Program of the DaFra seminar

Unramified and Tamely Ramified Geometric Class Field Theory

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Introduction

Gemetric class field theory gives a geometric formulation and categorification of classical class field theory. It can be considered as the Geometric Langlands Program for GL_1. In this seminar we focus on curves over finite fields as here the connection to classical number theory is best visible. One can formulate (and prove) similar statements for curves over other fields such as the complex numbers.

Let us make this somewhat more precise: Let C be a smooth proper geometrically connected curve over a field k. Attaching to its divisor D on C the line bundle $\mathscr{O}_C(D)$, yields the Abel Jacobi map

$$\mathrm{AJ}\colon C(k) \longrightarrow \mathrm{Pic}(C), \qquad c \mapsto \mathscr{O}_C([c]).$$

This can (and has to) be made into a morphism of schemes $C \longrightarrow \mathrm{Pic}_{C/k}$. Now geometric class field theory states that we obtain an equivalence of categories

$$\mathrm{AJ}^*\colon \mathrm{CharLoc}(\mathrm{Pic}(C)) \xrightarrow{\sim} \mathrm{Loc}_1(C),$$

where $\mathrm{Loc}_1(C)$ is the category of rank 1 local ℓ-adic systems on C and $\mathrm{CharLoc}(\mathrm{Pic}(C))$ are character local systems, i.e. rank 1 local systems compatible with the group structure, on $\mathrm{Pic}_{C/k}$. By passing to open subcurves of C one obtains also ramified versions of (geometric) class field theory.

If k is a finite field, then this equivalence of categories induces (by considering isomorphism classes on both sides) the unramified class field theory from number theory. For this one uses Weil's adelic interpretation of line bundles on curves and the sheaf-function dictionary of Grothendieck.

The goal of this seminar is to understand this fascinating melange of very different techniques. We will focus on the unramified geometric class field theory and the careful explanation of all technical notions mentioned above (first 9 talks). The last three talks give an idea how to proceed in the ramified case and will necessarily be somehwat more sketchy.

A nice short overview of these topics is [Bha]. Good references are also the master theses of Tóth [Tót] or of Tendler [Ten] which contain a lot of details[1]. For simplicity, a lot of references are given to [Tót], even if the statements there only give further references.

Prerequisites: Some familiarity with basic algebraic geometry (as in Chapter II of [Har]) and with basic number theory (as in Chapter 1 and 2 of [Neu]) is assumed. Everything beyond this (such as Reimann-Roch for curves) should be explained in the talks to make them accessible to all participants. In particular, every participant is assumed to have an at least vague familiarity with the topics of talks 1.

Setting: A *global field* is by definition a finite extension of $\mathbb{Q}$ (characteristic 0) or a finite extension of $\mathbb{F}_p(T)$ for some prime p (characteristic p). A *local field* is the completion of a global field for some non-trivial absolute value. In characteristic p, local fields are therefore finite extension of $\mathbb{F}_p((T))$ (which are all isomorphic to some $\mathbb{F}_q((T')))$.

Of course many of the results hold more generally and every speaker should feel free to work in a more general setting.

1 Motivation: Reminder on class field theory and outlook

Torsten Wedhorn, Darmstadt, 26. Oktober

Give a short reminder (without proofs) on global and local CFT focussing on the characterstic p setting:

(0) We assume that every participant is familiar with the theory of extensions of local or global fields. But feel free to recall some properties.

(1) Recall the notion of adeles (as topological ring) and ideles (as topological group) of a global field (focussing on global fields of characteristic p).

(2) Recall (e.g. by [GoWd, 15.22]) the equivalence of the category[2] of smooth proper geometrically connected curves C over finite extensions of a field k (e.g., $k = \mathbb{F}_p$) and the category[3] of finitely generated extension extensions K of k of transcendence degree 1 (e.g., global fields of characteristic p). Explain the dictionary linking (closed) points c on C and equivalence classes of (nontrivial) valuations v of K, in particular $\mathrm{Frac}(\hat{\mathscr{O}}_{C,c}) = K_v$.

(3) Recall the statement of global class field theory and its unramified version ([Bha, Theorem 1]).

(4) Explain the idea of geometrization as in [Bha] without making precise the occuring notions.

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[1] But beware that they also contain some gaps and mistakes.
[2] Morphisms are non-constant morphisms of k-schemes.
[3] Morphisms are homomorphisms of fields.

2 The algebraic fundamental group I

Jens Hesse, Darmstadt, 26. Oktober

The goal of this talk is to define the algebraic fundamental group of a connected scheme X[4]. It is suggested to follow [Sti, 2.1, 2.2] as guide line and to fill in details from [Stacks, Tag 0BQ6], [Mil], [Lenstra], [SGA1], or [Tót, 1.4]. Note that we will not need the general notion of a Galois category and the speaker should specialize statements on Galois categories to the case at hand.

