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Judit Abardia

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Flag area measures

A flag area measure on a finite-dimensional euclidean vector space is a continuous translation invariant valuation with values in the space of signed measures on the flag manifold consisting of a unit vector v and a $(p+1)$-dimensional linear subspace containing v, with $0 \leq p \leq n-1$.

In this talk, we will present a general construction of $\mathrm{SO}(n)$-covariant flag area measures, which generalizes an already known formula for flag area measures evaluated on polytopes. The construction involves arbitrary elementary symmetric polynomial in the squared cosines of the principal angles between two subspaces. The previously known case correspond to the special case where the elementary symmetric polynomial is the product.

We will also provide a classification result in the spirit of Hadwiger's theorem: After introducing a natural notion of smoothness, we show that every smooth $\mathrm{SO}(n)$-covariant flag area measure is a linear combination of the constructed ones. This is joint work with Andreas Bernig and Susanna Dann.

Semyon Alesker

Tel Aviv University

Theory of valuations and Monge-Ampère operators

Valuations on convex sets are finitely additive measures on convex compact sets. Supporting functional identifies convex bodies with 1-homogeneous convex functions. Under this identification, valuations on convex bodies get identified with valuations on 1-homogeneous convex functions.

The notion of a valuation on a class of functions was introduced only relatively recently. The goal of this talk is to study relations between valuations on convex bodies and on (not necessarily 1-homogeneous) convex functions. The main tool is the real Monge-Ampère operator which allows to produce many examples of continuous valuations on convex functions, and hence on convex bodies.

If time permits I will describe similar constructions obtained using complex Monge-Ampère operator, and also their quaternionic and octonionic versions introduced by the speaker.

Bo Berndtsson
Göteborgs Universitet

Symmetrization of plurisubharmonic and convex functions

If u is a real valued function in a domain in $\mathbb{R}^N$, its Schwarz symmetrization, $\hat{u}$, is a radial function, defined in a ball, that is equidistributed with u, so that for any real function

$$\int F(u) = \int F(\hat{u}).$$

The classical Polya-Szegö theorem says that the Newtonian energy, i e the L^2 norm of the gradient, decreases under symmetrization if u vanishes on the boundary of the domain. We will discuss similar results for plurisubharmonic or convex functions, with Monge-Ampere energy instead of Newtonian energy. The main result is - grosso modo - that symmetrization decresases Monge-Ampere energy if the domain we start with is a ball or an ellipsoid, but essentially only then. The proof of the symmetrization inequality uses (complex) Brunn-Minkowski inequalities. The proof of the converse direction uses a curious connection to complex geometry (Kähler-Einstein metrics) in the case of plurisubharmonic functions, and to the theorem of Jörgens-Calabi-Pogorelov in the case of convex functions.

This is joint work with Robert Berman.

Florian Besau
Goethe University Frankfurt

Weighted Floating Bodies of Polytopes

The floating body of a convex body is a classical affine construction that has proven very useful in different contexts. One particularly remarkable fact is, that the volume of the floating body gives rise to the *(equi-)affine surface area*. This fact was already established by Blaschke almost a century ago, when he introduced the concept of affine surface area for smooth convex bodies in two and three dimensions. Blaschkes result was gradually improved, until in 1990 C. Schütt and E. Werner were able to prove it for general convex bodies in all dimensions.

To construct the floating body of a convex body one has to consider the volume of caps, that is, intersections of the convex body with closed half-spaces. By replacing the volume with a weighted volume, by which we mean a measure with a positive and continuous density function, one obtains the *weighted floating body*.

For convex polytopes the affine surface area vanishes and C. Schütt was able to prove that the volume of the floating body of a convex polytope gives rise to another affine-invariant – the *flag-number* of the polytope, i.e., the number of all complete flags (or towers) that can be formed by the lower dimensional faces of the polytope. In this talk I will present our generalization of this result for the weighted floating body of a convex polytope.

Gabriele Bianchi
University of Florence

Existence and C^1 regularity for the L_p Minkowski problem, for $p < 1$

A convex body K (containing the origin o) solves the L_p Minkowski problem relative to some $p \in \mathbb{R}$ and to some measure μ on S^{n-1} if its L_p area measure is μ.

