is commensurable with a product $\Gamma_1\times\Gamma_2$ of two lattices $\Gamma_1,\Gamma_2 \leq \mathrm{PSL}_2(\mathbb{R})$, where we consider $\mathrm{PSL}_2(\mathbb{R})\times\mathrm{PSL}_2(\mathbb{R})$ via $\mathrm{Isom}^h(\mathbb{H})\cong\mathrm{PSL}_2(\mathbb{R})$ as a subgroup of $\mathrm{Isom}^h(\mathbb{H}\times\mathbb{H})$. In this case there are two curves C_1, C_2 of genus ≥ 2 and a finite group G such that $X = (C_1\times C_2)/G$. The first such surface has been given by Beauville in [Bea96], exercise X.4. A classification of such fake quadrics was proposed by Bauer, Catanese and Grunewald in [BCG08]. They identified 17 families of surfaces of the form $X = (C_1\times C_2)/G$ with $p_g(X) = q(X) = 0$, besides the quadric $\mathbb{P}^1\times\mathbb{P}^1$. Later, in [Fra13], Frapporti found one further family of such surfaces and hence completed the classification.

(2) *Irreducible fake quadrics,* i.e. those fake quadrics X_Γ that are not reducible. Here the group Γ arises from a quaternion algebra over a totally real number field. Such examples in stable case, i.e. Γ is contained in $\mathrm{PSL}_2(\mathbb{R})\times\mathrm{PSL}_2(\mathbb{R}) \leq \mathrm{Isom}^h(\mathbb{H}\times\mathbb{H})$, are given by Kuga and Shavel in [Sha78] as well as by Džambić in [Dža14], where the case of quaternion algebra over a real quadratic field has been established in details. The general case has been established by Linowitz, Stover and Voight in [LSV15] with a list of all possible number fields and quaternion algebras that can be used to construct a fake quadric.

In the case of positive characteristic, there are only few results known so far. The first known example is given by Stix and Vdovina in [SV13]. In this article, a class of algebraic surfaces is constructed by means of non-archimedean uniformisation based on a torsion-free lattice acting simply transitively on the product of two copies of the Bruhat-Tits-tree of $\mathrm{PGL}_2(\mathbb{F}_q((t)))$ for an odd prime power q. This yields a non-classical fake quadric in the case $q = 3$. Here "non-classical" means that the associated Picard-variety is non-trivial, i.e. $H^1(X,\mathscr{O}_X)\neq 0$, although the Albanese variety is trivial. This can occur only in a positive characteristic case.

We shall focus on the case characteristic 2 and explain the construction in the next sections.

5.2 The first steps in the construction

Throughout this section, let that R be a complete discrete valuation ring with uniformiser π, *finite* residue field $k = R/(\pi)$ with q elements and the field of fractions $K = R[\frac{1}{\pi}]$. We shall follow the construction of the "wonderful scheme" in [SV13, §7] as follows:

Definition 5.2. For each $L\in\mathrm{GL}_2(K)$, we define linear polynomials $\ell_i = \ell_i^L \in K[x_0,x_1]$ for $i = 1,2$ by

$$\begin{pmatrix}\ell_0\\ \ell_1\end{pmatrix} = L^t\begin{pmatrix}x_0\\ x_1\end{pmatrix}.$$

Considering $R\big[\frac{\ell_0}{\ell_1},\frac{\pi\ell_1}{\ell_0}\big]$ as a subring of $K\big(\frac{x_0}{x_1}\big)$, we define

$$\tilde{Y}_L := \mathrm{Spec}\, R\left[\frac{\ell_0}{\ell_1},\frac{\pi\ell_1}{\ell_0}\right] \cong \mathrm{Spec}\, R[u,v]/(uv-\pi).$$

This is a regular scheme of finite type over K. Its special fibre is a union of affine lines transversally glued in $(0,0)$, while its generic fibre is a complement of two K-rational points in a projective line $\mathrm{Proj}\, K[\ell_0,\ell_1]\cong\mathbb{P}^1_K$. Furthermore, we define the open subscheme

$$Y_L \subset \tilde{Y}_L$$

as the complement of all finitely many k-rational points on the special fibre of $\tilde{Y}_L$ up to the singular point $(0,0)$.

Note that Y_L defined above only depends on the image of L in $\mathrm{PGL}_2(K)$. The next definition tells us how to glue the schemes Y_L's for all $L \in \mathrm{PGL}_2(K)$ together.

Definition 5.3. We define the following gluing datum for the family $(Y_L)_{L\in\mathrm{PGL}_2(K)}$: For $L_1, L_2 \in \mathrm{PGL}_2(K)$, the schemes Y_{L_1}, Y_{L_2} are birationally equivalent since they have the same function field $K\left(\frac{x_0}{x_1}\right)$. Hence there exist maximal open subschemes $U_{L_1L_2} \subseteq Y_{L_1}$ and $U_{L_2L_1} \subseteq Y_{L_2}$ with an isomorphism $U_{L_1L_2} \xrightarrow{\cong} U_{L_2L_1}$ making the diagram

$$\begin{array}{ccc} & \operatorname{Spec} K\left(\frac{x_0}{x_1}\right) & \\ \swarrow & & \searrow \\ U_{L_1L_2} & \xrightarrow{\cong} & U_{L_2L_1} \\ \cap\!\!| & & \cap\!\!| \\ Y_{L_1} & & Y_{L_2} \\ \searrow & & \swarrow \\ & \operatorname{Spec} R & \end{array}$$

commutative. One can check that this defines a gluing datum and yields the scheme

$$Y = \bigcup_{L\in\mathrm{PGL}_2(K)} Y_L.$$

Remark 5.4. (1) The scheme Y obtained in this way is a seperated integral scheme over R and still has $K\left(\frac{x_0}{x_1}\right)$ as the function field.

(2) Using the homogeneous coordinate $(x_0 : x_1)$, we see that the generic fibre $Y_K = Y \otimes_R K$ is a projective line, i.e.
$$Y_K \cong \mathbb{P}^1_K = \operatorname{Proj} K[x_0, x_1].$$
Furthermore, each irreducible component of the special fibre $Y_s = Y \otimes_R k$ is a projective line over k, i.e. isomorphic to $\mathbb{P}^1_k$.

The next step is to define a group action of $\mathrm{PGL}_2(K)$ on the scheme Y. In what follows, let x denote the column vector $(x_0, x_1)^t$ consisting of the variables $x_0, x_1 \in K[x_0, x_1]$.

Definition 5.5. The action of $\mathrm{GL}_2(K)$ on $K[x_0, x_1]$ is defined on the variables x_0, x_1 by

$$S^*(x) = (S^*(x_0), S^*(x_1))^t = S^t x. \tag{5.3}$$

This is a left action since for $S_1, S_2 \in \mathrm{GL}_2(K)$, we have

$$(S_1S_2)^*(x) = (S_1S_2)^t x = S_2^t(S_1^t x) = S_2^t(S_1^* x) = S_1^*(S_2^t x) = S_1^*(S_2^*(x)).$$

Definition 5.6. The action of $\mathrm{PGL}_2(K)$ on Y is defined as follows: For $L \in \mathrm{PGL}_2(K)$ and $S \in \mathrm{GL}_2(K)$, we have

$$S^* \begin{pmatrix} \ell_0^L \\ \ell_1^L \end{pmatrix} = S^*(L^t x) = L^t(S^*(x)) = L^t(S^t x) = (SL)^t(x). \tag{5.4}$$

This hence defines an isomorphism from $R\left[\frac{\ell_0^L}{\ell_1^L}, \frac{\pi \ell_1^L}{\ell_0^L}\right]$ to $R\left[\frac{\ell_0^{SL}}{\ell_1^{SL}}, \frac{\pi \ell_1^{SL}}{\ell_0^{SL}}\right]$. Its inverse then yields an isomorphism

$$\sigma_{S,L} : Y_L \to Y_{SL}.$$

Gluing the isomorphisms $\sigma_{S,L}$ for all $L \in \mathrm{PGL}_2(K)$ together, we obtain the isomorphism

$$\sigma_S : Y \to Y.$$

Since this doesn't depend on scalar multiples of S, this yields the group action

$$\mathrm{PGL}_2(K) \longrightarrow \mathrm{Aut}_R(Y), \quad S \longmapsto \sigma_S.$$

Remark 5.7. This group action can be restricted to the group actions on the generic fibre and on the special fibre respectively. On the generic fibre, the action of $S \in \mathrm{PGL}_2(K)$ is given by

$$\sigma_S|_{\mathbb{P}^1_K}(x) = (S^t)^{-1} x$$

for homogeneous coordinates x_0, x_1 with $x = \left(\begin{smallmatrix} x_0 \\ x_1 \end{smallmatrix}\right)$.

To explain the group action on the special fibre Y_s better, we consider instead the dual graph of Y_s. This consists of a vertex v_C for each irreducible component C of Y_s and two vertices v_C and $v_{C'}$ are joined if and only if the corresponding irreducible components intersect (necessarily) in exactly one double point.

Lemma 5.8. *There is a* $\mathrm{PGL}_2(K)$*-equivariant bijection between the following sets:*

(1) *Irreducible components of the special fibre* Y_s*.*

(2) *Homothety classes of* R*-lattices in* $H^0(\mathbb{P}^1_K, \mathscr{O}(1)) = K \cdot x_0 \oplus K \cdot x_1$ *under the* $\mathrm{PGL}_2(K)$*-action defined on the coordinates* (x_0, x_1) *by* (5.3)*.*

(3) *Vertices of the Bruhat-Tits tree* $T_K = \Delta(\mathrm{PGL}_2(K))$*.*

In particular, there is a $\mathrm{PGL}_2(K)$*-equivariant isomorphism between the dual graph of the special fibre* Y_s *and* T_K*.*

Proof. Consider first the irreducible compenents of the special fibre Y_s and the homothety classes of R-lattices in $H^0(\mathbb{P}_K, \mathscr{O}(1))$. An irreducible component of Y_s is given via $L \in \mathrm{PGL}_2(K)$ as the strict transform of the special fibre of

$$Y \longrightarrow P_L := \mathrm{Proj}\, R[\ell_0^L, \ell_1^L] \cong \mathbb{P}^1_R.$$

This depends on L up to its class in

$$\mathrm{GL}_2(K)/(K^\times \cdot \mathrm{GL}_2(R)) = \mathrm{PGL}_2(K)/\,\mathrm{PGL}_2(R)$$

and corresponds to the homothety class of the lattice

$$M_L = R \cdot \ell_0^L \oplus R \cdot \ell_1^L = H^0(P_L, \mathscr{O}(1)) \subseteq H^0(\mathbb{P}^1_K, \mathscr{O}(1)).$$

This 1-1 correspondence is $\mathrm{PGL}_2(K)$-equivariant since under $S \in \mathrm{GL}_2(K)$, the component above is mapped under σ_L to the strict transform of the special fibre of $Y \longrightarrow P_{SL}$, while M_L is mapped to M_{SL} as shown in (5.4).

Now we come to the vertices of T_K. Notice first that the inclusion-preserving bijection between the R-lattices of $H^0(\mathbb{P}_K, \mathscr{O}(1))$ and those of K^2 is induced by the isomorphism

$$K^2 \xrightarrow{\cong} H^0(\mathbb{P}_K, \mathscr{O}(1)), \quad \begin{pmatrix} a \\ b \end{pmatrix} \longmapsto ax_0 + bx_1.$$

More precisely, each vertice $[v_1, v_2]$ of T_K corresponds to the homothety class of the lattice $M_{v_1,v_2} := R \cdot (x^t v_1) \oplus R \cdot (x^t v_2) \subseteq H^0(\mathbb{P}^1_K, \mathscr{O}(1))$. The action of $S \in \mathrm{GL}_2(K)$ maps the vertice $[v_1, v_2]$ to $[Sv_1, Sv_2]$, which corresponds to

$$M_{Sv_1,Sv_2} = R \cdot (x^t S v_1) \oplus R \cdot (x^t S v_2) = R \cdot (S^*(x^t v_1)) \oplus R \cdot (S^*(x^t v_2)) = S^*(M_{v_1,v_2}).$$

Since this obviously descends to an action of $\mathrm{PGL}_2(K)$, we get a $\mathrm{PGL}_2(K)$-equivariant bijection between these two sets as desired.