(1) Recall the notion of a finite étale morphism of schemes and introduce the category $\mathrm{Rev}(X)$ of (finite étale) coverings of X.

(2) Define the pro-finite fundamental group $\pi_1(X, \bar{x})$ (or the fundamental groupoid $\Pi_1(X)$ if you don't like base points) of X and show that $\mathrm{Rev}(X)$ is equivalent to the category of finite sets with continuous $\pi_1(X, \bar{x})$-action.

(3) Explain (and prove as much as possible of) the exact sequence

$$1 \to \pi_1(X \otimes_k k_s, \bar{x}) \to \pi_1(X, \bar{x}) \to \mathrm{Gal}(k_s/k) \to 1$$

for a quasi-compact geometrically integral scheme X over a field k with base point $\bar{x}$ ([Tót, 1.4.11], or all other references given above).

3 The algebraic fundamental group II

Michalis Neururer, Frankfurt, 2. November

This talk extends the results of the previous talk in less general situations. References are the same as for the previous talk.

(1) Show that for a normal integral scheme X its fundamental group is the Galois groups of the maximal X-unramified extension of the function field $K(X)$ (e.g., [Stacks, Tag 0BQM]). First explain all these notions. Again, feel free to consider only more special cases (still generalizing the case of a regular curve) if this makes things more transparent.

(2) Explain some examples from number theory, among them the special case $X = \mathrm{Spec}\, k$ for a field k and show that in this case the equivalence in (2) of Talk 2 yields the main theorem of Galois theory.

[4] Feel free to make simplifying assumptions on X as long as one can still specialize to connected curves over a field.

4 Representations of the fundamental group and ℓ-adic local systems

Matteo Costantini, Frankfurt, 2. November

The goal is to explain the equivalence of the category of continuous n-dimensional ℓ-adic representations of $\pi^1(C, \bar{c})$ and the category of n-dimensional ℓ-adic local systems, trivialized in $\bar{c}$. Reference is [Tót, 1.5], in particular Theorem 1.5.11.

(1) Recall the notion of an étale and an fppf-sheaf. Explain the notion of a locally constant étale sheaf. Show that for a connected scheme X and a geometric point $\bar{x}$ of X, the functor $\mathscr{F} \mapsto \mathscr{F}_{\bar{x}}$ yields an equivalence of the category of étale abelian sheaves with finite stalks and the category of finite abelian groups with a continuous action by $\pi_1(X, \bar{x})$ (e.g., [Mil] 6.16 or [Tót, 1.5]).

(2) Explain the notion of an ℓ-adic local system and (briefly!) some constructions (as in [Tót, 1.5.12]) and the above equivalence of categories.

(3) Specialize to $n = 1$ to obtain [Tót, 1.5.11].

5 Vector bundles on curves

Timo Henkel, Darmstadt, 23. November

Let C be a normal quasi-compact connected 1-dimensional scheme (a "curve"). The main goal of this talk is to explain for $n \geq 1$ the bijection

$$(*) \qquad \mathrm{GL}_n(K)\backslash \mathrm{GL}_n(\mathbb{A})/\mathrm{GL}_n(\mathcal{O}) \leftrightarrow \left\{ \begin{matrix} \text{isomorphism classes of} \\ \text{rank } n \text{ vector bundles on } C \end{matrix} \right\}.$$

References are for instance (in special cases, which easily can be generalized) [Tót, 1.1], [Fre, Lemma 2]. One can also follow the sketch below (or some mixture).

(1) Introduce the general setup. Let $|C|$ be the set of closed points of C. Recall $\mathscr{O}_c := \hat{\mathscr{O}}_{C,c}$, $K_c := \mathrm{Frac}(\mathscr{O}_c)$ (with $c \in |C|$), the ring of adeles $\mathbb{A} \subseteq \prod_{c \in |C|} K_c$ and its "ring of integers" $\mathcal{O} := \prod_{c \in |C|} \mathscr{O}_c$.

(2) Prove the Beauville-Laszlo theorem: Fix $c \in C$ a closed point and set $U := C \setminus \{c\}$. There is an equivalence of categories between the category of quasi-coherent modules $\mathcal{E}$ on C and the category of triples $(\mathcal{E}^\infty, \mathcal{E}_\infty, \alpha)$, where $\mathcal{E}^\infty$ is a quasi-coherent module on U, $\mathcal{E}_\infty$ is a quasi-coherent module over $\mathrm{Spec}(\hat{\mathscr{O}}_{C,c})$, and

$$\alpha \colon \mathcal{E}^\infty \times_{C \setminus \{c\}} \mathrm{Spec}(K_v) \xrightarrow{\sim} \mathcal{E}_\infty \otimes_{\hat{\mathscr{O}}_{C,c}} \mathrm{Spec}(K_v).$$

This equivalence induces an equivalence between vector bundles $\mathcal{E}$ of rank n and triples $(\mathcal{E}^\infty, \mathcal{E}_\infty, \alpha)$, where $\mathcal{E}^\infty$ and $\mathcal{E}_n\infty$ are vector bundles of rank n. One can deduce this from flat descent. Reference (in more general setting): [Stacks, Tag 05ES] (in the affine case) or [Stacks, Tag 0AF0] (in a very general situation).