When $-n < p < 0$ and μ has a density f with respect to H^{n-1} for which f does a solution exist ? The C^1 regularity and strict convexity of K is granted by a result by L. Caffarelli as soon as o is in the interior of K, but this need not be the case when $p < 1$. What can we say in this case?

The result presented in this talk have been obtained in collaboration with K. Boroczky, A. Colesanti and D. Yang.

Andrea Colesanti
Università degli Studi di Firenze

Hessian valuations

A new class of continuous valuations on the space of convex functions on $\mathbb{R}^n$ is introduced. On smooth convex functions, they are defined for $i = 0, \ldots, n$, by

$$u \mapsto \int_{\mathbb{R}^n} \zeta(u(x), x, \nabla u(x))\,[\mathrm{D}^2 u(x)]_i dx$$

where $\zeta \in C(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n)$ and $[\mathrm{D}^2 u]_i$ is the i-th elementary symmetric function of the eigenvalues of the Hessian matrix, $\mathrm{D}^2 u$, of u. Under suitable assumptions on ζ, these valuations are shown to be invariant under translations and rotations on convex and coercive functions.

The results presented in this talk are obtained in collaboration with Monika Ludwig and Fabian Mussnig.

Joseph Fu
University of Georgia, Athens

Riemannian curvature measures

A famous theorem of Weyl states that if M is a compact submanifold of euclidean space, then the volumes of small tubes about M are given by a polynomial in the radius r, with coefficients that are expressible as integrals of certain scalar invariants of the curvature tensor of M with respect to the induced metric. It is natural to interpret this phenomenon in terms of valuations and curvature measures canonically associated to the Riemannian structure of M. The resulting apparatus takes the form of a certain abstract module over the polynomial algebra $\mathbb{R}[t]$ that reflects the behavior of Alesker multiplication. Applying it to isotropic M, this module encodes a key piece of the array of its associated kinematic formulas. We illustrate this principle in precise terms in the case where M is a complex space form. This is joint work with Thomas Wannerer.

Dmitry Faifman
University of Toronto

Symplectic valuation theory and contact manifolds

The Euclidean intrinsic volumes are a pivotal example of valuations which led to numerous important notions in convex, differential and integral geometry. Analogous canonic families of valuations were constructed fairly recently in different linear spaces, such as complex Hermitian space (Alesker, Bernig, Fu, Tasaki), pseudo-Euclidean space (Alesker, Bernig, Faifman), some quaternionic spaces (Bernig, Solanes) and a few others, all associated to representations of certain Lie groups. In this talk we let the real symplectic group be our guide. It will lead us to the Heisenberg algebra, where we construct some natural valuations analogous to the intrinsic volumes; we apply them to construct new invariants of manifolds in contact geometry, as well as to recover a Crofton formula for the gaussian curvature. We consider also the linear symplectic space, where the symplectic volumes substitute for the Euclidean intrinsic volumes, and present a Crofton formula in this setting.

Daniel Hug

KIT Karlsruhe

Integral geometry of tensorial curvature measures and applications

In various settings, integral-geometric formulas have been thoroughly studied, partly in view of applications in stochastic geometry. In recent years, integral-geometric formulas for tensor valuations and their local generalizations have come into focus. We describe new intersectional kinematic and Crofton formulas for tensor valuations and (generalized) tensor-valued curvature measures (joint work with Jan Weis). The structural understanding of these formulas has been prepared by joint work with Rolf Schneider on local tensor valuations, joint work with Andreas Bernig on translation invariant tensor valuations, and is connected to various other contributions. While most of these use tools from Fourier analysis, representation theory, differential geometry or the theory of valuations, we follow a more direct approach.

Jin Li

TU Vienna

$C(\mathbb{R}^n)$ valued valuations

Since any convex body (star body) can be identified by its support function (radial function), valuations taking values in the space of convex bodies (star bodies) are often studied as valuations taking values in some function spaces.