It still remains to show that this induces a $\mathrm{PGL}_2(K)$-equivariant isomorphism between the dual graph of the special fibre Y_s and T_K. Indeed, two irreducible components C_1, C_2 of Y_s intersect at a point P if and only if Y_L is a neighbourhood of P and the closure of its special fibre is exactly the union of C_1 and C_2. This holds if and only if the corresponding lattices M_1, M_2 are of the form

$$M_1 = R \cdot \ell_0^L \oplus R \cdot \ell_1^L \quad \text{and} \quad M_2 = R \cdot \ell_1^L \oplus R \cdot \pi \ell_0^L.$$

This is the case if and only if $\pi M_1 \subsetneq M_2 \subsetneq M_1$, i.e. the corresponding vertices in T_K are joined by an edge. Hence the $\mathrm{PGL}_2(K)$-equivariant bijection between the irreducible components of Y_s and the vertices of T_K constructed above via the homothety classes of R-lattices in $H^0(\mathbb{P}^1_K, \mathscr{O}(1))$ induces a $\mathrm{PGL}_2(K)$-equivariant isomorphism between the dual graph of Y_s and the Bruhat-Tits tree T_K as desired. □

Since we are interested in a group action of an arithmetic lattice on the product of Bruhat-Tits trees, we consider here the product $Y \times_R Y$ of the wonderful scheme Y just constructed. It is locally given by an open subscheme of the affine spectrum of

$$R[u, v]/(uv - \pi) \otimes_R R[w, z]/(wz - \pi) = R[u, v, w, z]/(uv - \pi, wz - \pi).$$

which contains the point $(u, v, w, z) = (0, 0, 0, 0)$. Since $R[u, v]/(uv - \pi)$ is regular and smooth over R outside of $(0, 0)$, the scheme $Y \times_R Y$ is regular up to the points corresponding to $(u, v, w, z) = (0, 0, 0, 0)$ in the local chart given before. At such a point, $u, v, w + z$ form a regular sequence of length $3 = \dim Y \times_R Y$. This implies that $Y \times_R Y$ is Cohen-Macaulay and normal.

Now we shall focus on the special fibre $(Y \times_R Y)_s = (Y \times_R Y) \otimes_R k = Y_s \times_k Y_s$. Its dual complex Σ is given as follows:

(1) Vertices of Σ are irreducible components of $(Y \times_R Y)_s$. These are isomorphic to $\mathbb{P}^1_k \times \mathbb{P}^1_k$.

(2) Edges of Σ are irreducible curves in the intersection of two irreducible components of $(Y \times_R Y)_s$ and each edge joins the corresponding vertices. Here an irreducible component can, after identification with $\mathbb{P}^1_k \times \mathbb{P}^1_k$, intersect another component only on the grid lines

$$(\mathbb{P}^1_k(k) \times \mathbb{P}^1_k) \cup (\mathbb{P}^1_k \times \mathbb{P}^1_k(k)) \subseteq \mathbb{P}^1_k \times_k \mathbb{P}^1_k$$

and the intersection yields exactly a projective line $\mathbb{P}^1_k$.

(3) Squares of Σ are given by singular points of $(Y \times_R Y)_s$. Indeed, these are k-rational points on the special fibre, i.e. given by $\mathbb{P}_1(k) \times \mathbb{P}^1(k)$ on each irreducible component $\mathbb{P}^1_k \times \mathbb{P}^1_k$. At each such point P, the scheme $Y \times_R Y$ is locally isomorphic to $\mathrm{Spec}(R[u,v,w,z]/(uv = \pi = wz))$ with P mapping to the non-regular point $(0,0,0,0)$. Hence on the special fibre, P lies on exactly four irreducible components locally given by

$$\{u = w = 0\},\ \{u = z = 0\},\ \{v = z = 0\},\ \text{and}\ \{v = w = 0\}.$$

Among the six possible pairs of these components, there are exactly four of them whose intersection yields an irreducible curve, more precisely in such a way that the 2-cell corresponding to P is indeed a square.

The action of $\mathrm{PGL}_2(K)$ on Y then induces the action of $\mathrm{PGL}_2(K) \times \mathrm{PGL}_2(K)$ on $Y \times_R Y$. Its restriction on the special fibre $(Y \times_R Y)_s$ can then be described as in the following lemma:

Lemma 5.9. *There is a* $\mathrm{PGL}_2(K) \times \mathrm{PGL}_2(K)$*-equivariant isomorphism between the dual complex of* $(Y \times_R Y)_s$ *with the action described above and the product of Bruhat-Tits tree* $T_K \times T_K$ *with Bruhat-Tits-action.*

Proof. This follows immediately from Lemma 5.8 since the dual complex Σ is the product of the dual graph of Y_s with itself. □

5.3 The formal scheme and its quotient

Now we would like to build a quotient of the scheme $Y \times_R Y$ by an arithmetic subgroup of $\mathrm{PGL}_2(K) \times \mathrm{PGL}_2(K)$. One problem is that this scheme has too many closed points, namely the K-rational points on the generic fibre Y_K are also closed points in Y. To avoid this problem, we consider instead its formal completion as follows:

Definition 5.10. The formal scheme $\mathscr{Y}/\,\mathrm{Spf}(R)$ is defined as the formal completion of the scheme Y/R along its special fibre Y_s.

Remark 5.11. The formal scheme $\mathscr{Y}$ has the following properties:

(1) Its generic fibre in the sense of Raynaud's rigid analytic geometry is identified with the Drinfeld upper half plane Ω^1_K, the rigid analytic variety underlying the complement of $\mathbb{P}^1(K)$ in $\mathbb{P}^1_K$.

(2) It is also possible to construct $\mathscr{Y}$ by gluing as follows: For each $L \in \mathrm{PGL}_2(K)$, we let $\hat{Y}_L$ denote the formal completion of Y_L along its special fibre. All $\hat{Y}_L$'s are then glued together by functoriality from the gluing construction of Y to obtain

$$\mathscr{Y} = \bigcup_{L \in \mathrm{PGL}_2(K)} \hat{Y}_L.$$

(3) The formal scheme $\mathscr{Y}$ also carries a $\mathrm{PGL}_2(K)$-action obtained from the corresponding action on Y. Each $S \in \mathrm{GL}_2(K)$ sends the open subscheme $\hat{Y}_L$ for $L \in \mathrm{PGL}_2(K)$ to $\hat{Y}_{SL}$.

(4) The special fibre $\mathscr{Y}_s$ is isomorphic to Y_s. Its dual graph can be identified with the Bruhat-Tits tree T_K in a $\mathrm{PGL}_2(K)$-equivariant way according to Lemma 5.8.

(5) The product $\mathscr{Y} \times_R \mathscr{Y}$ over $\mathrm{Spf}(R)$ can also be obtained by completing of $Y \times_R Y$ along its special fibre. The generic fibre in the sense of rigid geometry is $\Omega^1_K \times \Omega^1_K$. The action of $\mathrm{PGL}_2(K) \times \mathrm{PGL}_2(K)$ on $Y \times_R Y$ extends to its completion. Its special fibre is the same as $(Y \times_R Y)_s$ and its dual square complex is $(\mathrm{PGL}_2(K) \times \mathrm{PGL}_2(K))$-equivariantly isomorphic to the product of Bruhat-Tits tree $T_K \times T_K$.

Now let $\Gamma \leq \mathrm{PGL}_2(K) \times \mathrm{PGL}_2(K)$ be a discrete torsion-free subgroup acting cocompactly on $T_K \times T_K$ via Bruhat-Tits action. This then induces a free and discontinuous action on the special fibre $(Y \times_R Y)_s$ with respect to Zariski topology, thus also on the formal scheme $\mathscr{Y} \times_R \mathscr{Y}$ over $\mathrm{Spf}(R)$. Hence it is possible to build the quotient

$$\mathscr{X}_\Gamma := \Gamma \backslash (\mathscr{Y} \times_R \mathscr{Y}).$$

The dual complex Σ_Γ of $\mathscr{X}_\Gamma$ is then the finite quotient complex $\Gamma\backslash\Sigma$. Hence the quotient $\mathscr{X}_\Gamma$ is finite. Moreover, we can assert the following fact about the quotient complex:

Lemma 5.12. *Suppose that the quotient square complex $\Sigma_\Gamma = \Gamma\backslash\Sigma$ has N vertices, then*

$$\#\mathbb{E}(\Sigma_\Gamma) = N(q+1) \text{ and } \#\mathbb{S}(\Sigma_\Gamma) = \frac{1}{4}N(q+1)^2.$$

Hence the topological Euler characteristic ist

$$\chi(\Sigma_\Gamma) = \frac{1}{4}N(q-1)^2.$$

Proof. [SV13, Lemma 46]. □

Now we want to transfer the quotient formal scheme back to an algebraic scheme over a field. This can be done by Grothendieck's formal GAGA principle. Its generic fibre is an algebraic surface and has several elementary properties as follows:

Proposition 5.13. *The formal scheme $\mathscr{X}_\Gamma$ over $\mathrm{Spf}(R)$ is a formal completion along the special fibre of a projective scheme X_Γ over R. Its generic fibre $X_{\Gamma,K} = X_\Gamma \otimes_R K$ is smooth projective with ample canonical line bundle. In particular, it is a minimal surface of general type without smooth rational curves with self-intersection number -1 or -2.*

Proof. This is done in [SV13, Prop.47]. In fact, the sheaf of relative log-differentials $\Omega^{2,\log}_{\mathscr{X}_\Gamma/R}$, obtained by descending the pull back $\Omega^{2,\log}_{Y\times_R Y/R}|_{\mathscr{Y}\times_R\mathscr{Y}}$ on $\mathscr{Y} \times_R \mathscr{Y}$ to the quotient $\mathscr{X}_\Gamma$, is an ample line bundle on $\mathscr{X}_\Gamma$, so that $\mathscr{X}_\Gamma$ as formal scheme over $\mathrm{Spf}(R)$ can be algebraised to a projective scheme X_Γ/R by Grothendieck's formal GAGA principle. Then it has been shown in loc.cit. that its generic fibre $X_{\Gamma,K}/K$ is smooth and the canonical sheaf $\omega_{X_{\Gamma,K}/K} = \Omega^{2,\log}_{X_\Gamma/R}|_{X_{\Gamma,K}}$ is ample. The last statement follows from the adjunction formula. □

5.4 Computing invariants of $X_{\Gamma,K}$

What we still have to do is to show that the surface $X_{\Gamma,K}$ obtained from the previous section is indeed a fake quadric for a certain choice of Γ. In order to do this, we need to compute the invariants in the definition, i.e. the Chern numbers and the Albanese variety.

Notation 5.14. We define the following maps:

- For each vertex $E \in \mathbb{V}(\Sigma_\Gamma)$, let $\pi_E : \tilde{E} \to E$ be the normalisation of an irreducible component $E \subseteq X_{\Gamma,s}$. In this case we have $\tilde{E} \cong \mathbb{P}^1_k \times_k \mathbb{P}^1_k$.
- For each edge $C \in \mathbb{E}(\Sigma_\Gamma)$, let $\pi_C : \tilde{C} \to C$ be the normalisation of an irredubible curve as a component in the intersection of two irreducible components of $X_{\Gamma,s}$. In this case we have $\tilde{C} \cong \mathbb{P}^1_k$.
- For each point $P \in \mathbb{S}(\Sigma_\Gamma)$, let $\iota_P : P \hookrightarrow X_{\Gamma,s}$ be the k-rational point in $X_{\Gamma,s}$ corresponding to the square P.

Now choosing an orientation on $\Sigma(\mathscr{X}_\Gamma)$, we obtain the following cellular cochain complex

$$0 \longrightarrow \bigoplus_{E \in \mathbb{V}(\Sigma(\mathscr{X}_\Gamma))} \mathbb{Z} \longrightarrow \bigoplus_{C \in \mathbb{E}(\Sigma(\mathscr{X}_\Gamma))} \mathbb{Z} \longrightarrow \bigoplus_{P \in \mathbb{S}(\Sigma(\mathscr{X}_\Gamma))} \mathbb{Z}. \tag{5.5}$$

Proposition 5.15. *Let N be the number of vertices of $\Sigma(\mathscr{X}_\Gamma)$. Then we have*

$$c_1(X_{\Gamma,K})^2 = 2N(q-1)^2 \quad \textit{and} \quad c_2(X_{\Gamma,K}) = N(q-1)^2.$$

Proof. Observe first that by flatness of X_Γ over R, we can compute the both Chern numbers as well as the Euler characteristic of its generic fibre $X_{\Gamma,K}$ via those of its special fibre $X_{\Gamma,s}$. The Euler characteristic can be computed using the exact sequence

$$0 \longrightarrow \mathscr{O}_{X_{\Gamma,s}} \longrightarrow \bigoplus_{E \in \mathbb{V}(\Sigma(\mathscr{X}_\Gamma))} \pi_{E,*}\mathscr{O}_{\tilde{E}} \longrightarrow \bigoplus_{C \in \mathbb{E}(\Sigma(\mathscr{X}_\Gamma))} \pi_{C,*}\mathscr{O}_{\tilde{C}} \longrightarrow \bigoplus_{P \in \mathbb{S}(\Sigma(\mathscr{X}_\Gamma))} \iota_{P,*}\mathscr{O}_P \longrightarrow 0,$$

where the maps are given by the sum of restrictions with signs coming from (5.5), see [SV13, Prop.48]. The first Chern number can be computed by $c_1(X_{\Gamma,K})^2 = c_1(\Omega^{2,\log}_{\mathscr{X}_\Gamma/R})^2$ and the second one by Noether's formula, compare the calculation after the proof of loc.cit. □