(3) Generalize the Beauville-Laszlo theorem to $U = C \setminus \{c_1, \dots, c_r\}$ and deduce for $U = \operatorname{Spec} A$ affine a bijection

$$\mathrm{GL}_n(A)\backslash \prod_{i=1}^{r} \mathrm{GL}_n(K_{c_i})/\prod_{i=1}^{r} \mathrm{GL}_n(\mathscr{O}_{c_i}) \leftrightarrow \left\{ \begin{matrix} \text{isomorphism classes of} \\ \text{rank } n \text{ vector bundles } \mathcal{E} \text{ on } C \\ \text{with } \mathcal{E}_{|U} \text{ trivial} \end{matrix} \right\}.$$

Deduce the bijection (*) by passing to the colimit over all finite subsets of $|C|$. State also the case $n = 1$ explicitly.

6 The Picard scheme and the Abel-Jacobi map

Adrian Zorbach, Darmstadt, 23. November

The main goal is the construction of the Picard scheme and the Abel-Jacobi map for curves over a field.

(1) Let S be a scheme. Define the notion of an fppf sheaf on the category of S-schemes. For an S-scheme X define $\mathrm{Pic}_{X/S}$ as the fppf-sheaification of the functor $T \mapsto \mathrm{Pic}(X \times_S T)$.

Reference: [BLR, 8.1]

(2) Now let $S = \operatorname{Spec} k$, k a field. Let $X \to \operatorname{Spec} k$ proper and geometrically connected and geometrically reduced (and hence $\Gamma(X, \mathscr{O}_X) = k$). Show that if $X(k) \neq \emptyset$, then

$$\mathrm{Pic}_{X/k}(T) = \mathrm{Pic}(X \times_k T)/\operatorname{Pic}(T).$$

[BLR, 8.1, Prop. 4]

(3) Construct $\mathrm{Pic}_{C/k}$ if C is a smooth proper curve over a field k and the open and closed subgroup schemes $\mathrm{Pic}^d_{C/k}$ together with the Abel-Jacobi morphism ([Tót, 1.3]).

(4) Show, that $\mathrm{Pic}^0_{C/k}$ is an abelian variety of dimension the genus of C.

7 Grothendieck's Sheaf-Function Correspondence

Jonathan Zachhuber, Frankfurt, 7. Dezember

From now on we restrict ourselves to schemes over finite fields (or their algebraic closures). Let $k = \mathbb{F}_q$ be a finite field. Let G be a connected commutative algebraic group over k. The main goal is to explain the bijection

$$(**) \qquad \left\{ \begin{matrix} \ell\text{-adic character} \\ \text{sheaves on } G \end{matrix} \right\} \leftrightarrow \mathrm{Hom}_{\mathrm{Grp}}(G(\mathbb{F}_q), \bar{\mathbb{Q}}_\ell^\times).$$

Main reference [Tót, 1.6] and the references therein.

(1) Explain the notion of an ℓ-adic character sheaf (and maybe compare to the notion of an equivariant ℓ-adic sheaf).
(2) Explain Lang's theorem: If G is a connected algebraic group over $\mathbb{F}_q$. Then $g \mapsto gF(g)^{-1}$ is a finite étale Galois covering with Galois group $G(\mathbb{F}_q)$. Here $F\colon G \to G$ is the q-Frobenius.
(3) Prove the bijection (**).

8 Proof of the Main Theorem of Unramified Geometric CFT I

Matthias Nickel, Frankfurt, 7. Dezember

The content of this and the following talk is [Tót, 2]. For a nice sketch see [Bha, Theorem 6].

9 Proof of the Main Theorem of Unramified Geometric CFT II

Tobias Schedlmeier, Darmstadt, 11. Januar

10 Statement of the tamely ramified geometric class field theory

???, Darmstadt, 11. Januar

Explain the statement of (geometric and adelic) class field theory in the tamely ramified case ([Tót, 3], [Bha, Theorem 9]). In particular explain the diagram [Tót, (3.1.2)].

11 Proof of the Main Theorem of Tamely Ramified Geometric CFT I

Martin Lüdtke, Frankfurt, 25. Januar

The content of this and the following talk is a proof of tamely ramified geometric class field theory ([Tót, 4] and [Bha, Theorem 9/11]).

12 Proof of the Main Theorem of Tamely Ramified Geometric CFT II

Yingkun Li, Frankfurt, 25. Januar

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