Especially, let $p \geq 1$ and $C_p(\mathbb{R}^n)$ be the space of p-homogeneous and continuous functions on $\mathbb{R}^n$. Associated with $\mathrm{SL}(n)$ contravariance or covariance, valuations taking values in $C_p(\mathbb{R}^n)$, where the addition is the ordinary function addition, are basically the same as L_p Minkowski valuations. Recent work of Ma and me showed that the Laplace transform on convex bodies is an additional example of $\mathrm{SL}(n)$ covariant valuations taking values in $C(\mathbb{R}^n)$, the space of continuous functions on $\mathbb{R}^n$. This suggest that there are more good operators which are $C(\mathbb{R}^n)$ valued valuations.

In this talk, I will show the classifications of measurable, $\mathrm{SL}(n)$ contravariant, $C(\mathbb{R}^n)$ valued valuations. A valuation Ψ mapping from convex bodies to $C(\mathbb{R}^n)$ is called $\mathrm{SL}(n)$ contravariant if $\Psi(\phi K)(x) = \Psi K(\phi^{-1}x)$ for any convex body K and $\phi \in \mathrm{SL}(n)$. All such valuations are linear combinations of $\mathrm{SL}(n)$ invariant real valued valuations, the support function of projection body and a new operator Ψ_ζ. Let $\zeta : \mathbb{R} \to \mathbb{R}$ be a continuous function and $N(P,o)$ be the normal cone of a polytope P at the origin (when P contains the origin in its interior, $N(P,o)$ is defined as the origin). Denote V_P be the cone-volume measure of P. The new operator

$$\Psi_\zeta(P)(x) = \int_{S^{n-1}\setminus N(P,o)} \zeta\left(\frac{x \cdot u}{h_P(u)}\right) dV_P(u),\ x \in \mathbb{R}^n$$

is the asymmetric extension of Orlicz mixed volume of the polytope P and the segment $[-x,x]$. When ζ is a p-homogeneous function on $\mathbb{R}$, $\Psi_\zeta(P)(\cdot)$ is the support function of asymmetric L_p projection body. It is also possible to extend Ψ_ζ continuously onto convex bodies when $\lim_{|t|\to\infty} \zeta(t)/t = 0$.

Monika Ludwig
TU Vienna

Affine Valuations on Function Spaces

A function Z defined on a space of real-valued functions $\mathcal{F}$ and taking values in an Abelian semigroup is called a *valuation* if

$$\mathrm{Z}(f \vee g) + \mathrm{Z}(f \wedge g) = \mathrm{Z}(f) + \mathrm{Z}(g) \qquad (1)$$

for all $f,g \in \mathcal{F}$ such that $f,g,f \vee g, f \wedge g \in \mathcal{F}$. Here $f \vee g$ is the pointwise maximum of f and g, while $f \wedge g$ is their pointwise minimum.

We discuss classification results of valuations with values in $\mathbb{R}$, in the space of convex bodies and in the space of measures with a focus on valuations that are $\mathrm{SL}(n)$ invariant or intertwine the $\mathrm{SL}(n)$.

(Based on joint work with Andrea Colesanti and Fabian Mussnig)

Emanuel Milman
Technion Haifa

Local L^p-Brunn–Minkowski inequalities for $p < 1$

The L^p-Brunn–Minkowski theory for $p \geq 1$, proposed by Firey and developed by Lutwak in the 90's, replaces the Minkowski addition of convex sets by its L^p counterpart, in which the support functions are added in L^p norm. Recently, Böröczky, Lutwak, Yang and Zhang have proposed to extend this theory further to encompass the range $p \in [0,1)$, conjecturing an L^p-Brunn–Minkowski inequality for origin-symmetric convex bodies in that range, which constitutes a strengthening of the classical Brunn-Minkowski inequality. Our main result confirms this conjecture locally for all (smooth) origin-symmetric convex bodies and $p \in [1 - \frac{c}{n^{3/2}}, 1]$.
Based on a joint work (very much in progress) with Alexander Kolesnikov.

Jan Rataj

Charles University Prague

Legendrian cycles, normal cycles and curvatures

Curvature measures of sufficiently regular subsets of the Euclidean space can be expressed by means of currents wihout boundary called normal cycles applied to certain universal differential forms (Zähle, 1986, for sets with positive reach). Fu observed that these currents have the important Legendrian property (they annihilate the contact 1-form), introduced the family of Legendrian cycles (currents) and formulated a very general definition of normal cycles attached to compact subsets of the Euclidean space. In the talk I will present an integral form of a Legendrian cycle involving local curvatures and mention some partial results concerning the first coordinate projection and support of a Legenrian cycle. The results were obtained jointly with Martina Zähle.