The next thing we shall focus on is the Albanese variety $\mathrm{Alb}_{X_{\Gamma,K}}$. This vanishes if and only if its étale fundamental group $\pi_1^{\text{ét}}(\mathrm{Alb}_{X_{\Gamma,K}})$ vanishes, or equivalently, its Tate module $T_\ell(\mathrm{Alb}_{X_{\Gamma,K}})$ vanishes for some prime $\ell \nmid \operatorname{char} K$. Now the latter one is a free $\mathbb{Z}_\ell$-module and sits in the exact sequence

$$0 \longrightarrow \mathrm{Hom}(\mathrm{NS}(X_{\Gamma,\bar{K}})\{\ell\}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \longrightarrow \pi_1^{\mathrm{ab},(\ell)}(X_{\Gamma,\bar{K}}) \longrightarrow T_\ell(\mathrm{Alb}_{X_{\Gamma,K}}) \longrightarrow 0, \tag{5.6}$$

where $\mathrm{NS}(X_{\Gamma,\bar{K}})\{\ell\}$ denotes the (finite) ℓ-primary torsion subgroup of the Neron-Severi group $\mathrm{NS}(X_{\Gamma,\bar{K}})$ and $\pi_1^{\mathrm{ab},(\ell)}(X_{\Gamma,\bar{K}})$ the maximal pro-ℓ-quotient of the abelianised fundamental group $\pi_1^{\mathrm{ab}}(X_{\Gamma,\bar{K}})$, see e.g. [Sza09, Cor. 5.8.10]. This leads us to study instead the structure of the group $\pi_1^{\mathrm{ab},(\ell)}(X_{\Gamma,\bar{K}})$ first. For this we consider

$$\mathrm{Hom}(\pi_1^{\mathrm{ab},(\ell)}(X_{\Gamma,\bar{K}}), \mathbb{Z}/\ell\mathbb{Z}) = \mathrm{Hom}(\pi_1^{\text{ét}}(X_{\Gamma,\bar{K}}), \mathbb{Z}/\ell\mathbb{Z}) = H^1_{\text{ét}}(X_{\Gamma,\bar{K}}, \mathbb{Z}/\ell\mathbb{Z}).$$

We shall compute $H^1_{\text{ét}}(X_{\Gamma,\bar{K}}, \mathbb{Z}/\ell\mathbb{Z})$, or more generally $H^1_{\text{ét}}(X_{\Gamma,\bar{K}}, \Lambda)$ for a finite commutative ring Λ of order prime to $\mathrm{char}(K)$, via Kummer étale cohomology as follows: The associated log-scheme is obtained from X_Γ/R by endowing this with the fs-log structure in the sense of Fontaine and Illusie determined by its special fibre. This is log-smooth and projective over

$\mathrm{Spec}(R)$. Hence by considering the log geometric fibre $X_{\Gamma,\tilde{s}}$, where $\tilde{s} \to \mathrm{Spec}(R)$ denote the log-geometric point over the closed point, we obtain an isomorphism

$$H^1_{\text{ét}}(X_{\Gamma,\bar{K}}, \Lambda) \cong H^1_{\text{két}}(X_{\Gamma,\tilde{s}}, \Lambda)$$

from the Kummer-étale cospecialisation map. Here $\bar{K}$ denotes an algebraic closure of K and $X_{\Gamma,\bar{K}}$ the corresponding geometric generic fibre. The group $H^1_{\text{két}}(X_{\Gamma,\tilde{s}}, \Lambda)$ can then be computed as follows:

Lemma 5.16. *The following sequence is exact:*

$$0 \longrightarrow H^1_{\text{Sing}}(\Sigma(\mathscr{X}_\Gamma), \Lambda) \longrightarrow H^1_{\text{két}}(X_{\Gamma,\tilde{s}}, \Lambda) \longrightarrow \ker(\partial) \tag{5.7}$$

where ∂ denotes sum of the restriction maps

$$\bigoplus_{E \in \mathbb{V}(\Sigma(\mathscr{X}_\Gamma))} H^1_{\text{két}}(\tilde{E}_{\tilde{s}}, \Lambda) \xrightarrow{\pm \operatorname{res}} \bigoplus_{C \in \mathbb{E}(\Sigma(\mathscr{X}_\Gamma))} H^1_{\text{két}}(\tilde{C}_{\tilde{s}}, \Lambda)$$

with signs coming from the corresponding cellular cochain complex in (5.5).

Proof. Define the following sheaves on the Kummer étale site $(X_{\Gamma,\tilde{s}})$:

$$\mathscr{F}^0 := \bigoplus_{E \in \mathbb{V}(\Sigma(\mathscr{X}_\Gamma))} \pi_{E,*}\Lambda, \quad \mathscr{F}^1 := \bigoplus_{C \in \mathbb{E}(\Sigma(\mathscr{X}_\Gamma))} \pi_{C,*}\Lambda \quad \text{and} \quad \mathscr{F}^2 := \bigoplus_{P \in \mathbb{S}(\Sigma(\mathscr{X}_\Gamma))} \iota_{P,*}\Lambda.$$

Then the constant sheaf Λ is quasi-isomorphic to $0 \to \mathscr{F}^0 \to \mathscr{F}^1 \to \mathscr{F}^2 \to 0$, where $\mathscr{F}^0 \to \mathscr{F}^1$ and $\mathscr{F}^1 \to \mathscr{F}^2$ are defined as the sum of the restriction maps with signs coming from the associated cochain complex for $\Sigma(\mathscr{X}_\Gamma)$. Hence we obtain the following hypercohomology spectral sequence

$$E_1^{p,q} = H^q_{\text{két}}(X_{\Gamma,\tilde{s}}, \mathscr{F}^p) \Longrightarrow E^{p+q} = H^{p+q}_{\text{két}}(X_{\Gamma,\tilde{s}}, \Lambda).$$

The first terms in the five-term exact sequence are then

$$\begin{aligned} E_2^{1,0} &= H^1_{\text{CW}}(\Sigma(\mathscr{X}_\Gamma), \Lambda) = H^1_{\text{Sing}}(\Sigma(\mathscr{X}_\Gamma), \Lambda), \\ E^1 &= H^1_{\text{két}}(X_{\Gamma,\tilde{s}}, \Lambda) \qquad \text{and} \\ E_2^{0,1} &= \ker\left(\bigoplus_{E \in \mathbb{V}(\Sigma(\mathscr{X}_\Gamma))} H^1_{\text{két}}(\tilde{E}_{\tilde{s}}, \Lambda) \xrightarrow{\pm \operatorname{res}} \bigoplus_{C \in \mathbb{E}(\Sigma(\mathscr{X}_\Gamma))} H^1_{\text{két}}(\tilde{C}_{\tilde{s}}, \Lambda) \right). \end{aligned}$$

Hence we obtain the exact sequence (5.7) as desired. □

From Now on we shall restrict to the case $K = \mathbb{F}_2(\!(z)\!)$ and Γ is the arithmetic lattice from Proposition 4.32. It has been shown there that Γ is isomorphic to the fundamental group $\pi_1(\Sigma_{\mathcal{A},\mathcal{B}}, s_{00})$ for the choice of $\mathcal{A}, \mathcal{B}$ as in Lemma 4.22, and that under this isomorphism, the Bruhat-Tits action of Γ on $T_3 \times T_3$ is the same as the action of $\pi_1(\Sigma_{\mathcal{A},\mathcal{B}})$ on $T_3 \times T_3$ as the universal covering of $\Sigma_{\mathcal{A},\mathcal{B}}$. Hence the dual complex $\Sigma(\mathscr{X}_\Gamma)$ is isomorphic to $\Sigma_{\mathcal{A},\mathcal{B}}$, i.e. has four vertices and a vertical-horizontal structure from the V_4-equivariant vertical-horizontal structure $(\mathcal{A}, \mathcal{B})$ in $G(R_1)$. Furthermore, we can determine $H^1_{\text{két}}(X_{\Gamma,\tilde{s}}, \Lambda)$ from the following lemma:

Lemma 5.17. *Let Λ be a finite commutative ring in which* 6 *is invertible. Then we have*

$$H^1_{\text{Sing}}(\Sigma(\mathscr{X}_\Gamma), \Lambda) \cong H^1_{\text{két}}(X_{\Gamma,\tilde{s}}, \Lambda).$$

Proof. From Lemma 5.16, it suffices to show that $\ker \partial$ is trivial. For this observe first that ∂ is the sum of the restriction maps $\mathrm{res}_C^E : H^1_{\text{két}}(\tilde{E}_{\tilde{s}}, \Lambda) \to H^1_{\text{két}}(\tilde{C}_{\tilde{s}}, \Lambda)$, for $E \in \mathbb{V}(\Sigma(\mathscr{X}_\Gamma))$ and $C \in \mathbb{E}(\Sigma(\mathscr{X}_\Gamma))$ such that $C \subseteq E$, with signs coming from a chosen orientation on $\Sigma(\mathscr{X}_\Gamma)$. In fact, we can choose the orientation in such a way that

$$\partial = \sum_{i,j \in \{0,1\}} \left((-1)^j \sum_{a \in \mathcal{A}} \mathrm{res}^{E_{ij}}_{C_{(a,i)}} + (-1)^i \sum_{b \in \mathcal{B}} \mathrm{res}^{E_{ij}}_{C_{(b,j)}} \right), \tag{5.8}$$

where each E_{ij} is the irreducible component of $X_{\Gamma,s}$ corresponding to the vertex $s_{ij} \in \mathbb{V}(\Sigma(\mathscr{X}_\Gamma))$, $C_{(a,i)}$ the irreducible curve corresponding to $(a,i) \in \mathcal{A} \times \{0,1\} = \mathbb{E}(\Sigma(\mathscr{X}_\Gamma))_v$ and $C_{(b,j)}$ the irreducible curve corresponding to $(b,j) \in \mathcal{B} \times \{0,1\} = \mathbb{E}(\Sigma(\mathscr{X}_\Gamma))_h$. We shall use the notation related to the square complex associated to a V_4-structure as in Section 3.6 and give the explicit computation of ∂ in the following steps:

STEP 1. *Compute $H^1_{\text{két}}(\tilde{E}_{\tilde{s}}, \Lambda)$ and $H^1_{\text{két}}(\tilde{C}_{\tilde{s}}, \Lambda)$.*

We begin with $H^1_{\text{két}}(\tilde{E}_{\tilde{s}}, \Lambda)$, where $E \in \mathbb{V}(\Sigma(\mathscr{X}_\Gamma))$. Let $E_{\text{Sing}} \subseteq E$ denotes the singular part of $E \subseteq X_{\Gamma,s}$. Then under the isomorphism $\tilde{E} \cong \mathbb{P}^1_k \times_k \mathbb{P}^1_k$, we have

$$E \setminus E_{\text{Sing}} \cong \mathbb{P}^1_k \times_k \mathbb{P}^1_k \setminus ((\mathbb{P}^1_k \times_k \mathbb{P}^1(k)) \cup (\mathbb{P}^1(k) \times_k \mathbb{P}^1_k)) = (\mathbb{P}^1_k \setminus \mathbb{P}^1(k)) \times (\mathbb{P}^1_k \setminus \mathbb{P}^1(k)).$$

This implies the following isomorphisms:

$$\begin{aligned} H^1_{\text{két}}(\tilde{E}_{\tilde{s}}, \Lambda) &\cong H^1_{\text{ét}}(E \setminus E_{\text{Sing}}, \Lambda) \cong H^1_{\text{ét}}((\mathbb{P}^1_k \setminus \mathbb{P}^1(k)) \times (\mathbb{P}^1_k \setminus \mathbb{P}^1(k)), \Lambda) \\ &\cong H^1_{\text{ét}}(\mathbb{P}^1_k \setminus \mathbb{P}^1(k), \Lambda) \oplus H^1_{\text{ét}}(\mathbb{P}^1_k \setminus \mathbb{P}^1(k), \Lambda), \end{aligned}$$

Here each of $H^1_{\text{ét}}(\mathbb{P}^1_k \setminus \mathbb{P}^1(k), \Lambda)$ in the direct sum above comes from the vertical and horizontal component and will be denoted in what follows by $H^1(E)_v$ and $H^1(E)_h$ respectively. By the excision principle, we have the following exact sequence

$$0 = H^1_{\text{ét}}(\mathbb{P}^1_k, \Lambda) \longrightarrow H^1_{\text{ét}}(\mathbb{P}^1_k \setminus \mathbb{P}^1(k), \Lambda) \longrightarrow H^2_{\text{ét},\mathbb{P}^1(k)}(\mathbb{P}^1, \Lambda) \longrightarrow H^2_{\text{ét}}(\mathbb{P}^1_k, \Lambda) = \Lambda(-1),$$

where $\Lambda(-1)$ denotes the inverse Tate-twist of Λ. Here $H^2_{\text{ét},\mathbb{P}^1(k)}(\mathbb{P}^1, \Lambda) \cong H^0_{\text{ét}}(\mathbb{P}^1(k), \Lambda(-1)) = \mathrm{Maps}(\mathbb{P}^1(k), \Lambda(-1))$, the group of mappings from $\mathbb{P}^1(k)$ to $\Lambda(-1)$, and the map $H^2_{\text{ét},\mathbb{P}^1(k)}(\mathbb{P}^1, \Lambda) \to H^2_{\text{ét}}(\mathbb{P}^1_k, \Lambda)$ is just the summation map, i.e.

$$H^1_{\text{ét}}(\mathbb{P}^1_k \setminus \mathbb{P}^1(k), \Lambda) \cong \mathrm{Maps}(\mathbb{P}^1(k), \Lambda(-1))^0,$$

where $\mathrm{Maps}(-, \Lambda(-1))^0$ denotes the subset of the mappings whose sum of the images of all elements from the domain is zero. The set $\mathbb{P}^1(k)$ can be interpreted as the set of vertical edges attachted to E in the case of $H^1(E)_h$ and horizontal edges in the case of $H^1(E)_v$, i.e. identified with $\mathcal{A}$ and $\mathcal{B}$ respectively. Hence we get

$$H^1(E)_h \cong \mathrm{Maps}(\mathcal{A}, \Lambda(-1))^0 \quad \text{and} \quad H^1(E)_v \cong \mathrm{Maps}(\mathcal{B}, \Lambda(-1))^0.$$

Similarly, we can show that

$$H^1_{\text{két}}(\tilde{C}_{\tilde{s}}, \Lambda) = \mathrm{Maps}(\mathbb{S}_C, \Lambda(-1))^0$$

for each $C \in \mathbb{E}(\Sigma(\mathscr{X}_\Gamma))$, where $\mathbb{S}_C \subseteq \mathbb{S}(\Sigma(\mathscr{X}_\Gamma))$ denotes the set of the squares in $\Sigma(\mathscr{X}_\Gamma)$ attached to the edge C, i.e. $\mathbb{S}_C = \mathbb{S}_{(a,i)}$ or $\mathbb{S}_{(b,j)}$ if C corresponds to the edge $(a,i) \in \mathcal{A} \times I$ or $(b,j) \in \mathcal{B} \times I$ in $\Sigma_{\mathcal{A},\mathcal{B}}$ respectively.

STEP 2. *Compute the map ∂.*

In virtue of (5.8), the map ∂ can be computed by understanding the maps

$$\begin{aligned}\mathrm{res}^{E_{ij}}_{C_{(a,i)}} &: H^1_{\text{két}}(\tilde{E}_{ij,\tilde{s}},\Lambda) \longrightarrow H^1_{\text{két}}(\tilde{C}_{(a,i),\tilde{s}},\Lambda) \quad \text{and}\\ \mathrm{res}^{E_{ij}}_{C_{(b,j)}} &: H^1_{\text{két}}(\tilde{E}_{ij,\tilde{s}},\Lambda) \longrightarrow H^1_{\text{két}}(\tilde{C}_{(b,j),\tilde{s}},\Lambda)\end{aligned}$$

for $i,j \in \{0,1\}$, $a \in \mathcal{A}$ and $b \in \mathcal{B}$. The latter one can be described as follows: Since $C_{(b,j)}$ corresponds to the horizontal edge $(b,j) \in \mathbb{E}(\Sigma_\Gamma)$, the restriction map $\mathrm{res}^{E_{ij}}_{C_{(b,j)}}$ vanishes on the vertical component $H^1(E_{ij})_v \subseteq H^1_{\text{két}}(\tilde{E}_{ij,\tilde{s}},\Lambda)$, so that we can first project $H^1_{\text{két}}(\tilde{E}_{ij,\tilde{s}},\Lambda)$ onto its horizontal component $H^1(E_{ij})_h$. On this component, the restriction map is, under the embeddings $H^1(E_{ij})_h \hookrightarrow \mathrm{Maps}(\mathcal{A}_i,\Lambda(-1))$ and $H^1_{\text{két}}(\tilde{C}_{(b,j),\tilde{s}},\Lambda) \hookrightarrow \mathrm{Maps}(\mathbb{S}_{(b,j)},\Lambda(-1))$, just the restriction of

$$\mathrm{Maps}(\mathcal{A}_i,\Lambda(-1)) \xrightarrow{-\circ t^i_{(b,j)}} \mathrm{Maps}(\mathbb{S}_{(b,j)},\Lambda(-1)).$$

Similarly, the restriction map $\mathrm{res}^{E_{ij}}_{C_{(a,i)}}$ is given by projection onto the horizontal component $H^1(E_{ij})_h \cong \mathrm{Maps}(\mathcal{A}_i,\Lambda(-1))^0$, followed by the restriction of

$$\mathrm{Maps}(\mathcal{B}_j,\Lambda(-1)) \xrightarrow{-\circ t^j_{(a,i)}} \mathrm{Maps}(\mathbb{S}_{(a,i)},\Lambda(-1)).$$

STEP 3. *Determine* $\ker(\partial)$ *in terms of invariants under group actions.*

From the last step and the formula in (5.8), we see that if

$$\xi = (\xi^h_{ij},\xi^v_{ij})_{i,j\in\{0,1\}} \in \bigoplus_{i,j\in\{0,1\}} H^1_{\text{két}}(\tilde{E}_{ij,\tilde{s}},\Lambda) = \bigoplus_{i,j\in\{0,1\}} \big(H^1(E_{ij})_h \oplus H^1(E_{ij})_v\big) \tag{5.9}$$

then we get

$$\partial(\xi) = \big((\xi^h_{0j}\circ t^0_{(b,j)} - \xi^h_{1j}\circ t^1_{(b,j)})_{(b,j)\in\mathcal{B}\times\{0,1\}}, (\xi^v_{i0}\circ t^0_{(a,i)} - \xi^v_{i1}\circ t^1_{(a,i)})_{(a,i)\in\mathcal{A}\times\{0,1\}}\big).$$

Hence $\partial(\xi)$ vanishes if and only if

$$\xi^h_{0j}\circ t^0_{(b,j)} = \xi^h_{1j}\circ t^1_{(b,j)} \quad\text{and}\quad \xi^v_{i0}\circ t^0_{(a,i)} = \xi^v_{i1}\circ t^1_{(a,i)} \quad \text{for all } i,j\in\{0,1\}, a\in\mathcal{A}, b\in\mathcal{B},$$

or equivalently,

$$\xi^h_{ij}\circ(\sigma^{\mathcal{A}}_{(b,j)}|_{\mathcal{A}_{\hat{i}}}) = \xi^h_{ij}\circ t^i_{(b,j)}\circ(t^{\hat{i}}_{(b,j)})^{-1} = \xi^h_{\hat{i}j} \qquad \text{and} \tag{5.10a}$$

$$\xi^v_{ij}\circ(\sigma^{\mathcal{B}}_{(a,i)}|_{\mathcal{B}_{\hat{j}}}) = \xi^v_{ij}\circ t^j_{(a,i)}\circ(t^{\hat{j}}_{(a,i)})^{-1} = \xi^v_{i\hat{j}} \tag{5.10b}$$

for all $a\in\mathcal{A}$, $b\in\mathcal{B}$ and $i,j\in\{0,1\}$. This leads us to consider the following group actions: For each $j\in I$, we let $P^{\mathcal{A}}_j$ acts on

$$H^1(E_{0j})_h \oplus H^1(E_{1j})_h \subseteq \mathrm{Maps}(\mathcal{A}_0,\Lambda(-1)) \oplus \mathrm{Maps}(\mathcal{A}_1,\Lambda(-1)) \cong \mathrm{Maps}(\mathcal{A}\times I,\Lambda(-1))$$

from the right by restricting the action of $P^{\mathcal{A}}_i \leq \mathrm{Sym}(\mathcal{A}\times I)$ on $\mathrm{Maps}(\mathcal{A}\times I,\Lambda(-1))$. Then (5.10a) is equivalent to

$$(\xi^h_{0j},\xi^h_{1j}).\sigma^{\mathcal{A}}_{(b,j)} = (\xi^h_{1j}\circ(\sigma^{\mathcal{A}}_{(b,j)}|_{\mathcal{A}_0}), \xi^h_{0j}\circ(\sigma^{\mathcal{A}}_{(b,j)}|_{\mathcal{A}_1})) = (\xi^h_{0j},\xi^h_{1j}),$$

i.e. $(\xi^h_{0j}, \xi^h_{1j}) \in H^1(E_{0j})_h \oplus H^1(E_{1j})_h$ is invariant under the action of $P^{\mathcal{A}}_j$ since this group is generated by $\sigma^{\mathcal{A}}_{(b,j)}$ for $b \in \mathcal{B}$.

Similarly, we can define the right group action of $P^{\mathcal{B}}_i$ on $H^1(E_{i0})_v \oplus H^1(E_{i1})_v$ for each $i \in I$. Then a similar argument as before shows that the condition (5.10b) holds if and only if $(\xi^h_{i0}, \xi^h_{i1}) \in H^1(E_{i0})_v \oplus H^1(E_{i1})_v$ is invariant under the action of $P^{\mathcal{B}}_i$. Summarising all these statements together, we get

$$\ker(\partial) = \left\{ \begin{array}{l} H^0\big(P^0_{\mathcal{A}}, H^1(E_{00})_h \oplus H^1(E_{10})_h\big) \oplus H^0\big(P^1_{\mathcal{A}}, H^1(E_{01})_h \oplus H^1(E_{11})_h\big) \\ \oplus H^0\big(P^0_{\mathcal{B}}, H^1(E_{00})_v \oplus H^1(E_{01})_v\big) \oplus H^0\big(P^1_{\mathcal{B}}, H^1(E_{10})_v \oplus H^1(E_{11})_v\big) \end{array} \right. .$$

STEP 4. *Conclude the result.*

Let $\xi = (\xi^h_{ij}, \xi^v_{ij})_{i,j\in\{0,1\}}$ be as in (5.9). Since by Proposition 4.31, the group $P^i_{\mathcal{A}}$ and $P^j_{\mathcal{B}}$ for all $i, j \in I$ acts on $\mathcal{A} \times I$ and $\mathcal{B} \times I$ respectively transitively, we see that ξ lies in $\ker(\partial)$ if and only if $\xi^h_{00} = \xi^h_{10}$, $\xi^h_{01} = \xi^h_{11}$, $\xi^v_{00} = \xi^v_{01}$ and $\xi^v_{10} = \xi^v_{11}$ are all constant mappings, i.e. there exist $\zeta^h_0, \zeta^h_1, \zeta^v_0, \zeta^v_1 \in \Lambda(-1)$ such that

$$\xi^h_{ij}(a) = \zeta^h_j \quad \text{and} \quad \xi^v_{ij}(b) = \zeta^v_i$$

for all $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $i, j \in I$. But the sum of the images of each map ξ^h_{ij} and ξ^v_{ij} must be zero, which implies that $3 \cdot \zeta^h_j = 0$ and $3 \cdot \zeta^v_i = 0$. Since 3 as a divisor of 6 is invertible in Λ, we have $\zeta^h_0 = \zeta^h_1 = \zeta^v_0 = \zeta^v_1 = 0$, i.e. $\xi \in \ker(\partial)$ only if all of its components vanish. In particular, $\ker(\partial)$ must be trivial. Hence the claim follows as mentioned at the beginning of the proof. □

Proposition 5.18. *The Albanese variety* $\mathrm{Alb}_{X_{\Gamma,K}}$ *is trivial.*

Proof. As mentioned before, it suffices to show that the Tate module $T_\ell(\mathrm{Alb}_{X_{\Gamma,K}})$ vanishes for some prime $\ell \nmid \mathrm{char}\, K$. From Lemma 5.17, we have

$$H^1_{\mathrm{Sing}}(\Sigma(\mathscr{X}_\Gamma), \mathbb{Z}/\ell\mathbb{Z}) \cong H^1_{\mathrm{k\acute{e}t}}(X_{\Gamma,\tilde{s}}, \mathbb{Z}/\ell\mathbb{Z}) \cong H^1_{\acute{e}t}(X_{\Gamma,\bar{K}}, \mathbb{Z}/\ell\mathbb{Z})$$

for $\ell \neq 2, 3$. Since $H^1_{\mathrm{Sing}}(\Sigma(\mathscr{X}_\Gamma), \mathbb{Z}/\ell\mathbb{Z}) \cong \mathrm{Hom}(\pi_1(\Sigma(\mathscr{X}_\Gamma)), \mathbb{Z}/\ell\mathbb{Z}) = \mathrm{Hom}(\Gamma, \mathbb{Z}/\ell\mathbb{Z})$ and the maximal abelian quotient of Γ^{ab} is $\mathbb{Z}/15\mathbb{Z}$, compare Corollary 4.34, the latter group is trivial for all $\ell \neq 3, 5$. It follows that

$$\mathrm{Hom}(\pi_1^{\mathrm{ab},(\ell)}(X_{\Gamma,\bar{K}}), \mathbb{Z}/\ell\mathbb{Z}) = H^1_{\acute{e}t}(X_{\Gamma,\bar{K}}, \mathbb{Z}/\ell\mathbb{Z}) = 0$$

for all prime $\ell \neq 2, 3, 5$. Therefore, $\mathrm{Hom}(\pi_1^{\mathrm{ab},(\ell)}(X_{\Gamma,\bar{K}}), \mathbb{Z}_\ell) = 0$ for such a prime ℓ, i.e. $\pi_1^{\mathrm{ab},(\ell)}(X_{\Gamma,\bar{K}})$ contains no free $\mathbb{Z}_\ell$-submodule, which implies that the Tate module $T_\ell(\mathrm{Alb}_{X_{\Gamma,K}})$ must vanish by the exact sequence in (5.6), and we are done. □

Having computed the necessary invariants for a fake quadric, we summarise the result in the following theorem.