Matthias Reitzner

University of Osnabrück

Monotonicity of Random Polytopes

Choose n random points independently and according to a given density function in $\mathbb{R}^d$. We call the convex hull P_n of these points a random polytope.

Various contributions deal with the expectation of functionals of P_n, e.g. volume, intrinsic volumes, number of faces. In this talk we will report on recent results concerning the monotonicity of the expectation of these functionals.

Boris Rubin

Louisiana State University

Weighted norm estimates for Radon transforms and geometric inequalities

We obtain sharp inequalities for the k-plane transforms and the "j-plane to k-plane" transforms acting in L^p spaces on $\mathbb{R}^n$ with a radial power weight. The corresponding operator norms are explicitly evaluated. The results extend to Funk-type transforms on the sphere and Grassmann manifolds. As a consequence, we obtain new weighted estimates of volumes of planar sections of arbitrary sets in $\mathbb{R}^n$, volumes of star bodies and their central sections.

Rolf Schneider

Freiburg

Conic curvature measures and their integral geometry

Several recent applications of random models in convex optimization and signal demixing have brought into focus the following geometric question. "When does a randomly oriented cone strike a fixed cone?"

More precisely, let C, D be polyhedral convex cones in $\mathbb{R}^d$ and let θ be a uniform random rotation of $\mathbb{R}^d$; what is the probability that $C \cap \theta D \neq \{o\}$?

The well-known explicit answer involves the conic (or spherical) intrinsic volumes and the spherical (or conic) kinematic formula. This recent interest in conic integral geometry motivates us to have a closer look at the localized versions of the conic intrinsic volumes and their kinematic formulas. First we observe that the 'Master Steiner Formula' of McCoy and Tropp can be localized, allowing a convenient approach to conic support and curvature measures. It is used (among other things) to prove a Hölder continuity property of the conic support measures. Then we sketch a new approach to the kinematic formula for the conic curvature measures. Finally, we obtain some generalized integral-geometric results for a class of random polyhedral cones.

Franz Schuster

TU Vienna

Affine vs. Euclidean isoperimetric inequalities

In this talk we explain how every even, zonal measure on the Euclidean unit sphere gives rise to an isoperimetric inequality for sets of finite perimeter which directly implies the classical Euclidean isoperimetric inequality. The strongest member of this large family of inequalities is shown to be the only affine invariant one among them – the Petty projection inequality. As application, a family of sharp Sobolev inequalities for functions of bounded variation is obtained, each of which is stronger than the classical Sobolev inequality. Moreover, corresponding families of L_p isoperimetric and Sobolev type inequalities are also presented. (joint work with Christoph Haberl)

Gil Solanes
UA Barcelona

Integral geometry of isotropic spaces

The classical integral geometry of Blaschke, Santal, Chern and Federer takes place in euclidean space, but it is well-known that kinematic formulas exist whenever the ambient geometry has enough symmetries. More precisely, there are kinematic formulas expressible in terms of valuations, and also at the level of curvature measures, on any isotropic space; i.e. a Riemannian manifold under the isometric action of a group which is transitive on points and directions. These spaces have been classified and there is an ongoing program which aims to completely describe their integral geometry.

Thanks to a new approach developed by Bernig and Fu, and based on fundamental results by Alesker, this program has progressed significantly over the last years. In the talk we will report on its current status. In particular, we will describe the cases of complex space forms, the quaternionic plane, and the exceptional spheres S^6 and S^7, which is joint work with Bernig, Fu and Wannerer. Emphasis will be put in some unexpected phenomena that have been observed, as well as the methods, ideas, and questions that have emerged during these investigations.