Theorem 5.19. *Let* $R = \mathbb{F}_2[\![t]\!]$, $K = \mathrm{Quot}(R) = \mathbb{F}_2(\!(t)\!)$, *and* $\Gamma \leq \mathrm{PGL}_2(K) \times \mathrm{PGL}_2(K)$ *be the lattice from Proposition 4.32. Then the surface* $X_{\Gamma,K}$ *is a fake quadric over* K.

Proof. By Proposition 5.13, the surface $X_{\Gamma,K}$ is smooth and projective over K and minimal with an ample canonical bundle. The last property implies that $X_{\Gamma,K}$ is a surface of general type. Since the dual square complex $\Sigma(\mathscr{X}_\Gamma)$ has four vertices, it follows from Proposition 5.15 that

$$c_1(X_{\Gamma,K})^2 = 2 \cdot 4 \cdot (2-1)^2 = 8 \quad \text{and} \quad c_2(X_{\Gamma,K}) = 4 \cdot (2-1)^2 = 4.$$

Furthermore, its Albanese variety is trivial by Proposition 5.18. Therefore, $X_{\Gamma,K}$ is a fake quadric over $K = \mathbb{F}_2(\!(z)\!)$ as desired. □

Remark 5.20. In contrary to the fake quadric over characteristic 3 constructed in [SV13, Thm.51], the maximal abelian quotient of Γ has order 15 which is prime to the characteristic of K. Hence it is not possible to show that $X_{\Gamma,K}$ has non-reduced Picard scheme by the same method as loc.cit. Indeed, it is still an open question whether the fake quadric $X_{\Gamma,K}$ we constructed here has reduced Picard scheme or not.

Remark 5.21. It is natural to ask whether this construction would yield further fake quadrics over $\mathbb{F}_2(\!(t)\!)$. In fact, even the question of whether this yields further algebraic surfaces X of general type with $c_1(X)^2 = 8$ and $c_2(X)$ remains open since it is still unknown whether there exists further torsion-free quaternionic arithmetic lattices over a global function field over $\mathbb{F}_2$ which produce quotient square complexes with four vertices. The heuristic of finding such quaternionic arithmetic lattices will be discussed in Appendix A.

Appendix A

Finding a quaternionic arithmetic lattice

In this thesis, we have introduced a quaternion algebra $Q = \left[\frac{z,1+z^3}{\mathbb{F}_2(z)}\right)$ and determined presentations of arithmetic subgroups in $Q^\times$ by means of the quotient square complex. It is quite natural to ask how to find such a quaternionic arithmetic lattice that yields a nice quotient square complex.

Since there are plenty of quaternion algebras over a global field over $\mathbb{F}_2$ with two unramified $\mathbb{F}_2$-rational places, it is extremely difficult to start with possible quaternion algebras, then determine for each of them a torsion-free arithmetic lattice and then build a quotient square complex. Our strategy is to start from the other end, i.e. to find first an appropriate square complex, as will be explain in this appendix. Also note that the method presented here is just a heuristic, i.e. this doesn't guarantee that we will definitely obtain an appropriate square complex and quaternion algebra.

We begin by the elementary properties such a square complex must have. Since it should be the quotient of the product of two Bruhat-Tits trees under the action of an arithmetic lattice arising from a quaternion algebra over a global function field over $\mathbb{F}_2$, i.e. $q = 2$, it must have $T_3 \times T_3$ as the universal covering. Furthermore, we wish our square complex to have the smallest possible number of vertices, namely 4 as discussed in the introduction. This leads to the following assertion:

Lemma A.1. *Let Σ be a square complex with four vertices and $T_3 \times T_3$ as the universal covering. Then we have*

$$\#\mathbb{E}(\Sigma)_v = \#\mathbb{E}(\Sigma)_h = 6 \quad \textit{and} \quad \#\mathbb{S}(\Sigma) = 9.$$

Proof. This follows easily by double counting. In fact, the condition of having $T_3 \times T_3$ as universal covering tells us that each vertex is attached to three vertical and three horizontal edges. Since each edge is attached to two vertices, we have

$$2 \cdot \#\mathbb{E}(\Sigma)_v = 2 \cdot \#\mathbb{E}(\Sigma)_h = 4 \cdot 3,$$

i.e. $\#\mathbb{E}(\Sigma)_v = \#\mathbb{E}(\Sigma)_h = 6$. Furthermore, each pair of a vertical and a horizontal edge attached to a vertex yields a square corner and each square is attached to four corners. This implies that

$$4 \cdot \#\mathbb{S}(\Sigma) = 4 \cdot 3 \cdot 3 = 36,$$

i.e. $\#\mathbb{S}(\Sigma) = 9$ as desired. □

An advantage of this strategy is that there are finitely many square complexes with four vertices and $T_3 \times T_3$ as the universal covering. However, there are still far too many such square complexes to go through all possibilities, so that it is advisable to restrict first to those square complexes with certain symmetry conditions. For example, we can ask such a square complex to carry a V_4-action, which respects its VH-structure and is simply transitive on its vertices, since then the fundamental group of the square complex can be extended to a group with a simply transitive action on the vertices of the universal covering $T_3 \times T_3$.

The next step is to determine whether each square complex we found from the previous step has the fundamental group that could be arithmetic subgroup of rank 2. The first criterion is based on the following theorem:

Theorem A.2. *Let G be a linear algebraic group over a global function field K, $S \subseteq \mathbb{P}_K$ a finite subset and Λ an S-arithmetic subgroup of G of rank ≥ 2. Then Λ has a finite abelianisation.*

Proof. This follows from [Mar91, Ch.VIII, Thm.A] since the commutator $[\Lambda, \Lambda]$ is not contained in the centre of G. □

This means that we compute the fundamental group for each square complex using Seifert-van Kampen Theorem, which is straightforward, and then compute its abelianisation to see whether it is a finite group. It is remarkable that, although the first criterion looks simple, many square complexes are already excluded at this stage. The next criterion is based on the following proposition:

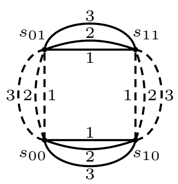
Proposition A.3. *Let K be a global function field, Q be a quaternion algebra over K, $S \subseteq \mathbb{P}_K$ be a finite subset containing all ramified places of Q and $G := \mathrm{PGL}_{1,Q}$ with $O_{K,S}$-integral structure. Then there exists a surjective homomorphism from $G(O_{K,S})$ to $\mathrm{PGL}_2(O_{K,S}/\mathfrak{p})$ for $\mathfrak{p} \in \mathbb{P}_K \setminus S$.*

Proof. In fact, the mapping $G(O_{K,S}) \to G(O_{K,S}/\mathfrak{p}) \cong \mathrm{PGL}_2(O_{K,S}/\mathfrak{p})$ is surjective. This follows from Strong Approximation Theorem, see [Mar91, Ch.II, Thm.6.8]. □

If K is a function field over $\mathbb{F}_2$, then for each $\mathfrak{p} \in \mathbb{P}_K \setminus S$, the factor ring $O_{K,S}/\mathfrak{p}$ is a finite field with 2^n elements for some $n \in \mathbb{N}$. Furthermore, this shouldn't produce any surjective homomorphism to the group $\mathrm{PGL}_2(\mathbb{F}_q)$ for any odd prime power q. This leads to the second criterion, namely to see if for a fixed square complex Σ, the statement

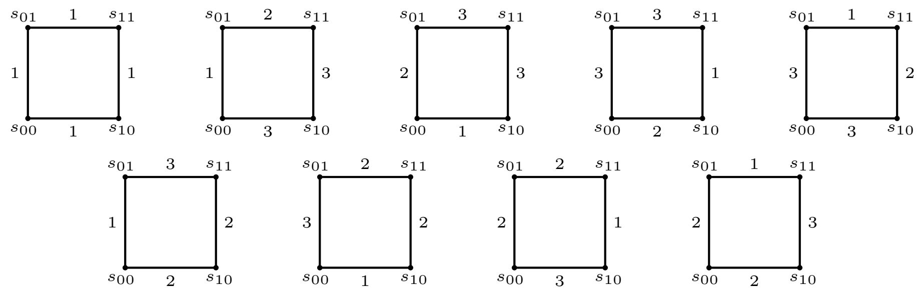
$$\#\,\mathrm{Hom}_{\mathrm{surj}}(\pi_1(\Sigma), \mathrm{PGL}_2(\mathbb{F}_q)) \begin{cases} > 0 & \text{if } q = 2^n \text{ for some } n \in \mathbb{N}, \\ = 0 & \text{otherwise.} \end{cases}$$

holds for almost all prime powers q. In fact, we could test this for several prime powers (e.g., all prime powers up to 70) to see if the statement above potentially holds. A computation by MAGMA shows that a square complex satisfying both criteria is the square complex Σ with $\mathbb{V}(\Sigma) = \{s_{00}, s_{01}, s_{10}, s_{11}\}$ whose 1-skelett (hence the set of edges) is given by the picture



,

where the solid lines indicate the horizontal edges, the dashed lines the vertical ones, and whose nine squares are given below.



The fundamental group $\Gamma := \pi_1(\Sigma, s_{00})$ has the following presentation:

$$\Gamma = \left\langle a_1, a_2 \,\middle|\, a_2^{-1}a_1a_2^{-2}a_1^{-1}a_2^{-1}a_1^{-1}a_2^{-2}a_1a_2^{-1}a_1^{-1}, a_1a_2a_1^{-1}a_2^2a_1^2a_2^{-1}a_1^{-1}a_2^{-2}a_1a_2^{-1} \right\rangle. \tag{A.1}$$

Here a_1 stands for the path from s_{00} along the edge 1 to s_{01} and then along the edge 2 back to s_{00}, and a_2 for the path along the edge 1 to s_{10} and then along the edge 2 back to s_{00}.

The next step is to find an appropriate quaternion algebra Q over an appropriate global function field K over $\mathbb{F}_2$ as well as a homomorphism $\Gamma \to Q^\times/K^\times$. The idea is that after a base extension to an algebraic closure of K, Q becomes the 2×2-matrix algebra. This leads us to consider the representation variety

$$X := \underline{\mathrm{Hom}}(\Gamma, \mathrm{PGL}_2)$$

over $\mathbb{F}_2$. This can be considered as a closed subvariety of $\mathrm{PGL}_2 \times \mathrm{PGL}_2$ via embedding

$$X \hookrightarrow \mathrm{PGL}_2 \times \mathrm{PGL}_2,\ \varphi \longmapsto (\varphi(a_1), \varphi(a_2)).$$

To find such a φ, we consider instead the quotient $\mathrm{PGL}_2 \times \mathrm{PGL}_2/\!\sim$, where $\sim$ stands for the equivalence relation obtained by conjugation, so that it suffices to consider certain matrices as explained by the following proposition:

Proposition A.4. *Let Ω be an algebraically closed field. Then the image of the morphism*

$$\iota : \mathbb{A}^3(\Omega) \longrightarrow \mathrm{PGL}_2(\Omega) \times \mathrm{PGL}_2(\Omega)/\!\sim,\ (x,y,z) \longmapsto \left(\begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ y & x \end{pmatrix} \right)$$

is a Zariski-open dense subset in $\mathrm{PGL}_2(\Omega) \times \mathrm{PGL}_2(\Omega)/\!\sim$

Proof. Suppose that $A, B \in \mathrm{GL}_2(\Omega)$. By scaling appropriately, we may assume that $\det(A) = -1$. If A is not a multiple of $\mathbf{1}_2$ (thus lies in an open dense subset of $\mathrm{PGL}_2(\Omega)$), there exists a vector $v \in \Omega^2$ such that Av is not a scalar multiple of v, i.e. the matrix $S := [v, Av]$ is invertible. It follows that

$$A' := S^{-1}AS = \begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix}$$

for some $z \in \Omega$. Now we come to the second matrix $B' := S^{-1}BS$. This must have an eigenvector $w = (w_1, w_2)^t \in \Omega^2$. Furthermore, we can choose $u := (u_1, u_2)^t \in \Omega$ such that

$$w_1u_1 + (w_1z - w_2)u_2 = 0 \quad \text{and} \quad w_2u_1 - w_1u_2 = w_1^2 + w_1w_2z - w_2^2. \tag{$*$}$$

It turns out that w, u are linearly independent iff $w_1^2 + w_1 w_2 z - w_2^2 \neq 0$, i.e. this holds for all (A, B) in an open dense subset of $\mathrm{PGL}_2(\Omega) \times \mathrm{PGL}_2(\Omega)/\sim$. In this case, the matrix $T := [u, w]$ is invertible and

$$\begin{aligned} T^{-1}A'T &= \frac{1}{u_1 w_2 - u_2 w_1} \begin{pmatrix} w_2u_2 - w_1u_1 - w_1u_2z & w_2^2 - w_1^2 - w_1w_2z \\ u_1^2 + u_1u_2z - u_2^2 & w_1u_1 + w_2u_1z - w_2u_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix} \quad \text{by using } (*). \end{aligned}$$

Furthermore, since w is an eigenvector of B', the matrix $T^{-1}BT$ must be a lower triangular matrix. This can be scaled so that the upper left entry is 1. Hence the claim follows. □

From this proposition, we can construct the homomorphism $\varphi : \Gamma \longrightarrow \mathrm{PGL}_2(\mathbb{F}_2(x, y, z))$ by first setting $\varphi(a_1) = A_1 := \begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix}$ and $\varphi(a_2) = A_2 := \begin{pmatrix} 1 & 0 \\ y & x \end{pmatrix}$. The relations in the presentation of Γ from (A.1) implies that

$$\begin{aligned} A_2^{-1}A_1A_2^{-2}A_1^{-1}A_2^{-1}A_1^{-1}A_2^{-2}A_1A_2^{-1}A_1^{-1} &= \mathbf{1}_2 \quad \text{and} \\ A_1A_2A_1^{-1}A_2^2A_1^2A_2^{-1}A_1^{-1}A_2^{-2}A_1A_2^{-1} &= \mathbf{1}_2 \end{aligned}$$

in $\mathrm{PGL}_2(\mathbb{F}_2(x, y, z))$. Also note that we can replace A_i^{-1} by the adjoint matrix $\mathrm{Adj}(A_i)$. To be equal to $\mathbf{1}_2$ in PGL_2 means that the upper left and lower right entries of the matrix coincide while the other entries vanish. By a computation, the preimage of $X/\sim\subseteq \mathrm{PGL}_2 \times \mathrm{PGL}_2/\sim$ under ι is given by the ideal $I \trianglelefteq \mathbb{F}_2[x, y, z]$ generated by the following polynomials

$$\begin{aligned} f_1 &= x^5y^2z^2 + x^4y^2z^2 + x^4y^2 + x^3y^4 + x^3y^3z + x^3y^2z^2 + x^3y^2 + x^3yz + x^3 + x^2y^2z^2 + x^2yz \\ &\quad + xy^4 + xy^3z, \end{aligned}$$

$$\begin{aligned} f_2 &= x^5yz^3 + x^5z^2 + x^4y^3z^3 + x^4y^2z^4 + x^4y^2z^2 + x^4yz^3 + x^4yz + x^4z^2 + x^4 + x^3y^3z^3 + x^3y^2z^4 \\ &\quad + x^3y^2z^2 + x^2y^5z + x^2y^4 + x^2y^2z^4 + x^2y^2z^2 + x^2y^2 + x^2z^2 + xy^3z^3 + xy^2z^4 + xy^2 \\ &\quad + xz^2 + y^5z + y^4 + y^3z^3 + y^2z^2, \end{aligned}$$

$$\begin{aligned} f_3 &= x^5y^2z^3 + x^4y^3z^2 + x^4y^2z + x^4yz^2 + x^3y^4z + x^3y^3 + x^3yz^2 + x^3y + x^3z + x^2y^5 + x^2y^2z \\ &\quad + x^2yz^2 + xy^4z + xy^3 + xy^2z^3 + xyz^2 + y^5 + y^3z^2, \end{aligned}$$

$$\begin{aligned} f_4 &= x^5yz^2 + x^4y^2z^3 + x^4y^2z + x^4yz^2 + x^4y + x^4z + x^3y^2z + x^3yz^2 + x^3y + x^3z + x^2y^4z \\ &\quad + x^2y^3 + x^2y^2z + x^2yz^2 + x^2y + x^2z + xy^3 + xy^2z + y^4z + y^2z^3, \end{aligned}$$

$$\begin{aligned} f_5 &= x^5y^2z^3 + x^5yz^2 + x^4y^3z^4 + x^4y^3z^2 + x^4yz^2 + x^4y + x^4z + x^3y^3z^2 + x^3z + x^2y^5z^2 + x^2y^3 \\ &\quad + x^2y^2z^3 + x^2y^2z + x^2y + x^2z + xy^4z + xy^3 + xy^2z^3 + xy^2z + y^5z^2 + y^4z + y^3z^4 + y^2z^3, \end{aligned}$$

$$\begin{aligned} f_6 &= x^5y^2z^2 + x^5yz^3 + x^5z^2 + x^4y^3z^3 + x^4y^3z + x^4y^2z^4 + x^4y^2 + x^4 + x^3y^3z + x^3y^2z^2 + x^3y^2 \\ &\quad + x^3yz^3 + x^3yz + x^3z^2 + x^3 + x^2y^5z + x^2y^4z^2 + x^2y^4 + x^2y^2 + x^2yz^3 + xy^4 + xy^3z + xy^2 \\ &\quad + xz^2 + y^5z + y^4z^2 + y^3z^3 + y^3z + y^2z^4 + yz^3 \end{aligned}$$

A computation by Magma shows that I has four irreducible components, but only one for which x doesn't belong to. Its associated prime ideal is given by the polynomials

$$xy + xz + x + y + z, \quad xz^2 + y + z^2 + z \quad \text{and} \quad y^2 + y + z.$$

This implies that the corresponding irreducible curve has function field equal to $\mathbb{F}_2(y)$ with

$$z = y^2 + y \quad \text{and} \quad x = \frac{y^2}{y^2+1}.$$

Since z is a quadratic polynomial in y, this leads us to consider a quaternion algebra defined over the subfield $K := \mathbb{F}_2(z)$ with $\mathbb{F}_2(y)$ as quadratic splitting field. Now we can substitute $x = \frac{y^2}{y^2+1}$ back to the matrix A_2 from above to obtain

$$A_2 = \frac{1}{1+y^2}\begin{pmatrix} 1+y^2 & 0 \\ y(1+y^2) & y^2 \end{pmatrix} \sim \begin{pmatrix} 1+y^2 & 0 \\ (1+y)z & y^2 \end{pmatrix}.$$

As discussed in Section 1.1, we wish to find $a, b \in K$ such that $Q = \left[\frac{a,b}{K}\right)$ is the quaternion algebra we are looking for. We could set $a = y$ by the discussion we just made and still wish to find b. The strategy is to conjugate A_1 and A_2 by the same invertible matrix to get matrices of the form as in Lemma 1.3. One possible attempt is to try with

$$S := \begin{pmatrix} 0 & 1 \\ 1 & (1+y)z \end{pmatrix}$$

to make A_2 diagonal. In fact, we have

$$S^{-1}A_1S = \begin{pmatrix} yz & 1+z^3 \\ 1 & (1+y)z \end{pmatrix} \quad \text{and} \quad S^{-1}A_2S = \begin{pmatrix} y^2 & 0 \\ 0 & 1+y^2 \end{pmatrix}.$$

These are indeed of the form we wish to have. Hence we can set $b = 1 + z^3$ and obtain the quaternion algebra

$$Q = \left[\frac{z, 1+z^3}{K}\right) = K\{I, J\}/(I^2 + I = z, J^2 = 1 + z^3, JI = (I+1)J).$$

It still remains to determine whether the homomorphism

$$\varphi : \Gamma \longrightarrow Q^\times/K^\times, \quad a_1 \longmapsto [A_1],\ a_1 \longmapsto [A_2]$$

is injective and makes Γ an S-arithmetic lattice of $\mathrm{PGL}_{1,Q}$ for a suitable finite subset $S \subseteq \mathbb{P}_K$. Lemma 1.27 shows that Q splits at the places $\{z = 0\}$ and $\{z = \infty\}$, which leads us to construct a φ-equivariant mapping

$$\tilde{\Sigma} = T_3 \times T_3 \longrightarrow \Delta(\mathrm{PGL}_2(T_0)) \times \Delta(\mathrm{PGL}_2(T_\infty)),$$

where $\tilde{\Sigma}$ stands for the universal covering of Σ. Here we have a difficulty that the action of Γ on the vertices of $T_3 \times T_3$ is not simply transitive. To avoid this difficulty, we consider instead a V_4-acion on Σ which is transitive on the vertices and determine the corresponding orbital fundamental group $\Lambda := \pi_1^{\mathrm{orb}}(\Sigma, V_4, s_{00})$ since this will act simply transitively on the universal covering $T_3 \times T_3$.

Since such an action necessarily has a fixed square, there is a section $V_4 = \langle \gamma_v, \gamma_h \rangle \to \pi_1^{\mathrm{orb}}(\Sigma, V_4, s_{00})$ to the exact sequence from Corollary 2.9. The images of the generators will be denoted by c_v and c_h. Our next goal is to determine their possible images in $Q^\times/K^\times$ under a possible extension of φ.

To this end, let $C_v, C_h \in Q$ be liftings of $\varphi(c_v), \varphi(c_h)$ respectively. Since c_v and c_h must have order 2, the elements C_v^2 and C_h^2 must lie in K. Furthermore, since c_v and c_h commute with each other, the products C_vC_h and C_hC_v can differ at most by a multiplication with a scalar $\lambda \in K$. The norm condition implies that $1 = \mathrm{n}(\lambda) = \lambda^2$, i.e. $\lambda = 1$ since $\operatorname{char} K = 2$. This implies that C_v and C_h must commute. We obtain the following lemma:

Lemma A.5. *Under the condition as above and after rescaling C_1, C_2 appropriately, there exist $f_v, f_h, g_1, g_2 \in K$ such that $(g_1, g_2) \neq (0,0)$ and*

$$C_v = f_v + g_1J + g_2IJ \quad \textit{and} \quad C_h = f_h + g_1J + g_2IJ.$$

Proof. Observe first that $C_v, C_h \notin K$ since their images in $Q^\times/K^\times$ can't vanish. The coefficients of I in both C_v and C_h are the traces of C_v and C_h respectively, thus vanish by the condition that $C_v^2, C_h^2 \in K$ since their minimal polynomials are then of the form $X^2 - f$ for some $f \in K$. Now C_h must lie in the centraliser of $K(C_v) \subseteq Q$, which is again $K(C_v)$ since this is a quadratic extension of K contained in Q. Hence the coefficients of J, IJ in C_h must be in the same ratio as in C_v. The claim then follows by rescaling C_h appropriately. □

Now we use the condition due to the conjugation of a_1, a_2 to find $f_v, f_h, g_1, g_2 \in K$. In fact, we have the equations

$$C_vA_i = \widehat{\varphi({}^{c_v}a_i)}C_v \quad \text{and} \quad C_hA_i = \widehat{\varphi({}^{c_h}a_i)}C_h \quad \text{for} \quad i = 1, 2.$$

Here ${}^ca$ stands for the conjugation cac^{-1} and $\widehat{\cdot}$ means an appropriate lifting in Q, which is uniquely determined in each case due to the norm condition. By comparing the coefficients of $1, I, J, IJ$ in each equations, we obtain a system of linear equations for which we can solve for $f_v, f_h, g_1, g_2 \in K$. In turns out that in our case, we obtain

$$C_v = 1 + z^2 + IJ \quad \text{and} \quad C_h = z + z^2 + IJ.$$

By considering their topological meaning in $\pi_1^{\mathrm{orb}}(\Sigma, V_4, s_{00})$, we conjecture that

$$\big(\{C_v, C_vA_1, (C_vA_1)^{-1}\}, \{C_h, C_hA_2, (C_hA_2)^{-1}\}\big)$$

forms a V_4-equivariant vertical-horizontal structure of an arithmetic subgroup of G in the sense of Definition 3.1, which is to be verified. Then we can show that the extension of $\varphi : \Gamma \longrightarrow Q^\times/K^\times$ to $\pi_1^{\mathrm{orb}}(\Sigma, V_4, s_{00})$ by $\varphi(c_v) = C_v$ and $\varphi(c_h) = C_h$ is indeed an embedding of $\pi_1^{\mathrm{orb}}(\Sigma, V_4, s_{00})$ as an arithmetic subgroup in G as we have done in our thesis.

We finish this appendix with the remark that so far we have found only one square complex that potentially gives a fundamental group with arithmetic origin, namely Σ. It turns out that this is really an arithmetic lattice. A computation shows that Σ is the only square complex that carries an action by $Z_4 = \mathbb{Z}/4\mathbb{Z}$ with this properties. Furthermore, another computation shows that the associated V_4-structure in the sense of Definition 3.1 is the only V_4-structure of size $(3, 3)$ such that the orbital fundamental group of the associated square complex can be an arithmetic lattice. However, it still remains an open question if there exists a further square complex with four vertices and $T_3 \times T_3$ such that its fundamental group has an arithmetic origin.

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Zusammenfassung (Deutsch)

In dieser Arbeit konstruieren wir ein torsionsfreies quaternionisches arithmetisches Gitter vom Rang 2 über einem globalen Körper der Charakteristik 2, dessen Wirkung auf dem Produkt zweier Bruhat-Tits-Bäume einen Quadratkomplex mit vier Eckpunkten als Quotienten liefert. In diesem Fall ist das Gitter genau die Fundamentalgruppe des Quotientenkomplexes, besitzt also eine endliche Präsentation. Es ist vor allem deswegen interessant, denn sehr selten kann die Fundamentalgruppe eines Quadratkomplexes einen arithmetischen Ursprung haben.