Christoph Thäle
Ruhr-Universität Bochum

New probabilistic results for convex bodies in high dimensions

In our modern world systems that involve a huge amount of parameters can be found in various different places and constitute a challenge in many different ways. In many situation it is helpful to assume that the different combinations of values of the parameters one observes are points in a high-dimensional convex body. The last decade has seen a tremendous growth in this area. Far reaching results were obtained and various powerful techniques, mainly of a probabilistic flavor, have been developed. In this talk we shall first summarize a number of (by now classical) results about the geometry of high-dimensional convex bodies. In the second part of the talk we present a number of recent results for a particularly interesting class of convex bodies, the ℓ_p-balls. In particular, we discuss new (quantitative) central limit theorems and large deviation principles together with selected applications.

Alina Stancu
Concordia University Montreal

The L_p-Minkowski problem revisited

I will report on some new results on the L_p-Minkowski problem for $p < 1$ and connections to other topics in PDEs and geometric inequalities.

Wolfgang Weil
KIT Karlsruhe

Integral representations of mixed volumes

The notion of mixed volumes $V(K_1, \ldots, K_d)$ of convex bodies $K_1, \ldots, K_d$ in Euclidean space $\mathbb{R}^d$ is of central importance in the Brunn-Minkowski theory. Representations for mixed volumes are available in special cases, for example as integrals over the unit sphere with respect to mixed area measures. More generally, in Hug-Rataj-Weil (2013) a formula for $V(K[n], M[d - n]), n \in \{1, \ldots, d-1\}$, as a double integral over flag manifolds was established which involved certain flag measures of the convex bodies K and M (and required a general position of the bodies). In the talk, we discuss the general case $V(K_1[n_1], \ldots, K_k[n_k]), n_1 + \cdots + n_k = d$, and show a corresponding result involving the flag measures $\Omega_{n_1}(K_1; \cdot), \ldots, \Omega_{n_k}(K_k; \cdot)$. For this purpose, we first establish a curvature representation of mixed volumes over the normal bundles of the bodies involved. We also point out a connection of the latter result to a combinatorial formula of R. Schneider, in the case of polytopes and we mention an application to Boolean models.

Joint work with Daniel Hug (Karlsruhe) and Jan Rataj (Prague).

Elisabeth Werner
Case Western University Cleveland

Everything floating

Based on joint work with Olaf Mordhorst, we discuss a duality relation between floating and illumination bodies.

The definitions of these two bodies suggest that the polar of the floating body should be similar to the illumination body of the polar. We consider this question for the class of centrally symmetric convex bodies.

We provide precise estimates for l_p^n-balls and for centrally symmetric convex bodies with everywhere positive Gauss curvature.

We also investigate the problem for the class of centrally symmetric polytopes.

Our estimates show that equality of the polar of the floating body and the illumination body of the polar can only be achieved in the case of ellipsoids.

Time permitting, we will further discuss the notion of a floating function for log concave functions. This is based on joint work with Ben Li and related to affine invariants for log concave functions.

Artem Zvavitch

Kent State University

The convexification effect of Minkowski summation

For a compact subset A of $\mathbb{R}^n$, let $A(k)$ be the Minkowski sum of k copies of A, scaled by $1/k$. It is well known that $A(k)$ approaches the convex hull of A in Hausdorff distance as k goes to infinity. In this talk we will discuss how exactly $A(k)$ approaches the convex hull of A, and more generally, how a Minkowski sum of possibly different compact sets approaches convexity, as measured by various indices of non-convexity.

The non-convexity indices considered will include the Hausdorff distance induced by most general norm, the volume deficit (the difference of volumes), a non-convexity index introduced by Schneider (1975), and the effective standard deviation or inner radius. We will present relationships between those indices and move to discussion of monotonicity of convergence of $A(K)$ with respect to those indices. In particular, we will present a conjecture proposed, a few years ago, by Bobkov, Madiman and Wang, that the volume of $A(k)$ is non-decreasing in k, or in other words, that when the volume deficit between the convex hull of A and $A(k)$ goes to 0, it actually does so monotonically. While this conjecture holds true in dimension 1, we show that it fails in dimension 12 or greater. For other indices of non-convexity, we will present several positive results, including a strong monotonicity of Schneiders index in general dimension, and eventual monotonicity of the Hausdorff distance and effective standard deviation.

Along the way we will demonstrate relations of our results to Combinatorial Discrepancy Theory and Information theory.

The talk is based on a joint work with Mokshay Madiman, Matthieu Fradelizi and Arnaud Marsiglietti.