Aus Sicht der arithmetischen Theorie algebraischer Gruppen sind bisher sehr wenige explizite Resultate über Präsentationen arithmetischer Gruppen bekannt. Zwar gibt es ein allgemeines Verfahren für Gitter in Lie-Gruppen, indem man einen Dirichlet-Fundamentalbereich bestimmt, sowie Kriterien nach [Beh98], wann eine arithmetische Gruppe endlich erzeugt bzw. endlich präsentiert ist, aber die bisher bekannten expliziten Resultate betreffen größtenteils den Fall vom Rang 1. Daher ist es bereits interessant, eine Präsentation für solche arithmetischen Untergruppen vom Rang 2 mit kleinerem Fundamentalbereich zu bestimmen.

Für quaternionische arithmetische Gitter vom Rang 2 über Funktionenkörpern können wir folgende Situation betrachten: Sei q eine Primzahlpotenz, K ein globaler Funktionenkörper einer Kurve über $\mathbb{F}_q$, sei Q eine Quaternionenalgebra über K mit zwei $\mathbb{F}_q$-rationalen unverzweigten Stellen $\mathfrak{p}, \mathfrak{p}'$ und $S \subseteq \mathbb{P}_K$ eine Teilmenge, die aus allen verzweigten Stellen von Q sowie $\mathfrak{p}$ und $\mathfrak{p}'$ besteht. Ferner sei Γ eine S-arithmetische Untergruppe der projektiv linearen Gruppen

$$G = \mathrm{PGL}_{1,Q} = \mathrm{GL}_{1,Q} / \mathbb{G}_m.$$

Dann wirkt Γ auf dem Produkt zweier Bruhat-Tits-Bäume für $\mathrm{PGL}_2(K_\mathfrak{p})$ und $\mathrm{PGL}_2(K_{\mathfrak{p}'})$ komponentenweise mittels der diagonalen Einbettung

$$\Gamma \hookrightarrow G(K_\mathfrak{p}) \times G(K_{\mathfrak{p}'}) \cong \mathrm{PGL}_2(K_\mathfrak{p}) \times \mathrm{PGL}_2(K_{\mathfrak{p}'}),$$

also auf dem Produkt $T_{q+1} \times T_{q+1}$ zweier Bäume derselben Valenz $q+1$. Ist Γ torsionsfrei, so ist die Wirkung frei und liefert einen kompakten Quadratkomplex als Quotienten. Zwar gibt es zahlreiche solche Gitter, aber sehr selten liefert ein solches Gitter einen kleinen Quadratkomplex. Daher stellt sich die Frage, wie klein ein solcher Quotientenkomplex sein kann.

Ist q ungerade, so gibt es nach Stix und Vdovina in [SV13] ein torsionsfreies quaternionisches Gitter über $\mathbb{F}_q(t)$, welches einfach transitiv auf den Ecken von $T_{q+1} \times T_{q+1}$ operiert. Insbesondere liefert dies einen Quotientenkomplex mit genau einen Eckpunkt. Ist dagegen q eine Potenz von 2, so ist es nicht möglich, ein torsionsfreies Gitter mit einfach transitiver Wirkung auf den Ecken von $T_{q+1} \times T_{q+1}$ zu finden. Der Grund hierfür ist, dass für die Anzahl der Quadrate im Quotientenkomplex gilt:

$$\#\mathbb{S}(\Gamma \backslash (T_{q+1} \times T_{q+1})) = \frac{1}{4} N(q+1)^2,$$

wobei N die Anzahl der Eckpunkte des Quotientenkomplexes bezeichnet, also Anzahl der Bahnen von Ecken auf $T_{q+1} \times T_{q+1}$. Da $q+1$ ungerade ist, muss N durch 4 teilbar sein. Insbesondere kann es kein torsionsfreies Gitter mit transitiver Wirkung auf den Eckpunkten geben. Nichtsdestotrotz können wir zeigen, dass es für $q = 2$ ein torsionsfreies arithmetisches Gitter Γ gibt, so dass $\Gamma\backslash(T_{q+1} \times T_{q+1})$ genau vier Eckpunkte hat.

Die Frage nach einem quaternionischen arithmetischen Gitter vom Rang 2 in Charakteristik 2, welches einen Quotientenkomplex unter der Bruhat-Tits-Wirkung auf $T_3 \times T_3$ mit vier Eckpunkten liefert, hat auch einen Ursprung aus der algebraischen Geometrie, nämlich die Konstruktion algebraischer Varietäten mittels nichtarchimedischer Uniformisierung. Diese Methode wurde von Mumford in [Mum79] verwendet, um eine falsche projektive Ebene zu konstruieren. Darunter versteht man eine minimale glatte projektive Fläche von allgemeinem Typ X mit $c_1(X)^2 = 9$, $c_2(X) = 3$ und trivialer Albanese-Varietät, also mit den gleichen Invarianten wie die projektive Ebene $\mathbb{P}^2$. Die Konstruktion basiert auf einem torsionsfreien arithmetischen Gitter, welches einfach transitiv auf den Eckpunkten des Bruhat-Tits-Gebäudes für $\mathrm{PGL}_3(\mathbb{Q}_2)$ operiert und einen Quotientenkomplex der Euler-Charakteristik 1 liefert.

Wir interessieren uns hier für falsche Quadriken. Dieser Begriff wird durch den Begriff einer falschen projektiven Ebene motiviert. Genau genommen ist eine falsche Quadrik eine minimale glatte projektive Fläche von allgemeinem Typ X mit $c_1(X)^2 = 8$, $c_2(X) = 4$ und trivialer Albanese-Varietät, also mit den gleichen Invarianten wie eine Quadrik. Alle bekannten falschen Quadriken in Charakteristik 0 werden mittels komplexer Uniformisierung konstruiert. Diese wurden bisher von Kuga und Shavel [Sha78], Bauer, Catanese and Grunewald [BCG08], Frapporti [Fra13], Džambić [Dža14] sowie auch Linowitz, Stover und Voight [LSV15] studiert.

Für falsche Quadriken über positiver Charakteristik dagegen sind bis heute sehr wenige Resultate bekannt. Ein erstes Beispiel wurde von Stix und Vdovina in [SV13] als Spezialfall der Klasse algebraischer Flächen X von allgemeinem Typ mit dem Chern-Verhältnis $c_1(X)^2/c_2(X) = 2$ und trivialer Albanese-Varietät konstruiert. Die Konstruktion basiert auf einem torsionsfreien quaternionischen arithmetischen Gitter, welches einfach transitiv auf den Eckpunkten des Produkts zweier Kopien des Bruhat-Tits-Baums für $\mathrm{PGL}_2(\mathbb{F}_q((t)))$ wirkt, wobei q eine ungerade Primzahlpotenz bezeichnet, und liefert speziell für $q = 3$ eine falsche Quadrik.

Diese Konstruktion funktioniert auch in einem allgemeineren Kontext, also mit einer beliebigen Primzahlpotenz q und einem torsionsfreien Gitter $\Gamma \leq \mathrm{PGL}_2(\mathbb{F}_q((t))) \times \mathrm{PGL}_2(\mathbb{F}_q((t)))$. Ist N die Anzahl der Eckpunkte des Quotientenkomplexes unter der Γ-Wirkung, so ist die Euler-Charakteristik der sich ergebenden Fläche X_Γ gleich

$$\chi(X_\Gamma) = \frac{1}{4}N(q-1)^2.$$

Daraus folgt, dass $\chi(X_\Gamma) = 1$ genau dann, wenn $N = 1, q = 3$ oder $N = 4, q = 2$ gilt. Der Fall $N = 1, q = 3$ wurde wie erwähnt von Stix und Vdovina untersucht. Der Fall $N = 4, q = 2$ hingegen führt zur Frage nach einem quaternionischen arithmetischen Gitter vom Rang 2 in Charakteristik 2, welches einen Quadratkomplex mit vier Eckpunkten als Quotienten unter der Bruhat-Tits-Wirkung auf $T_3 \times T_3$ liefert.

Um nun ein solches Gitter zu konstruieren, setzen wir $K := \mathbb{F}_2(z)$ und definieren folgende Quaternionenalgebra

$$Q := \left[\frac{z, 1+z^3}{K}\right) = K\{I, J\}/\{I^2 + I = z,\ J^2 = 1 + z^3,\ IJ = J(I+1)\}.$$

Diese Quaternionalgebra wird im Kapitel 1 dieser Arbeit untersucht, nachdem die benötigten Begriffe über Quaternionenalgebren eingeführt worden sind. Sie ist verzweigt genau an den Stellen $\{z = 1\}$ und $\{z = \zeta\}$, wobei $\zeta \in \mathbb{F}_2^{\mathrm{alg}}$ eine primitive dritte Einheitswurzel bezeichnet. Somit hat Q zwei $\mathbb{F}_2$-rationale unverzweigte Stellen, nämlich $\{z = 0\}$ und $\{z = \infty\}$ und wir können $S \subseteq \mathbb{P}_K$ als Menge der Stellen bei $0, 1, \infty, \zeta$ setzen. Für die lineare algebraische Gruppe

$$G := \mathrm{PGL}_{1,Q} = \mathrm{GL}_{1,Q} / \mathbb{G}_m$$

mit der durch die Basis $\{1, I, J, IJ\}$ definierten Ganzheitsstruktur über dem Ring $R = O_{K,S} = \mathbb{F}_2[z, \frac{1}{z+z^4}]$ ist somit die Gruppe $G(R)$ unter der diagonalen Einbettung

$$G(R) \longhookrightarrow G(K_0) \times G(K_\infty) \cong \mathrm{PGL}_2(\mathbb{F}_2((y))) \times \mathrm{PGL}_2(\mathbb{F}_2((t))) \qquad (\star)$$

ein kokompaktes arithmetisches Gitter. Dabei sind $y \in K_0$ bzw. $t \in K_\infty$ jeweils Uniformisierende, die so gewählt sind, dass $y^2 + y = z$ und $t^2 + t = z^{-1}$, und die Gruppenisomorphismen $G(K_0) \cong \mathrm{PGL}_2(\mathbb{F}_2((y)))$ bzw. $G(K_\infty) \cong \mathrm{PGL}_2(\mathbb{F}_2((t)))$ sind durch die Einbettungen

$$\rho_y : Q \longhookrightarrow \mathrm{M}_2(\mathbb{F}_2(y)) \quad \text{bzw.} \quad \rho_t : Q \longhookrightarrow \mathrm{M}_2(\mathbb{F}_2(t))$$

aus Lemma 1.33 bzw. 1.34 gegeben. Unter dieser Einbettung erhalten wir die Wirkung von $G(R)$ auf $T_3 \times T_3$ als Produkt zweier Bruhat-Tits-Bäume.

Darauf basierend wollen wir eine Präsentation für S-arithmetische Gitter von G bestimmen. Dafür erweitern wir den Begriff der Fundamentalgruppe eines topologischen Raums zur Fundamentalgruppe eines globalen Orbiraums. Dabei versteht man unter einem globalen Orbiraum den Quotienten unter einer Gruppenwirkung auf einem topologischem Raum. Es stellt sich heraus, dass sich einige Resultate aus der Überlagerungstheorie erweitern lassen. So wird eine Faser einer Überlagerung zu einer orbitalen Faser erweitert. Ferner wirkt die orbitale Fundamentalgruppe einfach transitiv auf der orbitalen Faser der universellen Überlagerung, so wie man es aus der Überlagerungstheorie topologischer Räume erwarten würde. Das ist der Hauptgegenstand des Kapitels 2.

Um nun die Struktur von S-arithmetischen Gittern von G mit der orbitalen Fundamentalgruppe eines gewissen Quadratkomplexes zu vergleichen, wird im Kapitel 3 der Begriff einer **V_4-äquivarianten vertikal-horizontalen Struktur** einer Gruppe, kurz **V_4-Struktur**, eingeführt. Darauf basierend konstruieren wir einen Quadratkomplex mit vier Eckpunkten und einer Wirkung der Klein'schen Vierergruppe $V_4 = Z_2^2$, wobei Z_2 die zyklische Gruppe $\mathbb{Z}/2\mathbb{Z}$ bezeichnet. Die zugehörige orbitale Fundamentalgruppe lässt sich dann anhand der exakten Sequenz

$$1 \longrightarrow \pi_1(\Sigma_{\mathcal{A},\mathcal{B}}, s_{00}) \longrightarrow \pi_1^{\mathrm{orb}}(\Sigma_{\mathcal{A},\mathcal{B}}, V_4, s_{00}) \longrightarrow V_4 \longrightarrow 1$$

berechnen. Genauer gesagt erhält man eine Präsentation für $\pi_1^{\mathrm{orb}}(\Sigma_{\mathcal{A},\mathcal{B}}, V_4, s_{00})$ aus derjenigen von V_4 sowie von $\pi_1(\Sigma_{\mathcal{A},\mathcal{B}}, s_{00})$, die sich wiederum aus dem Satz von Seifert-van Kampen ergibt. Dazu wird eine allgemeine Theorie, wie man eine Präsentation der Gruppenerweiterung bestimmt, im Abschnitt 3.3 ausführlich diskutiert. Damit hat $\pi_1^{\mathrm{orb}}(\Sigma_{\mathcal{A},\mathcal{B}}, V_4, s_{00})$ die Präsentation

$$\pi_1^{\mathrm{orb}}(\Sigma, V_4, s_{00}) = \left\langle \begin{array}{c} \alpha_a, \beta_b \\ \text{for } a \in \mathcal{A},\, b \in \mathcal{B} \end{array} \middle| \begin{array}{c} \alpha_a \beta_{b'} = \beta_b \alpha_{a'} \text{ für } a, a' \in \mathcal{A},\, b, b' \in \mathcal{B} \\ \text{mit } ab' = ba' \text{ in } \Lambda, \\ \alpha_a \alpha_{a^{-1}} = \beta_b \beta_{b^{-1}} = 1 \text{ für } a \in \mathcal{A},\, b \in \mathcal{B} \end{array} \right\rangle,$$

vgl. Proposition 3.15. Damit kommen wir dem Vergleich einer mit einer V_4-Struktur ausgestatteten Gruppe mit der orbitalen Fundamentalgruppe $\pi_1^{\mathrm{orb}}(\Sigma_{\mathcal{A},\mathcal{B}}, V_4, s_{00})$ nahe. In der Tat können wir folgenden Satz beweisen:

Satz 1 (Vergleichssatz, Theorem 3.17)**.** *Sei $(\mathcal{A}, \mathcal{B})$ eine V_4-Struktur einer Gruppe Λ und seien $m := \#\mathcal{A}$, $n := \#\mathcal{B}$. Die Wirkung von Λ auf dem Produkt $Y := T_m \times T_n$ erhalte die zelluläre Struktur. Des Weiteren gebe es einen Eckpunkt $v \in Y$ derart, dass die Bahnen $\mathcal{A}.v$ bzw. $\mathcal{B}.v$ genau die vertikalen bzw. horizontalen Nachbarn von v sind. Dann wirkt Λ einfach transitiv auf den Eckpunkten von Y. Ferner besteht ein Isomorphismus*

$$\varphi : \pi_1^{\mathrm{orb}}(\Sigma_{\mathcal{A},\mathcal{B}}, V_4, s_{00}) \longrightarrow \Lambda, \ \ \alpha_a \longmapsto a \ \textit{und} \ \beta_b \longmapsto b.$$

Unter diesem Isomorphismus ist $\pi_1(\Sigma_{AB}, s_{00})$ isomorph zu einem Normalteiler $\Gamma \trianglelefteq \Lambda$ mit V_4 als Quotienten. Für dieses Γ gilt: $\Gamma \backslash Y \cong \Sigma_{\mathcal{A},\mathcal{B}}$.

Um vor allem zu zeigen, dass φ ein Isomorphismus ist, konstruieren wir eine φ-äquivariante Abbildung von der zu $\Sigma_{\mathcal{A},\mathcal{B}}$ gehörigen universellen Überlagerung zum Produkt zweier Bäume $T_m \times T_n$, indem wir die Abbildung zunächst zwischen den Eckpunkten konstruieren, diese dann auf die 1-Skeletten und schließlich auf die beiden ganzen Räume erweitern. Es stellt sich heraus, dass die soeben konstruierte Abbildung eine Überlagerung ist, also ein Isomorphismus, da $T_m \times T_n$ einfach zusammenhängend ist. Daraus folgt die Behauptung.

Im Abschnitt 3.6 führen wir den Begriff der zur V_4-Struktur $(\mathcal{A}, \mathcal{B})$ gehörigen lokalen Permutationsgruppen ein. Es handelt sich hierbei um eine Modifikation des Begriffs desselben Namens in [SV13, §5.1]. In diesem Fall sind die lokalen Permutationsgruppen Untergruppen der Kranzprodukte $\mathrm{Sym}(\mathcal{A}) \wr \mathrm{Sym}(I)$ und $\mathrm{Sym}(\mathcal{B}) \wr \mathrm{Sym}(I)$, wobei $I = \{0, 1\}$. Diese Struktur wird vor allem zur Berechnung der Albanese-Varietät im Kapitel 5 benötigt.

Im Kapitel 4 bestimmen wir nun Präsentationen arithmetischer Untergruppen von G. Das wichtigste Werkzeug dieses Kapitels ist die Theorie von Bruhat-Tits-Bäumen, die im ersten Abschnitt behandelt wird. Die Strategie besteht darin, die Wirkung von $G(R)$ unter der Einbettung aus $(\star)$ auf dem Produkt der Bruhat-Tits-Bäume für $\mathrm{PGL}_2(\mathbb{F}_2((y)))$ und $\mathrm{PGL}_2(\mathbb{F}_2((t)))$ zu betrachten, und daraus den Stabilisator des Standardeckpunktes sowie eine Untergruppe, die einfach transitiv auf den Eckpunkten des Produkts wirkt, zu bestimmen.

Zum Stabilisator des Standardeckpunktes stellen wir zunächst fest, dass er als diskrete und kompakte Untergruppe von $\mathrm{PGL}_2(\mathbb{F}_2((y))) \times \mathrm{PGL}_2(\mathbb{F}_2((t)))$ endlich sein muss. Ferner kann ein Element endlicher Ordnung in $G(R)$ als Untergruppe von $\mathrm{PGL}_2(\mathbb{F}_2((y)))$ nur die Ordnung 1, 2 oder 3 haben. Eine weitere Überlegung allerdings schließt den Fall der Ordnung 3 aus, so dass der Stabilisator eine 2-elementarabelsche Gruppe ist. Mit Hilfe der Normabbildung stellen wir dann fest, dass das Bild von

$$D := (1 + z + z^2) + IJ$$

unter der Projektion auf $G(R)$ das einzige nichttriviale Element des Stabilisators ist. Um nun eine Untergruppe mit einfach transitiver Wirkung auf den Eckpunkten zu bestimmen, betrachten wir die Bilder von

$$B_1 := (1 + z)I + J, \qquad B_2 := z + z^2 + (1 + z)I + J + IJ,$$
$$C_1 := 1 + z^2 + IJ \quad \text{und} \quad C_2 := z + z^2 + IJ$$

unter der Projektion auf $G(R)$, die wir im Folgenden mit b_1, b_2, c_1, c_2 bezeichnen. Es stellt sich heraus, dass das Paar $(\mathcal{A}, \mathcal{B})$ mit

$$\mathcal{A} = \{b_1, b_1^{-1}, c_1\} \quad \text{und} \quad \mathcal{B} = \{b_2, b_2^{-1}, c_2\}$$

eine V_4-Struktur einer Untergruppe von $G(R)$ bildet. Genauer kann man zeigen, dass diese Untergruppe genau die Gruppe $G(R_1)$ ist, wobei $R_1 = \mathbb{F}_2[z, \frac{1}{z+z^2}]$. Ferner erfüllt diese V_4-Struktur die Bedingung des Vergleichssatzes, so dass $G(R_1)$ zur orbitalen Fundamentalgruppe $\pi_1^{\mathrm{orb}}(\Sigma_{\mathcal{A},\mathcal{B}}, V_4, s_{00})$ isomorph ist. Daraus ergibt sich folgende Präsentation von $G(R)$:

Satz 2 (Theorem 4.27)**.** *Das quaternionische arithmetische Gitter $G(R)$ besitzt folgende Präsentation:*

$$G(R) = \left\langle b_1, b_2, c_1, c_2, d \;\middle|\; \begin{matrix} c_1^2,\ c_2^2,\ d^2,\ c_1c_2 = c_2c_1,\ c_1d = dc_1,\ c_2d = dc_2, \\ b_1b_2c_1b_2,\ b_1c_2b_1b_2^{-1},\ db_1db_1,\ db_2db_2 \end{matrix} \right\rangle .$$

Ferner lässt sich $G(R)$ auch interpretieren als orbitale Fundamentalgruppe von $\Sigma_{\mathcal{A},\mathcal{B}}$ unter der Wirkung von Z_2^3, die eine Fortsetzung der Wirkung von V_4 ist. Eine Konsequenz der Existenz der Z_2^3-Wirkung ist, dass man die Struktur der lokalen Permutationsgruppen für die V_4-Struktur $(\mathcal{A}, \mathcal{B})$ relativ einfach beschreiben kann.

Um nun die Fundamentalgruppe $\pi_1(\Sigma_{\mathcal{A},\mathcal{B}})$ als arithmetisches Gitter zu interpretieren, betrachten wir die Elemente $a_1 = c_1b_1$ und $a_2 = c_2b_2$. Es stellt sich heraus, dass unter dem Isomorphismus $G(R_1) \cong \pi_1^{\mathrm{orb}}(\Sigma_{\mathcal{A},\mathcal{B}}, V_4, s_{00})$ die Einbettung $\Gamma := \langle a_1, a_2 \rangle \hookrightarrow G(R_1)$ zur Einbettung $\pi_1(\Sigma_{\mathcal{A},\mathcal{B}}) \hookrightarrow \pi_1^{\mathrm{orb}}(\Sigma_{\mathcal{A},\mathcal{B}}, V_4, s_{00})$ isomorph ist. Somit ist Γ ein torsionsfreies arithmetisches Gitter mit $\Gamma\backslash(T_3 \times T_3) \cong \Sigma_{\mathcal{A},\mathcal{B}}$, d.h. Γ ist in der Tat das gesuchte arithmetische Gitter.

Nachdem das torsionsfreie arithmetische Gitter Γ gefunden worden ist, widmen wir uns der Konstruktion einer falschen Quadrik mittels nichtarchimedischer Uniformisierung im Kapitel 5. Dazu konstruieren wir ein formales Schema $\mathscr{Y}$ über $\mathrm{Spf}(\mathbb{F}_2[\![t]\!])$, dessen generische Faser die Drinfeld'sche obere Halbebene ist und dessen spezielle Faser einen dualen Graphen hat, der zu T_3 isomorph ist. Anschließend bilden wir den Quotienten

$$\mathscr{X}_\Gamma = \Gamma\backslash(\mathscr{Y} \times \mathscr{Y}).$$

Dieser besitzt eine ample Garbe und kann daher nach Grothendiecks formalem GAGA-Prinzip zu einem projektiven Schema X_Γ über $\mathbb{F}_2[\![t]\!]$ algebraisiert werden. Damit ist die generische Faser $X_{\Gamma,K}$ eine algebraische Fläche von allgemeinem Typ über $K = \mathbb{F}_2(\!(t)\!)$. In der Tat gilt sogar folgender Satz:

Satz 3 (Theorem 5.19)**.** *Sei $R = \mathbb{F}_2[\![t]\!]$, $K = \mathrm{Quot}(R) = \mathbb{F}_2(\!(t)\!)$ und $\Gamma \leq \mathrm{PGL}_2(K) \times \mathrm{PGL}_2(K)$ das Gitter aus Proposition 4.32. Dann ist $X_{\Gamma,K}$ eine falsche Quadrik über K.*

Dabei wird die Euler-Charakteristik anhand der speziellen Faser $X_{\Gamma,s}$ berechnet, deren dualer Komplex zum Quotientenkomplex $\Gamma\backslash(T_3 \times T_3)$ isomorph ist. Die Chern-Zahl $c_1(X_{\Gamma,K})^2$ ergibt sich aus der Selbstschnittzahl der relativen log-Differentialgarbe $\Omega^{2,\log}_{\mathscr{X}_\Gamma/R}$, und $c_2(X_{\Gamma,K})$ aus der Noether-Formel für Flächen. Die Albanese-Varietät berechnet sich mittels Kummer-étaler Kohomologie unter Verwendung der im Abschnitt 3.6 eingeführten lokalen Permutationsgruppen. Damit haben wir eine falsche Quadrik in Charakteristik 2 konstruiert.

Im Anhang A wird eine heuristische Methode erklärt, wie man ein solches Gitter finden könnte. Dazu beginnt man mit möglichen Quadratkomplexen, die vier Eckpunkte und $T_3 \times T_3$ als universelle Überlagerung haben. Zu einem solchen Quadratkomplex berechnet man die Fundamentalgruppe. Diese soll dann gewisse Eigenschaften arithmetischer Gruppen erfüllen, ansonsten wird der zugehörige Quadratkomplex ausgeschlossen. Nachdem man einen geeigneten Quadratkomplex gefunden hat, bestimmt man einen Homomorphismus der Fundamentalgruppe in eine geeignete Quaternionengruppe. Damit bekommt man eine in Frage kommende Quaternionenalgebra und müsste untersuchen, ob ein zugehöriges arithmetisches Gitter die gewünschte Eigenschaft erfüllt, d.h. ob es torsionsfrei ist und einen Quotientenkomplex mit vier Eckpunkten als Quotienten liefert.

Bisher ist der Quadratkomplex $\Sigma_{\mathcal{A},\mathcal{B}}$ der einzige bekannte Quadratkomplex, dessen Fundamentalgruppe als quaternionisches arithmetisches Gitter aufgefasst werden kann. Es bleibt noch offen, ob es einen weiteren solchen Quadratkomplex sowie ein weiteres solches Gitter gibt. Insbesondere ist es noch unbekannt, ob es eine weitere falsche Quadrik in Charakteristik 2 geben kann.